Cuts in Feynman Parameter Space

Ruth Britto

Trinity College Dublin & Institute for Advanced Study

Amplitudes 2023

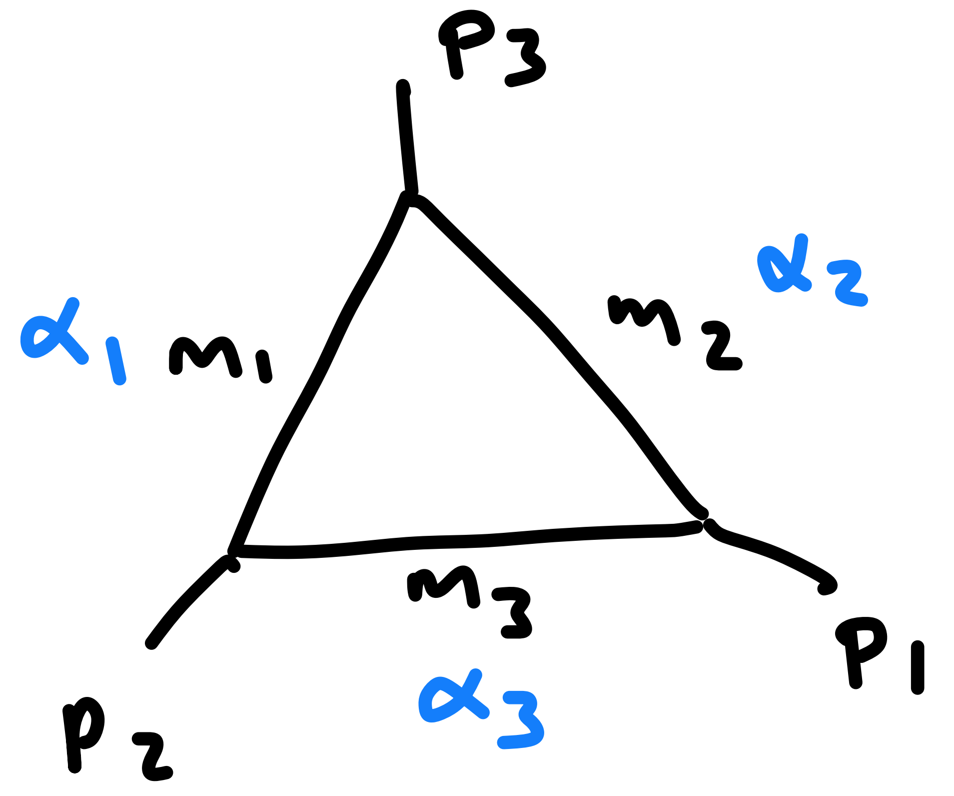
arXiv:2305.15369

Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

IAS | INSTITUTE FOR ADVANCED STUDY

- A Feynman integral is represented schematically as $I = \int_\Gamma \frac{\mathrm{d}\alpha}{\mathrm{GL}(1)} \mathscr{U}^\kappa(\alpha)\, \mathscr{F}^\lambda(\alpha)$.
- $\alpha = \{\alpha_1, \ldots, \alpha_E\}$ are the Feynman parameters.
- $\mathscr{U}$ and $\mathscr{F}$ are homogeneous graph polynomials in α.
- Prefactors and exponents κ, λ depend on dimensions, multiplicity of propagators, loop order.
- All kinematic dependence is in $\mathscr{F}$.
- The integration domain Γ is $\{\alpha_i \geq 0\}$. It is bounded by coordinate hyperplanes.
- Claim: cut integrals are obtained by changing the boundaries of Γ to include $\mathscr{F} = 0$.

Let's look at the triangle integral.



Feynman parameters $\alpha_1, \alpha_2, \alpha_3$.

$$\mathscr{F} = (\alpha_1 + \alpha_2 + \alpha_3)(\alpha_1 m_1^2 + \alpha_2 m_2^2 + \alpha_3 m_3^2) - \alpha_1\alpha_2 p_3^2 - \alpha_1\alpha_3 p_2^2 - \alpha_2\alpha_3 p_1^2$$

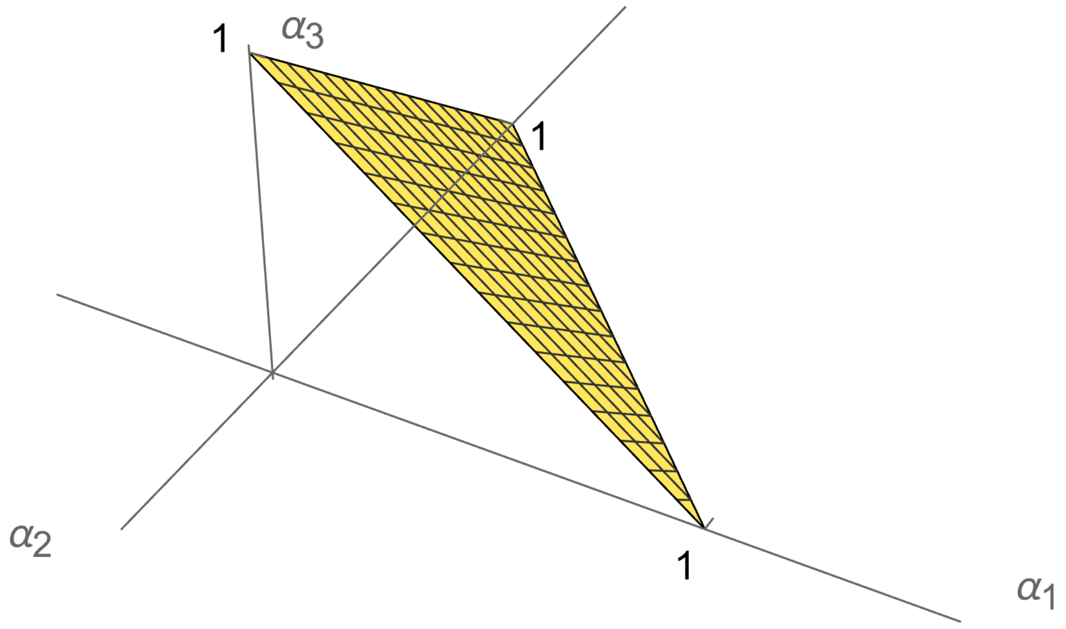
$$\mathscr{U} = \alpha_1 + \alpha_2 + \alpha_3$$

$$I_3^D = \int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2 \int_0^\infty \mathrm{d}\alpha_3\, \delta(1 - \alpha_1 - \alpha_2 - \alpha_3)\, \mathscr{U}^{3-D} \mathscr{F}^{D/2-3}$$

