Can 3-loop tadpoles be reduced to polylogarithms?

David Broadhurst, Open University, UK, 8 August 2023, at
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A massless **elliptic** 2-loop **lobster** read **George Orwell** and made up this ditty: *two legs bad, four legs good, ten legs better.* This talk concerns massive **legless** 3-loop **tadpoles**. I shall exhibit something that might surprise massless lobsters, namely that an integral of a **trilogarithm** against **elliptic integrals** reduces empirically to **classical polylogs** of weight 4, against strong expectation to the contrary.

1. **Comparison** of a 2-loop **lobster** with a 3-loop **tadpole**

2. **Fast** numerical algorithms for **kites** and **tadpoles**

3. **Surprising** empirical reductions to **polylogs**

4. **Rumination** on my title: **pro** and **contra**

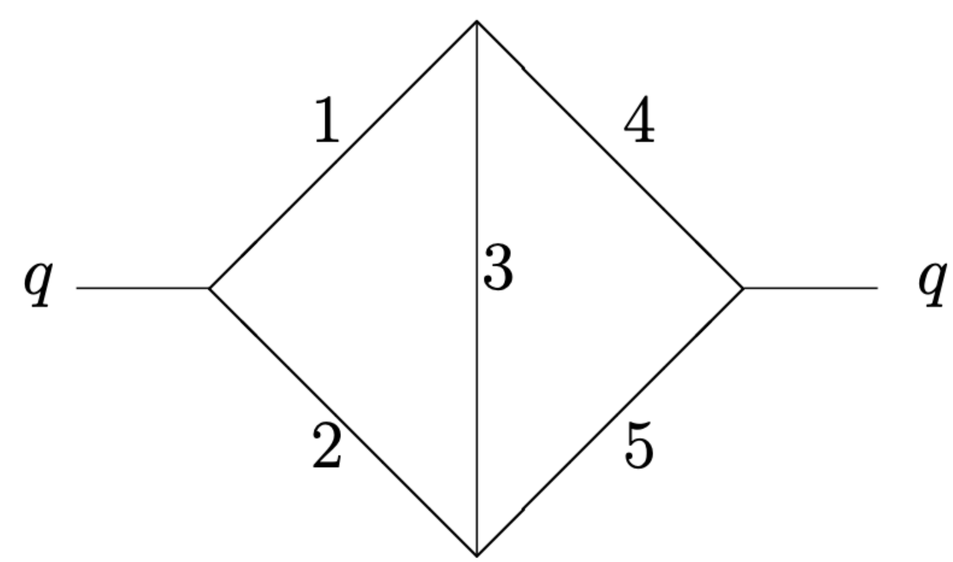
In memoriam, Gabriel Barton (25 February 1934 to 11 October 2022) and Donald Hill Perkins (15 October 1925 to 30 October 2022), trusted guides and mentors.

2-loop lobster: In 2012, Simon Caron-Huot, Kasper Larsen, Miguel Paulos, Marcus Spradlin and Anastasia Volovic found an **elliptic obstruction** to evaluation of a **massless double-box** integral with **10 legs**, suggesting a schematic form

$$\text{Lobster} \sim \int \frac{\mathrm{d}\alpha}{\sqrt{Q(\alpha)}}(\mathrm{Li}_3(\ldots)+\ldots)$$

