Towards Gravitational Scattering at the Fifth Order in G

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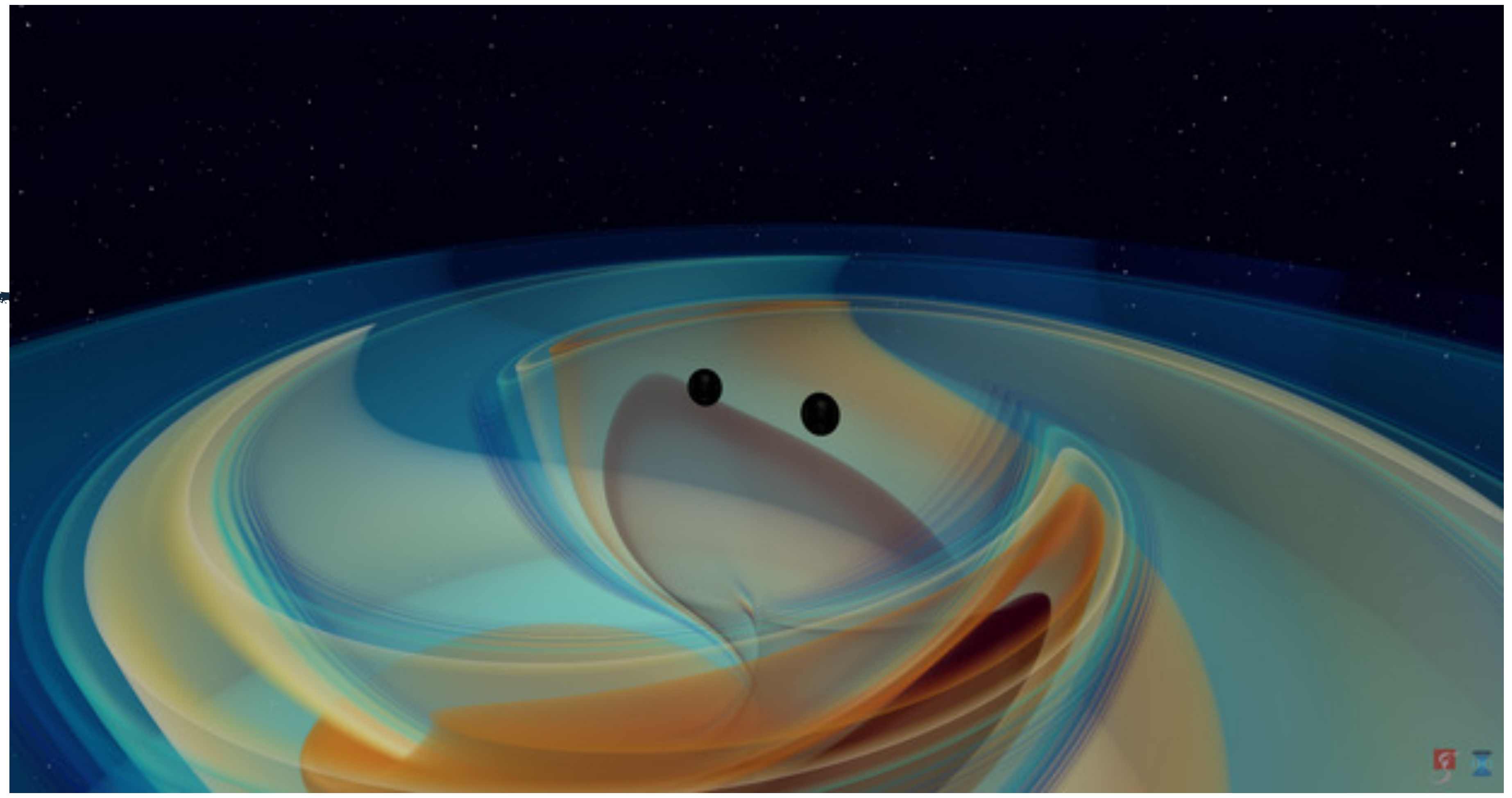
Amplitudes, CERN, Aug 8 2023

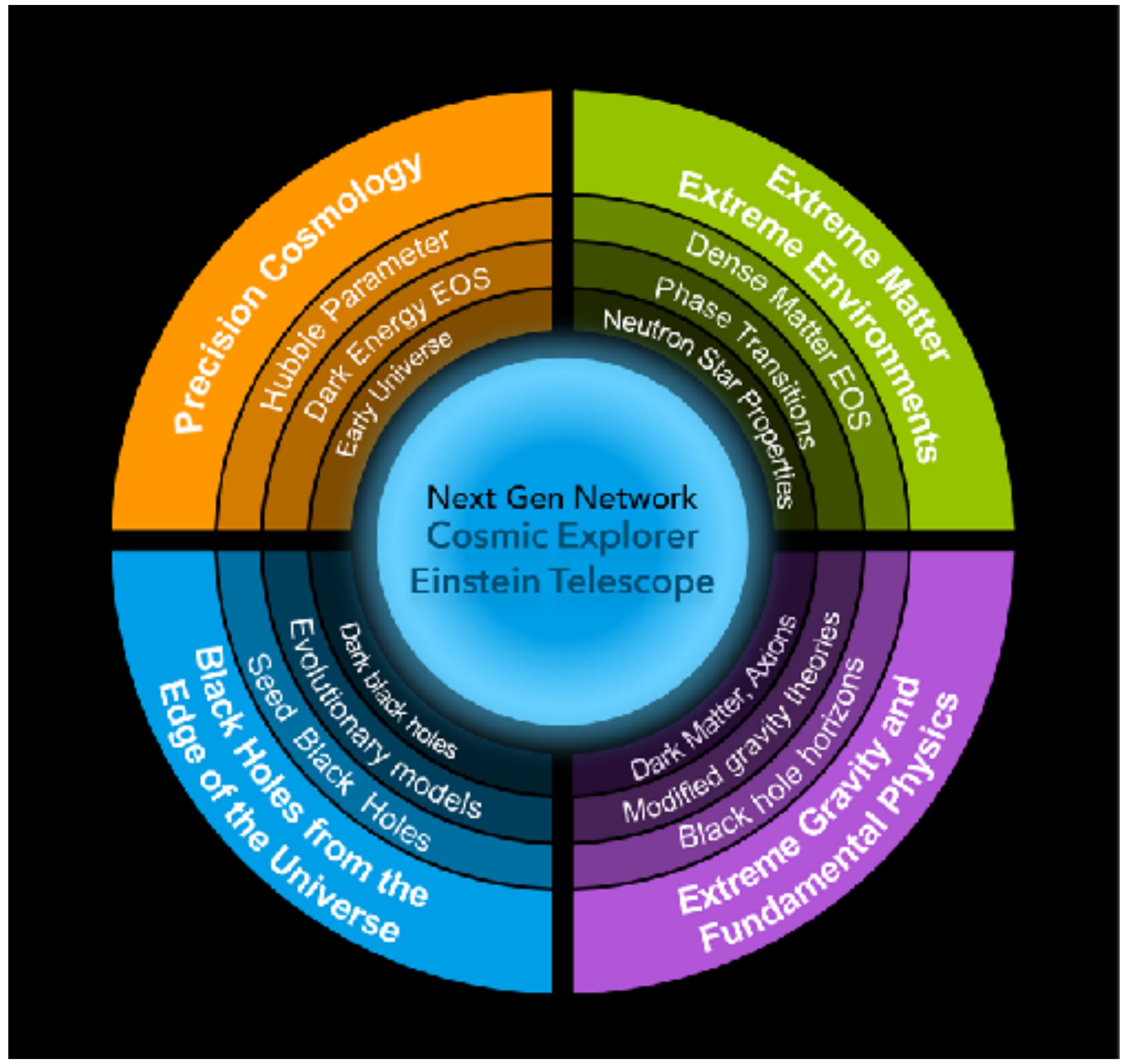
Based on work in collaboration w/ [Bern, Herrmann, Parra-Martinez, Roiban, A. Smirnov, V. Smirnov, Solon, Shen, Zeng]

Motivation

- GW abundant, important source: compact binary systems
- Physics goals:
  - Strong-field tests of GR, new physics
  - BH properties, abundance etc.
  - Ultra-dense matter (neutron star equation of state)
  - Multi-messenger astronomy
  - …

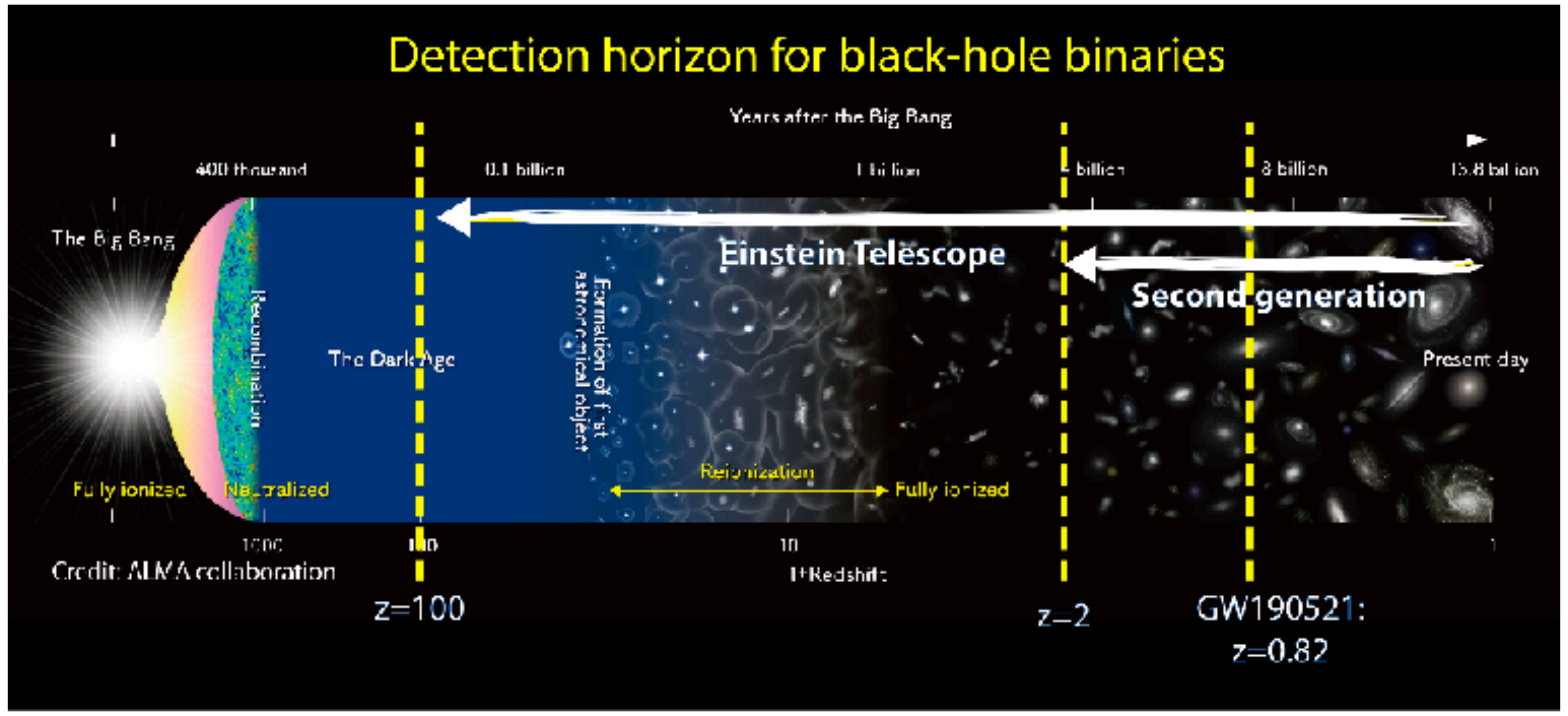


[GW190521, LIGO]

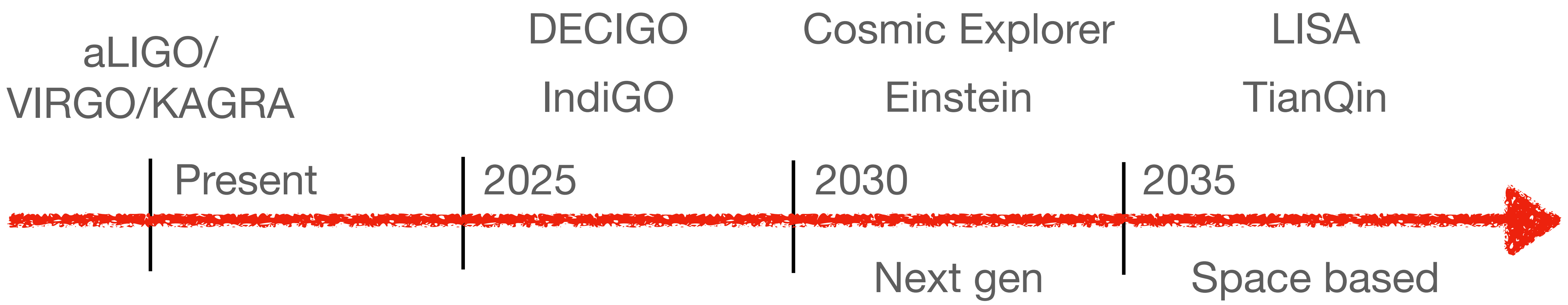


Motivation

- Present detectors (LIGO/VIRGO/KAGRA):
  - >100 events. O4: 150 events/year



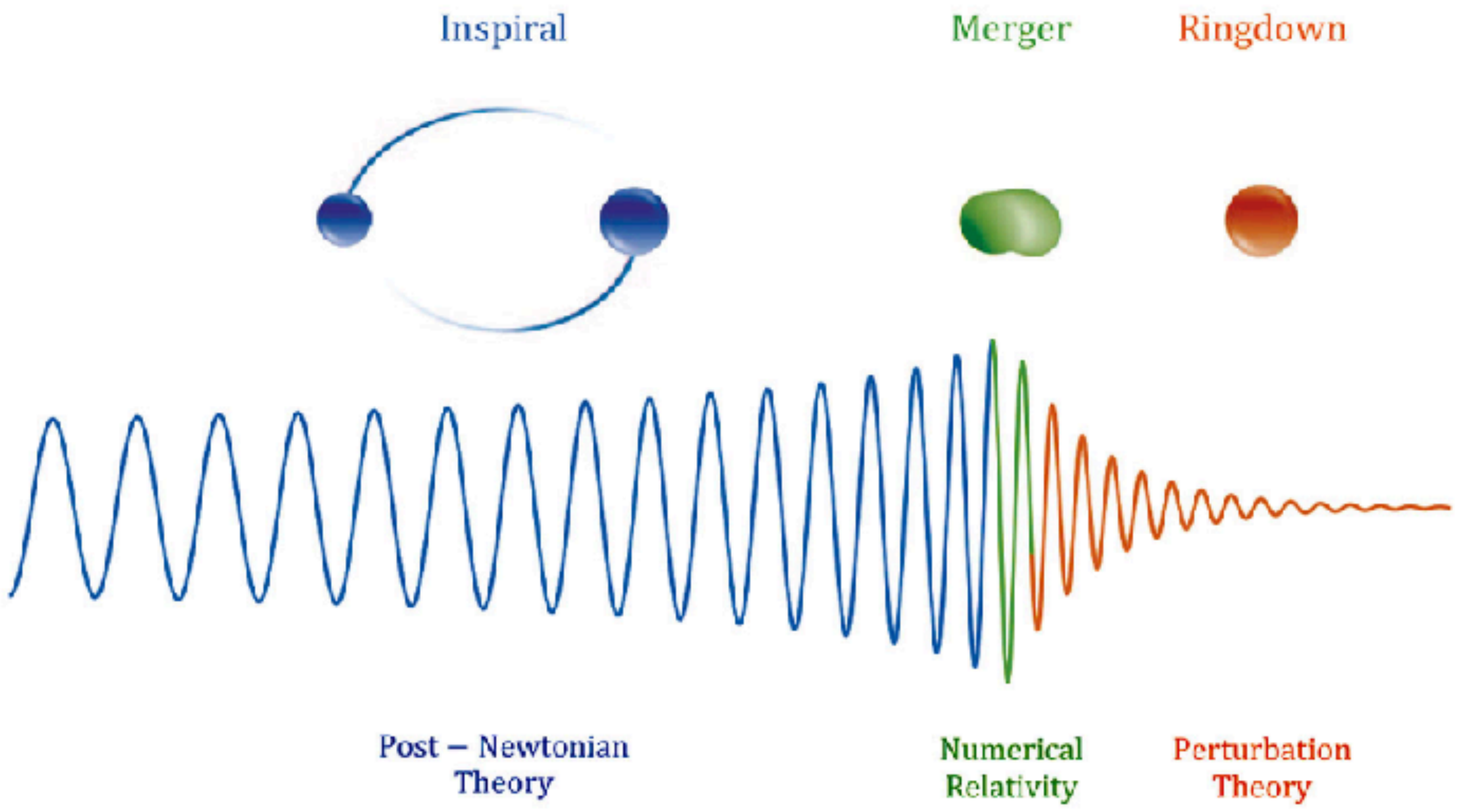
- Next-gen. experiments (ca. 2035):
  - More precision 10-100 x improved S/N, wider frequency band
  - More data (bigger reach + sensitivity), 20-60 events/day →pileup!
  - Extreme corners of parameter space, e.g. EMRI



Need high-precision wave-form modelling!

