Soft bootstrap: overview and new studies

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Outline

- EFTs and Adler zero
- NLSM and Classification
- Higher-order NLSM
- Loop level

Soft bootstrap at 10

collaboration with J.Trnka and J.Novotny '13

- motivated by amazing discoveries of amplitudes in gauge theories and gravity (e.g. BCFW)
- we wanted to focus on: Effective field theories
- motivated by theoretical considerations $\rightarrow$ taking something as simple as possible
- very broad subject
- focus on low energy dynamics of theories with SSB
- leading order, tree-level
- strictly massless limit

Leading order Lagrangian

- assume general simple compact Lie group G
- we will build a chiral non-linear sigma model, which will correspond to the spontaneous symmetry breaking ($G_L \simeq G_R \simeq G_V \simeq G$)

$$G_L \times G_R \rightarrow G_V$$

- consequence of the symmetry breaking: Goldstone bosons ($\equiv \phi$)

$$U = \exp\left(\sqrt{2}\frac{i}{F}\phi\right)$$

- their dynamics given by a Lagrangian (at leading order)

$$\mathcal{L} = \frac{F^2}{4}\langle \partial_\mu U \partial^\mu U^{-1} \rangle$$

- Using structure constants we can define ordered Feynman rule for the interaction vertices → stripped vertices

Stripping and ordering

Up to now general group: we didn't need any property of f^{abc} or t^i.
From now on: we will simplify the problem setting $G = SU(N)$.
Simplification due to the completeness relation:

$$\sum_{a=1}^{N^2-1} \langle X t^a \rangle \langle t^a Y \rangle = \langle XY \rangle - \frac{1}{N} \langle X \rangle \langle Y \rangle$$

- double trace has to cancel out
- two vertices are connected via a propagator (δ^{ab})
- ordering of t^{a_i} in the final single trace is conserved

The tree graphs built form the stripped vertices and propagators are decorated with cyclically ordered external momenta.

$G = U(N)$ – different parametrizations

General form of the parametrization $U(\phi) \to f(x)$

$$f(x) = \sum_{k=0}^{\infty} u_k x^k, \qquad f(-x)f(x) = 1$$

- "exponential": $f_{\rm exp} = {\rm e}^x$
- "minimal": $f_{\rm min} = x + \sqrt{1 + x^2}$
- "Cayley" $f_{\rm Caley} = \frac{1+x/2}{1-x/2}$

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- "exponential": $f_{\exp} = e^x$ $\to w_{k,n} = \frac{(-1)^k}{1+\delta_{kn}} \frac{1}{(2n+2)!} \binom{2n+2}{k+1}$
- "minimal": $f_{\min} = x + \sqrt{1+x^2}$ $\to w_{2k+1,n} = \frac{(-1)^n}{1+\delta_{2k+1,n}} \binom{k-\frac{1}{2}}{k+1} \binom{n-k-\frac{3}{2}}{n-k}$
- "Cayley" $f_{\mathrm{Caley}} = \frac{1+x/2}{1-x/2}$ $\to w_{k,n} = \frac{(-1)^k}{1+\delta_{kn}} \frac{1}{2^{2n}}$

The stripped Feynman rules can be written

$$V_{2n+2}(s_{i,j}) = (-1)^n \left(\frac{2}{F^2}\right)^n \sum_{k=0}^{n} w_{k,n} \sum_{i=1}^{2n+2} s_{i,i+k}$$

where $s_{i,j} \equiv (p_i + p_{i+1} + \ldots + p_j)^2$.

Explicit example: stripped 4pt amplitude

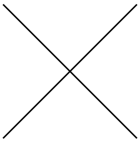
Natural parametrization for diagrammatic calculations: minimal

$$w^{\text{min}}_{2k,n} = 0$$

Thus off-shell and on-shell stripped vertices are equal.