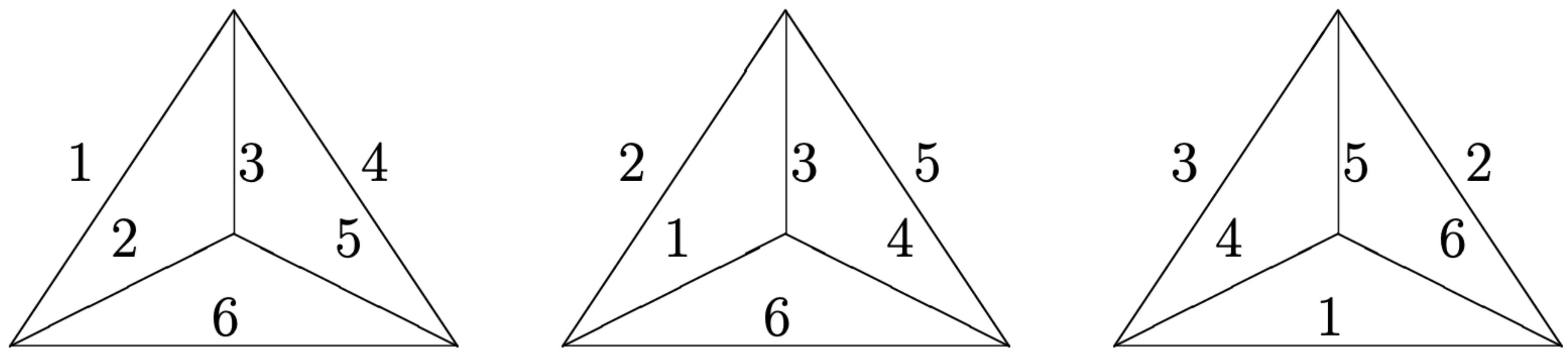
with a **quartic** $Q(\alpha)$ that seems to **frustrate** integration over **trilogs** of α and the external kinematics. In 2018, Jacob Bourjaily, Andrew McLeod, Marcus Spradlin, Matt von Hippel and Matthias Wilhelm gave an explicit result of this form.

3-loop tadpole: Consider the **generic** 2-loop scalar **kite** with 5 internal masses:



Since **1962** it was known to have **elliptic obstructions** from **3-particle cuts**.

Now **close the kite** with a **sixth** propagator $1/(q^2 - m_6^2)$ to obtain



tadpole **same** tadpole **same** tadpole

with the symmetry group S_4 of the **tetrahedron** giving **12 elliptic obstructions**. The tadpole has a logarithmic divergence that we regulate in $D = 4 - 2\varepsilon$ dimensions

$$T^{5,4,6}_{1,2,3} = \left(\frac{1}{3\varepsilon} + 1\right) 6\zeta_3 + 3\zeta_4 - F^{5,4,6}_{1,2,3} + O(\epsilon) \tag{1}$$

with a **finite part** F that depends on the **six ratios** m_k/μ, where μ is the **scale** of dimensional regularization. The rescaling $m_k \to \lambda m_k$ gives $F \to F + 12\zeta_3 \log(\lambda)$. Without loss of generality, choose m_6 to be the **largest** mass and set $\mu = m_6 = 1$.

With $\mu = m_6 = 1$, **Schwinger** parametrization gives the **5-dimensional** integral

$$F^{5,4,6}_{1,2,3} = \int_0^\infty \mathrm{d}x_1 \ldots \int_0^\infty \mathrm{d}x_5 \, \frac{1}{U^2} \log\left(1 + \sum_{k=1}^{5} x_k m_k^2\right) \tag{2}$$

after setting $x_6 = 1$ in the **Symanzik** polynomial of the **tetrahedron**

$$\begin{aligned} U = {} & x_3(x_1x_2 + x_4x_5) + x_6(x_1x_4 + x_2x_5) + x_3x_6(x_1 + x_2 + x_4 + x_5) \\ & + x_2x_4(x_1 + x_3 + x_5 + x_6) + x_1x_5(x_2 + x_3 + x_4 + x_6). \end{aligned} \tag{3}$$

I was able to reduce this to a **single** integral of a **dilogarithm** against the **derivative** of the **discontinuity** $I(s + \mathrm{i}\epsilon) - I(s - \mathrm{i}\epsilon) = 2\pi\mathrm{i}\sigma(s)$ of a **kite** integral:

$$F^{5,4,6}_{1,2,3} = -\int_{s_0}^\infty \mathrm{d}s \, \sigma'(s) \, \mathrm{Li}_2(1 - s), \tag{4}$$

$$I(q^2) = -\frac{q^2}{\pi^4} \int \mathrm{d}^4 l \int \mathrm{d}^4 k \prod_{j=1}^{5} \frac{1}{p_j^2 - m_j^2 - \mathrm{i}\epsilon} = \int_{s_0}^\infty \mathrm{d}s \, \sigma'(s) \log\left(1 - \frac{q^2}{s}\right), \tag{5}$$

$$(p_1,\, p_2,\, p_3,\, p_4,\, p_5) = (l,\, l - q,\, l - k,\, k,\, k - q),$$

$$s_0 = \min(s_{1,2}, s_{4,5}, s_{2,3,4}, s_{1,3,5}), \quad s_{j,k} = (m_j + m_k)^2, \quad s_{i,j,k} = (m_i + m_j + m_k)^2.$$