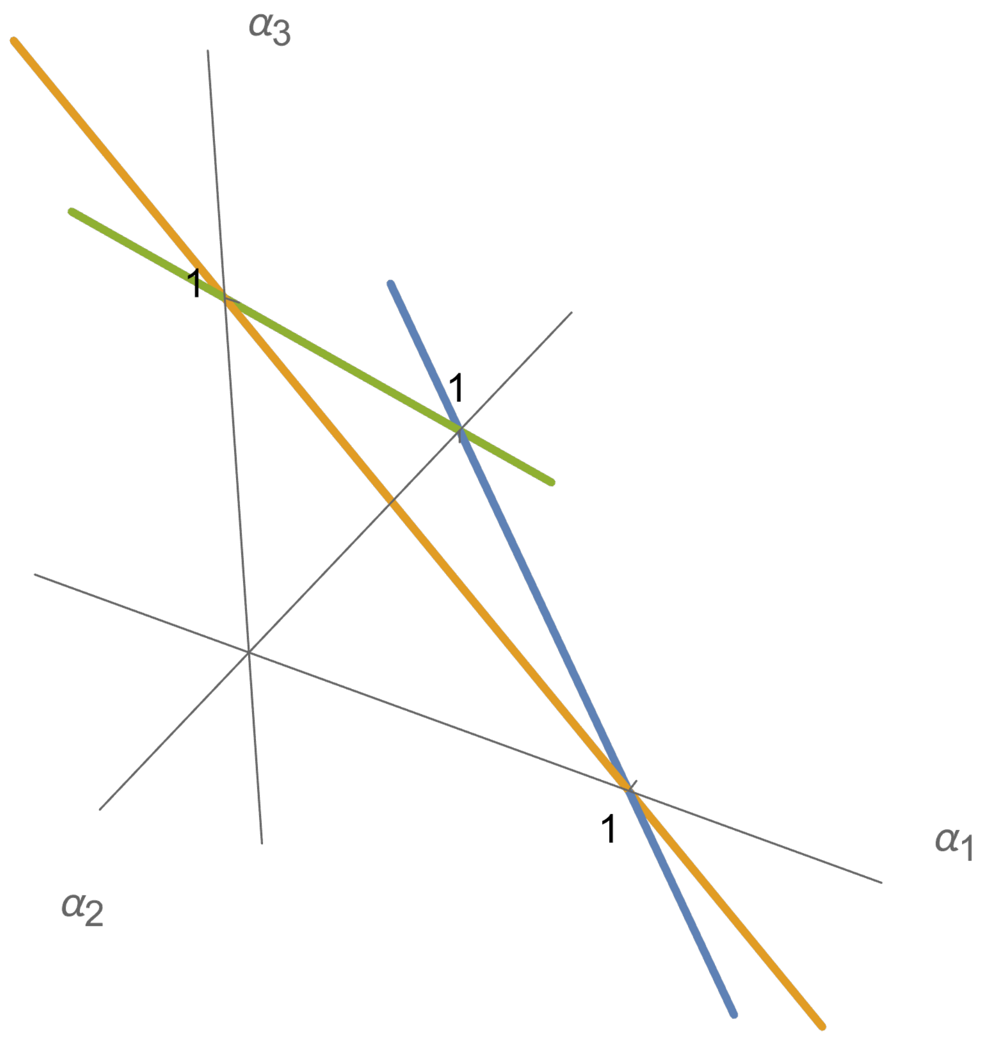
$$\equiv \int_\Gamma \mathrm{d}\alpha\, \mathscr{U}^\kappa \mathscr{F}^\lambda$$

$$I_3^D = \int_\Gamma \mathrm{d}\alpha\, \mathcal{U}^\kappa\, \mathcal{F}^\lambda$$

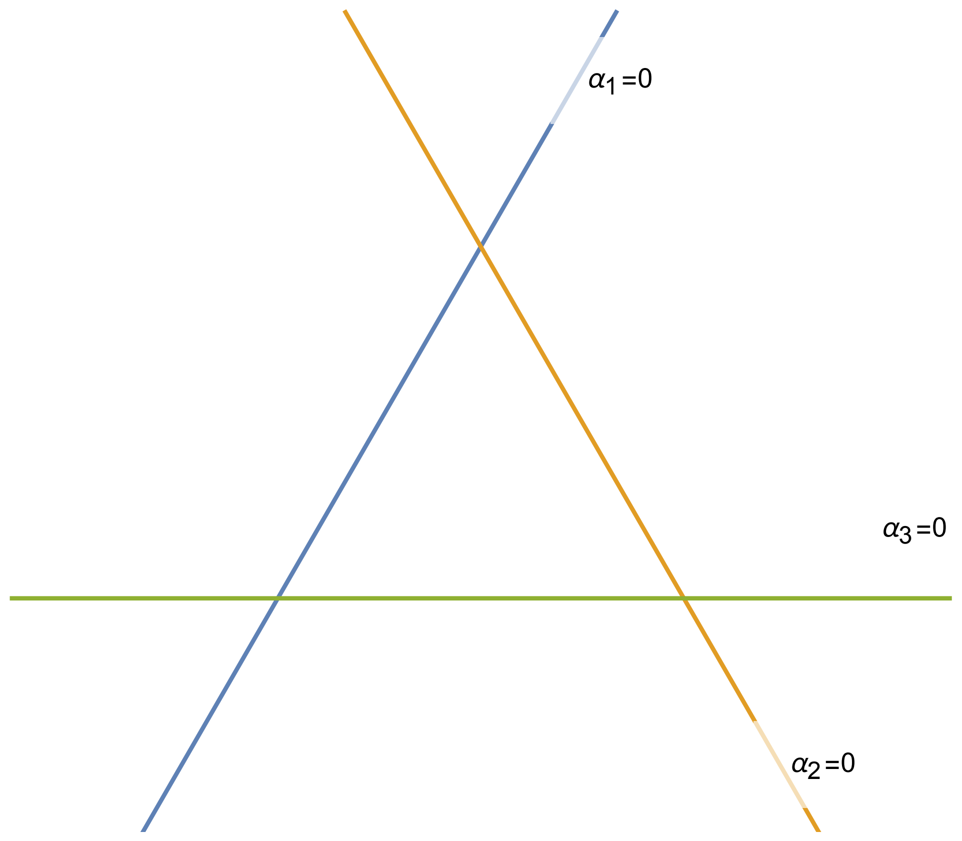
$$\Gamma = \{\alpha_i \geq 0, \sum \alpha_i = 1\}$$