Motivation

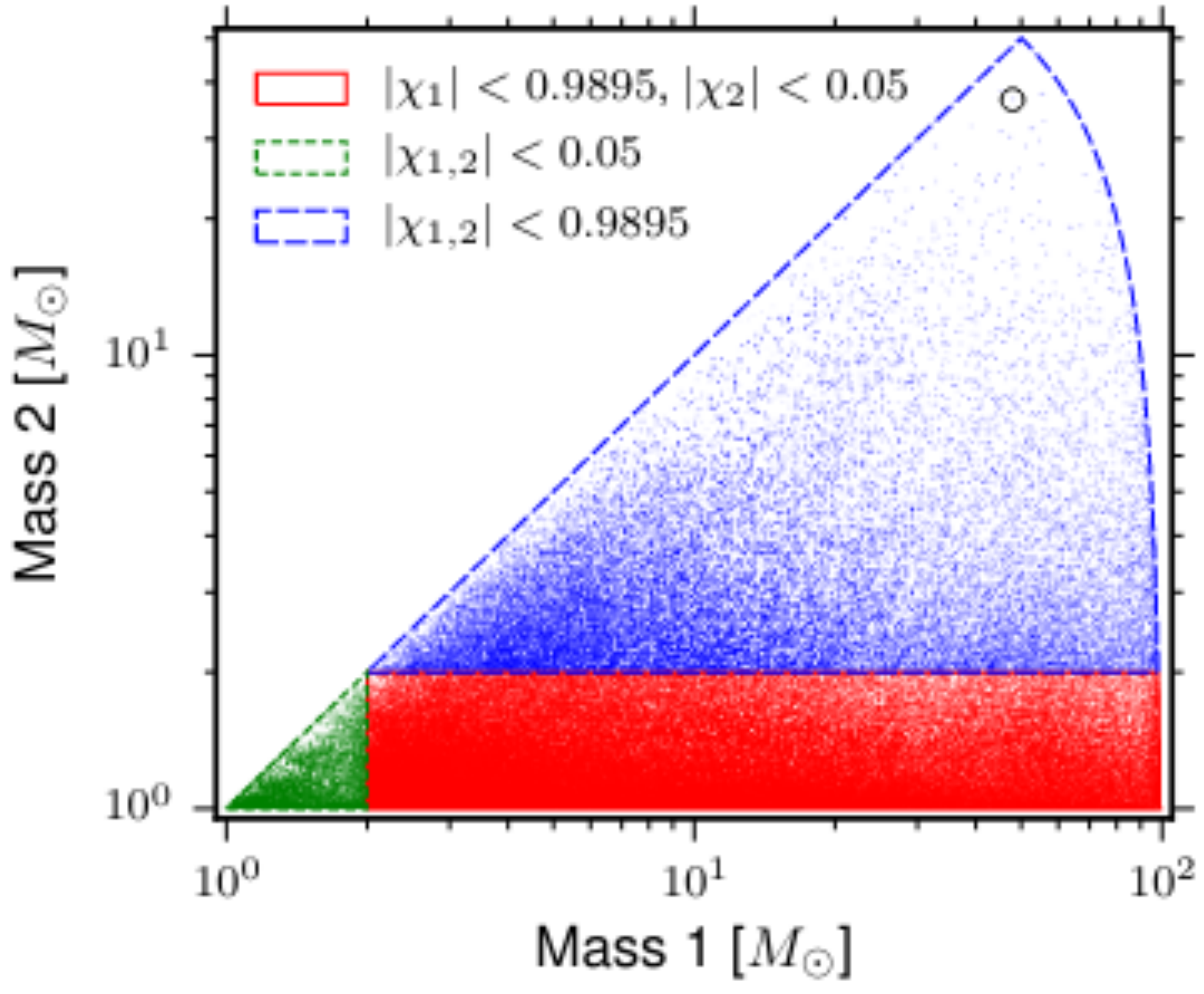
- Numerical waveform from Einstein eqn $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$



[1610.03567]

- Significant resource requirements:
  - $O(10^5)$ CPU h/ NR template
  - GW150914: 250k templates
  - Challenging in PS-corners: $m_1 \ll m_2, v \to c, |\vec{L}|/m \to 1$
- Solution: analytic and hybrid models (GW150914: post-Newtonian + effective-one-body)
- Corrections to Newton's potential to high orders

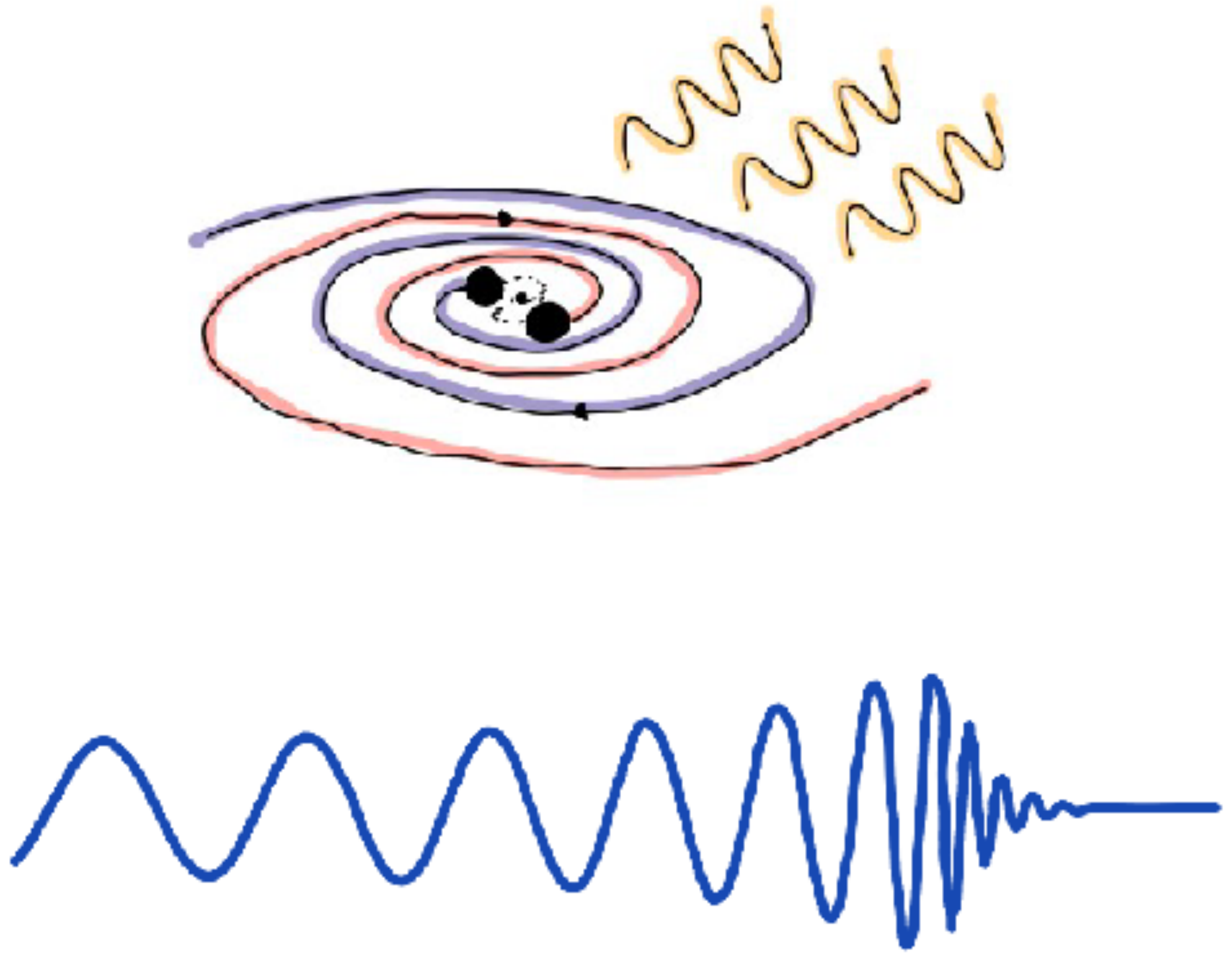
$$V(r) = -\frac{G\mu M}{r} + \frac{1}{c^2}\left[-\frac{3G\mu M \boldsymbol{v}^2}{2r} + \frac{G^2M^2}{r^2}\right] + \dots$$



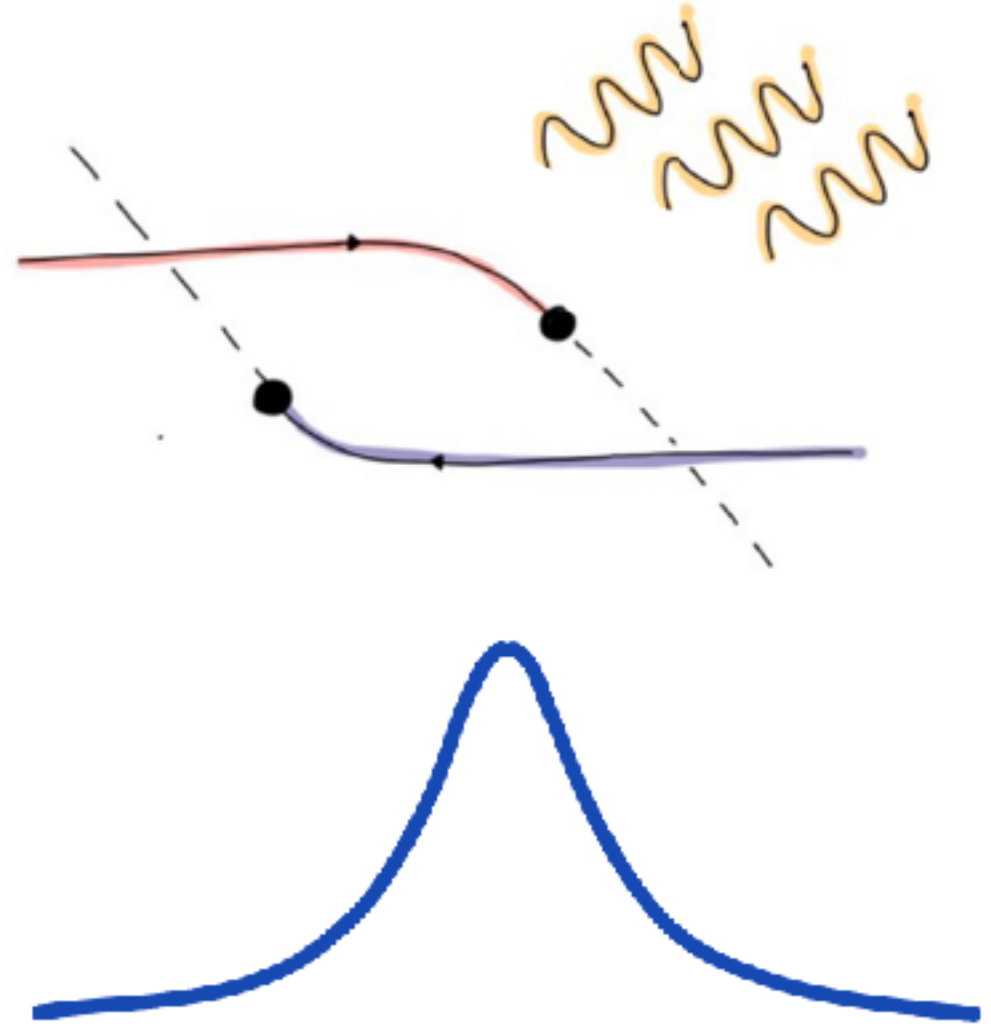
[GW150914, LIGO]

Gravitational Scattering

Process of interest: scattering of compact massive objects

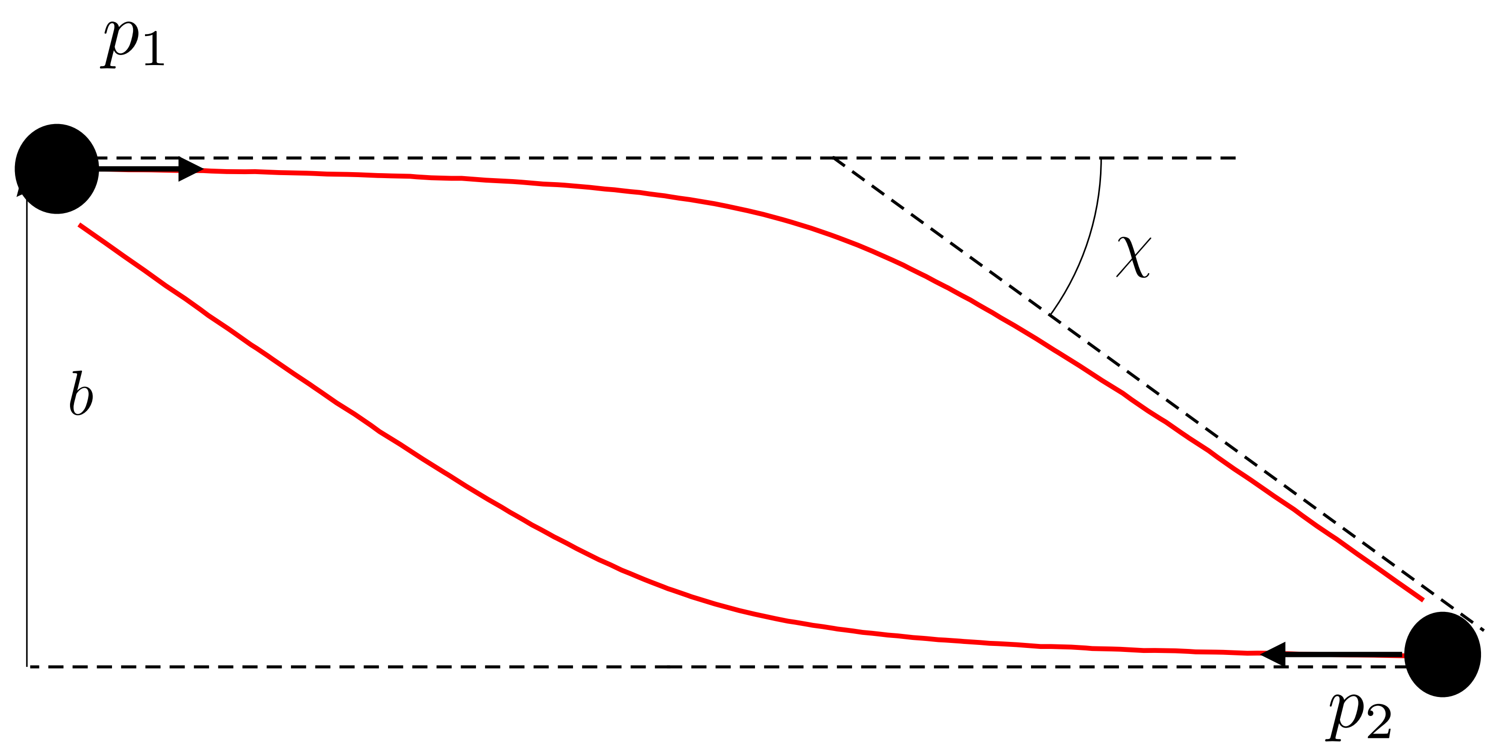


vs



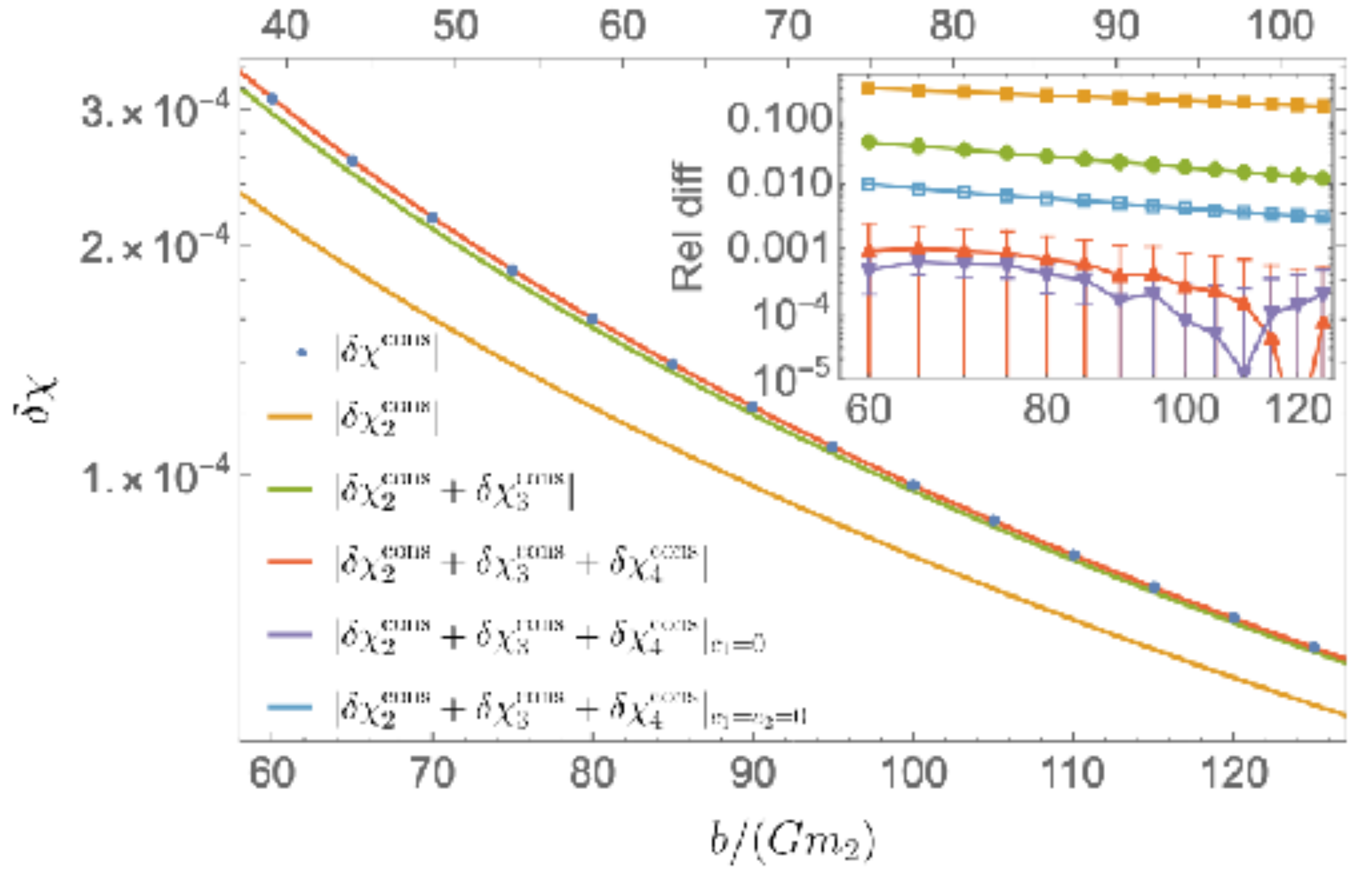
Hard to observe in GW observatories. Why bother?

