Mellin transforms, holonomic sequences & Invariants of Feynman graphs

Francis Brown
University of Oxford & Geneva

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Joint work in progress with Erik Panzer.

Earlier work with { O. Schnetz
K. Yeats.

I. Introduction.

G vacuum diagram in massless ϕ^4_4 theory.

We are interested in its residues in dimensional regularisation. Given by

$$I_G = \int_{[0,\infty]^{N-1}} \frac{d\alpha_1 \dots d\alpha_{N-1}}{\psi_G^2 \big|_{\alpha_N = 1}}$$

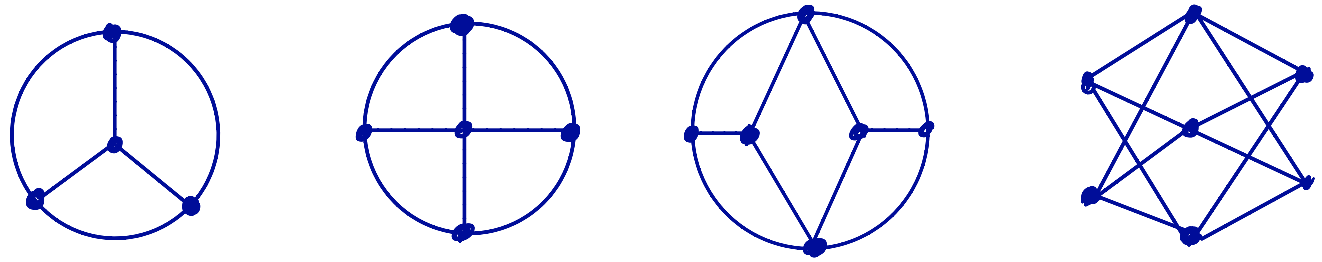
- Edges of G labelled $1, \dots, N$
- $\psi_G \in \mathbb{Z}[\alpha_1, \dots, \alpha_N]$ Kirchhoff 1847

$$\psi_G = \sum_{T \subseteq G} \prod_{e \notin T} \alpha_e$$ Symanzik

- I_G finite if $\begin{cases} N = 2\ell_G \\ \# E_\gamma > 2\ell_\gamma \quad \forall \gamma \subsetneq G \end{cases}$

Weinberg

G



I_G	$6\zeta(3)$	$20\zeta(5)$	$36\zeta(3)^2$	$\ldots\ \zeta(3,5)\ldots$
weight	3	5	6	8
$N_G - 3$	3	5	7	9

Broadhurst-Kreimer, Schnetz, ...

$\boxed{\text{What kind of graph invariants are there?}}$

- what properties do they have?
- How can we predict / compute I_G?
- Do there exist simpler, related invariants?

Properties:

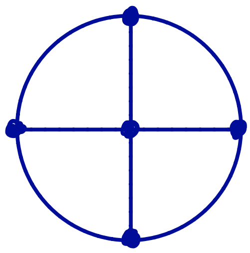
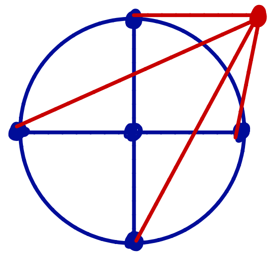
— **Multiplicative:** two-vertex join

$$I_{G_1 : G_2} = I_{G_1} I_{G_2}$$

— **Completion:**

$$I_{G_1} = I_{G_2} \quad \text{if} \quad \hat{G}_1 = \hat{G}_2$$

$\hat{G}$ = completion of G to a 4-regular graph

$W_4 =$  $\hat{W}_4 =$ 

— **Duality**

— **Twist** relation (Schnetz)

⋮

Related invariants of graphs.