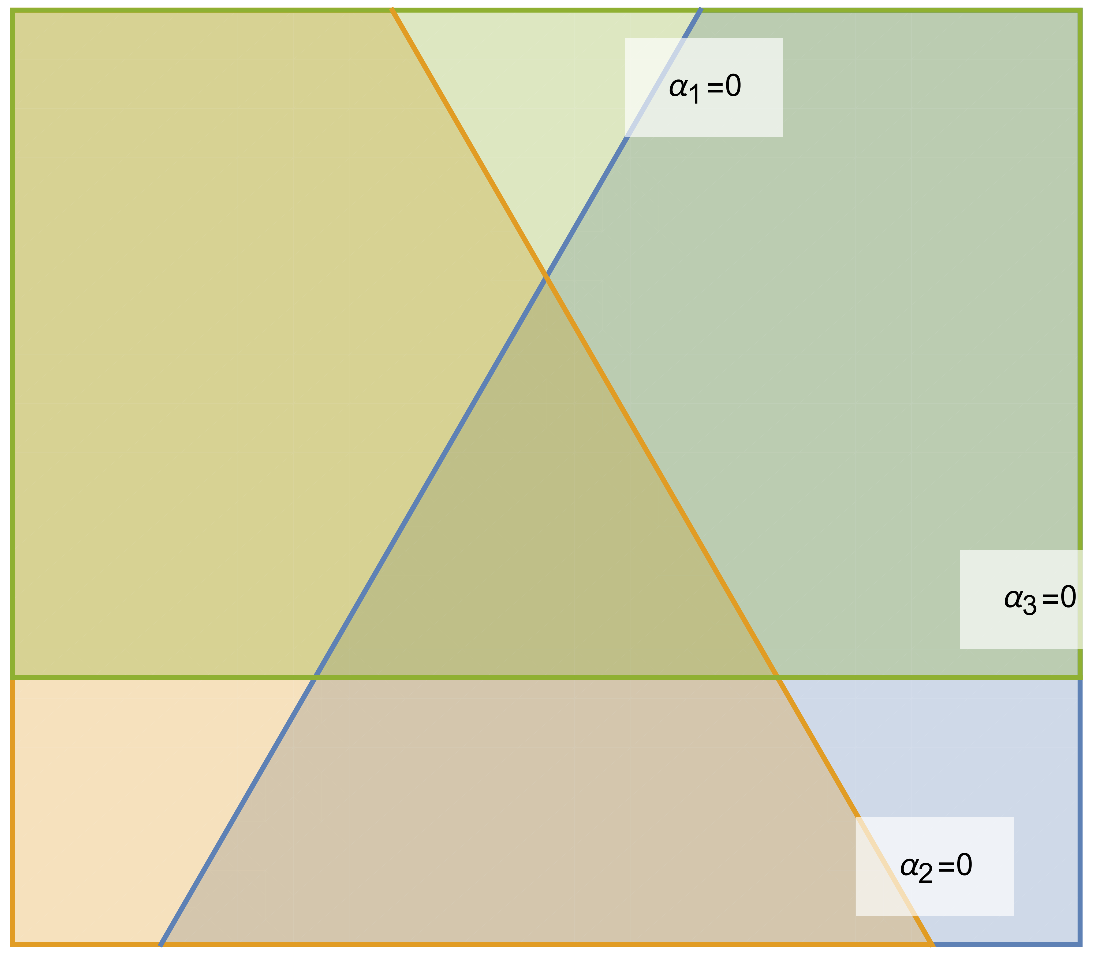


standard simplex



boundaries





What about cuts?

- Generalized cuts: propagators on shell
- Usually implemented by delta functions / residues
- Now: change contour Γ in parameter space
- Motivation from kinematic discontinuities in dimensional regularization