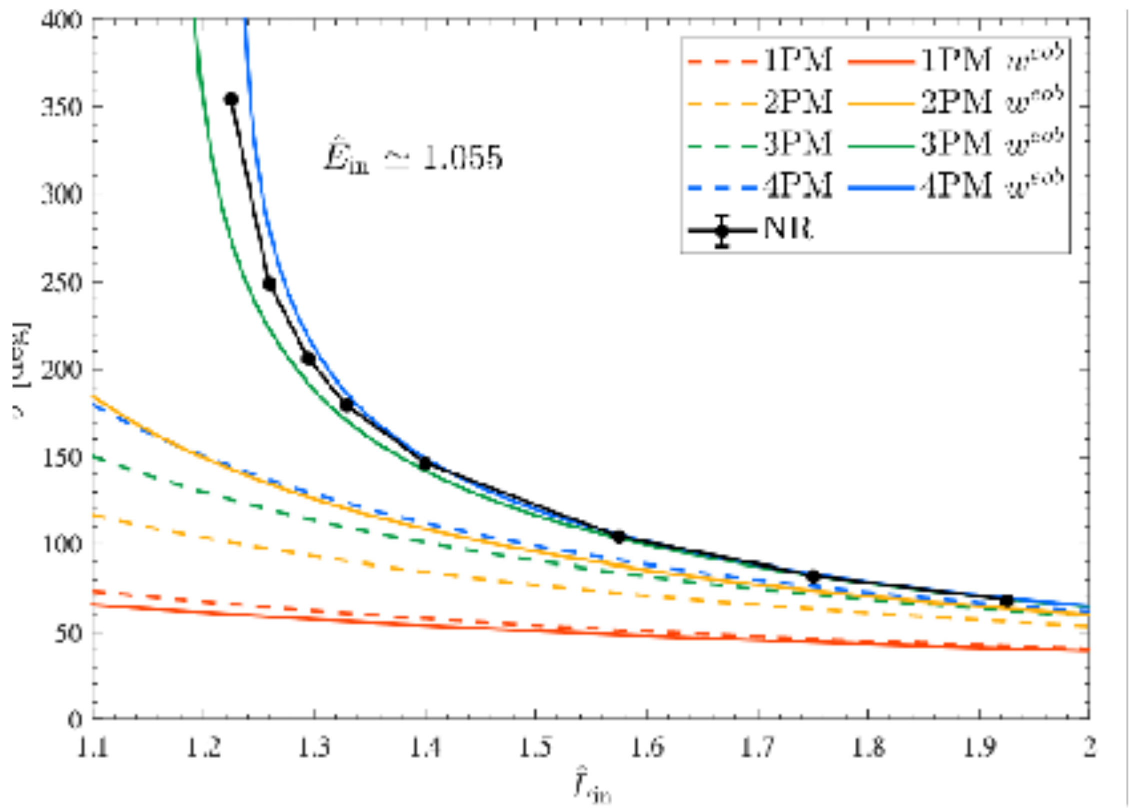
Gravitational Scattering



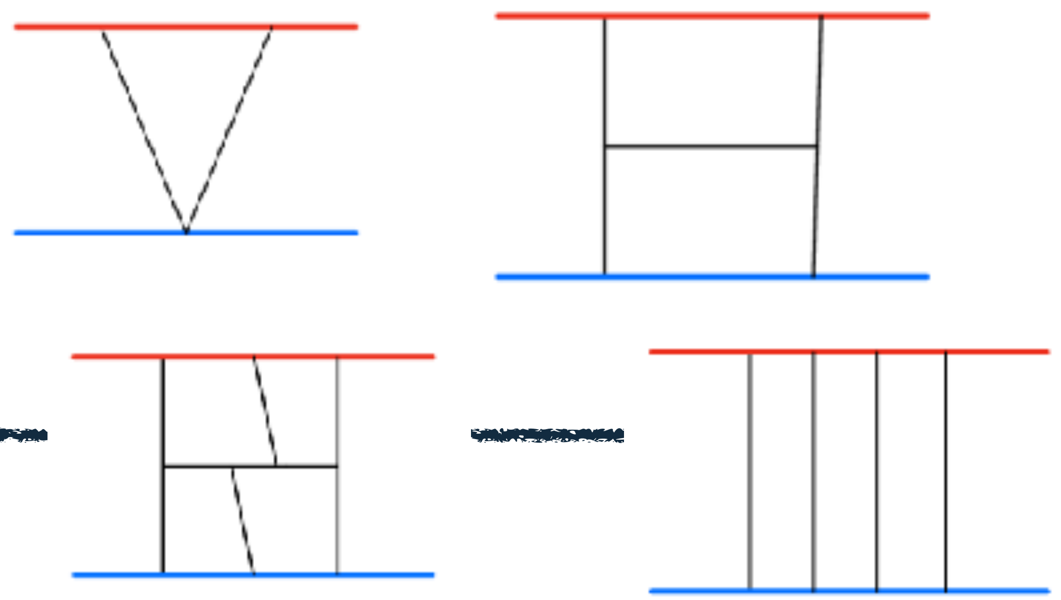
- Why bother?
  - Clean, determined by initial data $\{b, p_i\}$, no gauge dependence
  - Close relation to bound process
  - Relativistic treatment exposes structures, e.g. mass polynomiality [Damour]
  - Amplitudes

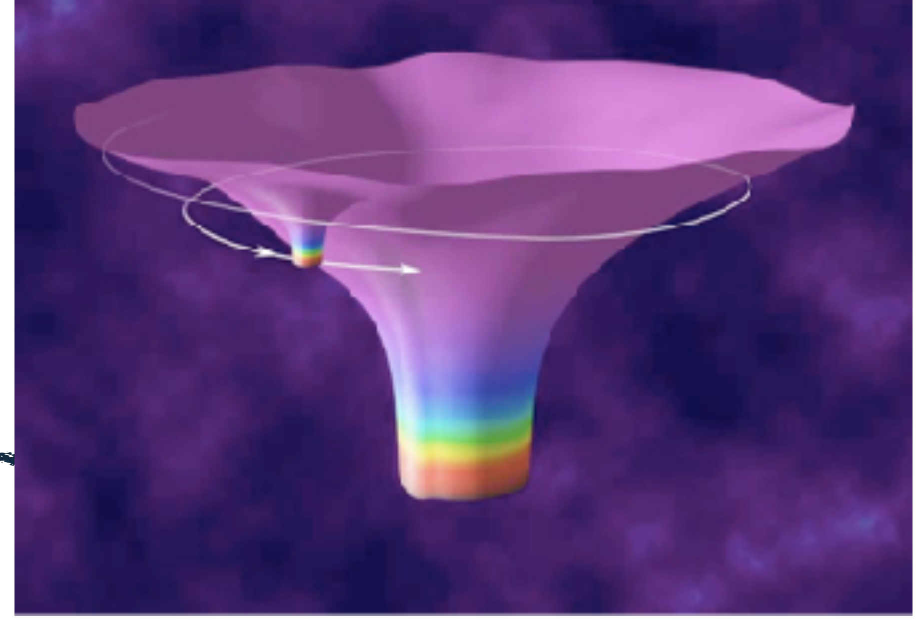
- Classical GR community is very interested in scattering
  [Barack, Berti, Bini, Buonanno, Cardoso, Damour, East, Geralico, Gralla, Guercilena, Hinder, Hinderer, Hopper, Khalil, Lobo, Long, van de Meent, Nagar, Pfeiffer, Pratten Pretorious, Pretorius, Rettegno, Rezzolla, Schmidt, Sperhake, Steinhoff, Thomas, Vines, Whittall, Yunes,. . .]





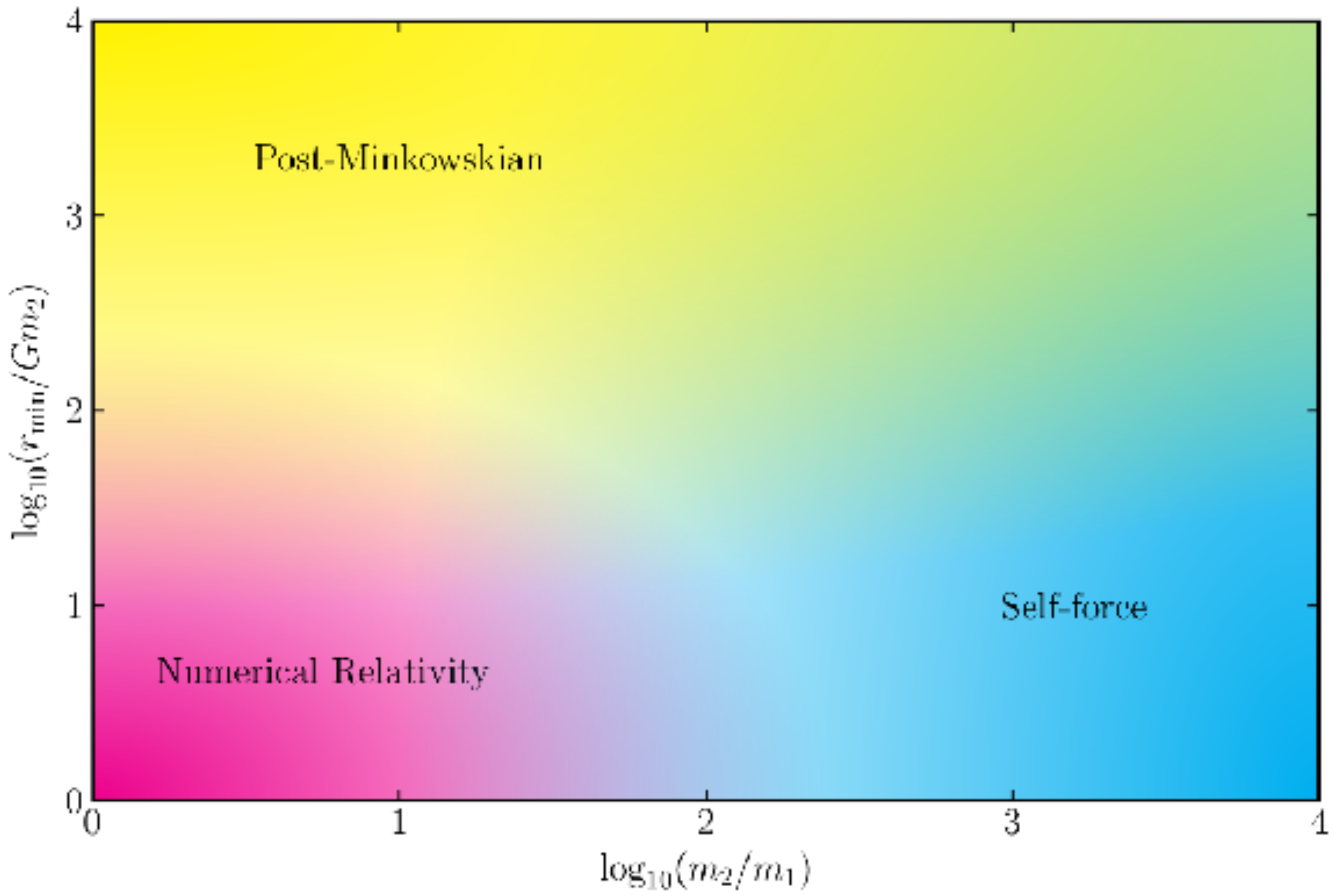
Gravitational Scattering



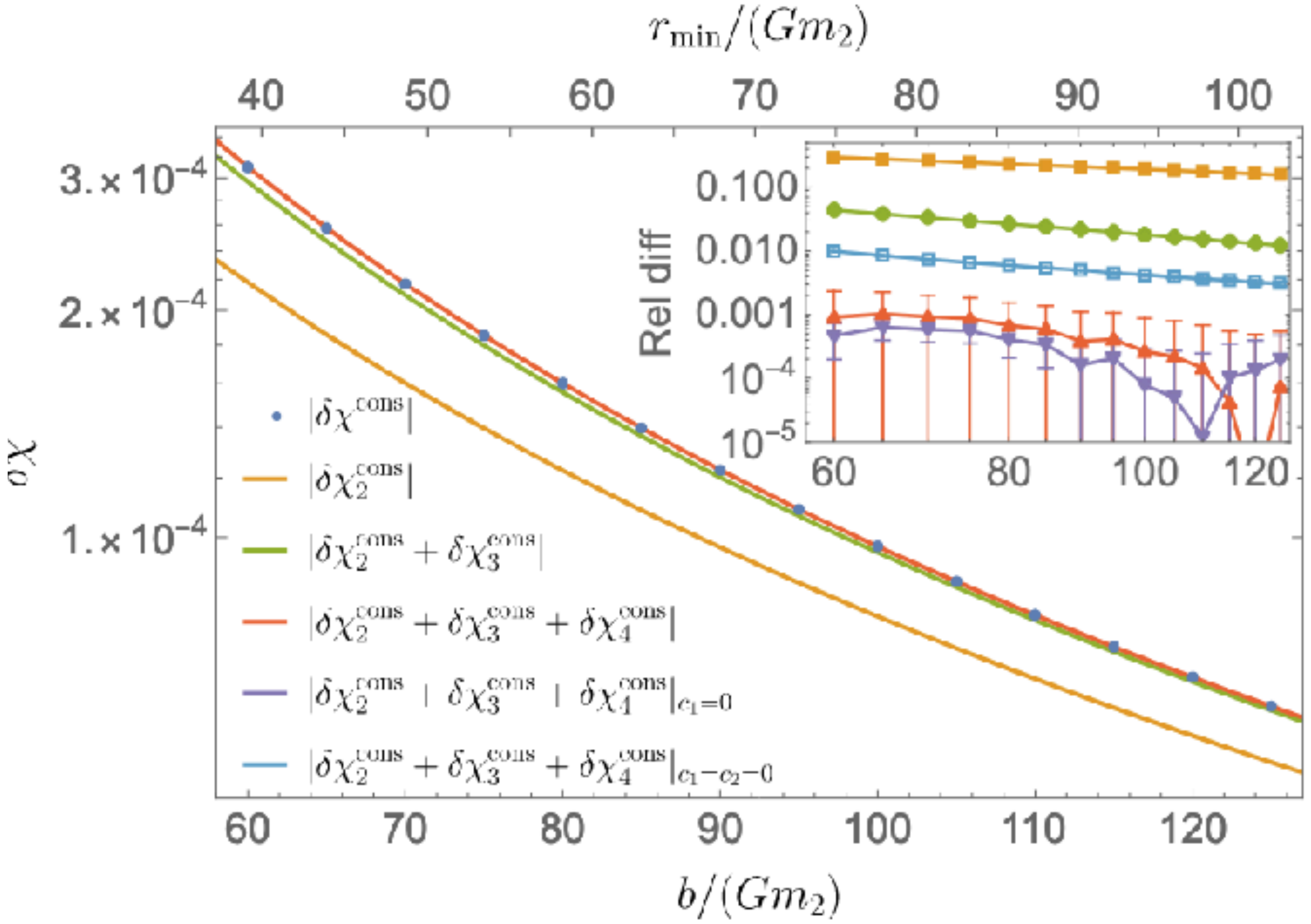


- Collaborative effort: weak field ($G \ll 1$) vs. 'gravitational self-force' ($m_1 \ll m_2$)
  [Barack, Bern, Herrmann, Long, Roiban, Parra-Martinez, **MSR**, Shen, Solon, Teng, Zeng]

- Complementary:
  - PM in weak field
  - SF/NR in strong field



- Goals:
  - Hybrid models
  - Resummations
  - Benchmarking



Percent-level agreement!

- Scalar toy model: physical effects enter at lower orders!
  - Finite size effects at $\mathcal{O}(Q^2G^3)$/3-loops!

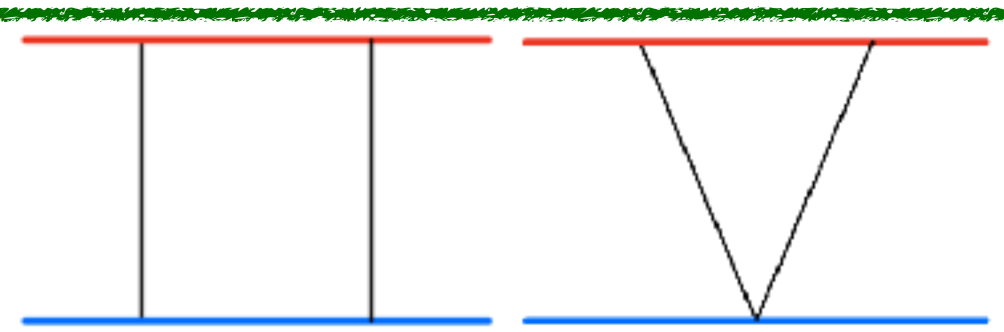
- Exiting prospects: Higher precision, Higher perturbative orders, comparisons in GR

Gravitational Scattering

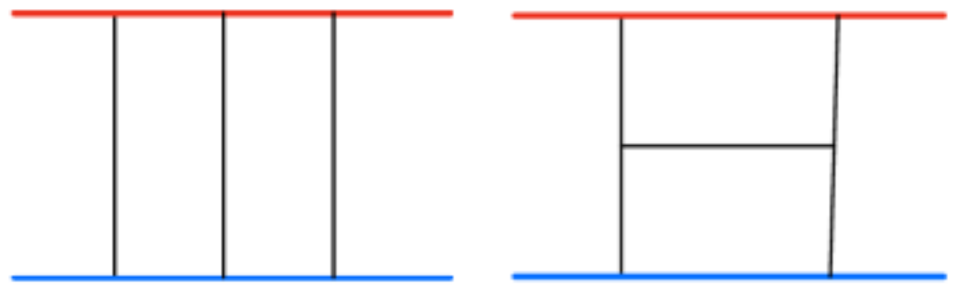
n-th post-Minkowskian order $= \mathcal{O}(G^n) = n$PM $= n - 1$ loops

1PM $G \times (1 + v^2 + v^4 + v^6 + v^8 + \ldots)$

2PM 1985 $G^2 \times (1 + v^2 + v^4 + v^6 + v^8 + \ldots)$ Westphal



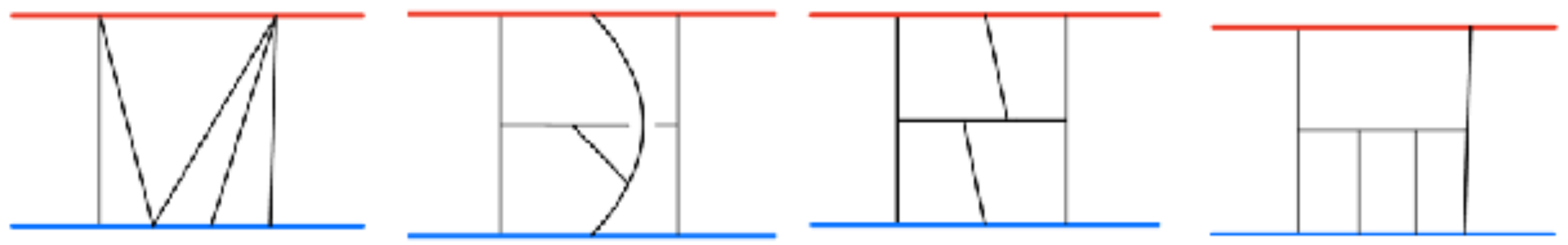
3PM 2019* $G^3 \times (1 + v^2 + v^4 + v^6 + v^8 + \ldots)$ Bern, Cheung, Roiban, Shen, Solon, Zeng



4PM 2021* $G^4 \times (1 + v^2 + v^4 + v^6 + v^8 + \ldots)$ Bern, Parra-Martinez, Roiban, **MSR**, Shen, Solon, Zeng

Important 4PM results: [Dlapa, Kalin, Liu, Neef, Porto; Bjerrum-Bohr, Plante, Vanhove; Jakobsen, Mogull, Plefka, Sauer, Xu]

5PM WIP $G^5 \times (1 + v^2 + v^4 + v^6 + v^8 + \ldots)$

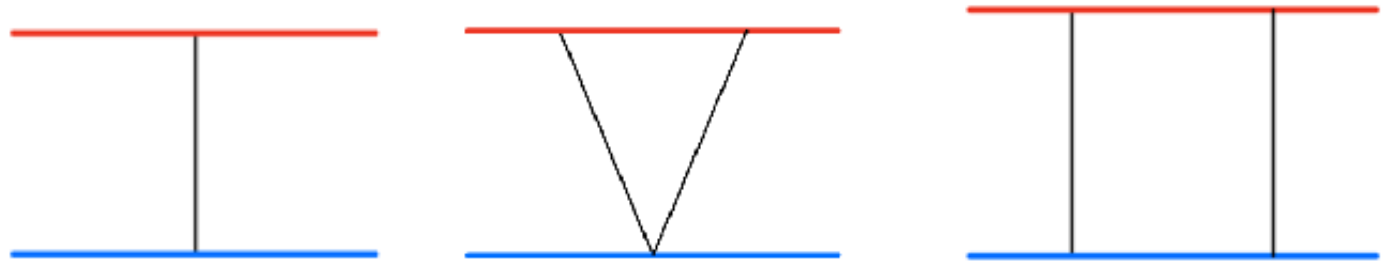


* conservative only

Gravitational Scattering

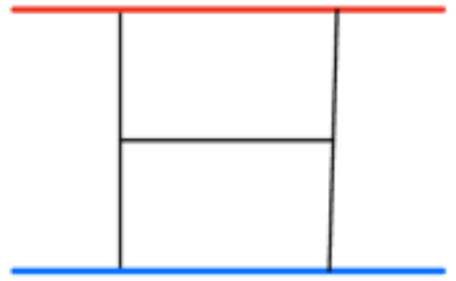
Every order exposes interesting features/puzzles!

1PM & **2PM** fixed by geodesic (zeroth-order self-force; 0SF)!



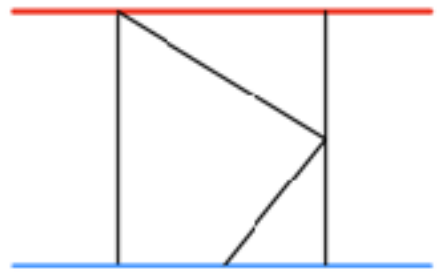
$$\longleftarrow \quad ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)\, d\phi^2$$

3PM: Excursions into high-energy limit

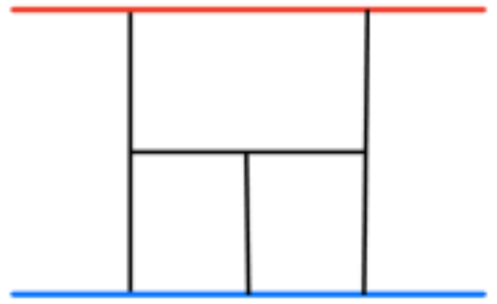


$$\sim \operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right) \to \log(E^2/m_1 m_2)$$

4PM: Elliptic functions, IR divergences/‘tail’ effect (analytic continuation?)

