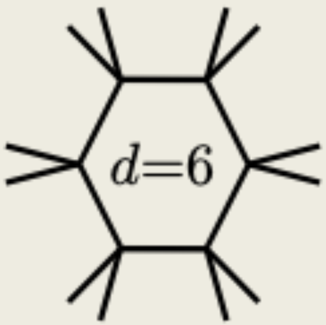
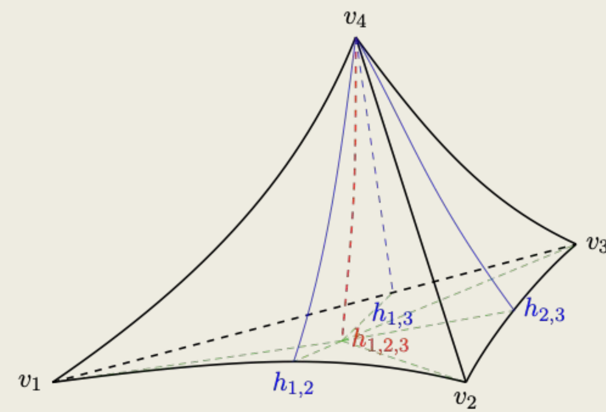
One-loop Integrals from Volumes of Orthoschemes



Anastasia Volovich

Brown and Harvard

Ren, Spradlin, Vergu



2306.04630

Amplitudes 2023, CERN, August 2023

I will explain how a recent mathematical result on volumes of orthoschemes by Rudenko 2012.05599 can be used to analytically evaluate one-loop scalar n-gon integrals in n-dimensions (for even n) with massless or massive internal and external edges.

Outline

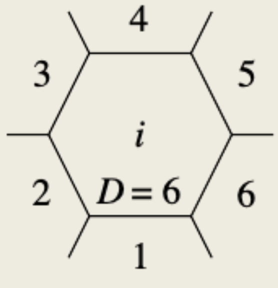
1. One-loop n-gon integral as a hyperbolic volume

2. Rudenko's formula for volume of hyperbolic orthoscheme

3. Dissecting a simplex into orthoschemes and our final formula

D=6 hexagon integral

Around a decade ago there was a brief flurry of interest in hexagon integrals in six dimensions.



Del Duca Duhr Smirnov 1104.2781

Dixon Drummond Henn 1104.2787

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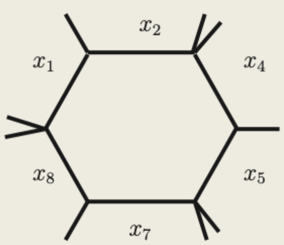


all with massless propagators

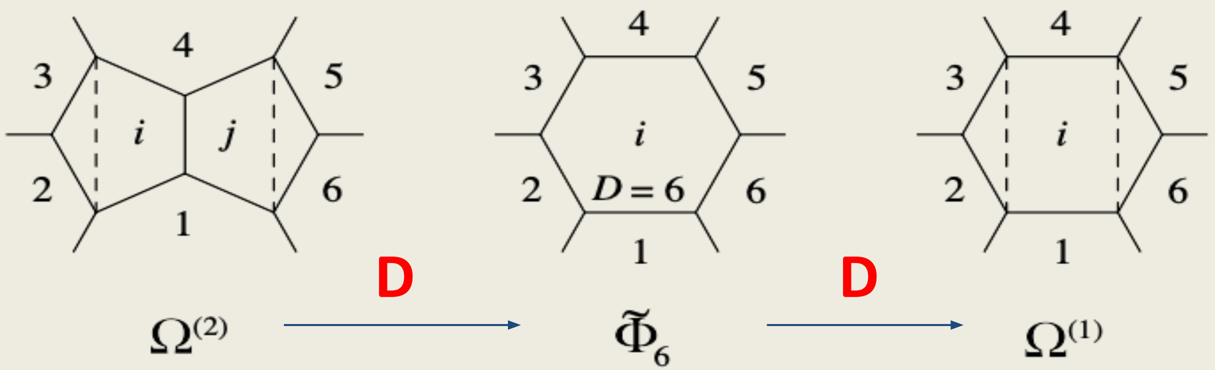
DDDDHS 1105.2011

Spradlin, AV 1105.2024

Hexagons

- Part of the motivation was simply that GSVV symbol technology 1006.5703 had made it possible to perform these previously difficult calculations–so people did them.
- But more practically, these integrals are related by simple differential equations to certain four-dimensional integrals of interest.

Dixon Drummond Henn 1104.2787



All-Mass n-gon Integrals in n Dimensions

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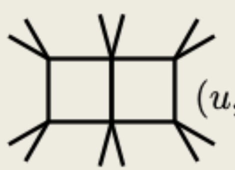
Abstract: We explore the correspondence between one-loop Feynman integrals and (hyperbolic) simplicial geometry to describe the *all-mass* case: integrals with generic external and internal masses. Specifically, we focus on n-particle integrals in exactly n space-time dimensions, as these integrals have particularly nice geometric properties and respect a dual conformal symmetry. In four dimensions, we leverage this geometric connection to give a concise dilogarithmic expression for the all-mass box in terms of the Murakami-Yano formula. In five dimensions, we use a generalized Gauss-Bonnet theorem to derive a similar dilogarithmic expression for the all-mass pentagon. We also use the Schläfli formula to write down the symbol of these integrals for all n. Finally, we discuss how the geometry behind these formulas depends on space-time signature, and we gather together many results related to these integrals from the mathematics and physics literature.