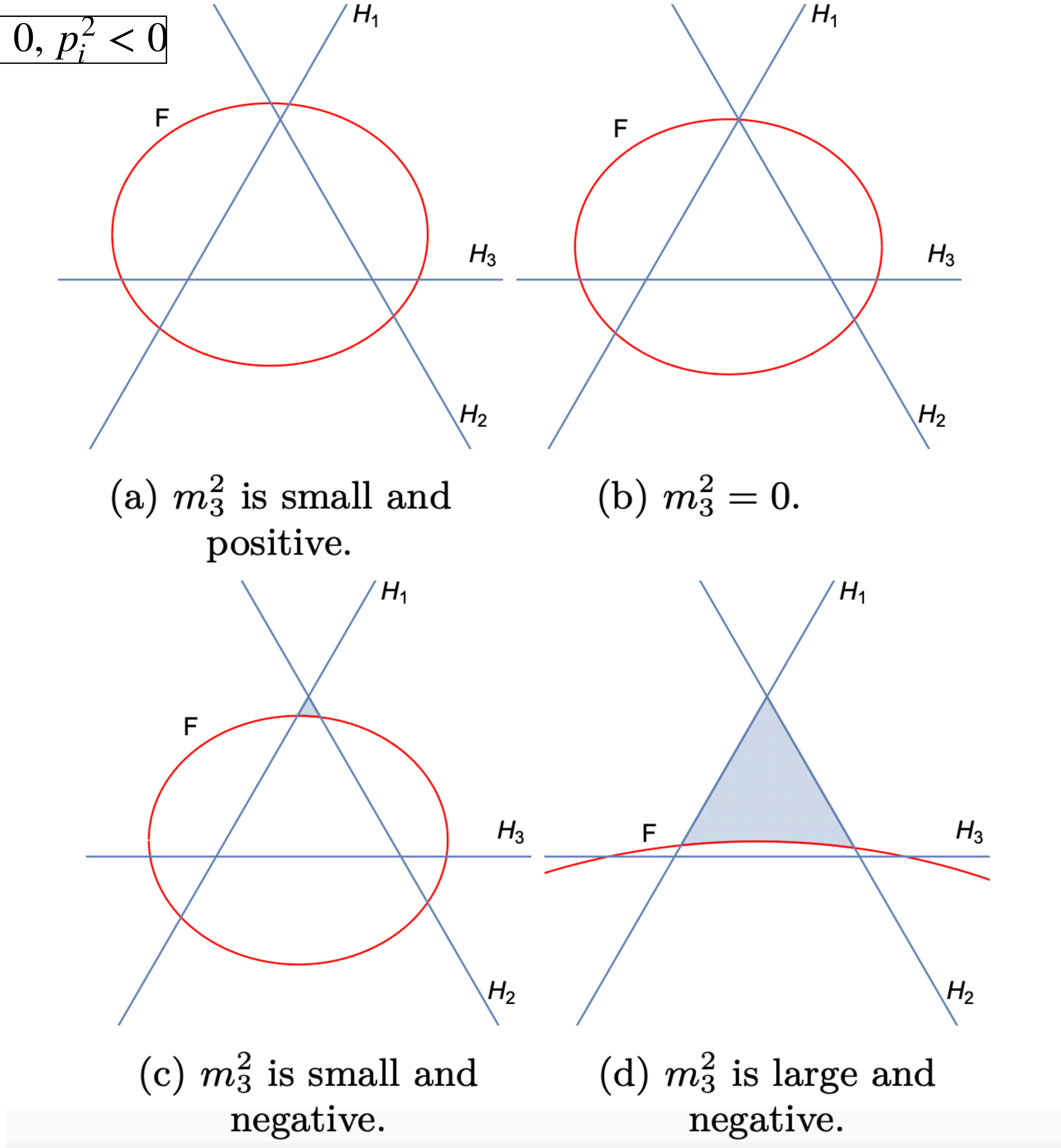
$$I = \int_\Gamma \mathrm{d}\alpha \, \mathcal{U}^\kappa \, \mathcal{F}^\lambda$$

$$\mathrm{Disc}[I] = \int_\Gamma \mathrm{d}\alpha \, \mathcal{U}^\kappa \, \mathrm{Disc}[\mathcal{F}^\lambda]$$

$$\begin{aligned} \mathrm{Disc}[\mathcal{F}^\lambda] &= (\mathcal{F} - i0)^\lambda - (\mathcal{F} + i0)^\lambda \\ &= -\,\theta[-\mathcal{F}]\,[-\mathcal{F}]^\lambda \, 2i\sin(\pi\lambda) \end{aligned}$$

m_3^2 discontinuity of the triangle

$m_1^2, m_2^2 > 0,\ p_i^2 < 0$

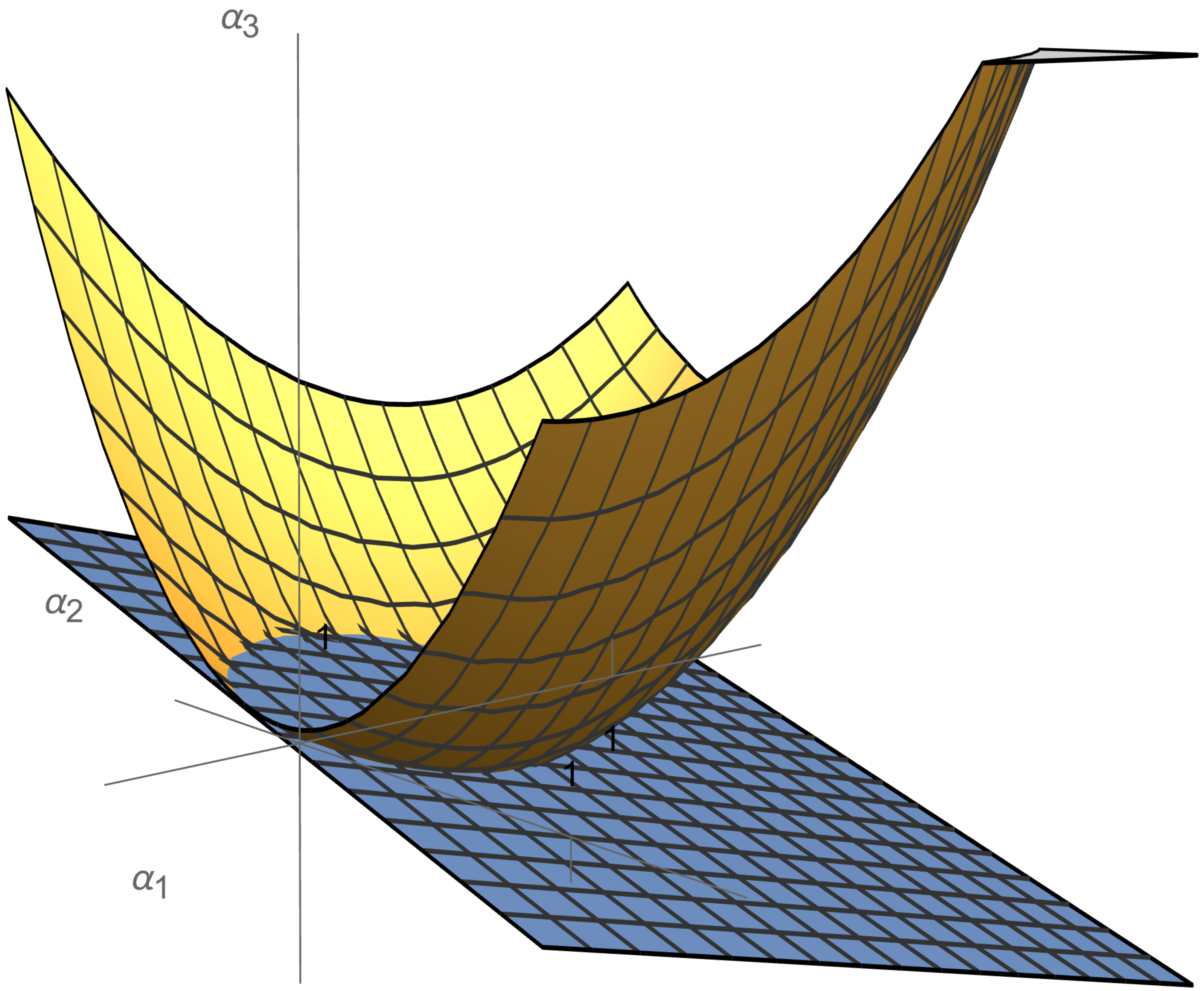


Disc has support where $\mathscr{F} < 0$.