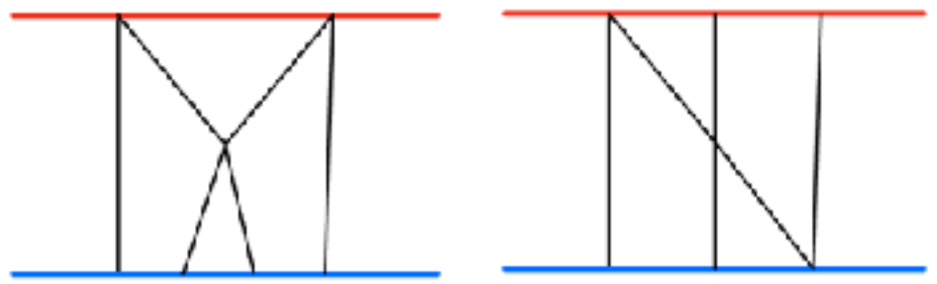


$$\sim K^2\left(\frac{\sigma-1}{\sigma+1}\right)$$

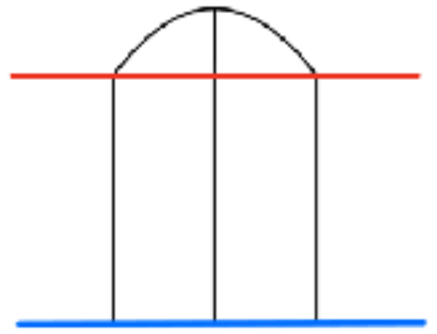


$$\sim \frac{1}{\epsilon}$$

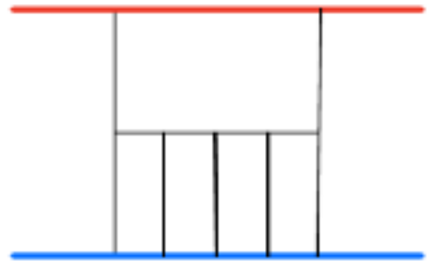
5PM: New functions? ’memory’ contributions? Second-order self-force, …

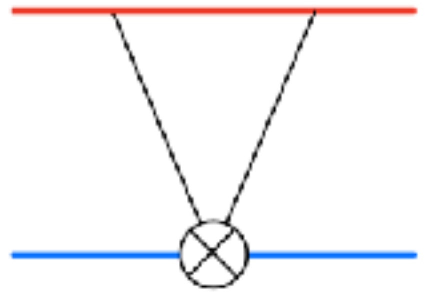


$$\sim ?$$



6PM: UV divergences, finite size effects, distinguish BH and NS!





Gravitational Scattering

- Other important directions besides higher orders:
  - Spin ~ higher spin particles → [talk by Jan Plefka]
    [Alessio, Aoude, Bautista, Bern, Cangemi, Chiodaroli, Chung, Damgaard, Di Vecchia, Febres Cordero, Guevara, Haddad, Helset, Hoogeveen, Huang, Jakobsen, Kim, Kosmopoulos, Krauss, Lee, Levi, Lin, Liu, Luna, Maybee, Mogull, **MSR**, Ochirov, O'Connel, Pichini, Plefka, Porto, Roiban, Sauer, Shen, Skvortsov, Steinhoff, Vines, Yang, Zeng,…]
  - Wave-forms ~ higher-point amplitudes → [talk by Donal O'Connell]
    [Brandhuber, Brown Chen, Christofoli, DeAngelis, Elkhidir, Georgoudis, Gonzo, Gowdy, Heissenberg, Herderschee, Kosower, OConnel, Roiban, Sergola, Teng, Travaglini, Vazquez-Holm,…]
  - Finite size effects ~ higher-dimensional operators
    [Bern, Cheung, Parra-Martinez, Roiban, Sawyer, Shen, Solon,…]
  - Radiation ~ interesting relation to soft theorems
    [Di Vecchia, Dlapa, Heissenberg, Herrman, Jakobsen, Källin, Liu, Manohar, Mogull, Neef, ParraMartinez, Plefka, Porto, Ridgway, **MSR**, Russo, Sauer, Shen, Veneziano, Zeng,…]

From Amplitudes to Observables

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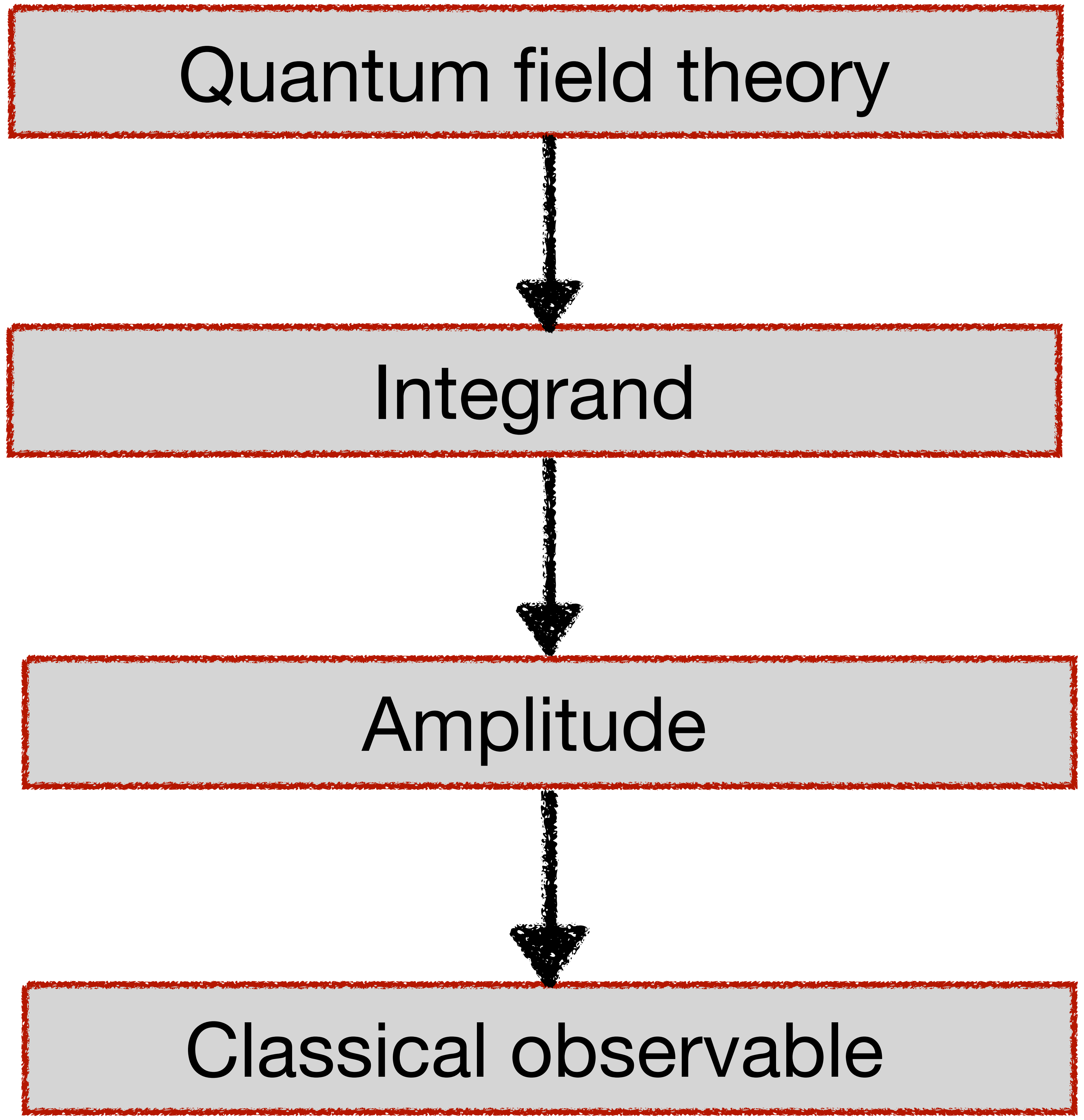

Feynman rules,
Generalised unitarity

Which field theory?
Algebraic complexity

Expansion in classical limit
IBP reduction
Evaluation of master integrals

Series expansions
Large IBP systems
Special functions beyond GPL’s

Stationary phase arguments,
Direct computation (KMOC),
Matching computations
(Non rel. EFT/Schrödinger)

Subtracting singular pieces

Scattering angle χ, ...

From Amplitudes to Observables

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Quantum field theory

Feynman rules,
Generalized unitarity

↓ Easy

Which field theory?
Algebraic complexity

Integrand

Expansion in classical limit
IBP reduction
Evaluation of master integrals

↓ Hard

Series expansions
Large IBP systems
Special functions beyond GPL’s

Amplitude

Stationary phase arguments,
Direct computation (KMOC),
Matching computations
(Non rel. EFT/Schrödinger)

↓ Easy

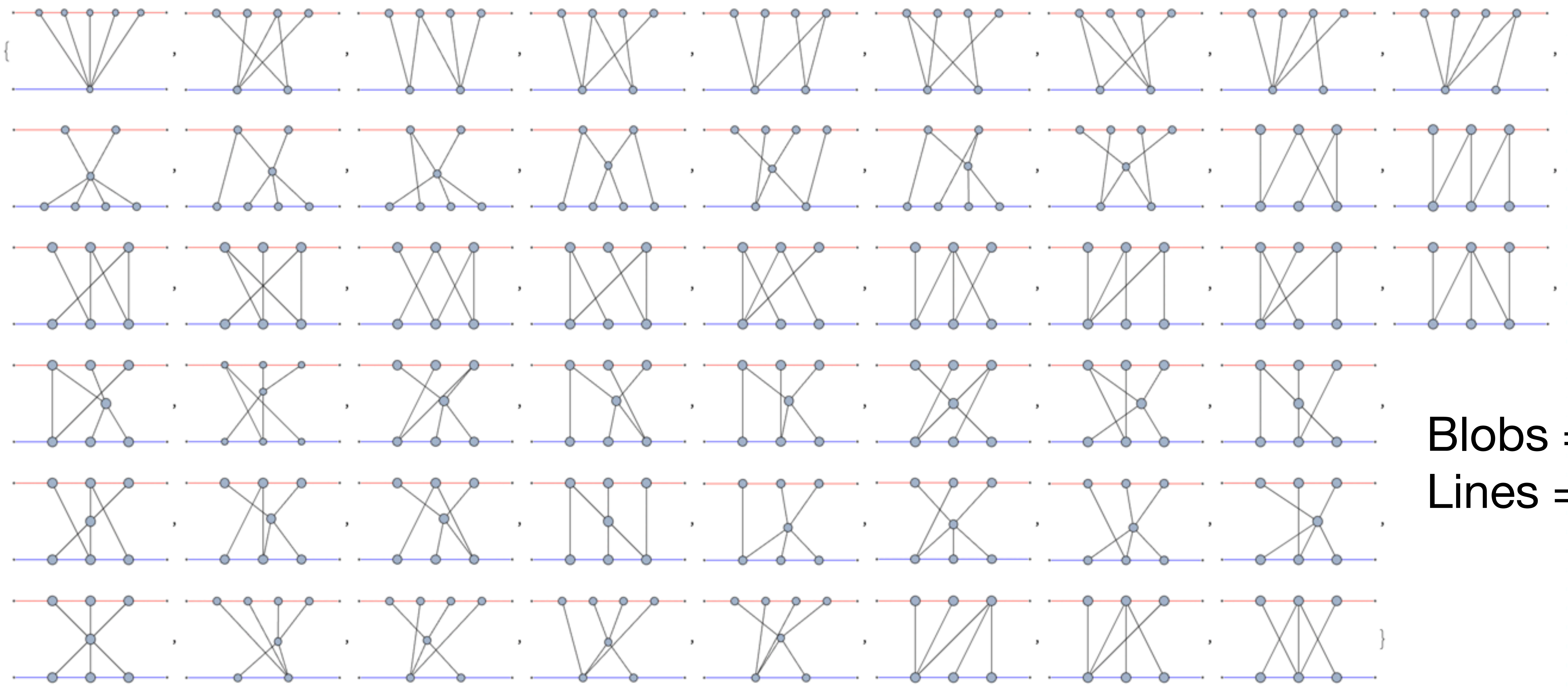
Subtracting singular pieces

Classical observable

Surprisingly, gravity interactions complicated but no bottleneck!

Integrand — Generalized unitarity

- Construct integrand from 51 unitary cuts (+images)



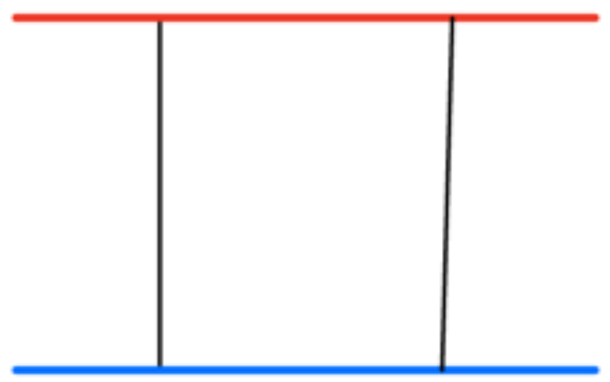
Blobs = tree amplitudes
Lines = on-shell gravitons

- Drastically simplified in classical limit!
  - No graviton loops, self energies, matter contacts
  - 1 matter line per loop

6PM/5-loops straightforward too!

Integrand

- Avoid construction of an off-shell integrand
  - High power-counting (large ansatz)
  - Need to impose additional constraints, e.g. scaling in classical limit
- Cut merging after IBP: determine individual master coefficients directly from cuts
- Expansion in classical/soft limit ($q \to 0$), efficient e.g. through shift operators
- Resulting in large number of HQET type integrals



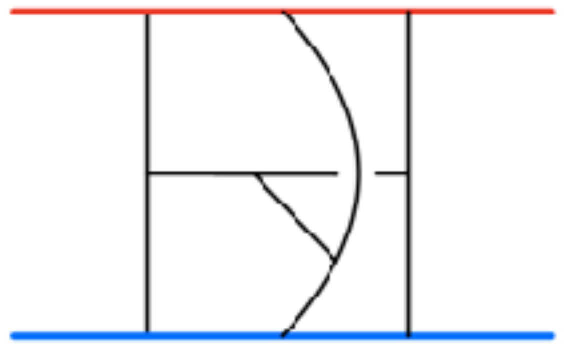
$$= \int \mathrm{d}^D\ell \frac{1}{\ell^2} \frac{1}{(\ell - q)^2} \frac{1}{2u_1 \cdot \ell} \frac{1}{2u_2 \cdot \ell}$$

Single-variable problem!

$$\sigma = u_1 \cdot u_2 = 1/\sqrt{1 - v^2}$$

Integral reduction

- Integral reduction using Laporta algorithm
  - 22 ‘indices’: 13 propagators and 9 irreducible scalar products



$$= \int \frac{d^{4D}k\,[u_2\cdot k_1]^{a_{-14}}[u_2\cdot k_4]^{a_{-15}}[u_1\cdot k_2]^{a_{-16}}[u_1\cdot k_3]^{a_{-17}}[k_1\cdot q]^{a_{-18}}[k_2\cdot q]^{a_{-19}}[k_1\cdot k_2]^{a_{-20}}[k_1\cdot k_4]^{a_{-21}}[k_2\cdot k_3]^{a_{-22}}}{[-2u_2\cdot k_2]^{a_1}[-2u_2\cdot k_{123}]^{a_2}[2u_1\cdot k_{234}]^{a_3}[2u_1\cdot k_{1234}]^{a_4}[k_1^2]^{a_5}[k_2^2]^{a_6}[k_3^2]^{a_7}[k_{13}^2]^{a_8}[k_4^2]^{a_9}[k_{34}^2]^{a_{10}}[k_{234}^2]^{a_{11}}[(k_{123}-q)^2]^{a_{12}}[(k_{1234}-q)^2]^{a_{13}}}$$

  - Tensor rank up to $a_i = 8$ + four doubled proagators
  - 394 families of integrals
  - 4000 master integrals
  - Up to 48 master integrals on maximal cut

- Explosion in the number of equations

IBP reduction is the main bottleneck of the 5PM computation!

Integral reduction

- Many straightforward improvements:
  - Filtering redundant equations (parity)
  - Choosing better basis of master integrals [Smirnov; Usovitch]
  - Identification of identical sectors/symmetries
  - Finite fields and functional reconstruction
  - Code improvements (upcoming version of FIRE)
- Improvement of several orders of magnitude!
- Reduction of low-rank tensors and simpler families feasible, especially differential equations!

