Tropical Feynman integration in Minkowski space

Amplitudes 2023 - CERN

arXiv:2008.12310 MB
arXiv:2204.06414 MB-Sattelberger-Sturmfels-Telen
arXiv:2302.08955 MB-Munch-Tellander

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Feynman Integrals

$$I_G = \int \frac{d^D k_1 \cdots d^D k_L}{\prod_e D_e}$$

Integrate over L copies of D dimensional Minkowski space

$$D_e = q_e^2 - m_e^2 + i\varepsilon$$

Momentum flowing through edge e

Motivating questions

1. What is an **effective way** to compute Feynman integrals?
2. What is the **computational complexity** of Feynman integration?

We look for <u>efficient</u> algorithms to compute I_G

What's the problem?

NIntegrate $\left[\int \frac{d^D k_1 \cdots d^D k_L}{\prod_e D_e}\right]$?

$$I_G = \int \frac{d^D k_1 \cdots d^D k_L}{\prod_e D_e}$$

Problem 0: Can be infinite → renormalization, subtraction, etc (different topic)

Here we assume I_G to be finite!

$$I_G = \int \frac{d^D k_1 \cdots d^D k_L}{\prod_e D_e}$$

Problem 1: non-bounded (and also non-standard if $D \notin \mathbb{N}$) integration domain

$$I_G = \int_{\mathbb{P}^E_{>0}} \frac{1}{U^{D/2}} \left(\frac{U}{F + i\varepsilon} \right)^{\omega} \Omega$$

- $\mathbb{P}^E_{>0}$: projective simplex (**positive** part of $(|E| - 1)$-dim. projective space)
- Ω: canonical volume form on $\mathbb{P}^E_{>0}$
- ω : superficial degree of divergence of G.
- U, F: Symanzik polynomials that depend on G and kinematics.

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⇒ Bounded integration domain and dimension is parameter in the integrand

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Problem 2: Integrand has poles in the integration domain

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E.g. $F = -Q^2 x_1 x_2 + m^2 (x_1 + x_2)^2 = 0$ **if** $\dfrac{x_1 x_2}{(x_1 + x_2)^2} = \dfrac{m^2}{Q^2}$

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These poles are 'regulated' by the causal $i\varepsilon$ prescription.

(Strictly speaking the integrand is just a distribution and no function)

NIntegrate $$\left[\int_{\mathbb{P}^E_{>0}} \frac{1}{U^{D/2}} \left(\frac{U}{F+i\varepsilon}\right)^{\omega} \Omega\right]$$?

NIntegrate

NIntegrate [f, {x, x_{min}, x_{max}}]
gives a numerical approximation to the integral $\int_{x_{min}}^{x_{max}} f\, dx$.

NIntegrate [f, {x, x_{min}, x_{max}}, {y, y_{min}, y_{max}}, …]
gives a numerical approximation to the multiple integral $\int_{x_{min}}^{x_{max}} d x \int_{y_{min}}^{y_{max}} dy \ldots f$.

NIntegrate [f, {x, y, …}∈reg]
integrates over the geometric region reg.

No option for $i\varepsilon$

Details and Options

- Multiple integrals use a variant of the standard iterator notation. The first variable given corresponds to the outermost integral and is done last.
- NIntegrate by default tests for singularities at the boundaries of the integration region and at the boundaries of regions specified by settings for the Exclusions option.
- NIntegrate [f, {x, x_0, x_1, …, x_k}] tests for singularities in a one-dimensional integral at each of the intermediate points x_i. If there are no singularities, the result is equivalent to an integral from x_0 to x_k. You can use complex numbers x_i to specify an integration contour in the complex plane.
- The following options can be given:

AccuracyGoal	Infinity	digits of absolute accuracy sought
EvaluationMonitor	None	expression to evaluate whenever *expr* is evaluated
Exclusions	None	parts of the integration region to exclude

Explicit, $i\varepsilon$-free representation is needed

$$I_G = \int_{\mathbb{P}^E_{>0}} \frac{1}{U^{D/2}} \left(\frac{U}{F + i\varepsilon} \right)^{\omega} \Omega$$

Plan:

Deform integration domain, such that $i\varepsilon$ is respected automatically.