Hexagons

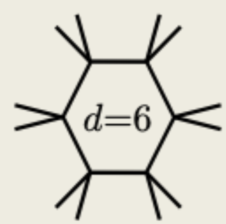
- **However, the problem of evaluating the "fully general" hexagon integral-with all external legs massive-remained unsolved; even with all propagators massless.**
- **This integral is of particular interest because it is related to a well studied example (the simplest with massless propagators) of an elliptic polylog integral.**

Paulos, Spradlin, AV 1203.6362

Morales, Spiering, Wilhelm, Yang, Zhang 2212.09762



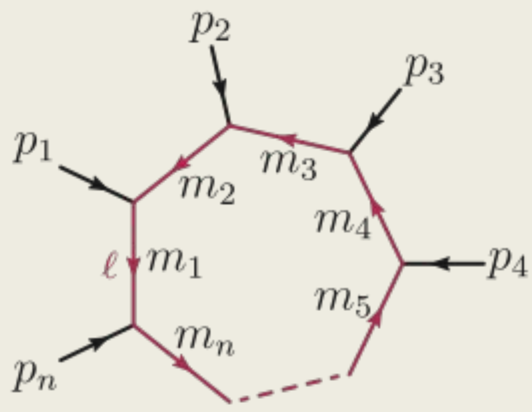
$$(u, \ldots) = -\frac{1}{2}\int_u^{+\infty}\frac{du'}{u'}$$



$$(u', \ldots)$$

n-gon integral in n-dimensions

the hexagon is a special case of
n-gon integral in n-dimensions



$$p_i^2 \neq 0,\ m_i \neq 0$$

$$\int d^n\ell \, \frac{1}{[\ell^2 + m_1^2]\,[(\ell - p_1)^2 + m_2^2] \cdots [(\ell - (p_1 + \cdots + p_{n-1}))^2 + m_n^2]}$$

n-gon integral in n-dimensions

The one-loop n-gon scalar integral in n-dimensions, with propagators having arbitrary masses, can be written in terms of dual momenta $p_i = x_{i+1} - x_i$

$$I_n(x_1, \ldots, x_n) = \int \frac{d^n x}{\pi^{n/2}} \prod_{i=1}^{n} \frac{1}{(x - x_i)^2 + m_i^2}$$

Introducing Feynman parameters and integrating x,

$$I_n(G_{ij}) = \Gamma\left(\frac{n}{2}\right) \int_0^\infty \prod_{i=1}^{n} d\alpha_i \, \delta(\alpha_n - 1) \left(\sum_{i,j=1}^{n} G_{ij} \alpha_i \alpha_j \right)^{-n/2}$$

$$G_{ij} = \frac{1}{2}((x_i - x_j)^2 + m_i^2 + m_j^2)$$

the integral can be viewed as function of G_{ij}.

n-gon integral as a hyperbolic volume

It has long been known [Davydychev, Delbourgo 9709216] that this integral computes the volume of a hyperbolic simplex

$$I_n(Q) = 2\,\Gamma\left(\frac{n}{2}\right)\frac{\mathrm{Vol}(Q)}{\sqrt{|\det Q|}}$$

$$\mathrm{Vol}(Q) = \frac{\sqrt{|\det Q|}}{2}\int_0^\infty \prod_{i=1}^n d\alpha_i\,\delta(\alpha_n - 1)\left(\sum_{i,j=1}^n Q_{ij}\alpha_i\alpha_j\right)^{-n/2}$$

where Q is the the Gram matrix associated to the simplex in n-1 dimensional hyperbolic space with vertices v_1, v_2… v_n

$$Q_{ij} = Q(v_i, v_j)$$

Outline

1. One-loop n-gon integral as a hyperbolic volume

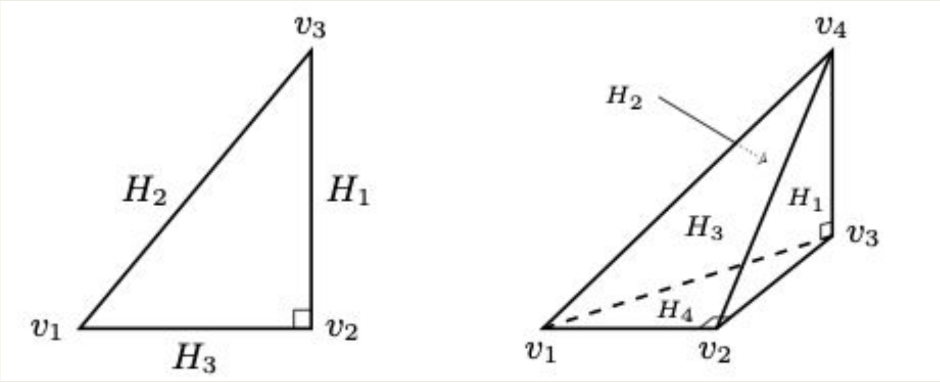
2. Rudenko's formula for volume of hyperbolic orthoscheme

3. Dissecting a simplex into orthoschemes and our final formula

Orthoscheme

An orthoscheme is a special kind of simplex that is basically a higher dimensional generalization of a right triangle:

An (n-1) dimensional hyperbolic orthoscheme is a hyperbolic symplex for which the bounding hyperplanes H_i can be ordered $(H_1,H_2,..H_n)$ in such a way that H_i is orthogonal to H_j for $|i-j|>1$



Volumes of Orthoschemes

Rudenko 2012.05599 gave an explicit formula for the volume of hyperbolic orthoschemes in terms of a new (to physics) class of functions called alternating polylogarithms.

$$\mathrm{ALi}_{m_1,\ldots,m_k}(\varphi_1,\ldots,\varphi_k) := \sum_{\epsilon_1,\ldots,\epsilon_k\in\{-1,1\}} \left(\prod_{i=1}^{k}\frac{\epsilon_i}{2}\right) \mathrm{Li}_{m_1,\ldots,m_k}(\epsilon_1\sqrt{\varphi_1},\ldots,\epsilon_k\sqrt{\varphi_k})$$

$$\mathrm{Vol}(Q) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\mathrm{per}\left(\mathrm{ALi}\big(T_{(0,\ldots,n+1)}(z)\big)\right)$$

Volumes of Orthoschemes

Rudenko 2012.05599 gave an explicit formula for the volume of hyperbolic orthoschemes in terms of a new (to physics) class of functions called alternating polylogarithms.