Integration [Parra-Martinez, MSR, Zeng '20]

- Master integrals satisfy Fuchsian differential equations (DE) [Kotikov '91]

$$\frac{\mathrm{d}}{\mathrm{d}x}\vec{I} = \sum_k \frac{\mathrm{d}\log(w_k)}{\mathrm{d}x} A_k(\epsilon)\vec{I}\,, \quad w_k \in \{x, 1 \pm x, 1 + x^2, \ldots\}\,, \quad \vec{I} = (I_1, \ldots, I_{3943})^T\,, \quad A_k(\epsilon) \in M_{3943}(\mathbb{Q})[\epsilon]$$

- Change of basis $\vec{I} \to \vec{J} = T\vec{I}$ to canonical form [Henn '13]

$$\frac{\mathrm{d}}{\mathrm{d}x}\vec{J} = \epsilon \sum_k \frac{\mathrm{d}\log(w_k)}{\mathrm{d}x} B_k \vec{J}$$

$$\sigma = \frac{1 + x^2}{2x}$$

$$1 < \sigma \leftrightarrow x \in (0,1)$$

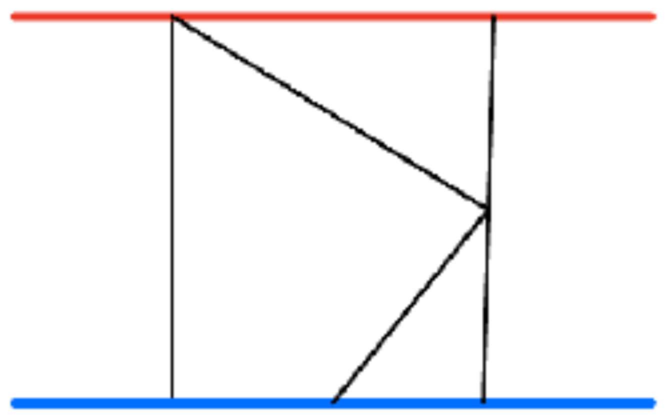
- Order-by-order solution in terms of (generalized) polylogarithms

$$\vec{J} = \sum_n \epsilon^n \vec{J}_n \quad \vec{J}_{n+1} = \sum_k A_k \int_0^x \mathrm{d}z \left[\frac{\mathrm{d}}{\mathrm{d}z}\log(w_k(z))\right] \vec{J}_n$$

- Boundary conditions: Regularity/scaling fixes most. Rest computed in the static limit $\sigma \to 1$

Integration — Elliptic sector

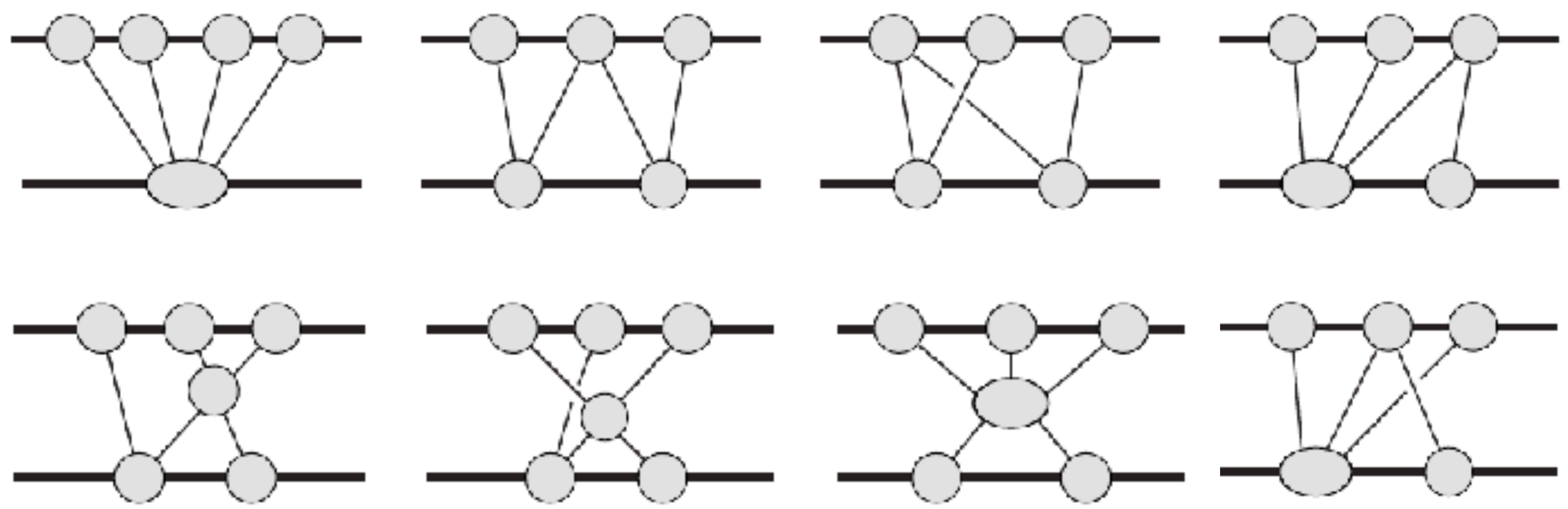
- Some cases: no canonical form



$$= \frac{1}{\epsilon^2}\frac{8}{\sigma+1}K^2\left(\frac{1-\sigma}{1+\sigma}\right) + \mathcal{O}(\epsilon^{-1})$$

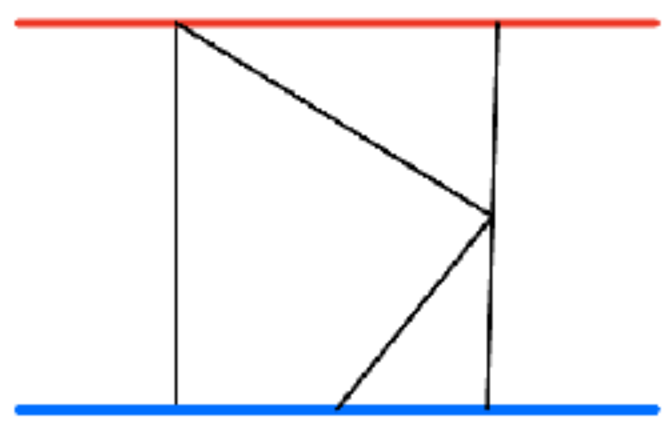
- **4PM**: Elliptics only in the potential region → hardest for integration!
- Strategy 1 [Bern, Parra-Martinez, Roiban, **MSR**, Shen, Solon, Zeng]:
  - Split amplitude $\mathcal{M} = \mathcal{M}_{\text{poly}} + \mathcal{M}_{\text{elliptic}}$
  - Solve $\mathcal{M}_{\text{poly}}$ through DE
  - For $\mathcal{M}_{\text{elliptic}}$ compute series to high orders and match to ansatz
- Strategy 2 [Dlapa et al.]: ϵ form with generalized kernels instead of $\frac{\mathrm{d}\log(w_k)}{\mathrm{d}x}$,

Integration — Elliptic sector



↑ 3 loop cuts

- Ansatz from contact integrals integrals



$$\sim \left\{ K^2\left(\frac{1-\sigma}{1+\sigma}\right), E\left(\frac{1-\sigma}{1+\sigma}\right) K\left(\frac{1-\sigma}{1+\sigma}\right), E^2\left(\frac{1-\sigma}{1+\sigma}\right) \right\}$$

Not allowed to collapse any line!

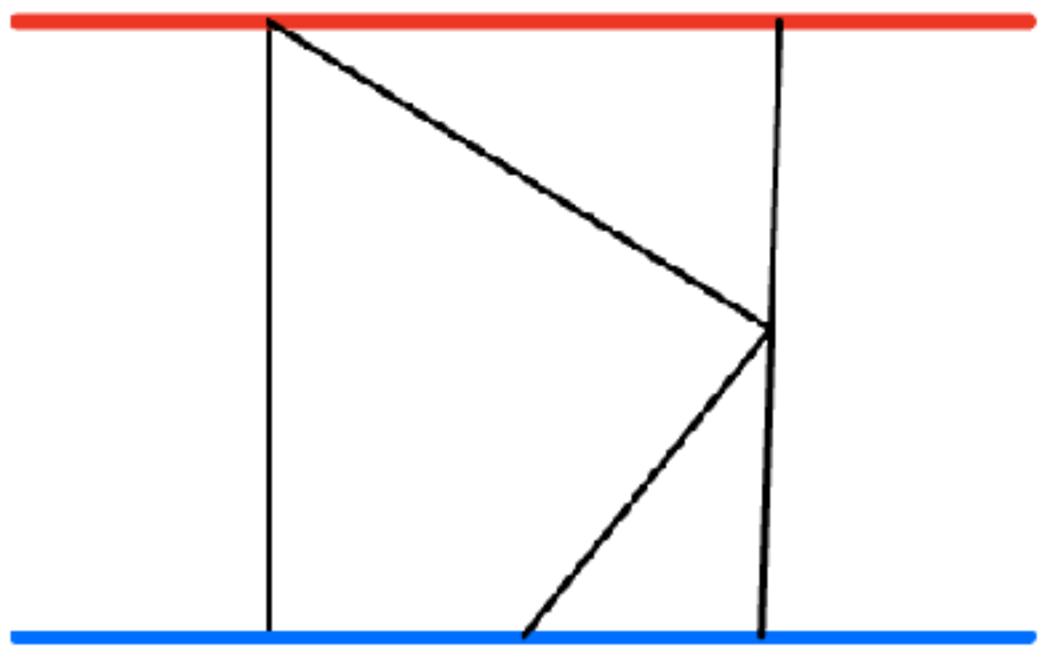
- Expand integrals/amplitude in v using DE and match

$$\mathscr{M}_{4,\text{elliptic}} \sim -\pi^2\left(\frac{41}{16}+\frac{33601v^2}{3072}+\ldots+\#v^{400}\right) = r_4\pi^2 + r_5K\left(\frac{1-\sigma}{1+\sigma}\right)^2 + r_6E\left(\frac{1-\sigma}{1+\sigma}\right)K\left(\frac{1-\sigma}{1+\sigma}\right) + r_7E\left(\frac{1-\sigma}{1+\sigma}\right)^2$$

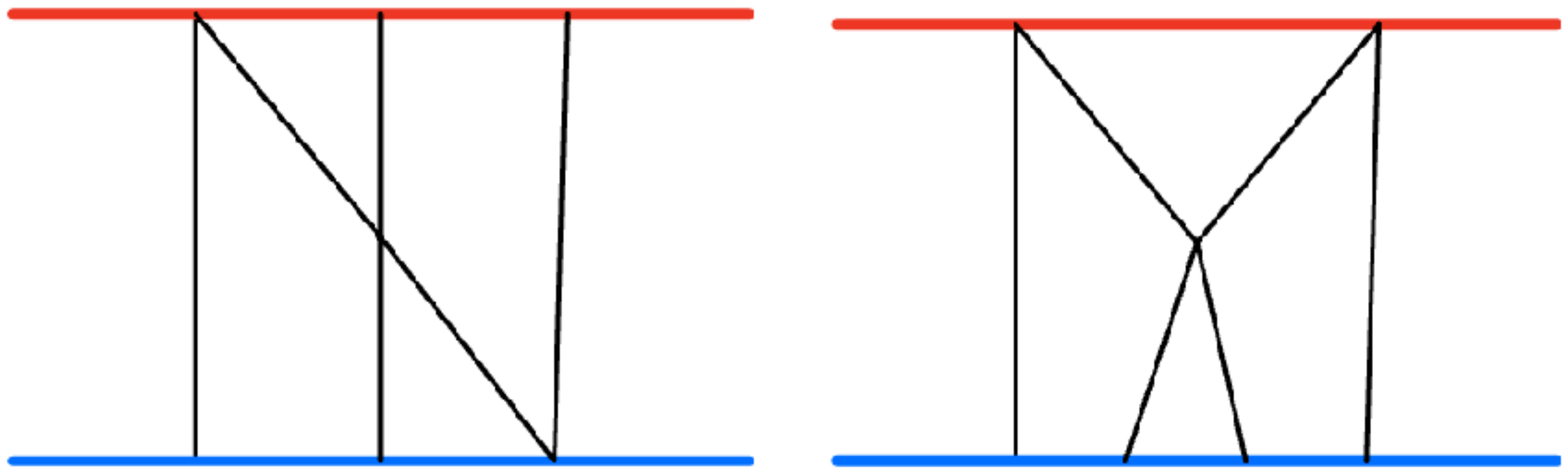
- 60 orders to fix r_i, 400 to check
- Avoids complicated integrals (elliptic polylogs) in intermediate steps
- (Pre-)Canonical form useful to make series expansion efficient

A First Glimpse at Integration at $\mathcal{O}(G^5)$

- Differential equations for all but a few sectors (completed soon)
- First step: study function space, 51 contact topologies first
- Integrals related to 3 loop elliptic sector



$$\sim \left\{ K^2\left(\frac{1-\sigma}{1+\sigma}\right), E\left(\frac{1-\sigma}{1+\sigma}\right)K\left(\frac{1-\sigma}{1+\sigma}\right), E^2\left(\frac{1-\sigma}{1+\sigma}\right) \right\}$$



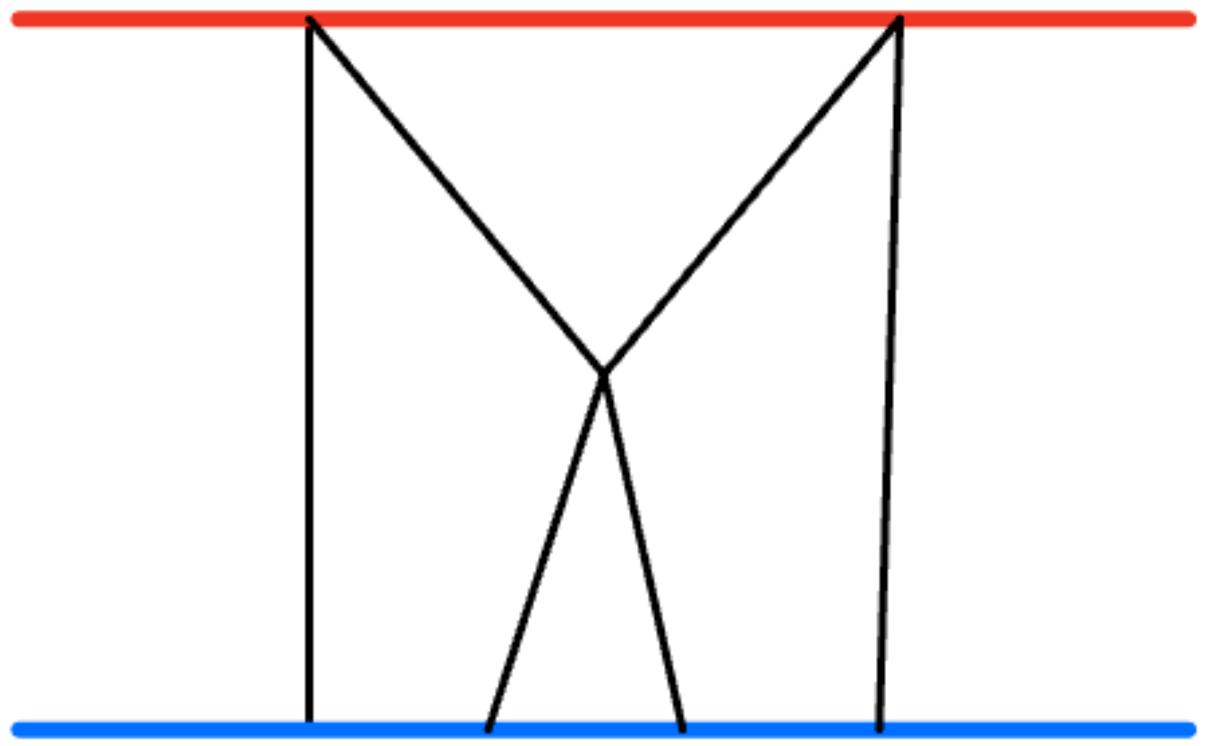
$$\sim \{\,?\,\}$$

A First Glimpse at Integration at $\mathcal{O}(G^5)$

Example 1

$$\mathrm{d}\begin{pmatrix} f_1 \\ \vdots \\ f_5 \end{pmatrix} = \epsilon \sum_k \mathrm{d}\log(w_k) A_k \begin{pmatrix} f_1 \\ \vdots \\ f_5 \end{pmatrix}, \quad w_k \in \{x, 1 \pm x, 1 + x^2\}$$

Same as at 3 loop



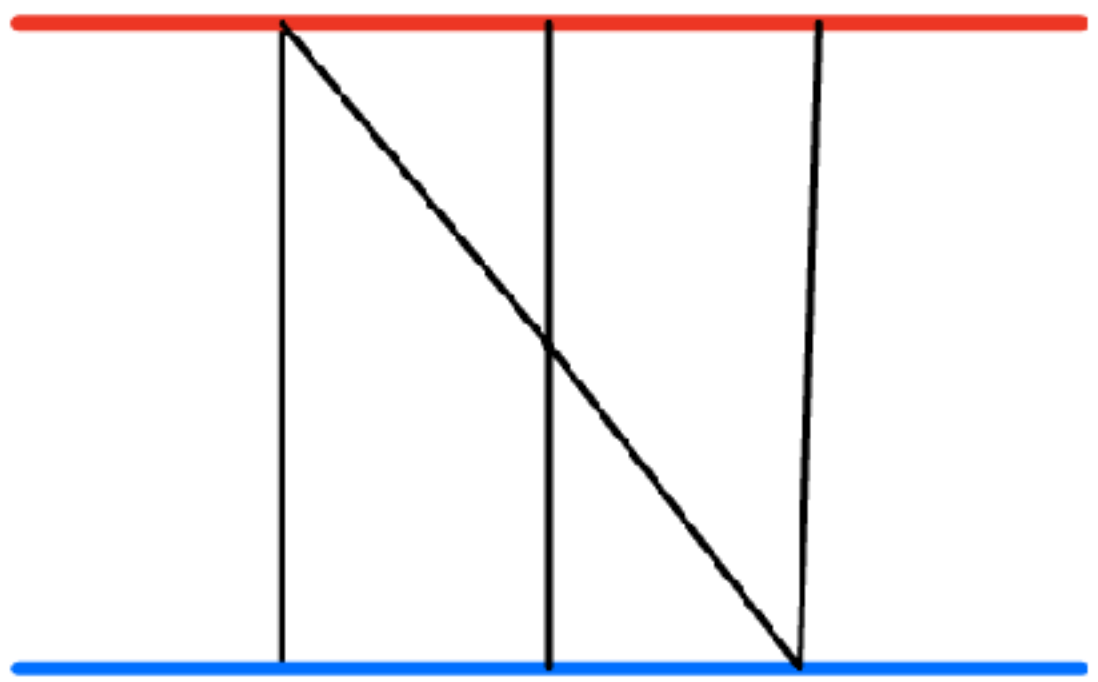
Evaluates to generalized Polylogs

A First Glimpse at Integration at $\mathcal{O}(G^5)$

Example 2

Beyond cyclotomic alphabet

$$\mathrm{d}\begin{pmatrix} f_1 \\ \vdots \\ f_9 \end{pmatrix} = \sum_k \mathrm{d}\log(w_k) A_k \begin{pmatrix} f_1 \\ \vdots \\ f_9 \end{pmatrix}, \quad w_k = \{x, 1 \pm x, 1 \pm x + x^2, 1 \pm 6x + x^2, 1 + 6x^2 + x^4\}$$



$$x = 3 - 2\sqrt{2} = 0.171... \in [0,1] \to \sigma = 3$$

$$x = -\mathrm{i}\sqrt{3 - 2\sqrt{2}} \to \sigma = \mathrm{i}$$

Inside the scattering region $\sigma > 1$!