4pt amplitude

$$2F^2\mathcal{M}(1,2,3,4) = -(s_{1,2} + s_{2,3})$$

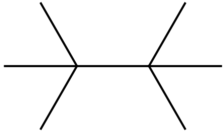
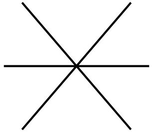


Explicit example: stripped 6pt amplitude

$$
\begin{aligned}
&4F^4\mathcal{M}(1,2,3,4,5,6) = \\
&= \frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,5})}{s_{1,3}} + \frac{(s_{1,4}+s_{2,5})(s_{2,3}+s_{3,4})}{s_{2,4}} \\
&\quad + \frac{(s_{1,2}+s_{2,5})(s_{3,4}+s_{4,5})}{s_{3,5}} - (s_{1,2}+s_{1,4}+s_{2,3}+s_{2,5}+s_{3,4}+s_{4,5})
\end{aligned}
$$

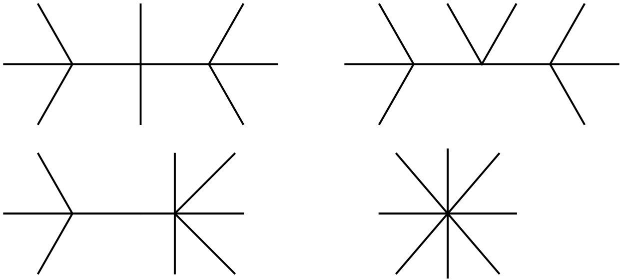
This can be rewritten as

$$
4F^4\mathcal{M}(1,2,3,4,5,6) = \frac{1}{2}\frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,5})}{s_{1,3}} - s_{1,2} + \text{cycl}\,,
$$

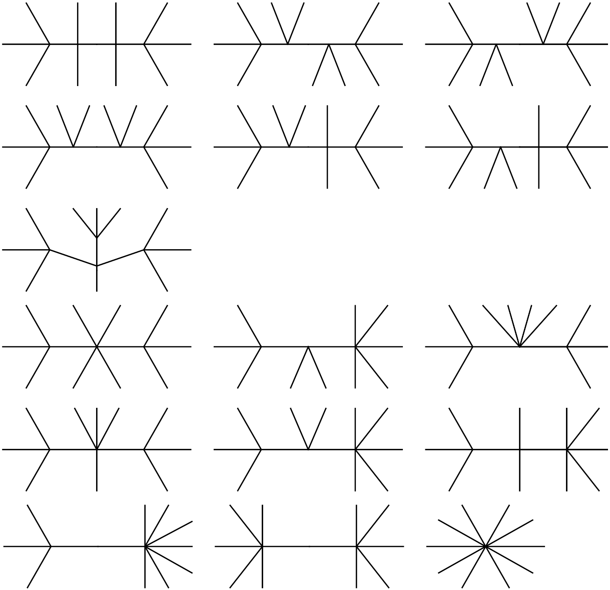
 

Explicit example: stripped 8pt amplitude

$$
\begin{aligned}
&8F^6\mathcal{M}(1,2,3,4,5,6,7) = \\
&= -\frac{1}{2}\frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,7})(s_{5,6}+s_{6,7})}{s_{1,3}s_{5,7}} - \frac{(s_{1,2}+s_{2,3})(s_{1,4}+s_{4,5})(s_{6,7}+s_{7,8})}{s_{1,3}s_{6,8}} \\
&\quad + \frac{(s_{1,2}+s_{2,3})(s_{4,5}+s_{4,7}+s_{5,6}+s_{5,8}+s_{6,7}+s_{7,8})}{s_{1,3}} - 2s_{1,2} - \frac{1}{2}s_{1,4} + \text{cycl}
\end{aligned}
$$

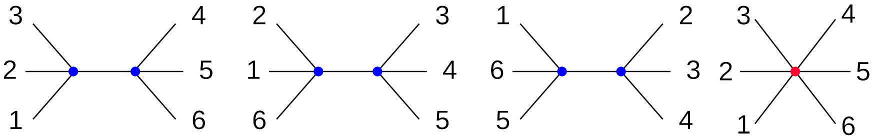


Explicit example: stripped 10pt amplitude



Bottom-up: amplitudes $\Rightarrow$ theory?