$\mathscr{F} = 0$ becomes a new boundary.

Some original boundaries are lost.

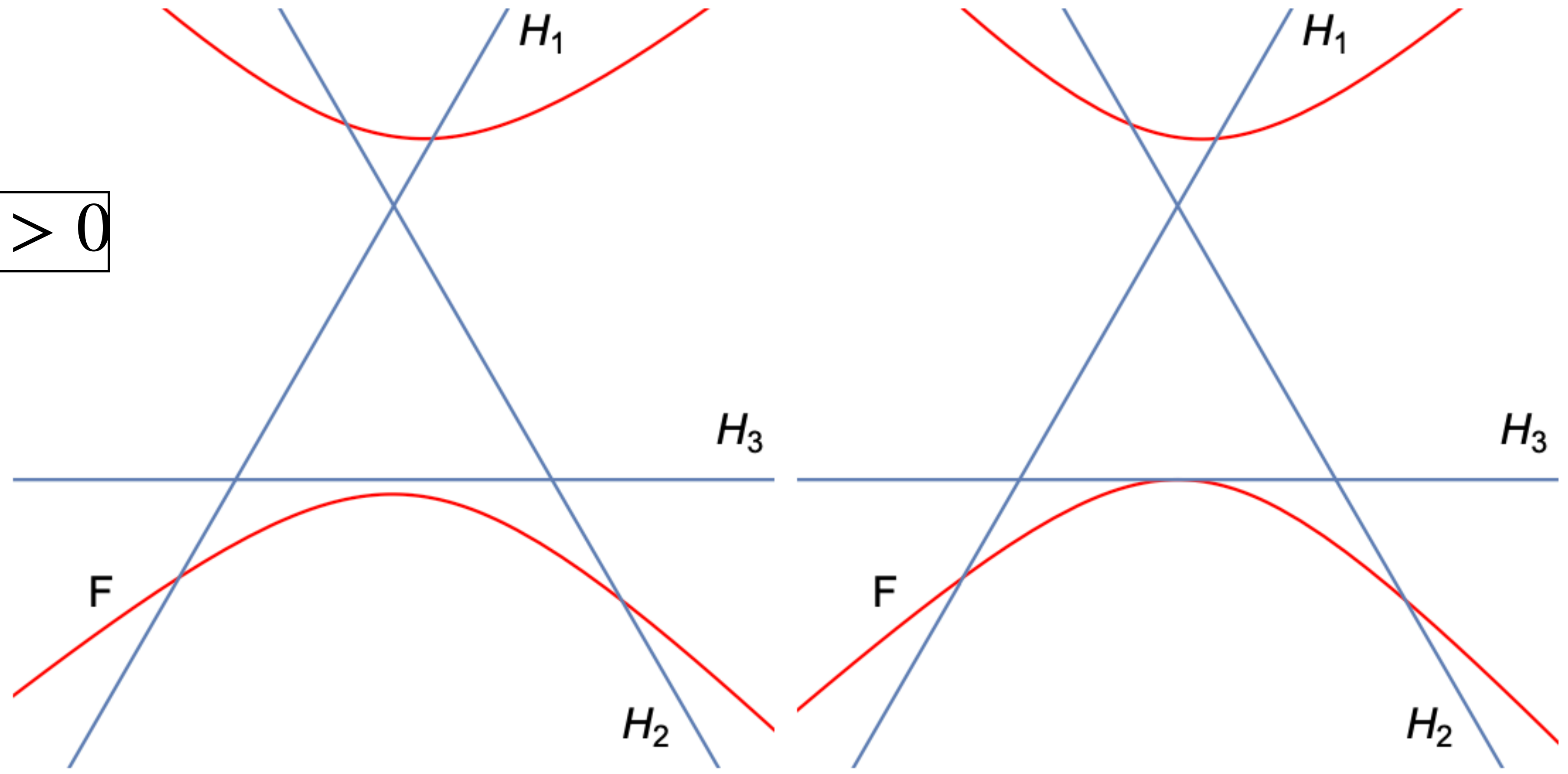
Implement this idea systematically.