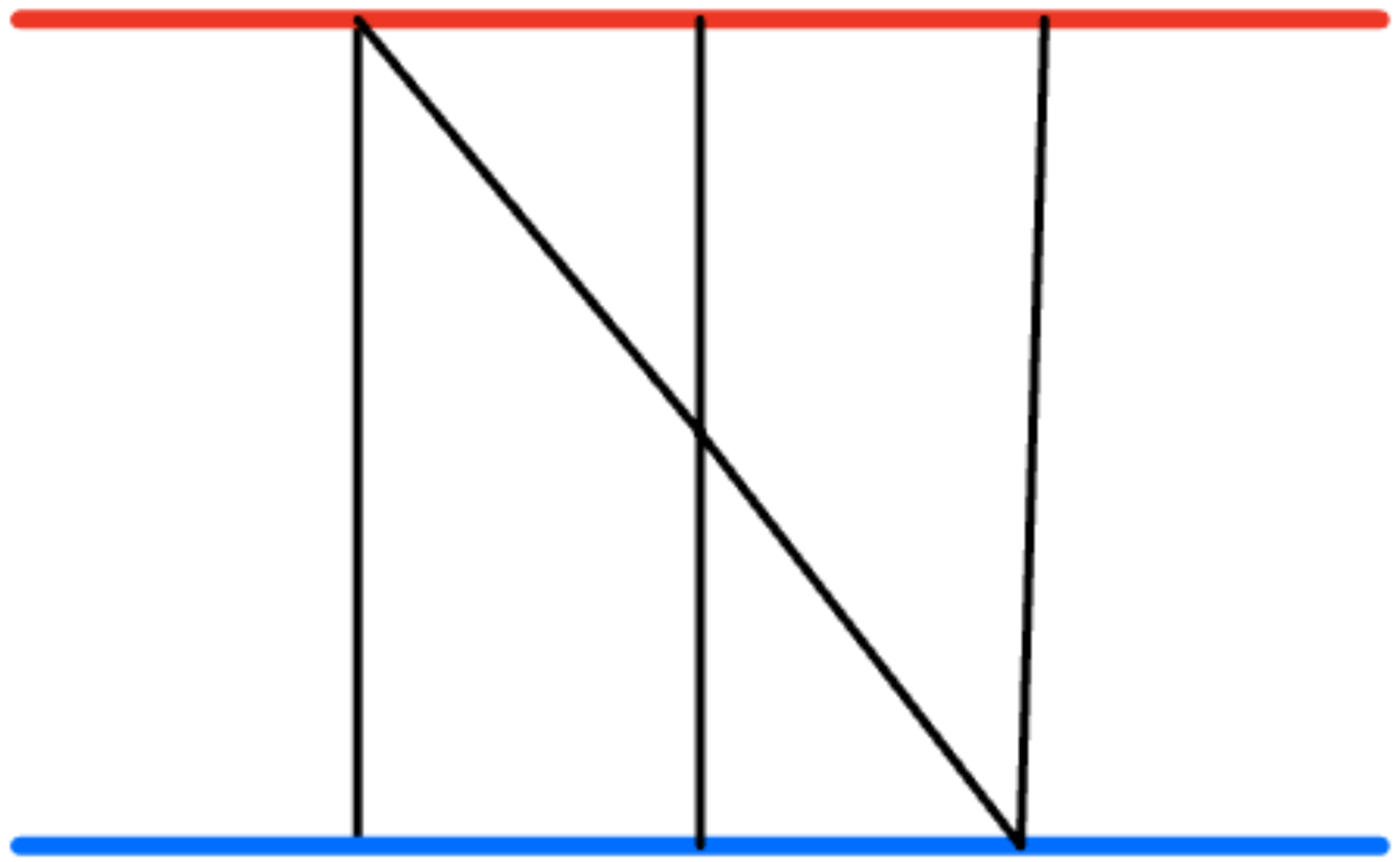
A_k eigenvalues $1/2 + a\epsilon, a \in \mathbb{Z}$ suggest elliptic (or more complicated) integrals, roots cannot be rationalized

A First Glimpse at Integration at $\mathcal{O}(G^5)$

- Equivalently solve 9th order ODE — hard

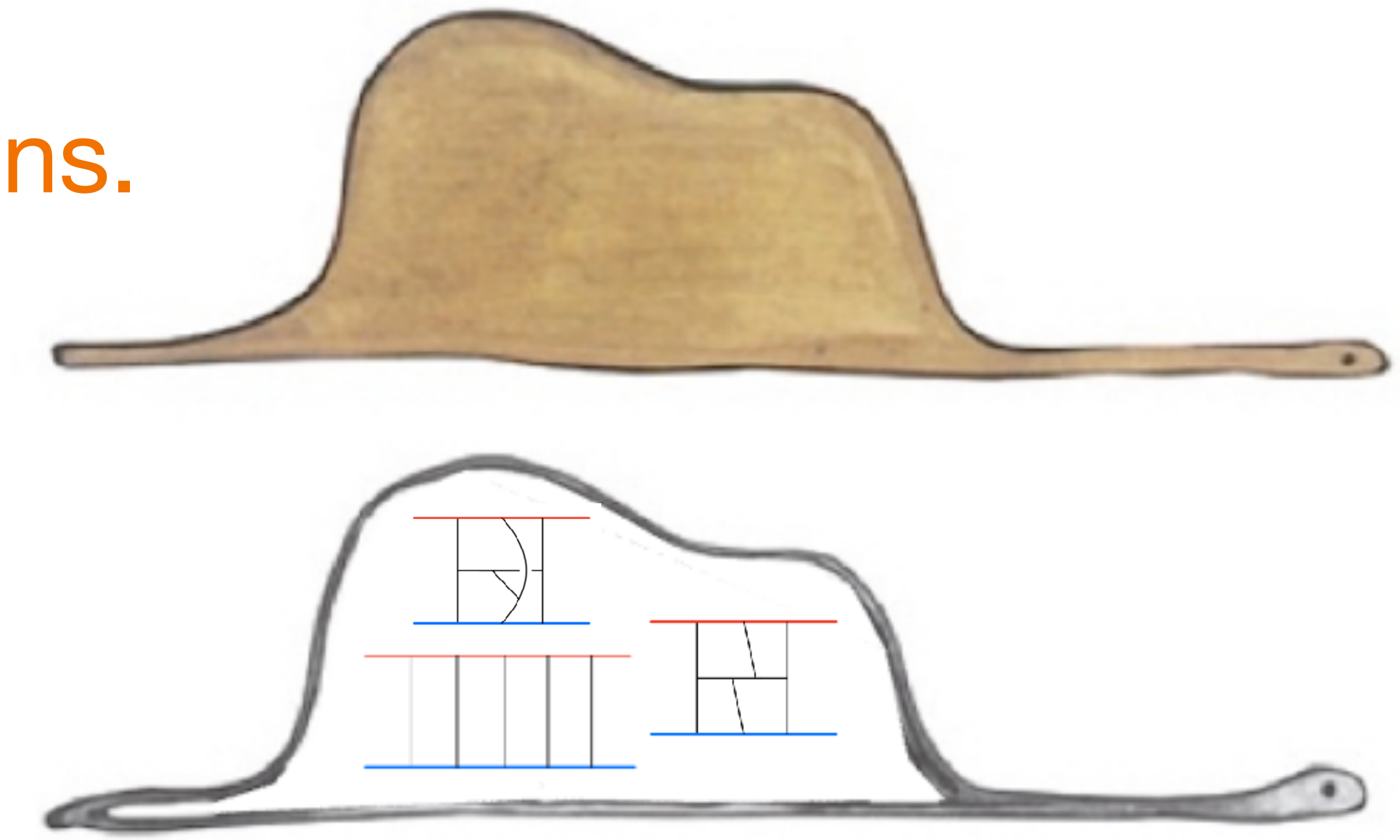
$$\mathrm{d}\begin{pmatrix} f_1 \\ \vdots \\ f_9 \end{pmatrix} = \epsilon \sum_k \mathrm{d}\log(w_k) A_k \begin{pmatrix} f_1 \\ \vdots \\ f_9 \end{pmatrix}, \quad 0 = \sum_{k=0}^{9} p_k(x,\epsilon) \frac{\mathrm{d}^k}{\mathrm{d}^k x} f_1$$

- Need detailed study of the geometry
- Might need to resort to series expansion



Amplitude computation

- Amplitude computation comes in parts
  - Constructing the integrand ✓ 51 cuts for 5PM, 6PM straightforward
  - Integral reduction ?
  - Evaluation of master integrals ?
- Integral reduction:
  - linear algebra problem, exponential growth in # eqns.
  - Intractable without major improvements
- Evaluating integrals:
  - DE's for almost all families
  - Series soon, analytic results harder



The 5PM problem is hard, don't try to swallow it whole!

Slicing the 5PM problem

Can we eat our cake one piece at a time? Expansions?

$$\mathscr{M}_{5\mathrm{PM}}(\mathrm{v}, m_1, m_2)$$

- PN expansion v $\to 0$
  - Complicated topologies suppressed
  - Important for phenomenology
- High-energy expansion v $\to 1$
  - Less well understood, important conceptual questions
- Numerics in v for masters (need numeric IBP) or amplitude (too many integrals?)
- Loose information on functional structure

Slicing the 5PM problem

Can we eat our cake one piece at a time? Expansions?

$$\mathscr{M}_{5\mathrm{PM}}(\mathrm{v}, m_1, m_2)$$

$$\nu = \frac{\mu}{M} = m_1 m_2/(m_1 + m_2)^2 \leq 1/4$$

- Hierarchical limit (SF) $m_1 \ll m_2$
  - Organization into gauge-invariant objects

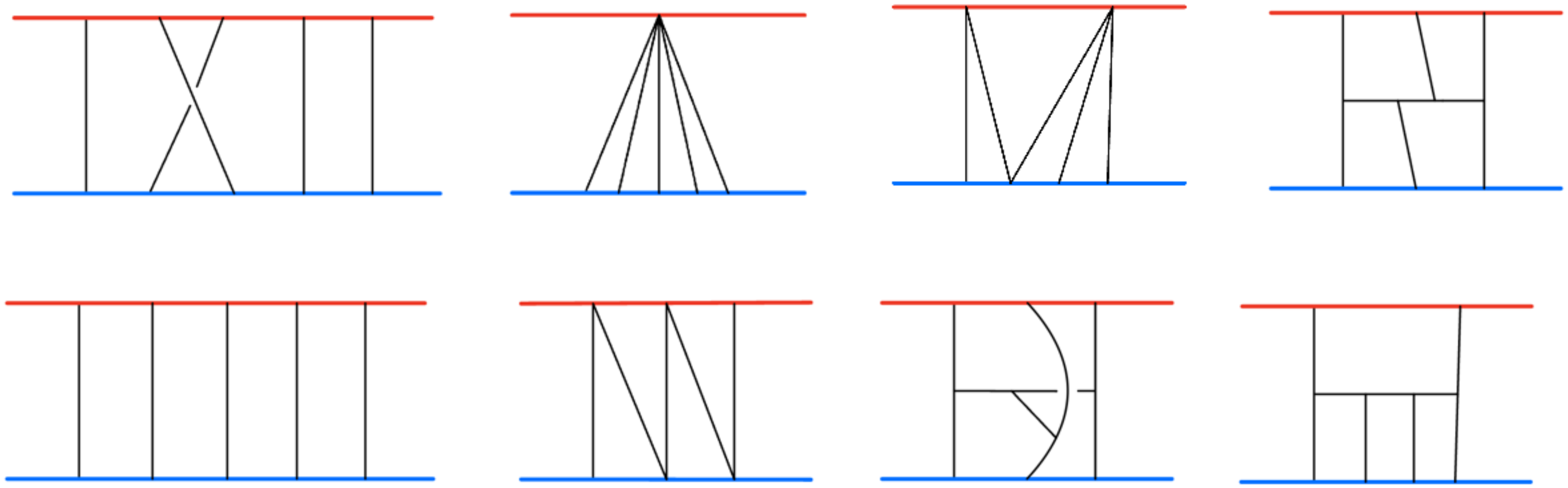
$$\mathscr{M}_{5\mathrm{PM}} = \mathscr{M}^{0\mathrm{SF}}_{5\mathrm{PM}} + \nu \mathscr{M}^{1\mathrm{SF}}_{5\mathrm{PM}} + \nu^2 \mathscr{M}^{2\mathrm{SF}}_{5\mathrm{PM}} \leftarrow \text{trivial from amplitudes!}$$

  - Useful expansion for equal-mass case! Similar to QCD $\frac{1}{N_c} = \frac{1}{3}$
  - Complicated integrals suppressed

Slicing the 5PM problem

Can we eat our cake one piece at a time? Expansions?

$$\mathscr{M}_{5\text{PM}} =$$

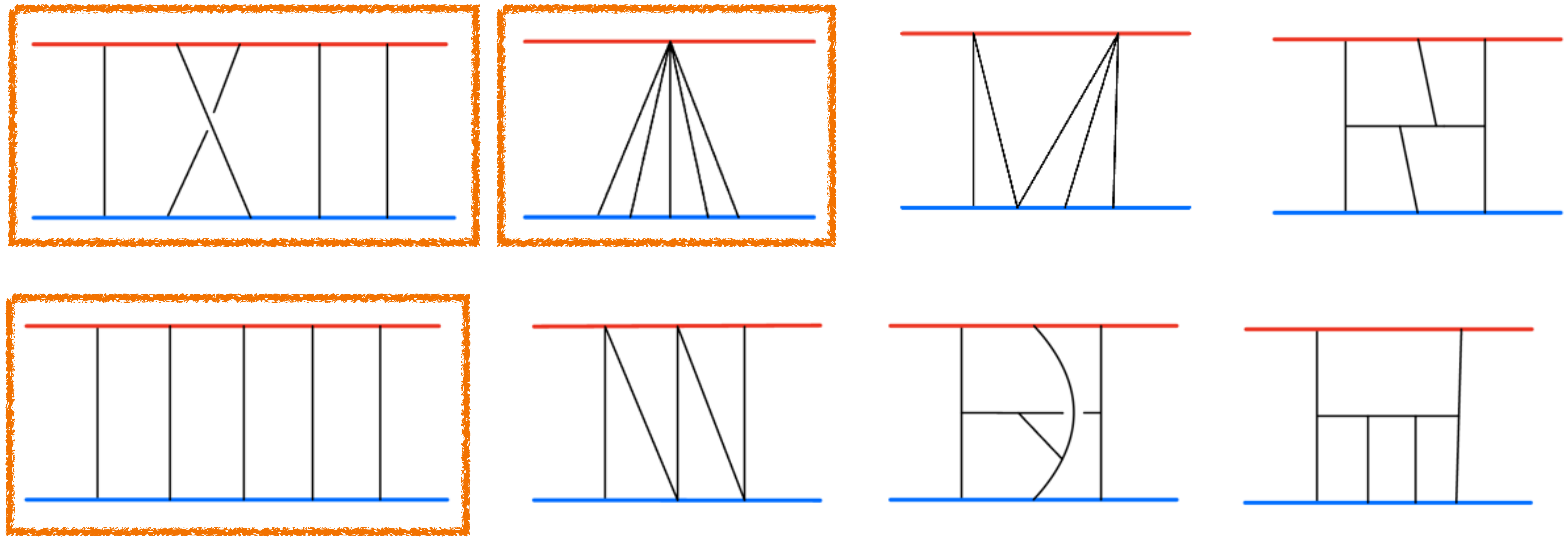


$+\ \mathcal{O}(10^4)$ more

Slicing the 5PM problem

Can we eat our cake one piece at a time? Expansions?

$$\mathscr{M}^{\mathrm{0SF}}_{\mathrm{5PM}} =$$



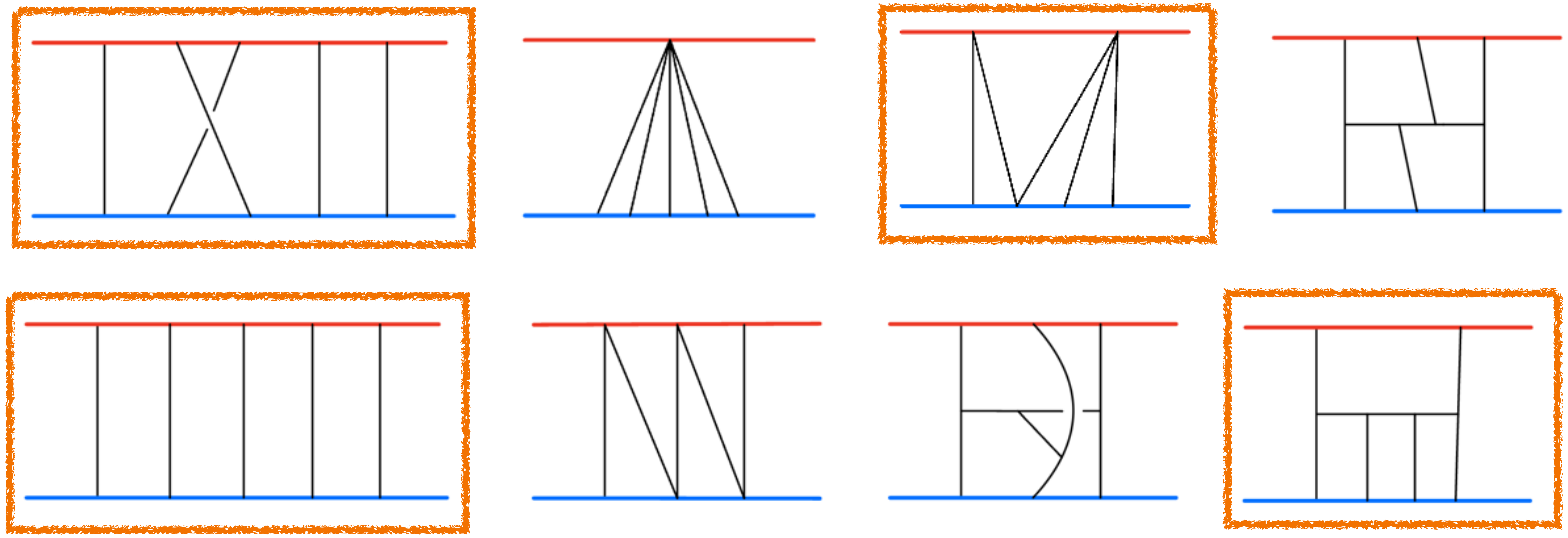
Fixed by geodesic!

$+\ \mathcal{O}(10^4)$ more

Slicing the 5PM problem

Can we eat our cake one piece at a time? Expansions?

$$\mathscr{M}^{\mathrm{1SF}}_{\mathrm{5PM}} =$$

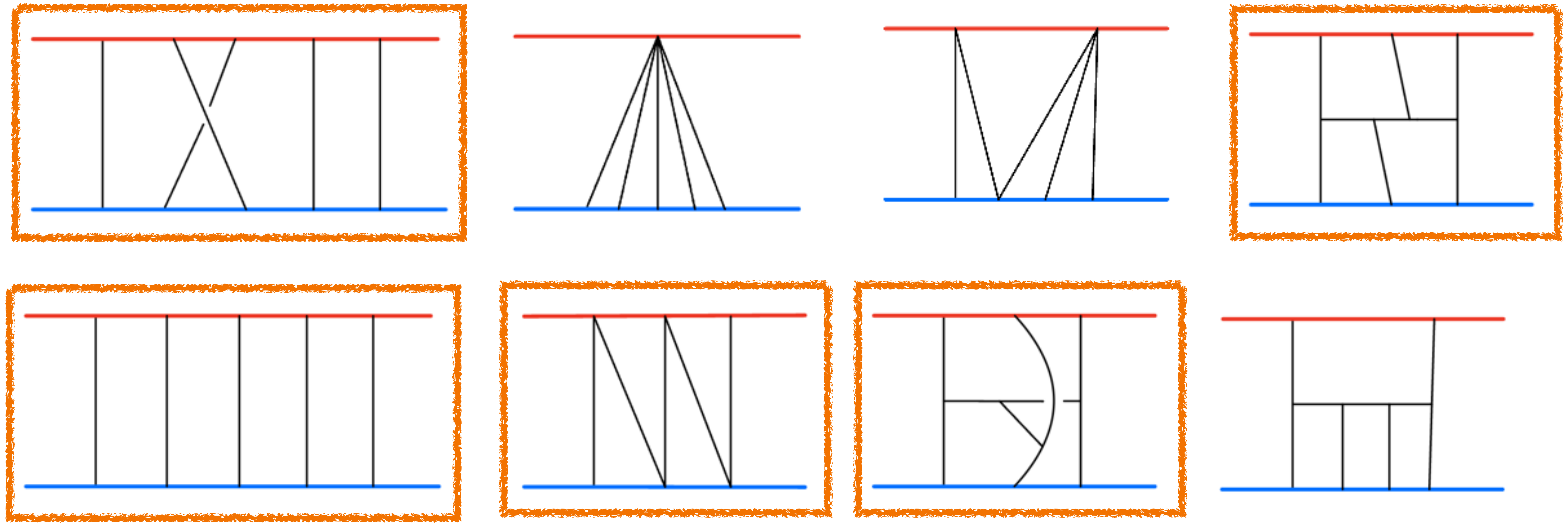


$+\ \mathcal{O}(10^4)$ more

Slicing the 5PM problem

Can we eat our cake one piece at a time? Expansions?