We can compute the volume of our full simplex by recursively slicing it into orthoschemes. [Spradlin, Ren, Vergu, AV]

Example 1:
3d orthoscheme/4d box

$$\mathrm{Vol}(\mathcal{Q}) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\mathrm{per}\left(\mathrm{ALi}\left(T_{(0,\ldots,n+1)}(z)\right)\right)$$

$$\mathrm{Vol}(\mathcal{Q}_{n=4}) = \frac{1}{4}\mathrm{per}\Big[\mathrm{ALi}_{1,1}\left(-\frac{z_{01}z_{25}}{z_{05}z_{12}}, -\frac{z_{23}z_{45}}{z_{25}z_{34}}\right) - \mathrm{ALi}_{1,1}\left(-\frac{z_{01}z_{45}}{z_{05}z_{14}}, -\frac{z_{14}z_{23}}{z_{12}z_{34}}\right) + \mathrm{ALi}_{1,1}\left(-\frac{z_{03}z_{45}}{z_{05}z_{34}}, -\frac{z_{01}z_{23}}{z_{03}z_{12}}\right)\Big]$$

Othoscheme geometry (z)

– Rudenko showed that there is a bijection between (n-1) dim orthoschemes and configurations $z=(z_0,z_1,..z_{n+1})$ in $M_{0,n+2}$; the moduli space of n+2 points in P^1.

– Under this bijection, Gram matrix of the orthoscheme can be expressed as

$$Q_{i,j} = Q_{j,i} = c_i c_j (z_0 - z_i)(z_j - z_{n+1}), \qquad 1 \leq i \leq j \leq n$$

– Inverting we can represent every cross-ratio in terms of Gram matrix

$$\frac{z_{a,b} z_{c,d}}{z_{a,d} z_{c,b}} = -\frac{Q_{b,c}^2 (Q_{a,b}^2 - Q_{a,a} Q_{b,b})(Q_{c,d}^2 - Q_{c,c} Q_{d,d})}{Q_{b,b} Q_{c,c} (Q_{a,d}^2 - Q_{a,a} Q_{d,d})(Q_{b,c}^2 - Q_{b,b} Q_{c,c})}$$

Arborification Map

A map T from z and any even subset P to the ordered set of words is defined recursively

$$T_P(z) = \begin{cases} \displaystyle\sum_{\substack{0<i<j<2n+1 \\ i \text{ is odd} \\ j \text{ is even}}} T_{(0,i,j,2n+1)} \otimes \left(T_{(0\cdots i)} \star T_{(i\cdots j)} \star T_{(j\cdots 2n+1)}\right) & \text{if } p_0 \text{ is even,} \\ \displaystyle\sum_{\substack{1<i<j<2n+2 \\ i \text{ is even} \\ j \text{ is odd}}} \Big\{ T_{(1,i,j,2n+2)} \otimes \left(T_{(1\cdots i)} \star T_{(i\cdots j)} \star T_{(j\cdots 2n+2)}\right) & \\ \qquad + T_{(1,i,j,2n+2)} \cdot \left(T_{(1\cdots i)} \star T_{(i\cdots j)} \star T_{(j\cdots 2n+2)}\right) \Big\} & \text{if } p_0 \text{ is odd,} \end{cases}$$

where

$$T_{(p_0,p_1,p_2,p_3)}(z) = \begin{cases} -\left[\dfrac{z_{p_0,p_1} z_{p_2,p_3}}{z_{p_0,p_3} z_{p_2,p_1}}, 1\right] & \text{if } p_0 \text{ is even,} \\ \left[\dfrac{z_{p_0,p_3} z_{p_2,p_1}}{z_{p_0,p_1} z_{p_2,p_3}}, 1\right] & \text{if } p_0 \text{ is odd} \end{cases}$$

$$T_{(p_0,p_1)}(z) = 1\,.$$

Aborification Map (T)

– **Weighted letters** $[\varphi, m]$

– **Words** $[\varphi_1, m_1] \otimes [\varphi_1, m_1]$, or $[\varphi_1, m_1 | \varphi_1, m_1]$

– **Star product**

$$([\varphi_1, m_1] \otimes \omega_1) \star ([\varphi_2, m_2] \otimes \omega_2) := [\varphi_1, m_1] \otimes ((\omega_1) \star ([\varphi_2, m_2] \otimes \omega_2)) + [\varphi_2, m_2] \otimes (([\varphi_1, m_1] \otimes \omega_1) \star (\omega_2)) + ([\varphi_1, m_1] \cdot [\varphi_2, m_2]) \otimes (\omega_1 \star \omega_2)$$

– **Alternating polylogs associated to a word**

$$\mathrm{ALi}([\varphi_1, m_1 | \cdots | \varphi_k, m_k]) := \sum_{\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}} \left(\prod_{i=1}^{k} \frac{\epsilon_i}{2} \right) \mathrm{Li}_{m_1, \dots, m_k}(\epsilon_1 \sqrt{\varphi_1}, \dots, \epsilon_k \sqrt{\varphi_k})$$

Example 1:
3d orthoscheme/4d box

$$P = (0,1,2,3,4,5)$$

$$T_P(x) = \left[\frac{z_{01}z_{25}}{z_{05}z_{21}}, 1 \,\middle|\, \frac{z_{23}z_{45}}{z_{25}z_{43}}, 1\right] - \left[\frac{z_{01}z_{45}}{z_{05}z_{41}}, 1 \,\middle|\, \frac{z_{14}z_{32}}{z_{12}z_{34}}, 1\right] + \left[\frac{z_{03}z_{45}}{z_{05}z_{43}}, 1 \,\middle|\, \frac{z_{01}z_{23}}{z_{03}z_{21}}, 1\right]$$

$$\mathrm{Vol}(\mathcal{Q}) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\mathrm{per}\left(\mathrm{ALi}\left(T_{(0,\dots,n+1)}(z)\right)\right)$$

$$\mathrm{Vol}(\mathcal{Q}_{n=4}) = \frac{1}{4}\,\mathrm{per}\Big[\mathrm{ALi}_{1,1}\left(-\frac{z_{01}z_{25}}{z_{05}z_{12}}, -\frac{z_{23}z_{45}}{z_{25}z_{34}}\right) - \mathrm{ALi}_{1,1}\left(-\frac{z_{01}z_{45}}{z_{05}z_{14}}, -\frac{z_{14}z_{23}}{z_{12}z_{34}}\right) + \mathrm{ALi}_{1,1}\left(-\frac{z_{03}z_{45}}{z_{05}z_{34}}, -\frac{z_{01}z_{23}}{z_{03}z_{12}}\right)\Big]$$