Combinatorial

- Point counts over finite fields $\mathbb{F}_q$, $q = p^n$, p prime

$$\# \{ \alpha_1, \ldots, \alpha_N \in \mathbb{F}_q \quad \text{st} \quad \psi_G(\alpha_1, \ldots, \alpha_N) = 0 \}$$

Kontsevich

- Graph permanents — Crump

Martin invariant — Panzer - Yeats

Tropical

- $$I_G^{trop} = \int_{[0,\infty]^{N-1}} \frac{d\alpha_1 \ldots d\alpha_{N-1}}{(\psi_G^{tr})^2} \Bigg|_{\alpha_N = 1} \in \mathbb{Q}$$

rational

Panzer's "Hepp bound", Borinsky ...

$$\psi_G^{trop} = \text{tropicalisation of } \psi_G$$

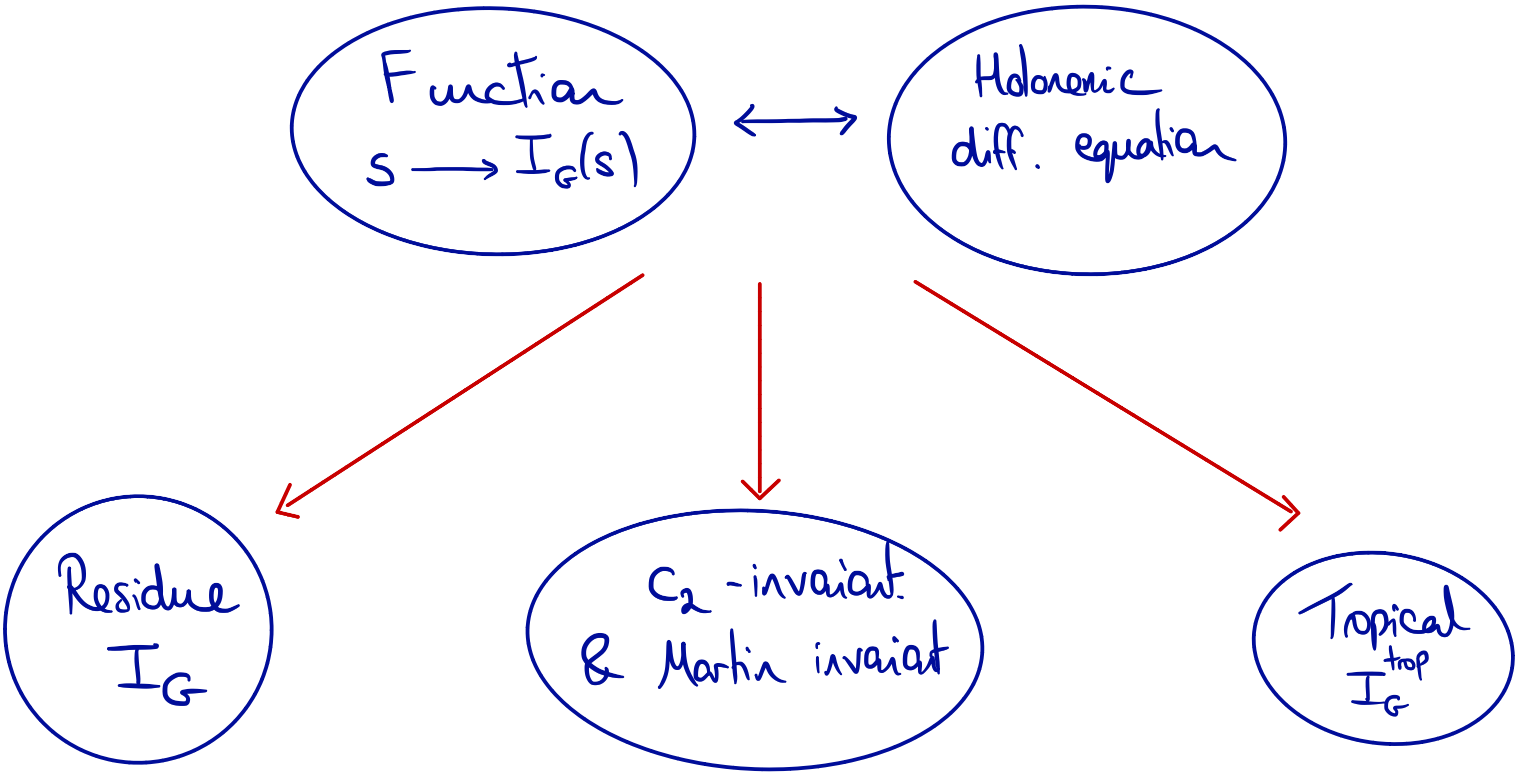
$$= \max_T \left(\prod_{e \notin T} \alpha_e \right)$$