Idea and setup go back to **Soper 2000; Binoth-Guillet-Heinrich-Pilon-Schubert 2005**

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Important requirement: Retain projective invariance

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Important requirement: Retain projective invariance

$$\iota : \mathbb{P}^E_{>0} \to \mathbb{P}\mathbb{C}^{|E|} \qquad x_e \mapsto x_e \exp\left(i\lambda \frac{\partial V}{\partial x_e} \right)$$

where $V = \dfrac{F}{U}$ **and** $\lambda > 0$

From Hannesdottir-Mizera 2023

$i\varepsilon$-free projective parametric representation

MB-Munch-Tellander 2023

$$I_G = \int_{\mathbb{P}^E_{>0}} \frac{J_\lambda}{\tilde{U}^{D/2}\tilde{V}\omega}\Omega$$

- Where J_λ is an **efficiently computable** rational function in $x_1, \ldots, x_{|E|}$
- $\tilde{U}, \tilde{V}$ are the deformed versions of U and $V = \dfrac{F}{U}$

NIntegrate $\left[\int_{\mathbb{P}^{E}_{>0}} \frac{J_\lambda}{\tilde{U}^{D/2}\tilde{V}\omega}\Omega\right]$?

Computer still says no…

$$I_G = \int_{\mathbb{P}^E_{>0}} \frac{J_\lambda}{\tilde{U}^{D/2}\tilde{V}\omega}\Omega$$

Problem 3: Integrand has poles on the boundary of the integration domain

E.g. $\frac{1}{\tilde{U}} \sim \frac{1}{U} = \frac{1}{x_1x_2 + x_1x_3 + x_2x_3} \to \infty$ **if** $x_1, x_2 \to 0$

Traditional solution:

Just look at all possible poles and perform a blowup (i.e. a local change of coordinates that removes the singularity):

Sector Decomposition

Binoth-Heinrich 2004

(Also gives an alternative solution to the $i\varepsilon$ problem)

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Caveat: **Computationally challenging / brute force**

Alternative: Tropical sampling

Tropical approximation

$$p(\boldsymbol{x}) = \sum_{\boldsymbol{\ell}\in J} a_{\boldsymbol{\ell}} \prod_{k=1}^{n} x_k^{\ell_k} \to p^{tr}(\boldsymbol{x}) = \max_{\boldsymbol{\ell}\in J} \prod_{k=1}^{n} x_k^{\ell_k}$$

Tropical approximation

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Theorem: MB 2020

Both $p(\boldsymbol{x})/p^{tr}(\boldsymbol{x})$ and $p^{tr}(\boldsymbol{x})/p(\boldsymbol{x})$ stay bounded on $\mathbb{P}^n_{>0}$.

(If $p(\boldsymbol{x})$ is completely non-vanishing on $\mathbb{P}^n_{>0}$.)

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Interesting mathematical applications e.g. to statistics:

MB-Sattelberger-Sturmfels-Telen 2022

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$$I_G = \int_{\mathbb{P}^E_{>0}} \frac{J_\lambda(x)}{\tilde{U}(x)^{D/2}\tilde{V}(x)^{\omega}}\Omega$$

$$= \int_{\mathbb{P}^E_{>0}} \underbrace{\frac{\Omega}{U^{tr}(x)^{D/2}V^{tr}(x)^{\omega}}}_{\text{TROPICAL VERSION OF } I_G} \cdot \underbrace{J_\lambda(x)\frac{U^{tr}(x)^{D/2}V^{tr}(x)^{\omega}}{\tilde{U}(x)^{D/2}\tilde{V}(x)^{\omega}}}_{\text{BOUNDED KERNEL}}$$