Frst non-trivial case – the 6pt amplitude:



schematically:

$$\lambda_4^2\ p^2 \frac{1}{p^2} p^2 + \lambda_6\ p^2$$

in order to combine the pole and contact terms we need to consider some limit. The most natural candidate: we will demand soft limit, i.e.

$$A \to 0, \qquad \text{for} \quad p \to 0$$

$$\Rightarrow \quad \lambda_4^2 \sim \lambda_6$$ corresponds to NLSM

How to extend it to all orders (n-pt)? $\rightarrow$ new recursion relations

New recursion relations: modification of BCFW

[Cheung, KK, Novotny, Shen, Trnka '15]

The high-energy behaviour forbids a naive Cauchy formula

$$A(z) \neq 0 \qquad \text{for} \quad z \to \infty$$

Can we instead use the soft limit directly?

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The high-energy behaviour forbids a naive Cauchy formula

$$A(z) \neq 0 \qquad \text{for} \quad z \to \infty$$

Can we instead use the soft limit directly? $\to$ yes!
The standard BCFW not applicable, we proposed new shifts:

$$p_i \to p_i(1 - z a_i) \qquad \text{on all external legs}$$

This leads to a modified Cauchy formula

$$\oint \frac{dz}{z} \frac{A(z)}{\Pi_i (1 - a_i z)} = 0$$

note there are no poles at $z = 1/a_i$ (by construction).

Natural classification: σ and ρ

Generalization of the soft limit:

$$A(tp_1, p_2, \ldots, p_n) = \mathcal{O}(t^{\sigma}), \qquad \text{as} \quad tp_1 \to 0$$

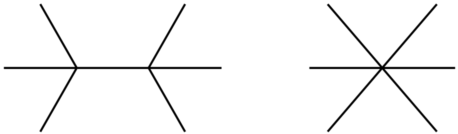
Interaction term

$$\mathcal{L} = \partial^m \phi^n$$

Then another natural parameter is:

$$\rho = \frac{m-2}{n-2} \qquad \text{“averaging number of derivatives”}$$

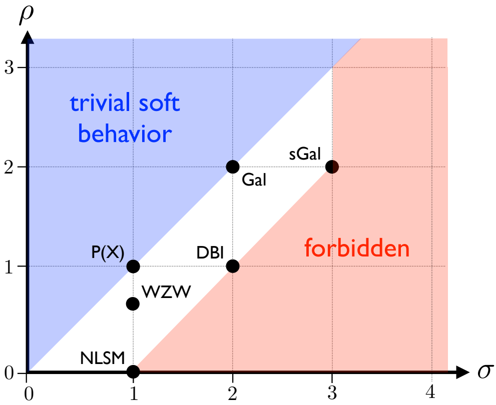
e.g. $\mathcal{L} = \partial^m \phi^4 + \partial^{\widetilde{m}} \phi^6$



so these two diagrams can mix if the same ρ

Summary of Classification of EFTs: “soft-bootstrap”

[C. Cheung, KK, J. Novotny, C. H. Shen and J. Trnka ’17]



- getting non-trivial and exceptional theories
- discovery of special galileon (proof given in Hinterbichler, Joyce ’15)
- independently by Cachazo, He, Yuan ’14

Many directions of the soft bootstrap:

selection of projects I was involved:

- vector effective field theories from soft limits [1801.01496]
- generalization for Adler zero [1910.04766]
- scalar-vector galileon [2104.10693]
- graded soft theorems [2107.04587]
- higher orders [2109.11574]
- NLSM at one-loop [2206.04694] $\longrightarrow$ see also Christoph Bartsch's poster
- GB on celestial sphere: [2303.14761]
- scalar BCJ bootstrap: [2305.05688]