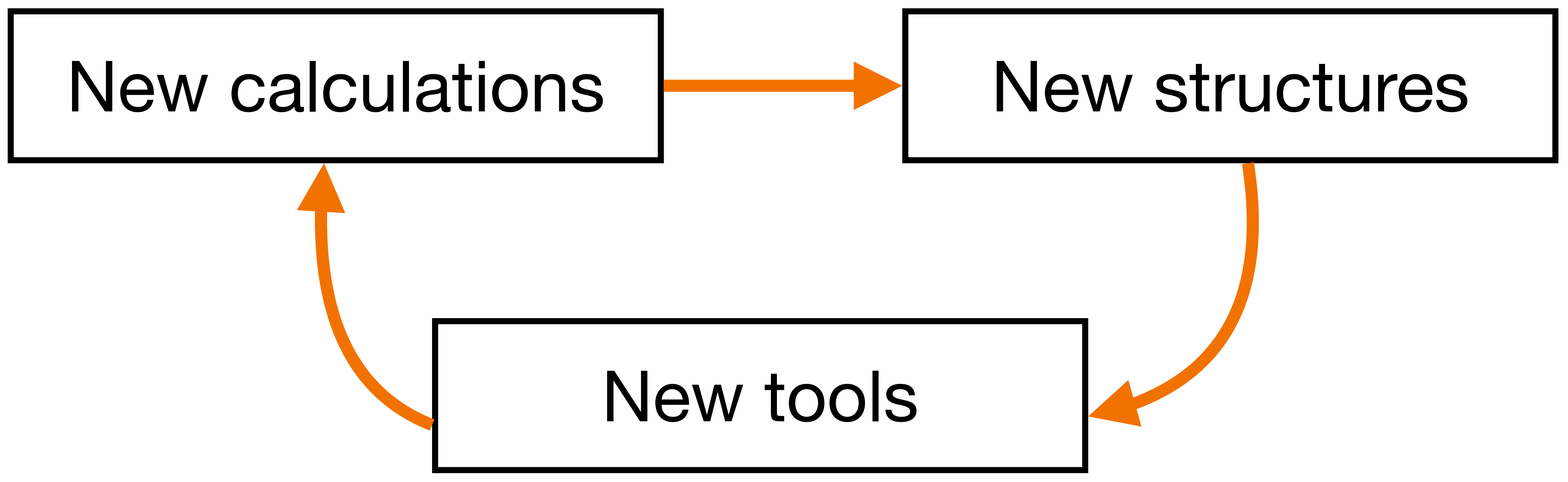
$$\mathcal{M}^{\mathrm{2SF}}_{\mathrm{5PM}} =$$



+ $\mathcal{O}(10^4)$ more

Slicing the 5PM problem

- Work on 1SF in progress
- Use a simpler model without approximations:
  - Maximal SUSY, scalar toy model,…
  - Electrodynamics



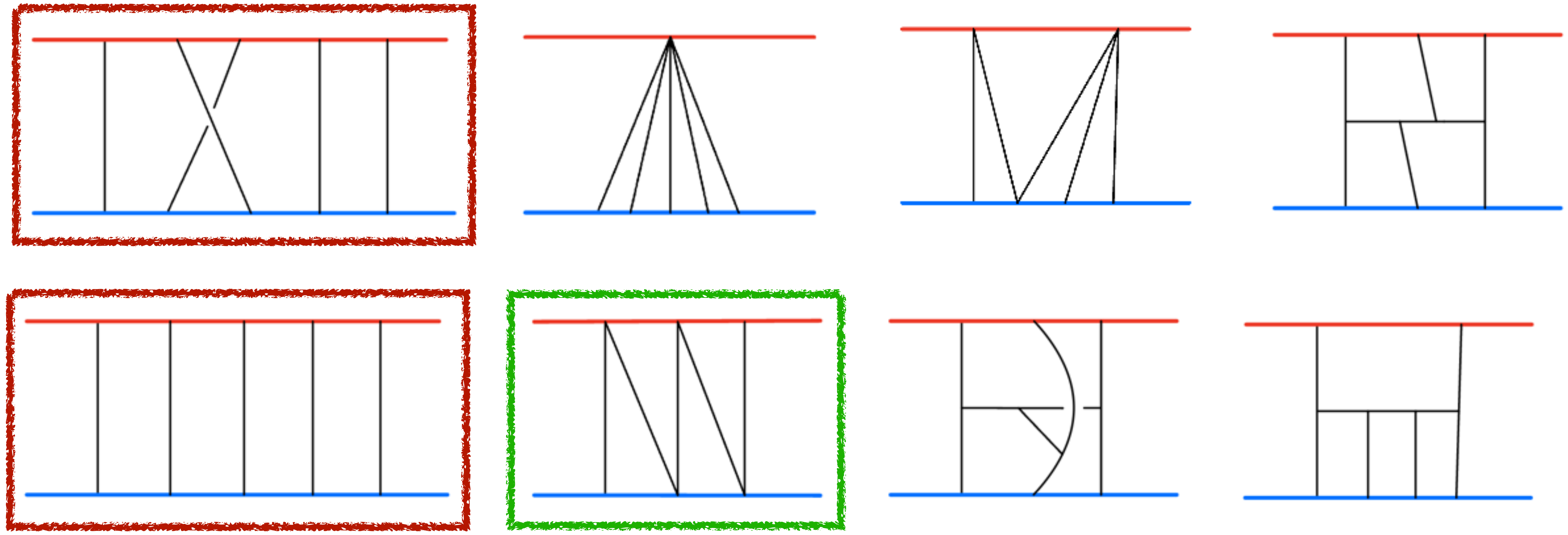
- Main criteria:
  - Sizeable overlap with GR
  - Significantly more complicated than 4PM
  - Real world system, applications to phenomenology

Electrodynamics checks all the boxes!

Slicing the 5PM problem

Can we eat our cake one piece at a time?

$$\mathscr{M}^{\mathrm{QED}}_{\mathrm{5PL}} =$$

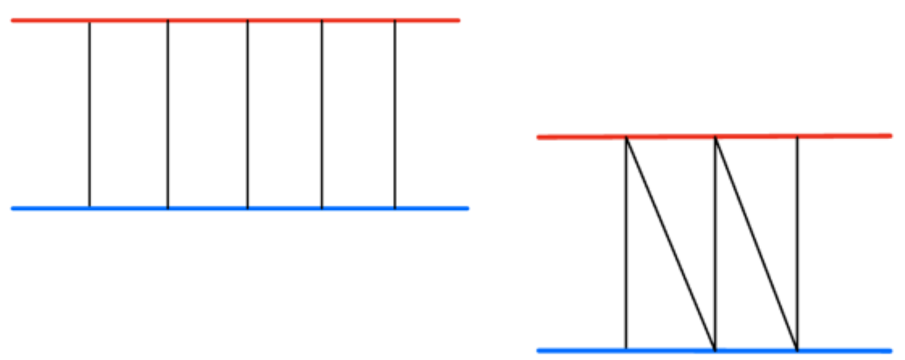


As complicated as in GR! 2SF graphs

$+ \mathcal{O}(10^4)$ more

Classical scattering at $O(\alpha^5)$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \sum_{i=1}^{2}\left[|D_\mu \phi_i|^2 - m_i^2 |\phi_i|^2\right]$$



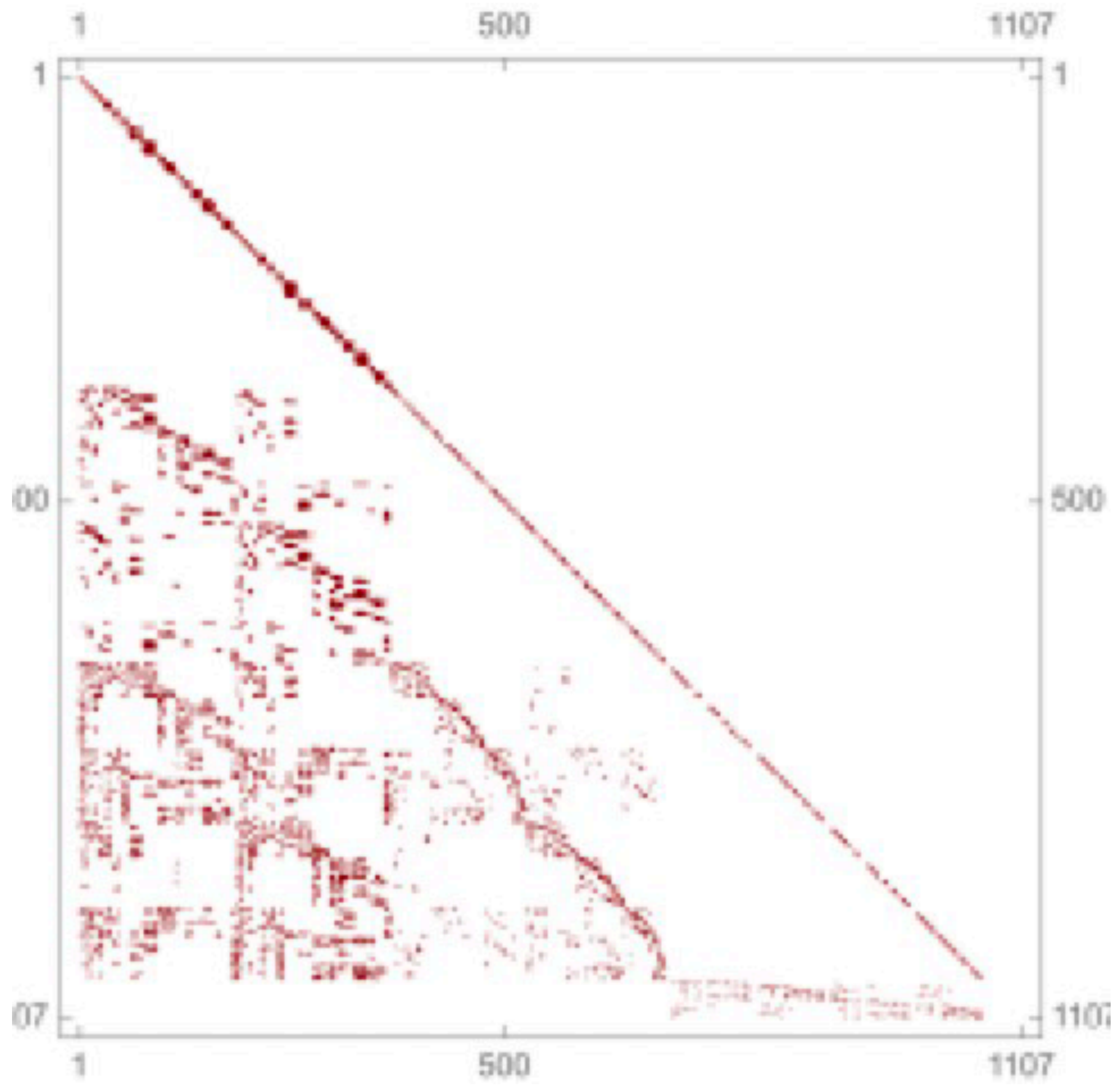
- QED integrand trivial: ~1000 Feynman diagrams
- Deep expansion in the classical limit $\mathscr{M}_4 \sim \frac{1}{\hbar^5} + \ldots + \frac{1}{\hbar}$
- Integral reduction: 10^6 integrals → 1107 masters, 23 families
- Improved version of FIRE+LiteRed for IBP reduction
- Reduction takes O(3 weeks) on Hoffman2 cluster

Classical scattering at $O(\alpha^5)$ — Integration

- No elliptic integrals for QED!

- Differential equations in canonical form

$$\partial_x \vec{I} = \epsilon \sum_{k,n} f_k^n(x) A_{k,n} \vec{I}, \quad \vec{I} = (I_1, \ldots, I_{1107})^T$$



- In terms of cyclotomic kernels

$$f_n^k(x) = \frac{x^k}{\Phi_n(x)}, \quad \Phi_{1,2} = x \pm 1\,, \Phi_4 = 1 + x^2\,, \Phi_{3,6} = 1 \pm x + x^2$$

Classical scattering at $O(\alpha^5)$

- Cyclotomic harmonic polylogs introduced in [Ablinger, Blumelein, Schneider 2011]
  - Manifestly real $$C_4^0(x) = \int \frac{\mathrm{d}x}{1+x^2} = \frac{\mathrm{i}}{2}\int \mathrm{d}x \left[\frac{1}{x+\mathrm{i}} - \frac{1}{x-\mathrm{i}}\right] = \frac{\mathrm{i}}{2}(\log(x+\mathrm{i}) - \log(x-\mathrm{i})) = \arctan(x)$$
  - Shuffle algebra (minimal basis)
  - Integration and differential rules
  - Series expansion & numerics

- Special combinations (amplitude) in terms of real Li's

$$C_{400}^{100}(x) - C_{000}^{200}(x) = -\frac{\mathrm{Li}_3\left(x^6\right)}{18} + \frac{1}{6}\mathrm{Li}_2\left(x^6\right)\log(x) + \frac{4\,\mathrm{Li}_3\left(x^3\right)}{9} - \frac{2}{3}\mathrm{Li}_2\left(x^3\right)\log(x) + \frac{\mathrm{Li}_3\left(x^2\right)}{2}$$
$$-\frac{1}{2}\mathrm{Li}_2\left(1-x^2\right)\log(x) - 4\,\mathrm{Li}_3(x) - 2\,\mathrm{Li}_2(-x)\log(x) - \log\left(1-x^2\right)\log^2(x) + \frac{1}{4}\pi^2\log(x) + \frac{707}{32}\zeta_3$$

Classical scattering at $O(\alpha^5)$ —Results

$$\chi_{\rm pot}^{\rm 5PL} = \frac{\alpha^5 (m_1 + m_2)^4}{30 J^5 E^4 \left(\sigma^2 - 1\right)^{5/2}} \times \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

Classical scattering at $O(\alpha^5)$ —Results

Fifth post-Lorentzian order

$$\chi_{\text{pot}}^{5\text{PL}} = \frac{\alpha^5 (m_1 + m_2)^4}{30 J^5 E^4 (\sigma^2 - 1)^{5/2}} \times \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

Classical scattering at $O(\alpha^5)$ —Results

$$\chi^{\mathrm{5PL}}_{\mathrm{pot}} = \frac{\alpha^5 (m_1 + m_2)^4}{30 J^5 E^4 (\sigma^2 - 1)^{5/2}} \times \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

Contribution from potential modes well-defined! No tail!

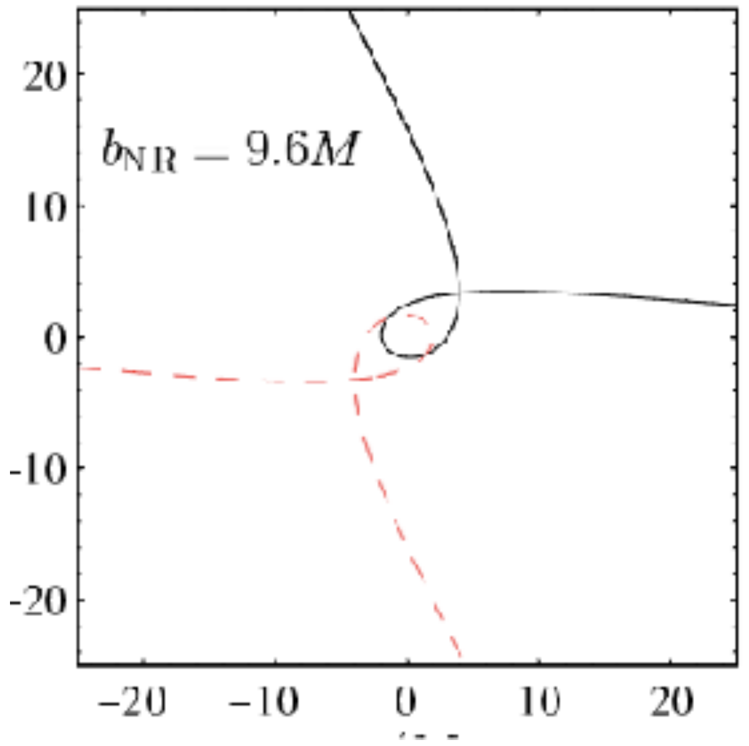
Classical scattering at $O(\alpha^5)$ —Results

$$\chi_{\text{pot}}^{5\text{PL}} = \frac{\alpha^5 (m_1 + m_2)^4}{30 J^5 E^4 (\sigma^2 - 1)^{5/2}} \times \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

Mass polynomiality

Second order E&M self-force!

Classical scattering at $O(\alpha^5)$ —Results



$$\chi^{\rm 5PL}_{\rm pot} = \left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

$$\begin{aligned}
r_9^{(1)} &= r_{12}^{(1)} = 240(\sigma^2 - 1)^2 \,, \\
r_{11}^{(1)} &= 120(\sigma^2 - 1)(\sigma^2 + 2\sigma - 1) \,, \\
r_6^{(1)} &= r_7^{(1)} = r_{10}^{(1)} = 0 \,, \\
r_1^{(2)} &= \frac{405\sigma\left(15-44\sigma^2\right)}{16\left(1-4\sigma^2\right)^2} - \frac{15\left(10\sigma^2+2\sigma-3\right)}{\sigma^3} \\
&\quad + \frac{-2048\sigma^7+6656\sigma^6+17872\sigma^5+20000\sigma^4}{16} \\
&\quad + \frac{-7740\sigma^3-22560\sigma^2-6635\sigma-2080}{16} \,, \\
r_2^{(2)} &= \sqrt{\sigma^2-1}\Bigg[\frac{45\left(1232\sigma^4-1168\sigma^2+287\right)}{16\left(4\sigma^2-1\right)^3} \\
&\quad + \frac{30\left(20\sigma^3-9\sigma^2-4\sigma+3\right)}{\sigma^4} \\
&\quad + \frac{5}{16}\left(1776\sigma^4+8192\sigma^3+10820\sigma^2+11776\sigma+3223\right)\Bigg] \,, \\
r_3^{(2)} &= -\frac{30\left(16\sigma^4+36\sigma^3-11\sigma^2-6\sigma+3\right)}{\sigma^5} \\
&\quad + 20\left(212\sigma^3+350\sigma^2+328\sigma+319\right) \,, \\
r_4^{(2)} &= \frac{2880(\sigma+1)(3\sigma+1)}{\sqrt{\sigma^2-1}} \,, \\
r_6^{(2)} &= 480\left(\sigma^2-1\right)^{3/2}\left(2\sigma^2-1\right) \,, \\
r_7^{(2)} &= 45\sigma\left(\sigma^2-1\right)^{5/2} \,, \\
r_9^{(2)} &= -480\left(\sigma^2-1\right)\left(\sigma^2-\sigma-1\right) \,, \\
r_{10}^{(2)} &= -135\left(\sigma^2-1\right)^2 \,, \\
r_{12}^{(2)} &= -480\left(\sigma^2-1\right)\left(\sigma^2-2\sigma-1\right) \,, \\
r_5^{(2)} &= r_8^{(2)} = r_{11}^{(2)} = 0 \,. \qquad (14)
\end{aligned}$$

Poles at $\sigma = 0, \pm 1/2, \pm 1$

Outside of the scattering region $1 < \sigma$

Implications for bound state?