Example 2:
5d orthoscheme/6d hexagon

$$
\begin{aligned}
\mathrm{Vol}(\mathcal{Q}_{n=6}) = & \frac{1}{16}\,\mathrm{per}\Bigg[\mathrm{ALi}_{1,2}\left(-\frac{z_{01}z_{47}}{z_{07}z_{14}}, \frac{z_{14}z_{23}z_{45}z_{67}}{z_{12}z_{34}z_{47}z_{56}}\right) - \mathrm{ALi}_{1,2}\left(-\frac{z_{01}z_{67}}{z_{07}z_{16}}, \frac{z_{16}z_{23}z_{45}}{z_{12}z_{34}z_{56}}\right) \\
& - \mathrm{ALi}_{1,2}\left(-\frac{z_{03}z_{47}}{z_{07}z_{34}}, \frac{z_{01}z_{23}z_{45}z_{67}}{z_{03}z_{12}z_{47}z_{56}}\right) + \mathrm{ALi}_{1,2}\left(-\frac{z_{03}z_{67}}{z_{07}z_{36}}, \frac{z_{01}z_{23}z_{36}z_{45}}{z_{03}z_{12}z_{34}z_{56}}\right) \\
& -\mathrm{ALi}_{1,1,1}\left(-\frac{z_{01}z_{27}}{z_{07}z_{12}}, -\frac{z_{23}z_{47}}{z_{27}z_{34}}, -\frac{z_{45}z_{67}}{z_{47}z_{56}}\right) + \mathrm{ALi}_{1,1,1}\left(-\frac{z_{01}z_{27}}{z_{07}z_{12}}, -\frac{z_{23}z_{67}}{z_{27}z_{36}}, -\frac{z_{36}z_{45}}{z_{34}z_{56}}\right) \\
& -\mathrm{ALi}_{1,1,1}\left(-\frac{z_{01}z_{27}}{z_{07}z_{12}}, -\frac{z_{25}z_{67}}{z_{27}z_{56}}, -\frac{z_{23}z_{45}}{z_{25}z_{34}}\right) + \mathrm{ALi}_{1,1,1}\left(-\frac{z_{01}z_{47}}{z_{07}z_{14}}, -\frac{z_{14}z_{23}}{z_{12}z_{34}}, -\frac{z_{45}z_{67}}{z_{47}z_{56}}\right) \\
& +\mathrm{ALi}_{1,1,1}\left(-\frac{z_{01}z_{47}}{z_{07}z_{14}}, -\frac{z_{45}z_{67}}{z_{47}z_{56}}, -\frac{z_{14}z_{23}}{z_{12}z_{34}}\right) - \mathrm{ALi}_{1,1,1}\left(-\frac{z_{01}z_{67}}{z_{07}z_{16}}, -\frac{z_{16}z_{23}}{z_{12}z_{36}}, -\frac{z_{36}z_{45}}{z_{34}z_{56}}\right) \\
& +\mathrm{ALi}_{1,1,1}\left(-\frac{z_{01}z_{67}}{z_{07}z_{16}}, -\frac{z_{16}z_{25}}{z_{12}z_{56}}, -\frac{z_{23}z_{45}}{z_{25}z_{34}}\right) - \mathrm{ALi}_{1,1,1}\left(-\frac{z_{01}z_{67}}{z_{07}z_{16}}, -\frac{z_{16}z_{45}}{z_{14}z_{56}}, -\frac{z_{14}z_{23}}{z_{12}z_{34}}\right) \\
& -\mathrm{ALi}_{1,1,1}\left(-\frac{z_{03}z_{47}}{z_{07}z_{34}}, -\frac{z_{01}z_{23}}{z_{03}z_{12}}, -\frac{z_{45}z_{67}}{z_{47}z_{56}}\right) - \mathrm{ALi}_{1,1,1}\left(-\frac{z_{03}z_{47}}{z_{07}z_{34}}, -\frac{z_{45}z_{67}}{z_{47}z_{56}}, -\frac{z_{01}z_{23}}{z_{03}z_{12}}\right) \\
& +\mathrm{ALi}_{1,1,1}\left(-\frac{z_{03}z_{67}}{z_{07}z_{36}}, -\frac{z_{01}z_{23}}{z_{03}z_{12}}, -\frac{z_{36}z_{45}}{z_{34}z_{56}}\right) + \mathrm{ALi}_{1,1,1}\left(-\frac{z_{03}z_{67}}{z_{07}z_{36}}, -\frac{z_{36}z_{45}}{z_{34}z_{56}}, -\frac{z_{01}z_{23}}{z_{03}z_{12}}\right) \\
& -\mathrm{ALi}_{1,1,1}\left(-\frac{z_{05}z_{67}}{z_{07}z_{56}}, -\frac{z_{01}z_{25}}{z_{05}z_{12}}, -\frac{z_{23}z_{45}}{z_{25}z_{34}}\right) + \mathrm{ALi}_{1,1,1}\left(-\frac{z_{05}z_{67}}{z_{07}z_{56}}, -\frac{z_{01}z_{45}}{z_{05}z_{14}}, -\frac{z_{14}z_{23}}{z_{12}z_{34}}\right) \\
& -\mathrm{ALi}_{1,1,1}\left(-\frac{z_{05}z_{67}}{z_{07}z_{56}}, -\frac{z_{03}z_{45}}{z_{05}z_{34}}, -\frac{z_{01}z_{23}}{z_{03}z_{12}}\right)\Bigg].
\end{aligned}
$$

$$\mathrm{Vol}(\mathcal{Q}) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\mathrm{per}\left(\mathrm{ALi}\left(T_{(0,\ldots,n+1)}(z)\right)\right)$$