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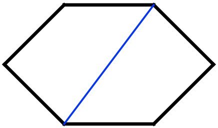
subleading soft theorem: using CHY – new bi-adjoint scalar, [Cachazo,Cha,Mizera'16]

$$\lim_{p_6 \to 0} A_6 = s_{26} A_5^{\phi^3 (521)} + s_{36} A_5^{\phi^3 (531)} + s_{46} A_5^{\phi^3 (541)}$$



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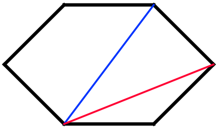

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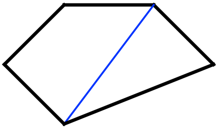
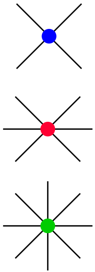


Higher-orders NLSM

40 years of ChPT: up to NNNLO $O(p^8)$

from the amplitude perspective?

yes!: [Dai, Low, Mehen, Mohapatra '20], [KK '21]



	#mesons	#terms
p^2	4	1
p^4	4	2
p^6	4	2
	6	5
p^8	4	3
	6	22
	8	17

Closed form for n-pions starting at $O(p^n)$:

$$\mathcal{L}^n_{\chi\mathrm{PT}} = \sum_{j=1}^{d_n} c_j \langle u_{\mu_{j_1}} \dots u^{\mu_{j_1}} \dots u_{\mu_{j_{n/2}}} \dots u^{\mu_{j_{n/2}}} \rangle$$

Higher-orders NLSM: scalar BCJ bootstrap

[Brown,KK,Oktem,Paranjape, Trnka '23]

BCJ

$$\sum_{i=2}^{n-1}(s_{12}+\ldots+s_{1i})A_n(2,\ldots,i,1,i+1,\ldots,n)=0\,,$$

plays an important role in double copy (e.g. in our context $\text{NLSM}^2 =$ sGal [Cheung, Shen, Wen '17])
We focused on the statement [Gonzalez, Penco, Trodden'19]:

$$\text{BCJ} \;\Rightarrow\; \text{Adler}.$$

For recent studies of the KLT bootstrap see also [Chi, Elvang, Herderschee, Jones, Paranjape '21], [Chen, Elvang, Herderschee '23]

Higher-orders NLSM: scalar BCJ bootstrap

[Brown,kk,Oktem,Paranjape, Trnka '23]

- 4pt

$\mathcal{O}(p^{\#})$	2	4	6	8	10	12	14	16	18
Soft amplitudes	1	2	2	3	3	4	4	5	5
BCJ amplitudes	1	0	1	1	1	1	2	1	2

not the final answer!

Higher-orders NLSM: scalar BCJ bootstrap

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Soft amplitudes	1	2	2	3	3	4	4	5	5
BCJ amplitudes	1	0	1	1	~~1~~ **0**	1	~~2~~ **1**	1	~~2~~ **1**

not the final answer!

- analysis of 6pt (up to $O(p^{18})$ and 8pt (up to $O(p^{10})$): many surprised relations among coefficients of different orders, e.g.

$$\alpha^{(10)} \sim \left(\alpha^{(6)}\right)^2$$

- what are "BCJ Lagrangians"?
  - NLSM
  - Z-theory [Broedel, Schlotterer, Stieberger '13], [Carrasco, Mafra, Schlotterer'16]
- very fresh work on the extension to the gauged NLSM: [Li, Roest, ter Veldhuis '23]

Loop level: motivation

- NLSM (ChPT) at the loop level phenomenologically important.
- State of the art: two-loop calculations ($\pi\pi$ scattering, $\eta \to 3\pi$) - but these are "only" 4pt
- 6pt pion scattering analysis at one-loop level only recently [Bijnens, Husek, Sjö '21-'22]
- on-going plan to use it in the lattice analysis