The **non-elliptic** contribution from **2-particle** intermediate states has the form

$$\sigma'_{\mathrm{N}}(s) = \Theta(s - s_{1,2})\sigma'_{1,2}(s) + \Theta(s - s_{4,5})\sigma'_{4,5}(s). \tag{6}$$

Denote the **square root** of the symmetric **Källén function** by

$$\Delta(a, b, c) = \sqrt{a^2 + b^2 + c^2 - 2(ab + bc + ca)} \tag{7}$$

with **abbreviations** $\Delta_{j,k}(s) = \Delta(s, m_j^2, m_k^2)$ and $\Delta_{i,j,k} = \Delta_{j,k}(m_i^2)$. Then

$$D_{j,k}(s) = \frac{r}{s - (m_j - m_k)^2} \log\left(\frac{1 + r}{1 - r}\right), \quad r = \left(\frac{s - (m_j - m_k)^2}{s - (m_j + m_k)^2}\right)^{1/2} \tag{8}$$

provides the **logarithms** in

$$\Delta_{1,2}(s)\sigma'_{1,2}(s) = \Re\left((s + \alpha)D_{4,5}(s) + L_{4,5} + \sum_{i=0,+,-} C_i \frac{D_{4,5}(s) - D_{4,5}(s_i)}{s - s_i}\right) \tag{9}$$

with **constants**

$$C_0 = -(m_1^2 - m_2^2)(m_4^2 - m_5^2), \quad C_\pm = \alpha s_\pm + \beta, \quad L_{4,5} = \log\left(\frac{m_4 m_5}{m_3^2}\right),$$

$$\alpha = \frac{(m_1^2 - m_4^2)(m_2^2 - m_5^2)}{m_3^2} - m_3^2, \quad \beta = \frac{(m_1^2 m_5^2 - m_2^2 m_4^2)(m_1^2 - m_2^2 - m_4^2 + m_5^2)}{m_3^2},$$

$$s_0 = 0, \quad s_\pm = \frac{m_1^2 + m_2^2 - 2m_3^2 + m_4^2 + m_5^2 - \alpha}{2} \pm \frac{\Delta_{1,3,4}\Delta_{2,3,5}}{2m_3^2}$$

where $s_\pm$ locate **leading Landau singularities** of triangles that form the kite.

Elliptic contribution: This comes from **3-particle** intermediate states, giving

$$\sigma'_{\mathrm{E}}(s) = \Theta(s - s_{2,3,4})\sigma'_{2,3,4}(s) + \Theta(s - s_{1,3,5})\sigma'_{1,3,5}(s). \tag{10}$$

It contains **complete** elliptic integrals of the **third kind** of the form

$$P(n,k) = \frac{\Pi(n,k)}{\Pi(0,k)}, \quad \Pi(n,k) = \int_0^{\pi/2} \frac{d\theta}{(1 - n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}} \tag{11}$$

with $\Pi(0,k) = (\pi/2)/\mathrm{AGM}\,(1, \sqrt{1-k^2})$ given by an **arithmetic-geometric mean**.

With $s = w^2$, an integration over the phase space of particles 2, 3 and 4 determines

$$k^2 = 1 - \frac{16 m_2 m_3 m_4 w}{W}, \quad W = (w_+^2 - m_+^2)(w_-^2 - m_-^2) \tag{12}$$

with $w_\pm = w \pm m_2$ and $m_\pm = m_3 \pm m_4$. Then I obtain

$$\sigma'_{2,3,4}(w^2) = \frac{4\pi m_3 m_4}{\mathrm{AGM}\left(\sqrt{16 m_2 m_3 m_4 w}, \sqrt{W}\right)} \Re \left(\sum_{i=+,-} E_i \frac{P(n_i, k) - P(n_1, k)}{t_i - t_1} \right) \tag{13}$$