Rational coefficients

Classical scattering at $O(\alpha^5)$ —Results

$$\chi^{5PL}_{pot} = \begin{aligned}
&f_1 = 1\,,\ f_2 = C^0_0(x)\,,\ f_3 = C^{0,0}_{0,0}(x)\,,\ f_4 = C^{0,0,0}_{0,0,0}(x)\,,\\
&f_5 = -C^{0,0}_{1,0}(x) + C^{0,0}_{2,0}(x) + \frac{\pi^2}{4}\,,\\
&f_6 = -C^{0,0}_{2,0}(x) + C^{1,0}_{4,0}(x) - \frac{\pi^2}{16}\,,\\
&f_7 = C^{0,0}_{3,0}(x) + 2C^{1,0}_{3,0}(x) + C^{0,0}_{6,0}(x) - 2C^{1,0}_{6,0}(x) + \frac{\pi^2}{6}\,,\\
&f_8 = -C^{0,0,0}_{0,1,0}(x) + C^{0,0,0}_{0,2,0}(x) + \frac{\pi^2}{4}C^0_0(x) + \frac{7\zeta_3}{2}\,,\\
&f_9 = -C^{0,0,0}_{0,2,0}(x) + C^{0,1,0}_{0,4,0}(x) - \frac{\pi^2}{16}C^0_0(x) - \frac{21\zeta_3}{16}\,,\\
&f_{10} = C^{0,0,0}_{0,3,0}(x) + 2C^{0,1,0}_{0,3,0}(x) + C^{0,0,0}_{0,6,0}(x) - 2C^{0,1,0}_{0,6,0}(x)\\
&\qquad + \frac{1}{6}\pi^2 C^0_0(x) + \frac{28\zeta_3}{9}\,,\\
&f_{11} = -C^{0,0,0}_{1,0,0}(x) + C^{0,0,0}_{2,0,0}(x) - \frac{7\zeta_3}{4}\,,\\
&f_{12} = -C^{0,0,0}_{2,0,0}(x) + C^{1,0,0}_{4,0,0}(x) + \frac{21\zeta_3}{32}\,. \qquad (15)
\end{aligned}$$

$$\left[r_0^{(0)} + \sum_{k=1}^{12} \left(\nu r_k^{(1)} + \nu^2 r_k^{(2)} \right) f_k \right]$$

Transcendental functions

Functions are special:

No ζ-values $f_k = \sum_{n,r} \log^r(1-x) a_n^r (1-x)^n\,, \quad a_n^r \in \mathbb{Q}$

Only specific contributions of indices (symbology)

Alternative form: polylogs with real arguments

Conclusion

- Amplitudes program for GR observables to high perturbative orders: loops → precision
- Scattering in QED at α^5 important case study
  - Proof of principle computation, potential phone applications
  - Large overlap with GR computation ~25% of the master integrals, including 2SF integrals
  - Identified bottlenecks and improved setup in integral reduction
- Progress towards scattering at G^5
  - Integrand constructed ✓
  - Differential equations for all but a few families (hopefully completed soon) ✓
  - Integral reduction challenging **?** New ideas from collider physics/amplitudes could help!
- Slice the problem into digestible pieces, gauge-invariant organisation by self-force (cf. $1/N_c$ expansion)
- Eventually reconnect with the bound problem

Optimistic on near-term progress — 5PM hard but doable!

Backup

Symbology

$$f_1 = 1\,,\ f_2 = C_0^0(x)\,,\ f_3 = C_{0,0}^{0,0}(x)\,,\ f_4 = C_{0,0,0}^{0,0,0}(x)\,,$$
$$f_5 = -C_{1,0}^{0,0}(x) + C_{2,0}^{0,0}(x) + \frac{\pi^2}{4}\,,$$
$$f_6 = -C_{2,0}^{0,0}(x) + C_{4,0}^{1,0}(x) - \frac{\pi^2}{16}\,,$$
$$f_7 = C_{3,0}^{0,0}(x) + 2C_{3,0}^{1,0}(x) + C_{6,0}^{0,0}(x) - 2C_{6,0}^{1,0}(x) + \frac{\pi^2}{6}\,,$$
$$f_8 = -C_{0,1,0}^{0,0,0}(x) + C_{0,2,0}^{0,0,0}(x) + \frac{\pi^2}{4}C_0^0(x) + \frac{7\zeta_3}{2}\,,$$
$$f_9 = -C_{0,2,0}^{0,0,0}(x) + C_{0,4,0}^{0,1,0}(x) - \frac{\pi^2}{16}C_0^0(x) - \frac{21\zeta_3}{16}\,,$$
$$f_{10} = C_{0,3,0}^{0,0,0}(x) + 2C_{0,3,0}^{0,1,0}(x) + C_{0,6,0}^{0,0,0}(x) - 2C_{0,6,0}^{0,1,0}(x) + \frac{1}{6}\pi^2 C_0^0(x) + \frac{28\zeta_3}{9}\,,$$
$$f_{11} = -C_{1,0,0}^{0,0,0}(x) + C_{2,0,0}^{0,0,0}(x) - \frac{7\zeta_3}{4}\,,$$
$$f_{12} = -C_{2,0,0}^{0,0,0}(x) + C_{4,0,0}^{1,0,0}(x) + \frac{21\zeta_3}{32}\,. \qquad (15)$$

$$\mathcal{S}_1=1 \quad \mathcal{S}_2=x\,, \quad \mathcal{S}_3=x\otimes x\,, \qquad (17)$$
$$\mathcal{S}_4=x\otimes x\otimes x\,, \quad \mathcal{S}_5=\frac{1}{2}x\otimes\alpha_2\,, \quad \mathcal{S}_6=\frac{1}{2}x\otimes\alpha_3\,,$$
$$\mathcal{S}_7=x\otimes\alpha_4\,, \quad \mathcal{S}_8=\frac{1}{2}x\otimes\alpha_2\otimes x\,, \quad \mathcal{S}_9=\frac{1}{2}x\otimes\alpha_3\otimes x\,,$$
$$\mathcal{S}_{10}=x\otimes\alpha_4\otimes x\,, \quad \mathcal{S}_{11}=\frac{1}{2}x\otimes x\otimes\alpha_2\,, \quad \mathcal{S}_{12}=\frac{1}{2}x\otimes x\otimes\alpha_3\,.$$

$$\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\} = \left\{x, \frac{(1+x)^2}{(1-x)^2}, \frac{1+x^2}{(1+x)^2}, \frac{1+x+x^2}{1-x+x^2}\right\} = \left\{x, \frac{\sigma+1}{\sigma-1}, \frac{\sigma}{\sigma+1}, \frac{2\sigma+1}{2\sigma-1}\right\}\,. \qquad (16)$$

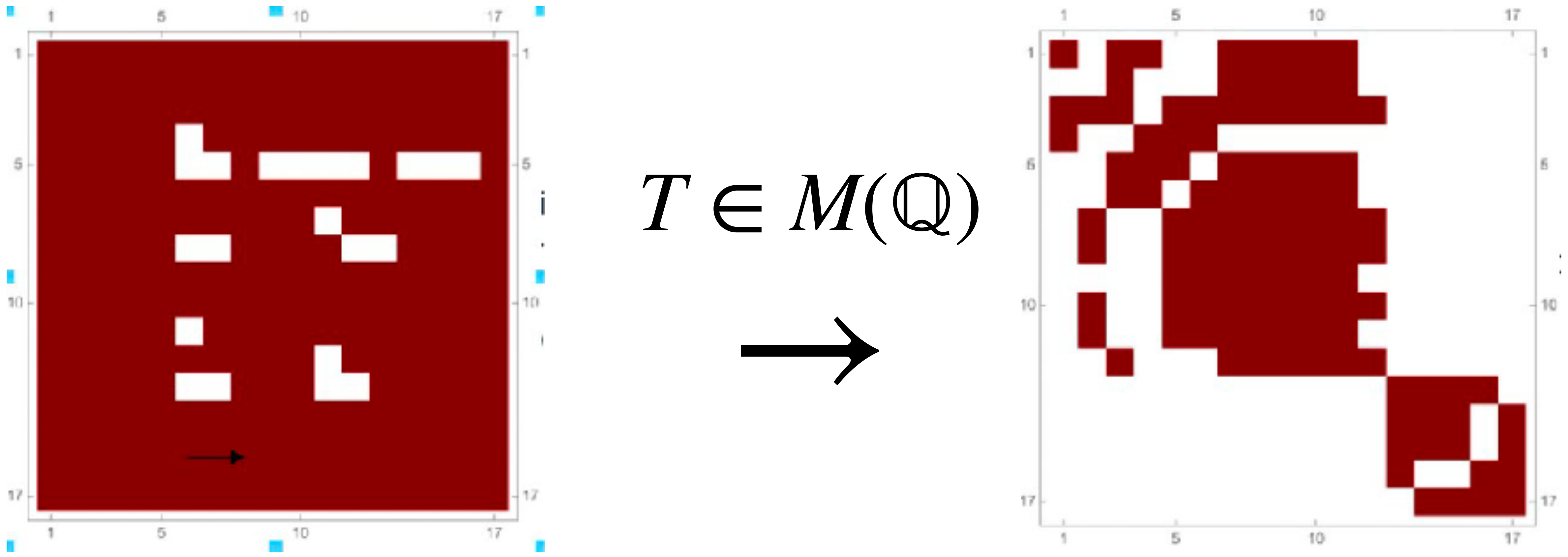
Classical scattering at $O(\alpha^5)$ —Lessons

- Computations at fifth order in perturbation theory are possible!
- E&M scattering at large impact parameter — heavy ion scattering
- First glimpse at the function space, organization in terms of CHPL useful
- Understand which parts of the computation are hardest $\rightarrow$ 2SF graphs
- QED is an useful playground for bootstrap ideas
  - High post-Coulombian orders from Fokker action
  - Simple space of transcendental function
  - Additional relation to energy loss
- Performed lots of checks of the result

Classical scattering at $O(\alpha^5)$ — Integration

- DE has additional structure: sparse, top sectors don't talk to bottom
- Canonical DE invariant under rational transformation: factorize

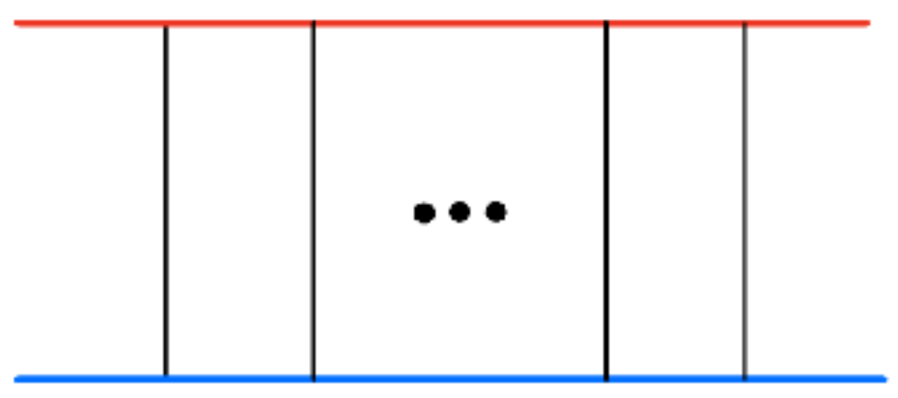
$$\epsilon \sum_{k,n} f_k^n(x) A_{k,n} \rightarrow \epsilon \sum_{k,n} f_k^n(x) A'_{k,n}$$



- Bonus relations between integrals, linked to special structure of eikonal integrals

Amplitudes to observables

- Amplitude not observable: $\mathcal{M}_{L\text{-loop}} \sim \frac{1}{\hbar^{L+1}}, \quad \mathcal{M}_{L\text{-loop}} \sim \frac{1}{\epsilon^{L}}$



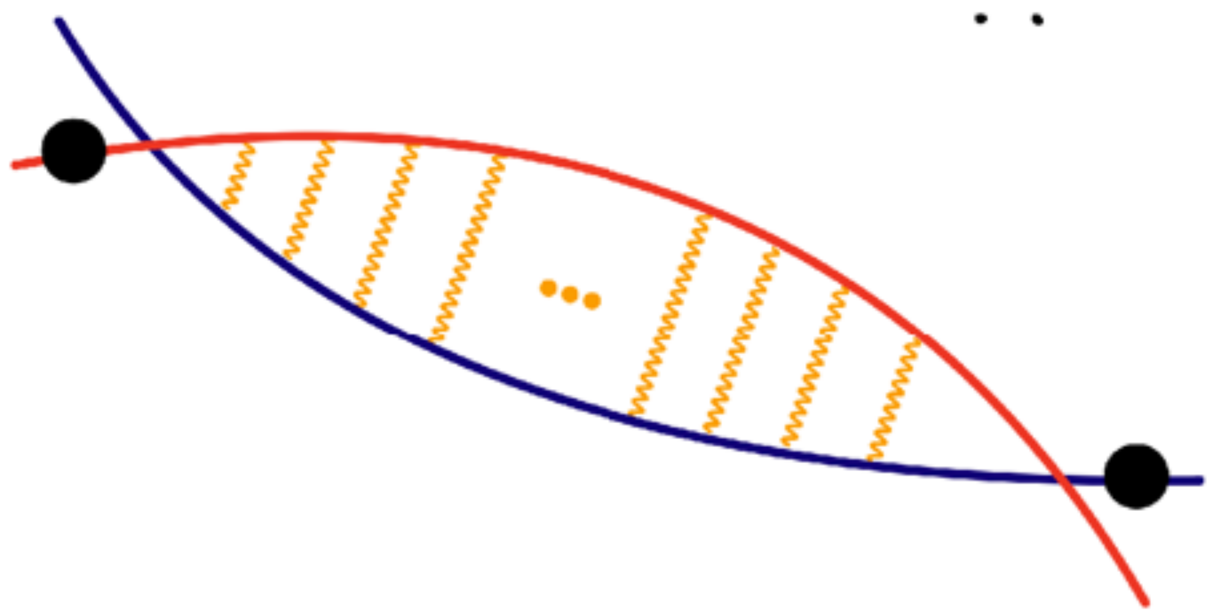
- Observables through
  - Direct computation [Kosower, Maybee, O'Connell; Damgaard, Hansen, Planté, Vanhove (4PM)]
  - Hamiltonian (Schrödinger eqn. or EFT matching [Rothstein,Neill; Cheung, Solon, Rothstein])
  - Stationary phase/generating functionals (eikonal, partial waves, heavy particle phase,…)
- Amplitude ↔ radial action [Bern, Parra-Martinez, Roiban, **MSR**, Shen, Solon, Zeng]

$$\mathcal{M} = \mathrm{i}\int_J (e^{\mathrm{i}I_r(J)/\hbar} - 1), \quad I_r(J,E) = \int_{\text{trajectory}} p_r(J,E)\mathrm{d}r \qquad \chi(J,E) = -\partial_J I_r(J,E)$$

Gauge invariants

Very efficient extraction that meshes with relativistic integration

Classical Limit



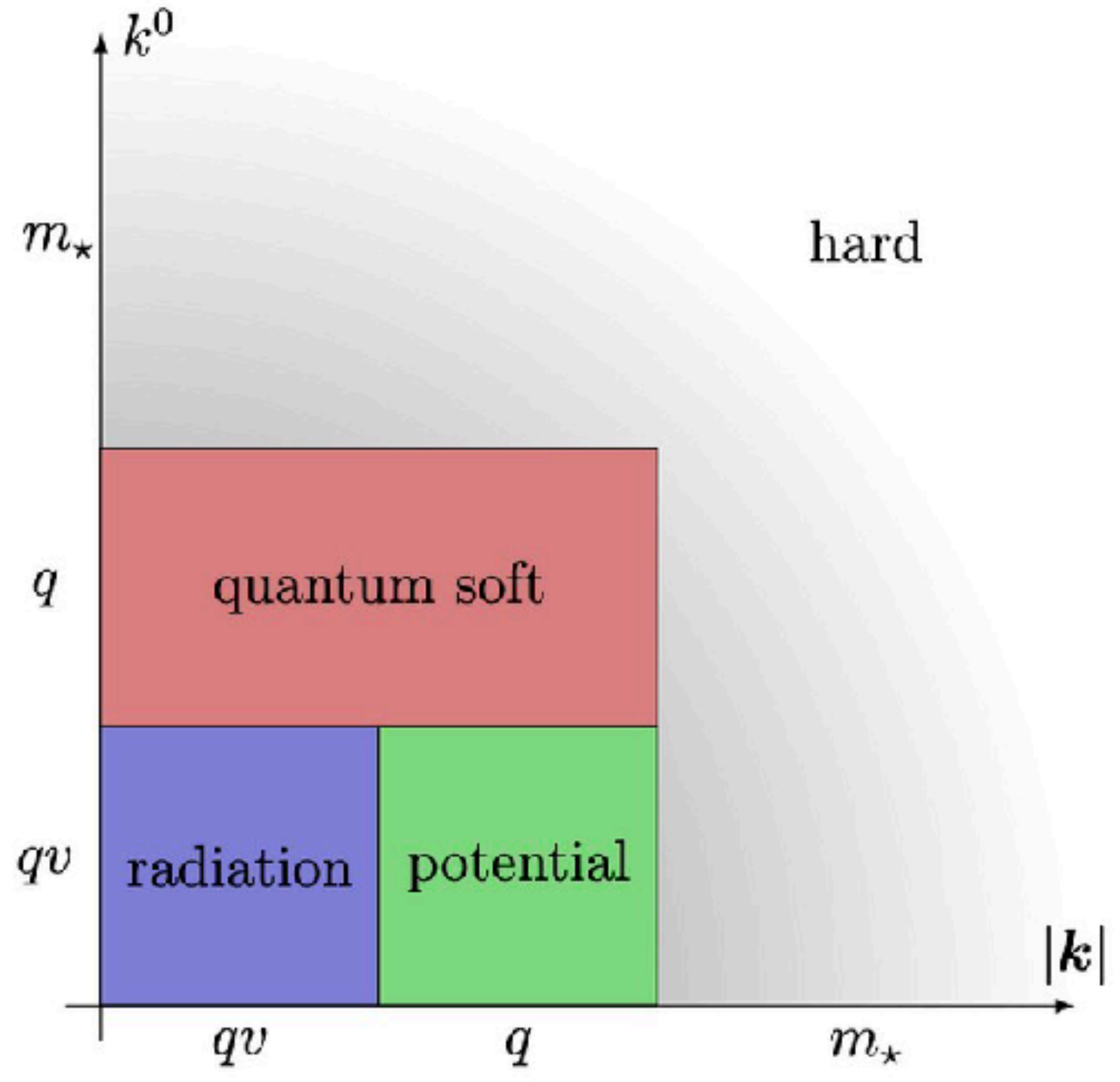
- Classical physics: Large number of soft exchanges $q = \hbar \bar{q}$

$$1 \ll J^2 \sim \frac{s}{q^2} \sim \frac{m_i^2}{q^2} \to q^2 \ll m_i^2 \sim s$$

$$l_{\text{compton}} \sim \frac{1}{M} \ll R_S \sim GM \ll b$$

- Relativistic regions: [Benecke, Smirnov]
  - Hard (h): $\ell \sim m \leftarrow$ UV, quantum $\lambda_{\text{compton}} \sim b$
  - Soft (s): $\ell \sim q \leftarrow$ long range $\lambda_{\text{compton}} \ll b$
- Threshold expansion: $v \sim |\vec{p}_{\text{COM}}|/\sqrt{s}$
  - Potential (p): $(\omega, \vec{\ell}) \sim (|q|v, |q|) \leftarrow$ instantaneous
  - Radiation (r): $(\omega, \vec{\ell}) \sim (|q|v, |q|v)$
- Classical physics (p)+(r), not well-defined separately
- Formally $v \ll 1$, resumption to $v \sim \mathcal{O}(1)$

$$v + \frac{v^3}{3} + \frac{v^5}{5} + \ldots = \text{arctanh}(v)$$



- $C^{b_1,\ldots,b_n}_{a_1,\ldots,a_n}(x) = \int_0^x \mathrm{d}z\, f^{b_1}_{a_1}(z)\, C^{b_2,\ldots,b_n}_{a_2,\ldots,a_n}(z)\,, \quad f^1_4 = \dfrac{x}{1+x^2}$

$$\mathbb{W} = \left\{ \frac{1}{x}, \frac{1}{1+x}, \frac{1}{x-1}, \frac{2x}{1+x^2}, \frac{1+2x}{1+x+x^2}, \frac{2x-1}{1-x+x^2} \right\}$$