Today:

A single object, Mellin-Feynman integral

$$I_G(s) = \int_{[0,\infty]^N} \left(\frac{\prod \alpha_i}{\psi_G^2} \right)^s \frac{d\alpha_1 \ldots d\alpha_{N-1}}{\psi_G^2}$$

determines all the others:



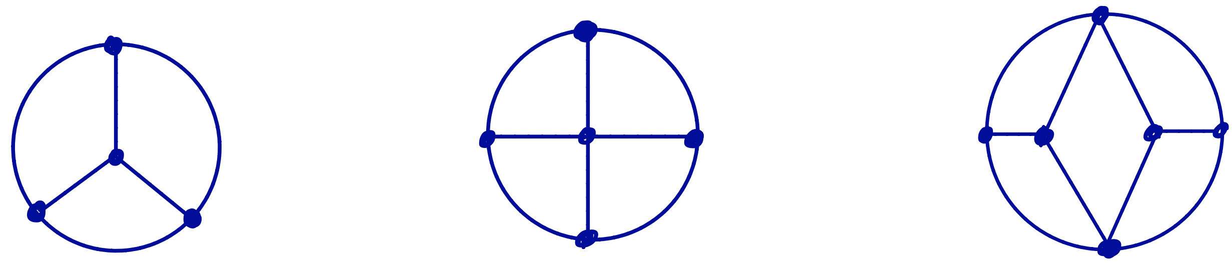
II. c_2 invariants

$[G] : \{\text{prime powers}\} \longrightarrow \mathbb{N}$ — Kontsevich, Stanley

$$q \longmapsto \#\{\alpha_1,\dots,\alpha_N \in \mathbb{F}_q : \psi_G(\alpha_1,\dots,\alpha_N) = 0\}$$

Stembridge: $[G]_q$ polynomial in q if $\#E_G \leq 12$

Belkale-Brosnan: false for general G $\#E_G >> 0$

G 

I_G	$6\zeta(3)$	$20\zeta(5)$	$36\zeta(3)^2$
$[G]_q$	$q^5 - q^3 - q^2$	$q^7 + 3q^5 - 6q^4 + 4q^3 - q^2$	$q^9 + 4q^7 - 7q^6 + 3q^5$

"c_2-invariant"

$$c_2^{(q)}(G) := \frac{[G]_q}{q^2} \pmod q$$

B.-Schnetz

$$\in \prod_q \mathbb{Z}/q\mathbb{Z}$$

$c_2(G)$ contains the right information. B.-Schnetz-Yeats
B.-Doryn

Example: $c_2(G) \equiv 0 \quad \longleftrightarrow \quad I_G$ has lower than expected transcendental weight

Conjecture (B.-Schnetz): $c_2(G_1) = c_2(G_2)$ if $\hat{G}_1 = \hat{G}_2$.

Hu-Yeats $p = 2$
Panzer-Yeats all primes p.
unknown for prime powers.

B.-Schnetz: Using c_2, exhibited graphs with 8 loops with modular point counts.
$\Longrightarrow$ Not MZV's!

The c_2-invariant encodes information about the top weight part of the residue I_G.

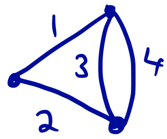
III. Diagonal sequence

Define:

$$\delta_n := \text{coefficient of } \alpha_1^n \alpha_2^n \cdots \alpha_N^n \text{ in } \psi_G^{2n}$$

Defines a sequence of integers:

$$\Delta_G = (\underset{\substack{\| \\ 1}}{\delta_0}, \delta_1, \delta_2, \delta_3, \ldots)$$

Eg: $G =$  $\psi_G = \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4$

$$\Delta_G = (1, 4, 36, 400, 4900, 63504, 853776, \ldots)$$

$\binom{2n}{n}^2$

$$\boxed{\delta_{p-1} \equiv -3p^2 c_2^{(p)} \pmod{p^3}}$$

(Chevalley-Waring Theorem)