TROPICAL VERSION OF I_G

BOUNDED KERNEL

$$
\begin{aligned}
I_G &= \int_{\mathbb{P}^E_{>0}} \frac{J_\lambda(x)}{\tilde{U}(x)^{D/2}\tilde{V}(x)^{\omega}}\Omega \\
&= \int_{\mathbb{P}^E_{>0}} \frac{\Omega}{U^{tr}(x)^{D/2}V^{tr}(x)^{\omega}} \cdot J_\lambda(x)\frac{U^{tr}(x)^{D/2}V^{tr}(x)^{\omega}}{\tilde{U}(x)^{D/2}\tilde{V}(x)^{\omega}} \\
&= Z\int_{\mathbb{P}^E_{>0}} \mu^{tr} \cdot J_\lambda(x)\frac{U^{tr}(x)^{D/2}V^{tr}(x)^{\omega}}{\tilde{U}(x)^{D/2}\tilde{V}(x)^{\omega}}
\end{aligned}
$$

$$\mu^{tr} = \frac{1}{Z} \frac{\Omega}{U^{tr}(x)^{D/2} V^{tr}(x)^{\omega}} \quad \textbf{s.t.} \quad 1 = \int_{\mathbb{P}^E_{>0}} \mu^{tr}$$

Theorem MB 2020:

For ‘tame’ kinematics, there is a fast algorithm to sample from the probability distribution μ^{tr}.

$$I_G = Z \int_{\mathbb{P}^E_{>0}} \mu^{tr} \cdot J_\lambda(x) \frac{U^{tr}(x)^{D/2} V^{tr}(x)^{\omega}}{\tilde{U}(x)^{D/2} \tilde{V}(x)^{\omega}}$$

We get an algorithm that evaluates I_G up to δ accuracy in runtime

$$O(n2^n + n^3\delta^{-2})$$

where $n = |E|$.

"Exponential wall" starts at around $n = 30$ edges
⇒ Exponential term is negligible for loop order ≤ 10

Under the hood

- Algorithm makes heavy use of algebraic and convex geometry of U, F
- Works thanks to well-understood analytic structure in the UV **Speer, Brown, …**
- Key structure: **generalised permutahedra** (related to Lorentzian polynomials)
- Problems due to failure of this structure with IR divergences.
- Findings of **Arkani-Hamed, Hillman, Mizera 2022** helpful to resolve this partially.
- Implementation: `https://github.com/michibo/feyntrop`

Conclusion

- Tropical sampling + new $i\varepsilon$ free projective parametric representation

  $\Rightarrow$ Fast method to integrate Feynman integrals: **Code, feyntrop on github**

- Exceptional kinematics are problematic (IR singularities)

  $\Rightarrow$ More information on pole structure of integrands needed

- Extensions necessary: Numerators of Feynman integrals and divergences
- Question: Is there are polynomial time algorithm for Feynman integration?
- Question: Is there an algorithm for amplitudes faster than the naive one?