with coefficients and arguments given, as **compactly** as possible, by

$$E_\pm = \frac{m_2^2 - m_3^2 + m_5^2}{2m_5^2} \pm \left(\frac{m_4^2 - m_5^2 - w^2}{2m_5^2} \right) \frac{\Delta_{2,3,5}}{\Delta_{4,5}(w^2)},$$

$$t_\pm = \frac{\gamma \pm \Delta_{2,3,5} \Delta_{4,5}(w^2)}{2m_5^2}, \quad t_1 = m_1^2, \quad n_i = \frac{(w_-^2 - m_+^2)(t_i - m_-^2)}{(w_-^2 - m_-^2)(t_i - m_+^2)},$$

$$\gamma = (m_2^2 + m_3^2 + m_4^2 - m_5^2 + w^2) m_5^2 + (m_2^2 - m_3^2)(m_4^2 - w^2).$$

An **AGM procedure** speedily evaluates $P(n,k) = \Pi(n,k)/\Pi(0,k)$ to high precision.

1. **Initialize** $[a,\, b,\, p,\, q] = [1,\, \sqrt{1-k^2},\, \sqrt{1-n},\, n/(2-2n)]$. Then set $f = 1+q$.
2. Set $m = ab$ and then $r = p^2 + m$. **Replace** $[a,\, b,\, p,\, q]$ by a vector of **new values** as follows: $[(a+b)/2,\, \sqrt{m},\, r/(2p),\, (r-2m)q/(2r)]$. Add q to f.
3. If $|q/f|$ is sufficiently **small**, return $P(n,k) = f$, else go to step 2.

On the cut with $n \geq 1$, the **principal value** is $\Re P(n,k) = 1 - P(k^2/n, k)$.

Criterion for any anomalous contribution: Suppose that $s_{4,5} \geq s_{1,2}$. Then

$$\sigma'(s) = \sigma'_{\mathrm{N}}(s) + \sigma'_{\mathrm{E}}(s) + C_{\mathrm{A}} \frac{\Theta(s - s_{4,5})}{\Delta_{4,5}(s)} \Re\left(\frac{2\pi \mathrm{i} \Delta_{4,5}(s_-)}{s - s_-}\right) \tag{14}$$

with $C_{\mathrm{A}} \neq 0$ **if and only if** $(m_1 + m_2)(m_3^2 + m_1 m_2) < m_1 m_5^2 + m_2 m_4^2$ and at least one of $\Delta_{1,3,4}$ and $\Delta_{2,3,5}$ is imaginary, in which case $C_{\mathrm{A}} = \pm 1$ is the sign of $\Im \Delta_{4,5}(s_-)$.

This value of $C_{\mathrm{A}} \in \{0, 1, -1\}$ is determined by the **high-energy** behaviour

$$s^2 \sigma'(s) = 2L_3 + \sum_{k=1,2,4,5} (L_k + m_k^2) + O\left(\frac{\log(s)}{s}\right), \quad L_k = m_k^2 \log(s/m_k^2). \tag{15}$$