α3
α2
α1
1
1
1

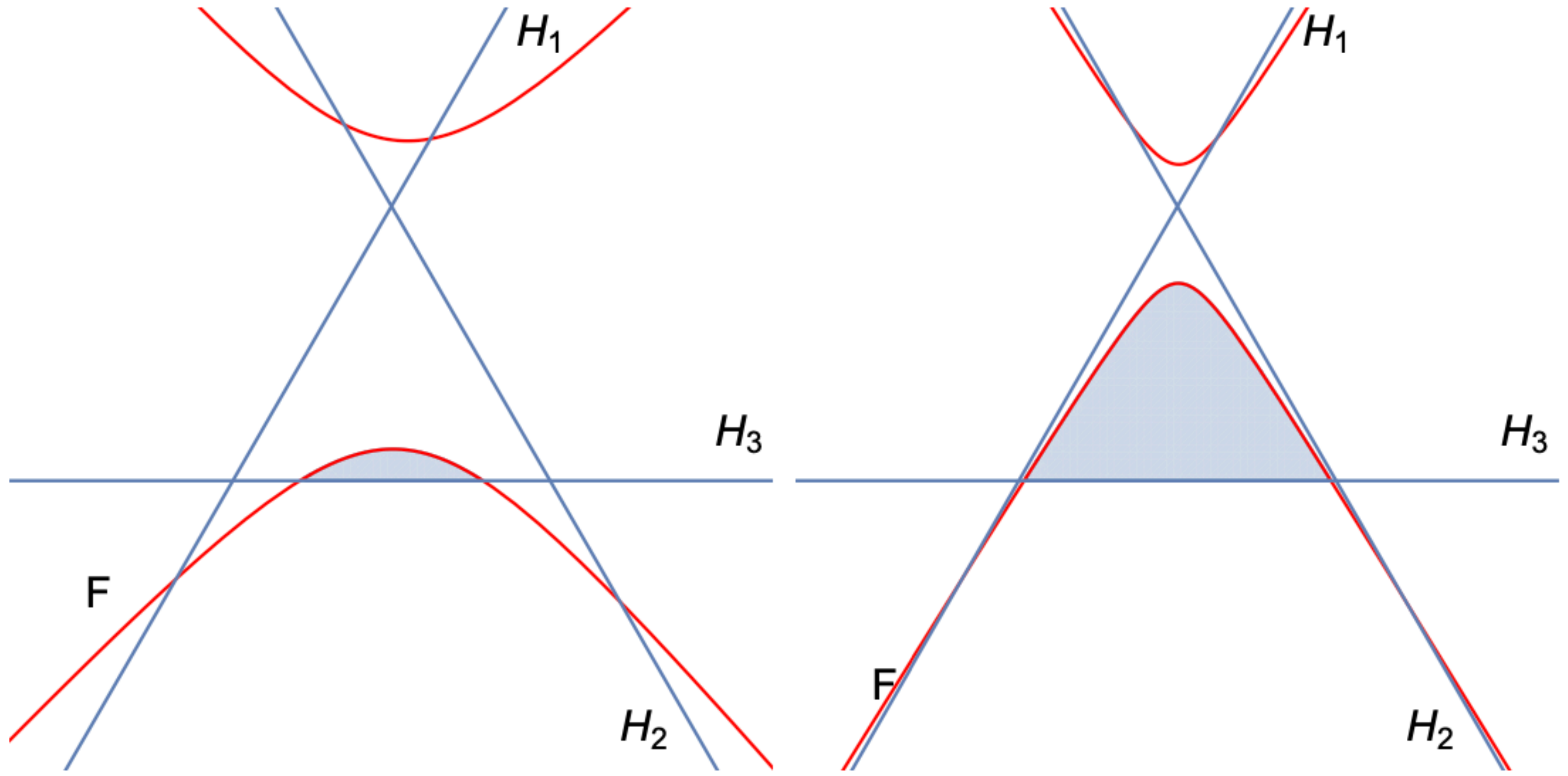
p_3^2 discontinuity of the triangle

$p_1^2, p_2^2 < 0,\ m_i^2 > 0$



(a) p_3^2 just below threshold.

(b) $p_3^2 = (m_1 + m_2)^2$.



(c) p_3^2 just above threshold.

(d) p_3^2 large compared to other scales.

Disc has support where $\mathscr{F} < 0$.

$\mathscr{F} = 0$ becomes a new boundary.

Some original boundaries are lost.

Implement this idea systematically.

Cuts change boundaries in parameter space

Justification from Landau conditions required for a singularity to occur.

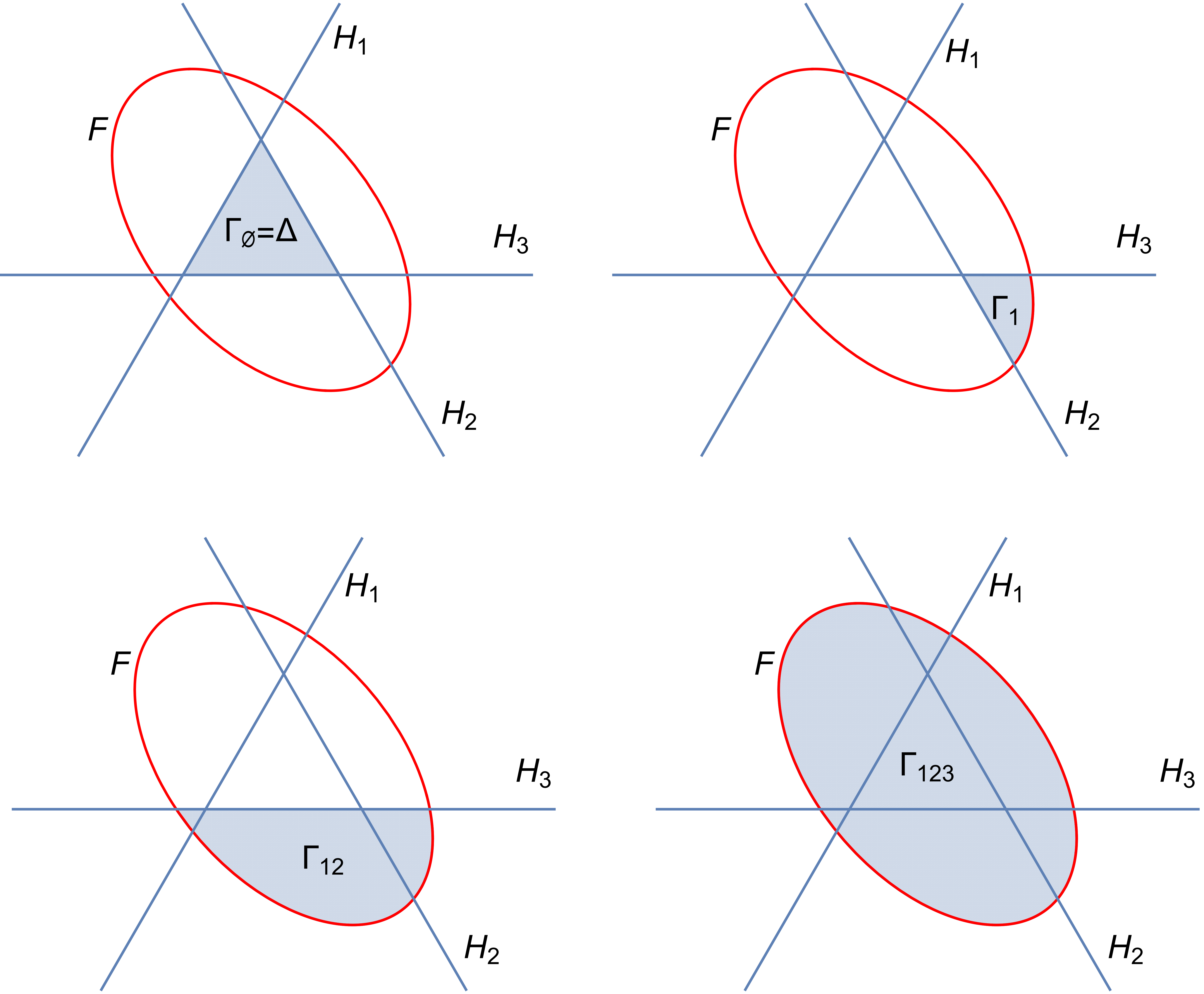
Inverse propagators: $A_i = m_i^2 - q_i^2 - i0$.