"Diagonal determines c_2"

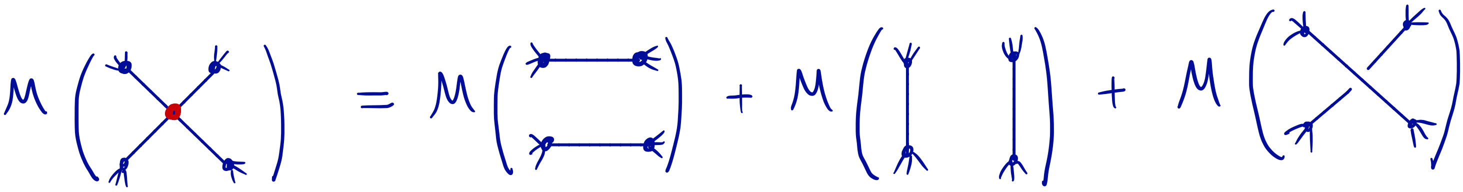
Martin invariant and δ_2

(Panzer – Yeats)

- $\delta_1(G)$ only depends on $\hat{G}$, completion.

$$\delta_1(G) = 2 \cdot M(\hat{G})$$

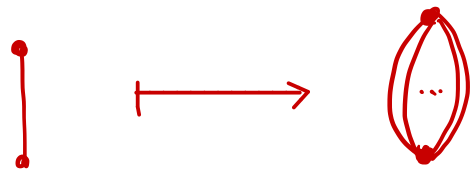
$\hat{G}$ 4-regular graph. Martin invariant.



STU relation.

- $\delta_n(G) = (*) \quad M(\hat{G}^{[n]})$

replace each edge with n copies.



- Diagonal sequence $\delta_{\frac{p-1}{2}}(G) \pmod{p}$
related to graph permanent sequence.
(I. Crump)

IV Integral interpretation.

δ_n = coefficient of $\alpha_1^n \alpha_2^n \cdots \alpha_N^n$ in ψ_G^{2n}

$$= \frac{1}{(2\pi i)^N} \oint \left(\frac{\psi_G^2}{\alpha_1 \cdots \alpha_N}\right)^n \frac{d\alpha_1 \cdots d\alpha_N}{\alpha_1 \cdots \alpha_N}$$

Cauchy

$\Rightarrow$ Generating function of diagonal sequence

$$\sum_{n\geq 0} \delta_n t^n = \frac{1}{(2\pi i)^N} \int_{(S^1)^N} \frac{d\alpha_1 \cdots d\alpha_N}{\alpha_1 \alpha_2 \cdots \alpha_N - t\psi_G^2}$$

Study the family of hypersurfaces

$$\boxed{\prod_{e\in E_G} \alpha_e - t\psi_G^2 = 0}$$

V. Mellin – Feynman

(G primitive log-divergent)

$$I_G(s) = \int_0^\infty \left(\frac{\alpha_1 \cdots \alpha_N}{\psi_G^2}\right)^s \Bigg|_{\alpha_N=1} \frac{d\alpha_1 \cdots d\alpha_{N-1}}{\psi_G^2} \qquad \mathrm{Re}\, s \geq 0$$

- $\alpha_1 \ldots \alpha_N$ and ψ^2 are flipped
- Domain of integration is different.

Proposition: $I_G(s)$ is uniquely determined by $\{ I_G(n)$, for integers $n \geq 0 \}$

$$I_G(0) = I_G \quad \underline{\text{residue}}$$

$$I_G(1) = \int_0^\infty \left(\frac{\alpha_1 \cdots \alpha_{N-1}}{\psi_G^4}\right) d\alpha_1 \cdots d\alpha_{N-1}$$

$\vdots$

Feynman periods.