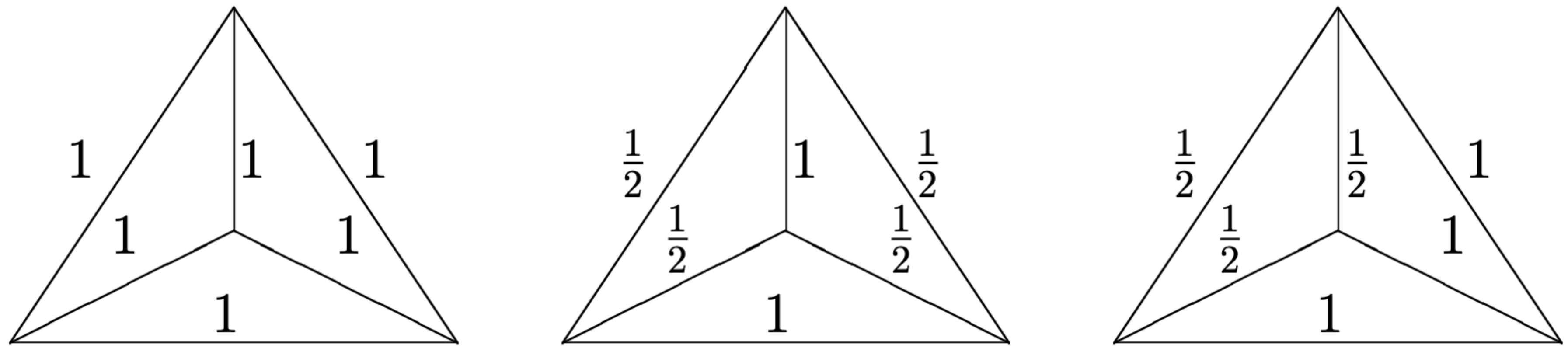
Stringent tests for kites and tadpoles

1. Elliptic terms do **not** depend on the **order** of phase-space integrations.
2. The derivative of the discontinuity of a kite satisfies the **sum rule**

$$\int_{s_0}^{\infty} \mathrm{d}s\, \sigma'(s) \log\left(\frac{s}{s_0}\right) = 6\zeta_3. \tag{16}$$

3. High-energy behaviour of $\sigma'(s)$ holds **irrespective** of anomalous thresholds.
4. **Benchmarks** for **kites** given by Stefan **Bauberger** and Manfred **Böhm**, to 6 decimal digits, and by Stephen **Martin**, to 8 decimal digits, are confirmed and then extended to **100 digits** in less than a **second**.
5. The **same tadpole** is obtained by integrating over **6 distinct kites**.
6. The **binary** tadpoles with $m_k \in \{0, 1\}$ agree with my previous reductions to poloylogs of **sixths roots of unity**.

Surprising reductions to polylogs: When all **6** masses are non-zero, there is no non-elliptic route. Yet in 3 cases, I found **empirical** reductions to **polylogs**.



binary perfect imperfect

A binary surprise: Dressings of the tetrahedron with zero or unit masses give rational linear combinations of **4 constants**: $\zeta_4 = \pi^4/90$, $\mathrm{Cl}_2^2(\pi/3)$, $U_{3,1}$ and $V_{3,1}$, with $\mathrm{Cl}_2(\pi/3) = \Im\mathrm{Li}_2(\lambda)$, $\lambda = (1+\sqrt{-3})/2$, and **reducible** double sums

$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \tfrac{1}{2}\zeta_4 + \tfrac{1}{2}\zeta_2 \log^2(2) - \tfrac{1}{12}\log^4(2) - 2\,\mathrm{Li}_4(\tfrac{1}{2}), \tag{17}$$

$$V_{3,1} = \sum_{m>n} \frac{(-1)^m \cos(2\pi n/3)}{m^3 n} = -\tfrac{145}{432}\zeta_4 + \tfrac{1}{8}\zeta_2 \log^2(3) - \tfrac{1}{96}\log^4(3) + \tfrac{1}{32}\mathrm{Li}_4(\tfrac{1}{9}) - \tfrac{3}{4}\mathrm{Li}_4(\tfrac{1}{3}) + \tfrac{1}{3}\mathrm{Cl}_2^2(\pi/3). \tag{18}$$

With 5 unit masses, there was a **non-elliptic** route to my result

$$F_5 = \tfrac{550}{27}\zeta_4 + 16V_{3,1} - \tfrac{8}{3}\mathrm{Cl}_2^2(\pi/3) \tag{19}$$

which **Yajun Zhou** and I have now proved, using `HyperInt` from **Erik Panzer**. More **surprising** is my very simple **empirical** result for the **totally massive** case

$$F_6 \stackrel{?}{=} 16\zeta_4 + 8U_{3,1} + 4\mathrm{Cl}_2^2(\pi/3). \tag{20}$$

The closest we recently got to a proof involves a **double** integral of products of logs, for which `HyperInt` gives **1300** multiple polylogarithms of **12th roots** of unity. We use powerful software from **Kam Cheong Au** to hand 12th roots, yet still fall far short of proving (20).