Landau conditions: $\alpha_i A_i = 0$ for each i, and $\sum_i \alpha_i \frac{\partial A_i}{\partial k_\ell} = 0$.

First condition: $A_j = 0,\ j \in J$ for some subset of edges J, and $\alpha_k = 0$ for $k \notin J$.

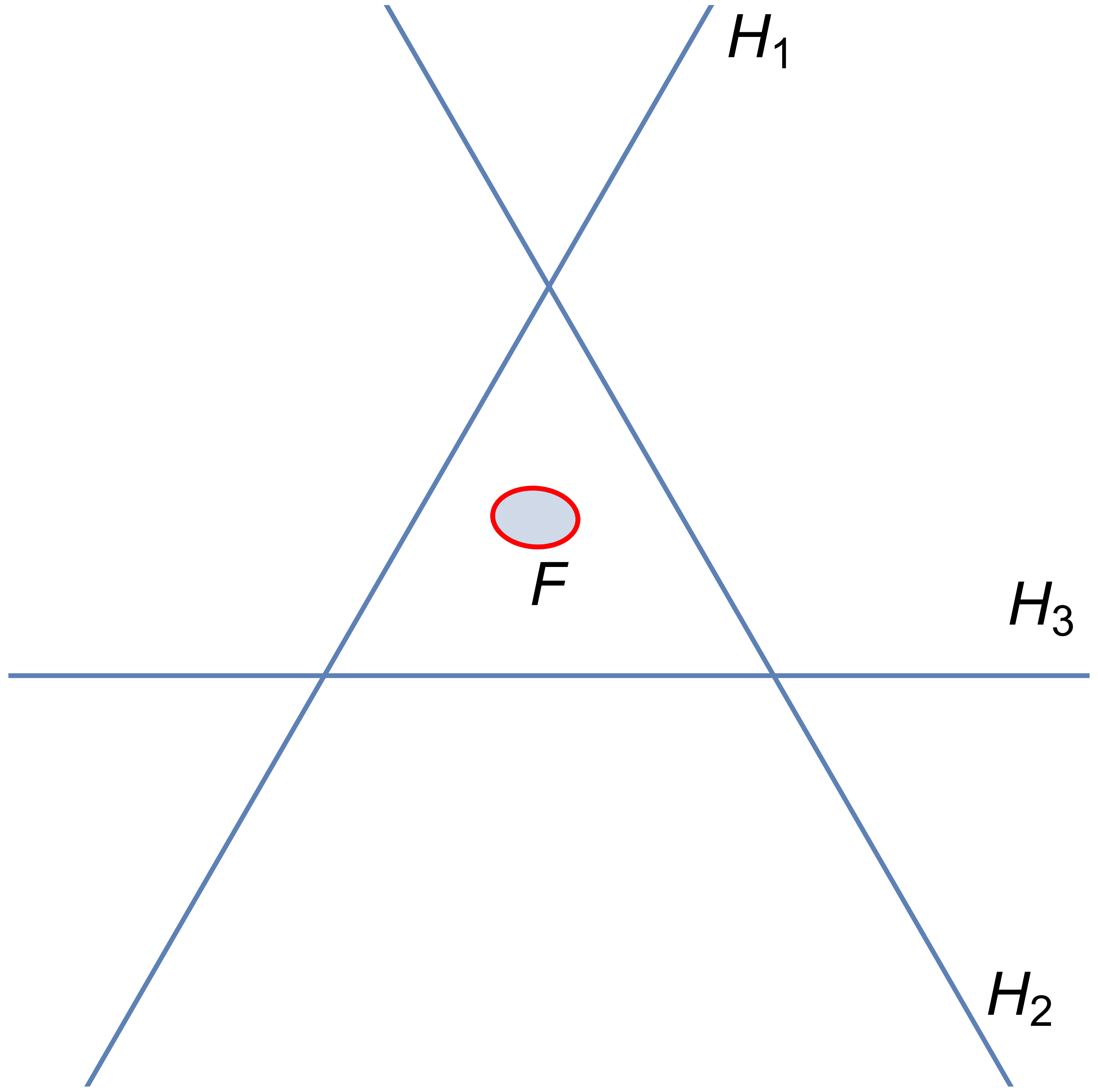
Parametrically: $\mathscr{F} = 0$, $\alpha_k = 0$ for $k \notin J$, $\frac{\partial \mathscr{F}}{\partial \alpha_j} = 0$ for $j \in J$.

Interpret the parametric condition in terms of boundaries!



- Γ_J is the domain of integration for the cut of propagators $j \in J$.
- Γ_J is bounded by the coordinate hyperplanes H_i for $i \notin J$, and by F.
- (H_i is $\alpha_i = 0$, and F is $\mathscr{F} = 0$).
- Plotted here in the Euclidean region.