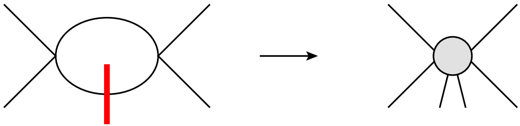
Loop level: our strategy

focus on the integrand, follow the logic from the tree-level

- simplification: planar limit → cyclically ordered
- can calculate the n-loop integrand recursively using e.g. the minimal parametrization
  (for a novel approach see Nima's talk tomorrow, and see also 2 posters by him and Cao, Dong, Figueiredo, He)
- our plan: determine the integrand based on
  - knowledge of the factorizations
  - Adler zero in all external legs
- we can use
  - different parametrization (unphysical but visible!)
  - tadpoles (integrate to zero)

One-loop level: summary of our result [Bartsch, KK, Trnka '22]

- natural $(n+2)$-pt object to deal with, after the single cut:



- we define the B-function (it is not an amplitude!)

$$B_{n+2}(p_1, \ldots, p_n, -\ell, \ell) := \underset{\ell^2=0}{Res}\, I_n(\ell, p_1, \ldots, p_n)$$

- B_{n+2} can be obtained recursively based on the two properties:
  1. by consistent factorization on poles $\ell_{1\ldots i}^2 = 0$
  2. by its soft limit in all but two legs ($i = 1, n$)
- summary

> double bootstrap: Modulo tree-level amplitudes the one-loop n-pt integrand is obtained recursively using the lower-pt integrand <u>and</u> B_{n+2} which is given recursively using the lower-pt B.

- price to pay: our tadpoles don't correspond to any parametrization

Two-loop level [Bartsch, KK, Novotny, Trnka in progr.]

- possible to extend to higher orders?

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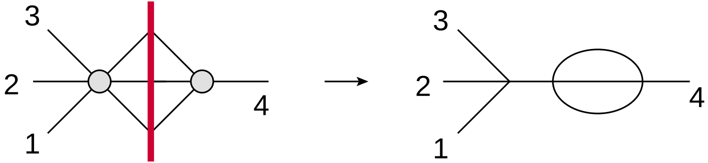
- possible to extend to higher orders? $\rightarrow$ no!
- already two-loop 4pt amplitude is problematic! e.g. following cut leads to (6pt)x(4pt), one of which is not one-particle irreducible

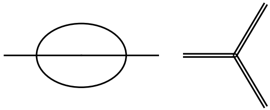


- way out? $\rightarrow$ we have to give up on the Adler zero!
- or better: we have to modify it

Soft theorem of integrand [Bartsch, KK, Novotny, Trnka in progr.]

- reduced non-vanishing soft theorem for the two-loop level, e.g. 4pt

$$\lim_{p_4 \to 0} I_4(\ell_1, \ell_2, p_1, p_2, p_3, p_4) = \sum_{1,3}$$



- appearance of the extended theory already at the leading order!
- similarly for the higher n-pt two-loop integrands
- why reduced?
  - because of tadpoles $\rightarrow$ they can go to the two-point function
  - this changes the form of tadpoles in integrand (they can now agree with the minimal parametrization) $\Rightarrow$ already at one loop!

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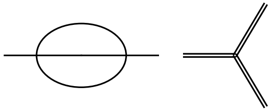

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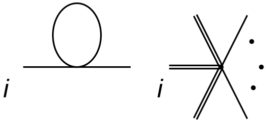
- complete non-vanishing one-loop soft-theorem

$$\lim_{p_n \to 0} I_n(\ell, p_1, \dots, p_n) = \sum_{i=1,n-1}$$



Summary

- short overview of ten years of the soft bootstrap
- focused on three subjects:
  - higher power-counting orders in NLSM
    we can generalize the soft bootstrap
  - scalar BCJ bootstrap
    surprising consistent relations from higher orders
  - loop orders of NLSM
    non-trivial soft theorems of integrands: soft limit of the integrand is not zero, but proportional to the extended amplitudes

thank you!