Real periods (per)

The real period can be defined via coproduct structure [Goncharov 1996]

$$\mathrm{per}(G) := (2\pi)^{w(G)} \sum_{k=1}^{w(G)} \sum_{a_1+\cdots+a_k=w(G)} (-1)^{k-1} \nabla(\Delta_{a_1,\cdots,a_k}(G))$$

$$\nabla\left(\bigotimes_{i=1}^{m} f_i\right) := \mathrm{Re}\left(\prod_{i=1}^{m-1} \frac{f_i}{(2\pi i)^{w(f_i)}}\right) \mathrm{Im}\left(\frac{f_m}{(2\pi i)^{w(f_m)}}\right)$$

w is the weight

coproduct $\Delta_{w(G)}(G) = G$

Examples

Weight One

$$\Delta_1(\log(x)) = \log(x)$$

$$\mathrm{per}(\log(x)) = -2\pi\nabla(\log(x)) = 2\pi \ \mathrm{Im}\left(\frac{\log(x)}{2\pi i}\right) = -\,\mathrm{Re}(\log(x)) = -\log(|x|)$$

Weight Two

$$\Delta_2(\mathrm{Li}_2(x)) = \mathrm{Li}_2(x)\,, \qquad \Delta_{1,1}(\mathrm{Li}_2(x)) = \mathrm{Li}_1(x)\otimes\log(x)$$

$$\begin{aligned}\mathrm{per}(\mathrm{Li}_2(x)) &= 4\pi^2\left\{\nabla(\mathrm{Li}_2(x)) - \nabla\left(\mathrm{Li}_1(x)\otimes\log(x)\right)\right\}\\ &= 4\pi^2\left\{\mathrm{Im}\left(\frac{\mathrm{Li}_2(x)}{(2\pi i)^2}\right) - \mathrm{Re}\left(\frac{\mathrm{Li}_1(x)}{2\pi i}\right)\ \mathrm{Im}\left(\frac{\log(x)}{2\pi i}\right)\right\}\\ &= -\,\mathrm{Im}\ \mathrm{Li}_2(x) + \mathrm{Im}\ \mathrm{Li}_1(x)\ \mathrm{Re}\log(x)\end{aligned}$$

$$\begin{aligned}\mathrm{per}(\mathrm{Li}_{1,1}(x,y)) = -\,&\mathrm{Im}\ \mathrm{Li}_{1,1}(x,y) - \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Im}\ \mathrm{Li}_1(xy)\\ &+ \mathrm{Im}\ \mathrm{Li}_1(y)\ \mathrm{Re}\ \mathrm{Li}_1(x) + \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Im}\ \mathrm{Li}_1(xy)\end{aligned}$$

Weight Three

$$\mathrm{per}(\mathrm{Li}_3(x)) = \mathrm{Re}\ \mathrm{Li}_3(x) + (\mathrm{Re}\log(x))^2\ \mathrm{Re}\ \mathrm{Li}_1(x) - \mathrm{Re}\log(x)\ \mathrm{Re}\ \mathrm{Li}_2(x)\,,$$