Outlook: Amplitudes on moduli spaces

$$A_L = \sum_G \frac{I_G}{|\mathrm{Aut}(G)|}$$

Sum over graphs with L loops of shape determined by the QFT

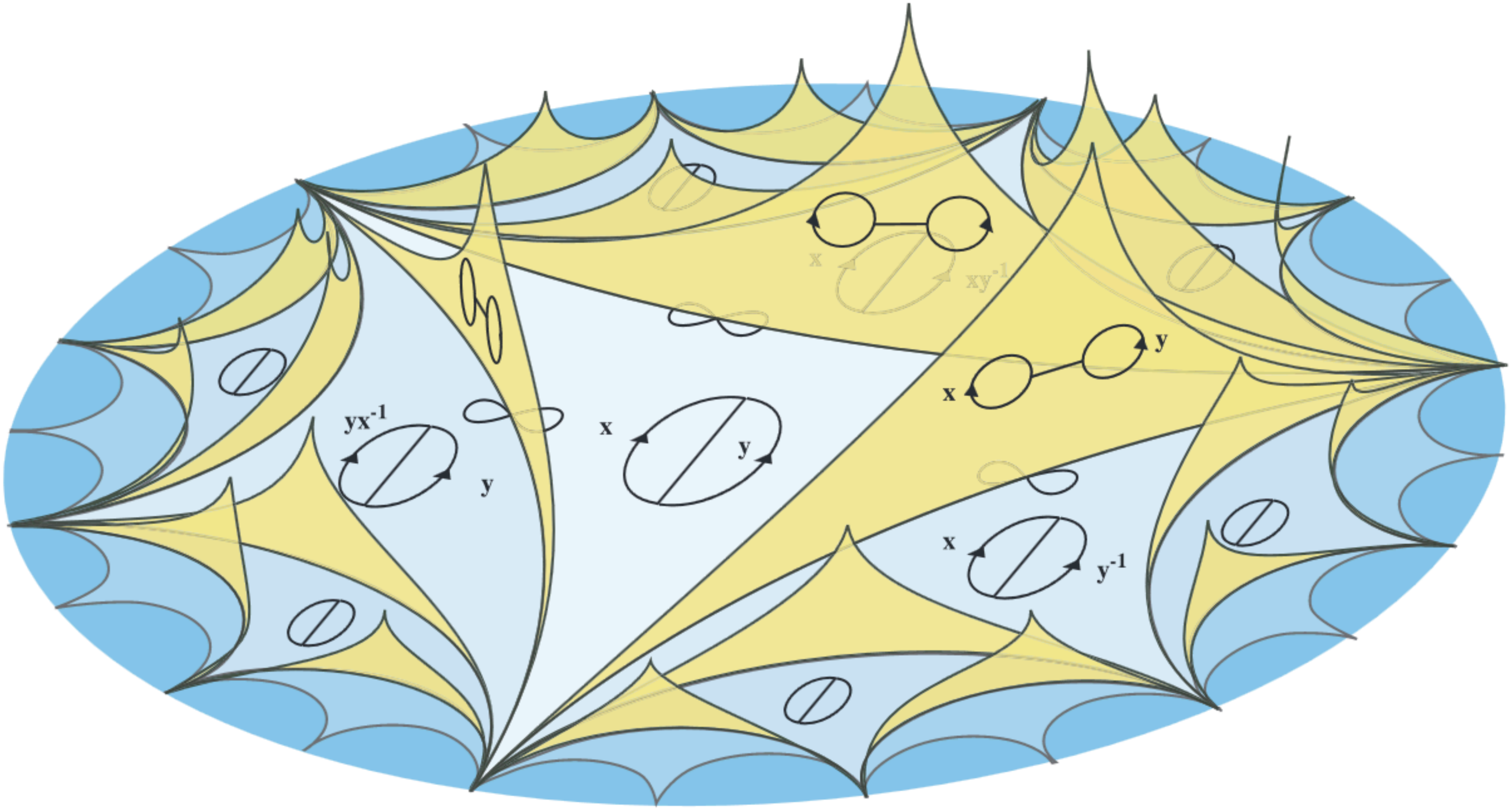
Feynman Integral for each graph weighted by symmetry factor

Outlook: Amplitudes on moduli spaces

$$A_L = \sum_G \frac{I_G}{|\mathrm{Aut}(G)|}$$

$$= \underbrace{\sum_G \frac{1}{|\mathrm{Aut}(G)|} \int_{\mathbb{P}^E_{>0}}}_{} \frac{\Omega}{U^{D/2} V^{\omega}}$$

Integral over moduli space of graphs $\mathcal{MG}_g$



x
xy-1
y
x
yx-1
y
x
y
x
y-1

QFT is very useful to study this moduli space

arXiv:1907.03543 MB-Vogtmann
arXiv:2202.08739 MB-Vogtmann
arXiv:2301.01121 MB-Vogtmann

Theorem MB-Vogtmann 2023

$$\chi(\mathcal{MG}_g) \sim -e^{-1/4} \left(\frac{g}{e}\right)^g / (g \log g)^2 \qquad\qquad g \to \infty$$

- Related to result by **Harer-Zagier 1986** on the moduli space of curves $\mathcal{M}_g$.
- The moduli space of graphs $\mathcal{MG}_g$ is a tropicalization of $\mathcal{M}_g$.
- Feynman type integrals on $\overline{\mathcal{MG}}_g$ certify classes in $\mathcal{M}_g$ **Brown 2021**
- Long story…

$\Rightarrow$Use integrals over the moduli space to study/evaluate amplitudes

Bloch-Kreimer, Berghoff

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