A perfect surprise: Dirk Kreimer and I agreed that the next cases to investigate should be **perfect** tetrahedra with $\Delta_{i,j,k} = 0$ at all 4 vertices, eliminating all resolutions square roots. Here I also found an **empirical** reduction to **classical polylogs**, with help from **Steven Charlton**. Promoting subscripts and superscripts to mass values, I **conjecture** that, with $L = \log(2)$,

$$F_{(\frac{1}{2},\frac{1}{2},1)}^{(\frac{1}{2},\frac{1}{2},1)} \stackrel{?}{=} B = 6\left(2\zeta_4 - 3\mathrm{Li}_4(\tfrac{1}{4})\right) + 8\left(2\zeta_3 - 3\mathrm{Li}_3(\tfrac{1}{4})\right)L - 12\,\mathrm{Li}_2(\tfrac{1}{4})L^2 - 4L^4. \tag{21}$$

This is **equivalent** to an evaluation in **classical polylogs** of the integral of a **trilog** against **complete** elliptic integrals of the **first** and **second** kinds:

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{-1/2} \mathrm{d}\theta, \quad E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{1/2} \mathrm{d}\theta, \tag{22}$$

$$Z(y) = \frac{y(1+y)K(k) + E(k)}{(1+y+y^2)\sqrt{1+y}}, \quad k^2 = 1 - y^3, \tag{23}$$

$$T(y) = \mathrm{Li}_3(u) - \tfrac{1}{2}\,\mathrm{Li}_2(u)\log(u), \quad u = \frac{y}{(1+y)^2}, \tag{24}$$

$$4\int_0^1 \mathrm{d}y \left(\frac{1}{y} - 1\right) T(y)Z(y) \stackrel{?}{=} B + 16\,\zeta_4 + 32\,U_{3,1} - 30\,\zeta_3 \log(2). \tag{25}$$

A third surprise: In an **imperfect** case, I found **empirically** that

$$F^{(1,1,1)}_{(\frac{1}{2},\frac{1}{2},\frac{1}{2})} \stackrel{?}{=} 10\zeta_4 - 4U_{3,1} + 10\mathrm{Cl}_2^2(\pi/3) + 3\zeta_3\log(2) - \tfrac{1}{2}B \tag{26}$$

also has a remarkable reduction to **classical polylogs**.

Combining the perfect and imperfect cases, I arrive at the conjecture

$$4\int_2^\infty \frac{\mathrm{d}w}{w}\left(\mathrm{Li}_2\left(1-\frac{1}{w^2}\right)-\zeta_2\right)Y(w) \overset{?}{=} \zeta_4 - 4U_{3,1} + 7\zeta_3\log(2) \tag{27}$$

$$Y(w) = \frac{\Pi(0,k) - \Pi(n,k) - 6\,\Pi(\widehat{n},k)}{(w-1)\sqrt{w^2+2w}}, \tag{28}$$

$$k^2 = 1 - \frac{4}{(w-1)^2(w+2)}, \quad n = 1 - \frac{1}{(w-1)^2}, \quad \widehat{n} = 1 - \frac{2}{w(w-1)}. \tag{29}$$

with an integral of a **dilogarithm** against **complete** elliptic integrals of the **third** kind reduced to **classical polylogs** in a spectacularly **simple** result.

Comment: I was guided by Feynman's skepticism, imagining him to say: *ignore fancy reasons for this integral being impossible; just try to guess it.*

Can 3-loop tadpoles be reduced to polylogarithms?

Conservative answer: some can, some cannot.

Bold (or foolish?) suggestion: **every** 3-loop tadpole with rational masses reduces to multiple polylogs in an alphabet with **algebraic** letters.

Contra hyp: With 5 distinct mass ratios there are **12 elliptic obstructions**.

Chinks of light:

1. Three seemingly impossible cases reduce **empirically** to polylogs.
2. The **Schwinger** parametrization does not too frightening.
3. A **double** integral over a product of logs with **rational** arguments is possible. The obstructing **quartics** can be rationalized by a **pair** of Euler substitutions.

Caveat: Regarding massless 2-loop **lobsters**, I am both ignorant and **agnostic**.