$$\begin{aligned}
\mathrm{per}(\mathrm{Li}_{1,2}(x,y)) &= \mathrm{Re}\ \mathrm{Li}_{1,2}(x,y) + \mathrm{Re}\log(1/y)\ \mathrm{Re}\ \mathrm{Li}_{1,1}(x,y)\\
&- \mathrm{Im}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\log(1/y)\ \mathrm{Im}\ \mathrm{Li}_1(xy) + \mathrm{Im}\ \mathrm{Li}_1(y)\ \mathrm{Re}\log(1/y)\ \mathrm{Im}\ \mathrm{Li}_1(xy)\\
&+ \mathrm{Im}\ \mathrm{Li}_1(1/x)\ \mathrm{Im}\ \mathrm{Li}_1(xy)\ \mathrm{Re}\log(1/xy) - \mathrm{Im}\ \mathrm{Li}_2(1/x)\ \mathrm{Im}\ \mathrm{Li}_1(xy)\\
&+ \mathrm{Im}\ \mathrm{Li}_2(y)\ \mathrm{Im}\ \mathrm{Li}_1(xy) - \mathrm{Re}\ \mathrm{Li}_1(x)\ \mathrm{Re}\ \mathrm{Li}_2(y) + \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\ \mathrm{Li}_2(xy)\\
&- 2\ \mathrm{Re}\ \mathrm{Li}_1(x)\ \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\log(1/y) + \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\log(1/y)\ \mathrm{Re}\ \mathrm{Li}_1(xy)\\
&+ \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\ \mathrm{Li}_1(xy)\ \mathrm{Re}\log(1/xy) - \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\log(1/y)\ \mathrm{Re}\ \mathrm{Li}_1(xy)\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{per}(\mathrm{Li}_{2,1}(x,y)) &= \mathrm{Re}\ \mathrm{Li}_{2,1}(x,y) - \mathrm{Re}\log(1/y)\ \mathrm{Re}\ \mathrm{Li}_{1,1}(x,y)\\
&+ \mathrm{Re}\log(1/xy)\ \mathrm{Re}\ \mathrm{Li}_{1,1}(x,y) - \mathrm{Im}\ \mathrm{Li}_1(x)\ \mathrm{Im}\ \mathrm{Li}_1(y)\ \mathrm{Re}\log(1/y)\\
&+ \mathrm{Im}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\log(1/y)\ \mathrm{Im}\ \mathrm{Li}_1(xy) - \mathrm{Im}\ \mathrm{Li}_1(y)\ \mathrm{Re}\log(1/y)\ \mathrm{Im}\ \mathrm{Li}_1(xy)\\
&+ \mathrm{Im}\ \mathrm{Li}_1(x)\ \mathrm{Im}\ \mathrm{Li}_1(y)\ \mathrm{Re}\log(1/xy) + \mathrm{Im}\ \mathrm{Li}_1(1/x)\ \mathrm{Im}\ \mathrm{Li}_1(xy)\ \mathrm{Re}\log(1/xy)\\
&+ \mathrm{Im}\ \mathrm{Li}_2(x)\ \mathrm{Im}\ \mathrm{Li}_1(y) - \mathrm{Im}\ \mathrm{Li}_2(y)\ \mathrm{Im}\ \mathrm{Li}_1(xy) + \mathrm{Im}\ \mathrm{Li}_1(xy)\ \mathrm{Im}\ \mathrm{Li}_2(1/x)\\
&- \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\ \mathrm{Li}_2(xy) + \mathrm{Re}\ \mathrm{Li}_1(x)\ \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\log(1/y)\\
&- \mathrm{Re}\ \mathrm{Li}_1(x)\ \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\log(1/xy) - \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\log(1/y)\ \mathrm{Re}\ \mathrm{Li}_1(xy)\\
&+ \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\ \mathrm{Li}_1(xy)\ \mathrm{Re}\log(1/xy) + \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\log(1/y)\ \mathrm{Re}\ \mathrm{Li}_1(xy)\\
&- 2\ \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\ \mathrm{Li}_1(xy)\ \mathrm{Re}\log(1/xy)\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{per}(\mathrm{Li}_{1,1,1}(x,y,z)) &= \mathrm{Im}\ \mathrm{Li}_1(z)\ \mathrm{Im}\ \mathrm{Li}_{1,1}(x,y) - \mathrm{Im}\ \mathrm{Li}_{1,1}(y,1/xy)\ \mathrm{Im}\ \mathrm{Li}_1(xyz)\\
&+ \mathrm{Im}\ \mathrm{Li}_{1,1}(y,z)\ \mathrm{Im}\ \mathrm{Li}_1(xyz) + \mathrm{Re}\ \mathrm{Li}_1(1/y)\ \mathrm{Re}\ \mathrm{Li}_{1,1}(x,yz)\\
&- \mathrm{Re}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_{1,1}(x,yz) - \mathrm{Re}\ \mathrm{Li}_1(x)\ \mathrm{Re}\ \mathrm{Li}_{1,1}(y,z)\\
&+ \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\ \mathrm{Li}_{1,1}(xy,z) - \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\ \mathrm{Li}_{1,1}(xy,z) + \mathrm{Re}\ \mathrm{Li}_{1,1,1}(x,y,z)\\
&+ \mathrm{Im}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Im}\ \mathrm{Li}_1(xy) - \mathrm{Im}\ \mathrm{Li}_1(y)\ \mathrm{Im}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_1(x)\\
&- \mathrm{Im}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\ \mathrm{Li}_1(1/y)\ \mathrm{Im}\ \mathrm{Li}_1(xyz)\\
&+ \mathrm{Re}\ \mathrm{Li}_1(1/y)\ \mathrm{Im}\ \mathrm{Li}_1(yz)\ \mathrm{Im}\ \mathrm{Li}_1(xyz) - \mathrm{Im}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Im}\ \mathrm{Li}_1(xy)\\
&+ \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Im}\ \mathrm{Li}_1(1/xy)\ \mathrm{Im}\ \mathrm{Li}_1(xyz) - \mathrm{Im}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Im}\ \mathrm{Li}_1(xyz)\\
&+ \mathrm{Im}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\ \mathrm{Li}_1(1/xy)\ \mathrm{Im}\ \mathrm{Li}_1(xyz) - \mathrm{Re}\ \mathrm{Li}_1(z)\ \mathrm{Im}\ \mathrm{Li}_1(yz)\ \mathrm{Im}\ \mathrm{Li}_1(xyz)\\
&+ \mathrm{Re}\ \mathrm{Li}_1(x)\ \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\ \mathrm{Li}_1(z) - \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_1(xy)\\
&+ \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_1(xy) - 2\ \mathrm{Re}\ \mathrm{Li}_1(x)\ \mathrm{Re}\ \mathrm{Li}_1(1/y)\ \mathrm{Re}\ \mathrm{Li}_1(yz)\\
&+ 2\ \mathrm{Re}\ \mathrm{Li}_1(x)\ \mathrm{Re}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_1(yz) + \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\ \mathrm{Li}_1(1/y)\ \mathrm{Re}\ \mathrm{Li}_1(xyz)\\
&- \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\ \mathrm{Li}_1(1/xy)\ \mathrm{Re}\ \mathrm{Li}_1(xyz)\\
&+ \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\ \mathrm{Li}_1(1/xy)\ \mathrm{Re}\ \mathrm{Li}_1(xyz)\\
&- 2\ \mathrm{Re}\ \mathrm{Li}_1(1/x)\ \mathrm{Re}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_1(xyz) + \mathrm{Re}\ \mathrm{Li}_1(y)\ \mathrm{Re}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_1(xyz)\\
&- \mathrm{Re}\ \mathrm{Li}_1(1/y)\ \mathrm{Re}\ \mathrm{Li}_1(yz)\ \mathrm{Re}\ \mathrm{Li}_1(xyz) + \mathrm{Re}\ \mathrm{Li}_1(z)\ \mathrm{Re}\ \mathrm{Li}_1(yz)\ \mathrm{Re}\ \mathrm{Li}_1(xyz)\,.
\end{aligned}$$

$$\mathrm{per}(G(\vec{a};z)) = \begin{cases} -\dfrac{i}{2}\,\mathrm{Re}\,\mathrm{sv}(G(\vec{a};z))\,, & |\vec{a}| \text{ is odd,} \\ -\dfrac{i}{2}\,\mathrm{Im}\,\mathrm{sv}(G(\vec{a};z))\,, & |\vec{a}| \text{ is even.} \end{cases}$$