$$\boxed{I_G^{trop} = \lim_{s \to -1}{}^{*} \; I_G(s)}$$

Tropical/ Hepp bound

New sequence of real numbers

$$\eta_G = \left(I_G, I_G(1), I_G(2), \ldots\right)$$

Theorem: The sequences η_G & Δ_G are dual holonomic. More precisely, there exist polynomials $p_i \in \mathbb{Q}[n]$, $1 \leq i \leq k$ such that the polynomial recurrence

$$\boxed{p_0(n)\, u_n + p_1(n)\, u_{n+1} + \ldots + p_k(n)\, u_{n+k} = 0}$$

is satisfied by

$$u_n = I_G(n-1) \qquad \forall\ n \gg 0$$

&

$$u_{-n} = \Delta_G(n) \qquad n \geq 1.$$

More precisely, p_n obtained from Picard – Fuchs equation of

$$\prod \alpha_e - t\,\psi_G^2 = 0$$

i.e.,

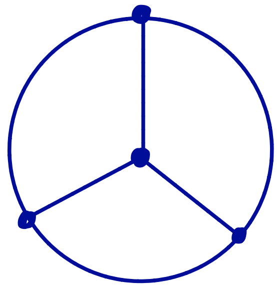
Differential operator $D_t \in \mathbb{Q}[t, \partial/\partial t]$ such that

$$D_t \left(\frac{d\alpha_1 \cdots d\alpha_N}{\alpha_1 \cdots \alpha_N - t\,\psi_G^2} \right) = d\eta_t$$

is <u>exact</u>.

(Subtle point : no domain of integration !)

Example:

$G =$ 

$\circledast$

$$n^5 u_n$$
$$- \, 2(2n+1)(65n^4 + 130n^3 + 105n^2 + 40n)\, u_{n+1}$$
$$+ \, (4n+2)(4n+3)(4n+4)(4n+5)(4n+6)\, u_{n+2}$$

(B. 2014)

$$\eta_G = \left(6\zeta(3), \ \frac{6}{60}\zeta(3) - \frac{7}{360}, \ \frac{6}{900}\zeta(3) - \frac{173}{129600}, \dots \right)$$

$\circledast$ has two fundamental solutions

$$a_\cdot = \left(1, \ \frac{1}{60}, \ \frac{1}{900}, \ \frac{47}{400400}, \dots \right)$$

$$b_\cdot = \left(0, \ \frac{-7}{360}, \ \frac{-173}{129600}, \dots \right)$$

Corresponding to two fundamental periods $1, 6\zeta(3)$.

Coefficients of $6J(3)$

$$a_\cdot = \left(1, \frac{1}{60}, \frac{1}{900}, \frac{47}{400400}, \dots\right)$$

Diagonal coefficients:

$$\Delta_G = (1, 12, 756, 78960, \dots)$$

Splice them together

$$u_n = \begin{cases} a_{n-1} & : \quad n \geq 1 \\ \delta_{-n} & : \quad n \leq 0 \end{cases}$$

$$(u_n)_{n\in\mathbb{Z}} = \Big(\underbrace{\dots, 756, 12, 1}_{\substack{\text{Diagonals} \\ \Delta_G(-n)}}, \underbrace{1, \frac{1}{60}, \frac{1}{900}, \dots}_{\substack{\text{Coefficients of} \\ 6J(3) \text{ in } I_G(n)}} \Big)$$

Is a solution to $(*)$ for all $n \in \mathbb{Z}$.

Apéry limits :

$$I_G(n) = a_n\, 6\,\zeta(3) + b_n$$

$$\sim \lambda^n \qquad \text{exponential decay.}$$

λ smallest root of characteristic polynomial of $(*)$

$$\chi_G = 1024\,\lambda^2 - 260\,\lambda + 1$$

$$= (256\,\lambda - 1)(4\lambda - 1)$$

$\Longrightarrow$ The residue $I_G = 6\,\zeta(3)$

may be obtained as the limit

$$\boxed{I_G = \lim_{n\to\infty}\left(-\frac{b_n}{a_n}\right)}$$

a_n, b_n solutions to $(*)$ Convergence $\left(\frac{1}{64}\right)^n$.

Summary

Mellin–Feynman Integrals

$I_G(s)$

P.F. operator

Interpolation of minimal solution

Polynomial recurrence

$$\sum_{i=0}^{k} p_i(n)\, u_{n+i} = 0$$

Apéry limit

Residue

I_G

Unique solution

$u_{-n} \in \mathbb{Z}$, $n \geq 0$

c_2- invariant

Martin, permanent