Brown 2004

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Outline

1. One-loop n-gon integral as a hyperbolic volume

2. Rudenko’s formula for volume of hyperbolic orthoscheme

3. Dissecting a simplex into orthoschemes and our final formula

Dissecting a simplex into orthoschemes

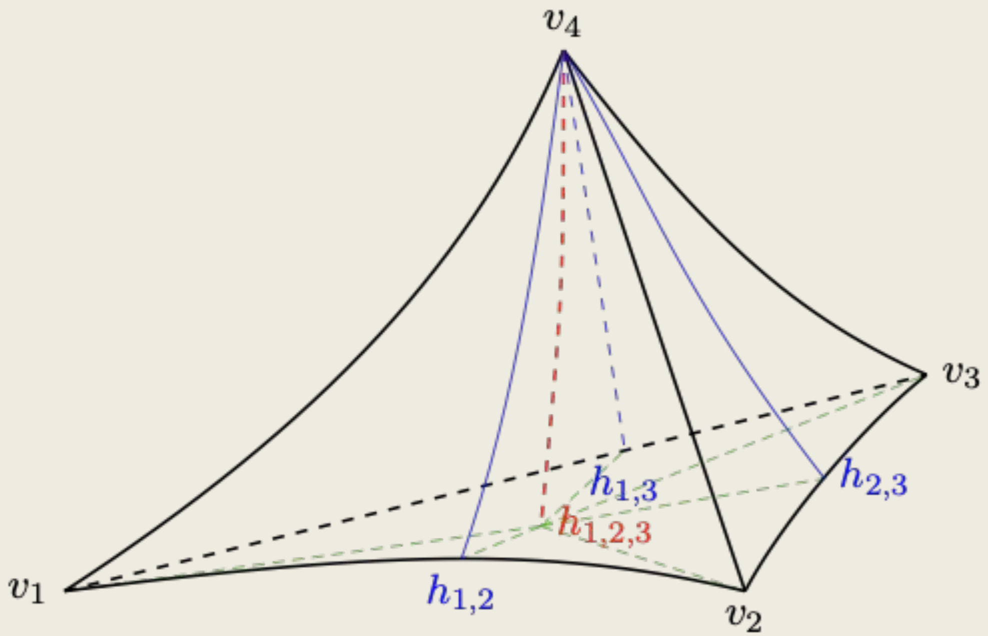

Figure 1. Dissection of a hyperbolic tetrahedron into six orthoschemes. The point h_I denotes the point where the altitude from v_4 intersects the boundary $\langle I \rangle$, defined as the convex hull of vertices $\{v_i | \, i \in I\}$. The six orthoschemes are $\{\mathrm{conv}(v_4, h_{1,2,3}, h_{i,j}, v_i) | \, i, j \in \{1, 2, 3\}, \, i \neq j\}$.

Explicit Formula

[Spradlin, Ren, Vergu, AV]

$$\mathrm{Vol}(Q) = \sum_{\sigma \in S_{n-1}} \frac{\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\det Q^\sigma)}{\mathrm{sgn}(\det Q)} \mathrm{Vol}(Q^\sigma)$$

$$Q^\sigma := \mathrm{conv}(v_{\sigma(1)}, h_{\sigma(1),\sigma(2)}, h_{\sigma(1),\sigma(2),\sigma(3)}, \ldots, h_{\sigma(1),\sigma(2),\cdots,\sigma(n-2)}, h_{\sigma(1),\sigma(2),\cdots,\sigma(n-1)} = h_{1,2,\cdots,n-1}, v_n)$$

$$z_i^\sigma = 1 - \frac{\det Q[\sigma(1),\ldots,\sigma(i),n]}{\det Q[\sigma(1),\ldots,\sigma(i)]\, Q_{n,n}}, \qquad 1 \leq i \leq n-1$$

$z_0^\sigma = 0$, $z_n^\sigma = 1$, and $z_{n+1}^\sigma = \infty$

Box Integral

$$\mathrm{Vol}(Q_{n=4}) = \frac{1}{4}\Bigg\{\mathrm{sgn}\left(\frac{(Q_{1,2}Q_{1,4}-Q_{1,1}Q_{2,4})\bar{Q}_{3,4}}{(Q_{1,2}Q_{1,4}-Q_{2,2}Q_{1,4}-Q_{1,1}Q_{2,4}+Q_{1,2}Q_{2,4})(\bar{Q}_{1,4}+\bar{Q}_{2,4}+\bar{Q}_{3,4})}\right)$$

$$\times\,\mathrm{per}\Bigg(\mathrm{ALi}_{1,1}\left[1-\frac{Q_{4,4}\bar{Q}_{4,4}}{\Delta^2},\frac{Q_{1,4}^2\bar{Q}_{3,4}^2}{(Q_{1,2}Q_{1,4}-Q_{1,1}Q_{2,4})^2(\Delta^2-Q_{4,4}\bar{Q}_{4,4})}\right]$$

$$+\mathrm{ALi}_{1,1}\left[\frac{Q_{1,4}^2(Q_{1,2}^2-Q_{1,1}Q_{2,2})}{(Q_{1,2}Q_{1,4}-Q_{1,1}Q_{2,4})^2},\frac{\bar{Q}_{3,4}^2}{\Delta^2(Q_{1,2}^2-Q_{1,1}Q_{2,2})}\right]$$

$$-\mathrm{ALi}_{1,1}\left[\frac{Q_{1,4}^2}{Q_{1,4}^2-Q_{1,1}Q_{4,4}},\frac{(Q_{1,4}^2-Q_{1,1}Q_{4,4})\bar{Q}_{3,4}^2}{\Delta^2(Q_{1,2}Q_{1,4}-Q_{1,1}Q_{2,4})^2}\right]\Bigg)\Bigg\}$$

$$+\,(\mathrm{perm}\ 1,2,3)\,,$$

$$\bar{Q}_{i,j} := (-1)^{i+j}\det Q[1\cdots\hat{\imath}\cdots n;1\cdots\hat{\jmath}\cdots n]\,,\quad \Delta := \det Q$$

In the massless limit: $Q_{i,i} \longrightarrow 0$

$$\mathrm{Vol}(Q^{\mathrm{ideal}}_{n=4}) = \frac{1}{4}\left\{ \mathrm{sgn}\left(\frac{Q_{1,4}\bar{Q}_{3,4}}{(Q_{1,4}+Q_{2,4})(\bar{Q}_{1,4}+\bar{Q}_{2,4}+\bar{Q}_{3,4})}\right) \mathrm{per}\left(\mathrm{ALi}_{1,1}\left[1, \frac{\bar{Q}^2_{3,4}}{Q^2_{1,2}\,\Delta}\right]\right)\right\}$$

$$+(\mathrm{perm}\ 1,2,3)\,.$$

it evaluates to
the usual four-mass box function

$$\frac{1}{2i}\left[\mathrm{Li}_2\left(\frac{w+v-u+\sqrt{\Delta}}{w+v-u-\sqrt{\Delta}}\right) - \mathrm{Li}_2\left(\frac{w+v-u-\sqrt{\Delta}}{w+v-u+\sqrt{\Delta}}\right)\right] + (\mathrm{cyclic}\ u,v,w)$$

$$u = Q_{1,2}Q_{3,4}\,, \quad v = Q_{1,4}Q_{2,3}\,, \quad w = Q_{1,3}Q_{2,4}\,, \quad \Delta = u^2+v^2+w^2-2(uv+uw+vw)$$

Hexagon integral: numerical check

```
(* Choose Random Kinematics and Propagator Masses *)

Set[x[#], Table[Random[Integer, {-9, 9}], {6}]] & /@ Range[6]

{{8, 3, 4, 2, -5, 4}, {-3, 5, -4, 8, 1, 5}, {-9, -9, 0, -3, -1, 3}, {3, 2, 3, 3, -9, 9}, {-4, 4, -2, -5, 5, 5}, {4, -8, -2, 4, 4, 1}}

Set[m[#], Random[Integer, {0, 9}]] & /@ Range[6]

{9, 4, 6, 9, 2, 6}

(* Construct the Gram Matrix *)

Q = Table[(x[i] - x[j]).(x[i] - x[j]) + m[i]^2 + m[j]^2, {i, 6}, {j, 6}] / 2
```

$$\left\{\left\{81, \frac{359}{2}, 304, \frac{231}{2}, 208, 192\right\}, \left\{\frac{359}{2}, 16, \frac{429}{2}, 166, \frac{211}{2}, \frac{315}{2}\right\}, \left\{304, \frac{429}{2}, 36, \frac{527}{2}, 141, 162\right\}, \left\{\frac{231}{2}, 166, \frac{527}{2}, 81, \frac{439}{2}, \frac{477}{2}\right\}, \left\{208, \frac{211}{2}, 141, \frac{439}{2}, 4, 173\right\}, \left\{192, \frac{315}{2}, 162, \frac{477}{2}, 173, 36\right\}\right\}$$

```
(* Numerically Integrate the Feynman Integral *)

A = {a1, a2, a3, a4, a5, 1};
NIntegrate[Sqrt[-Det[Q]] / 2 (A.Q.A) ^ (-3), {a1, 0, Infinity}, {a2, 0, Infinity}, {a3, 0, Infinity}, {a4, 0, Infinity}, {a5, 0, Infinity}

0.00335374

(* Our Analytic Formula *)

HexagonIntegral[Q]
```

$$-\frac{1}{64}\pi\,\mathrm{Im}\left[\mathrm{PolyLog}\left[2, \frac{35\,702\,096\,579\,409\,608\sqrt{\frac{13}{6\,077\,417\,669\,446\,330\,823\,944\,861\,449\,165}}}{49}\right]\right] + \frac{1}{64}\pi\,\mathrm{Im}\left[\mathrm{PolyLog}\left[2, \frac{5\,051\,691\,588\,837\,010\,109\sqrt{\frac{3}{153\,879\,312\,497\,380\,722\,601\,745\,988\,319}}}{13\,555}\right]\right] +$$

$$\cdots 475763 \cdots + \frac{1}{128}\mathrm{Re}\left[\mathrm{PolyLog}\left[3, -\frac{i\sqrt{\frac{667\,357\,738\,058\,073\,850\,949}{16\,812\,997\,771}}\left(-1-\frac{32}{\sqrt{943}}\right)\left(-1+\frac{109\,973\,644\,626\,397\,280\,i}{\sqrt{92\,746\,494\,059\,638\,514\,546\,039\,818\,214\,341\,041}}\right)}{72\,355\left(-1+\frac{i\sqrt{\frac{667\,357\,738\,058\,073\,850\,949}{16\,812\,997\,771}}}{72\,355}\right)\left(-1+\frac{703\,831\,325\,608\,942\,592}{14\,471\sqrt{2\,203\,411\,690\,545\,077\,279\,261\,284\,977}}\right)}\right]\right]$$

Full expression not available (original memory size: 495.5 MB)

```
N[%]

0.00335343
```

Conclusion

We used Rudenko's formula for the volume of hyperbolic orthoschemes to provide an explicit analytic result for the one-loop scalar n-gon integral in n-dimensions.

Open Questions

- Is there a more “canonical” formula for the n-gon: one that doesn’t rely on an ad hoc dissection?
- Do individual terms in the orthoscheme dissection relate to any meaningful pieces of other integrals via differential equations?
- Are there other applications of alternating polylogs in amplitudes?

Thank you!