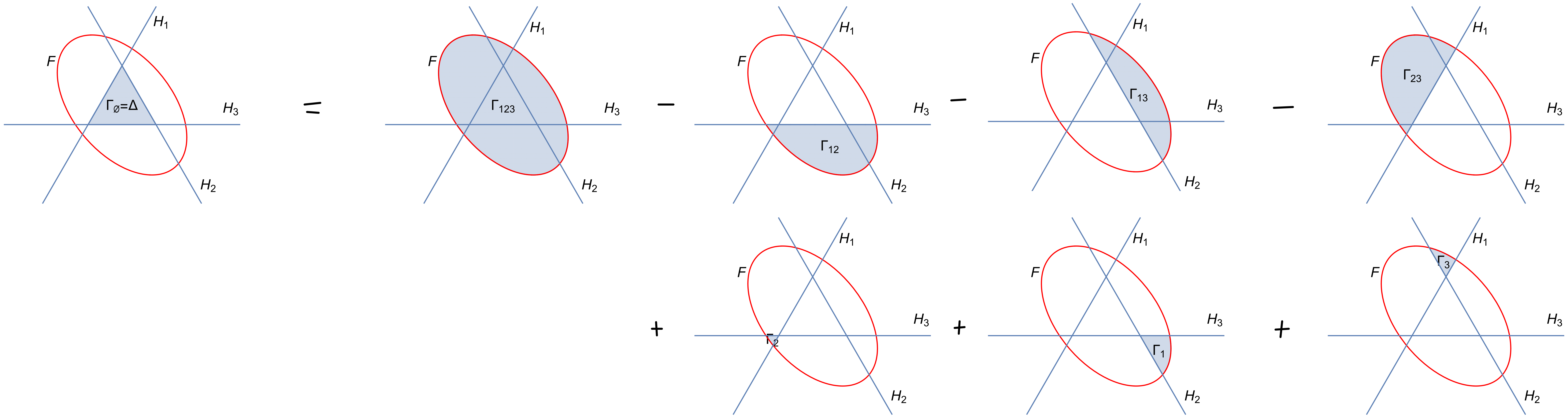
Maximal cut domains are bounded only by $\mathscr{F} = 0$!



$$\int_{\alpha^T Y \alpha > 0} \mathrm{d}^n\alpha\, \delta(1 - e^T\alpha)\left(\alpha^T Y \alpha\right)^{\frac{D}{2}-n} = \frac{1}{\sqrt{\det Y}} \left(e^T Y^{-1} e\right)^{\frac{n-D}{2}} \pi^{\frac{n-1}{2}} \frac{\Gamma\left(-n + \frac{D}{2} + 1\right)}{\Gamma\left(\frac{1+D-n}{2}\right)}$$

$$\alpha = (\alpha_1, \ldots, \alpha_n)^T, \quad e = (1,\ldots,1)^T, \quad Y_{ij} = \frac{1}{2}\left(m_i^2 + m_j^2 - q_{ij}^2\right), \quad e^T Y^{-1} e = \frac{\mathrm{Gram}_n}{\det Y}\,.$$

Linear relation among cuts



$$\sum_{j=1}^{n} \mathcal{C}_{\{j\}}\mathcal{I} + \sum_{\{j,k\}\subseteq[n]} \mathcal{C}_{\{j,k\}}\mathcal{I} \equiv -\epsilon\mathcal{I} \mod i\pi\,.$$

[Abreu, RB, Duhr, Gardi]

Originally derived from decomposition theorem in homology.

Normalization of cuts explains ϵ, signs, terms dropped modulo $i\pi$.

Domain relations provide an exact version of the relation.

General Feynman integrals

- Parametric representation of an L-loop Feynman integral with E edges in D dimensions:

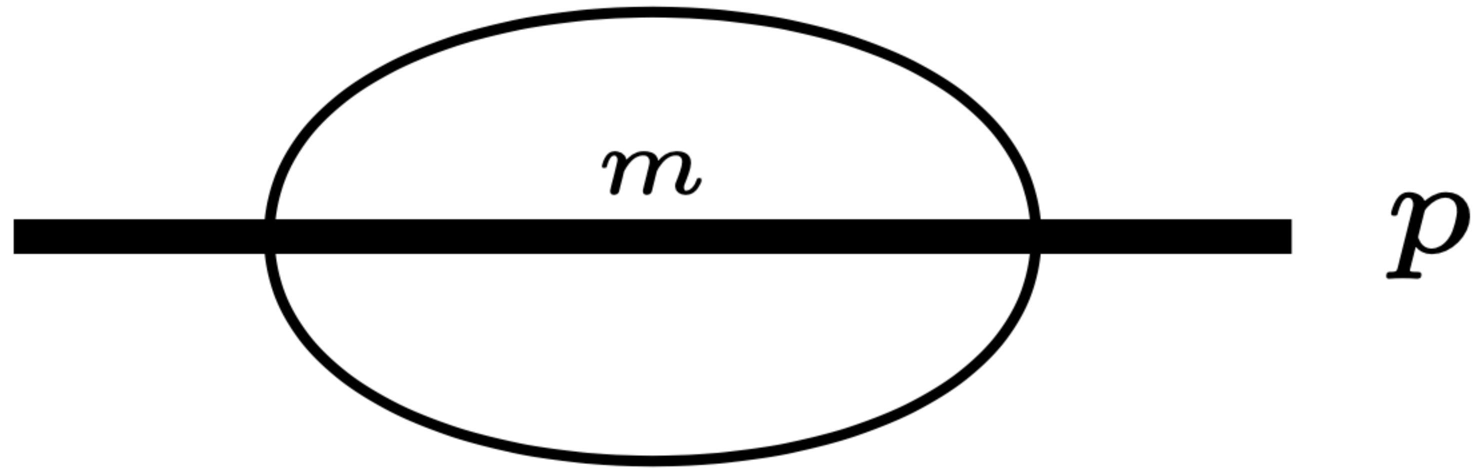
$$I = \Gamma\left(\nu - \frac{LD}{2}\right) \int_{\alpha_i \geq 0} d^E\alpha\, \delta\left(1 - \sum_{i \in S} \alpha_i\right) \left(\prod_{i=1}^{E} \alpha_i^{\nu_i - 1}\right) \frac{\mathcal{U}^{\nu-(L+1)D/2}}{\mathcal{F}^{\nu - LD/2}}$$

  where S is a subset of edges.

- Parametric Landau equations (for $\mathcal{U} \neq 0$):

$$\mathcal{F} = 0\,, \quad \alpha_k = 0 \text{ for } k \notin J\,, \quad \frac{\partial \mathcal{F}}{\partial \alpha_j} = 0 \text{ for } j \in J.$$

Simple 2-loop example



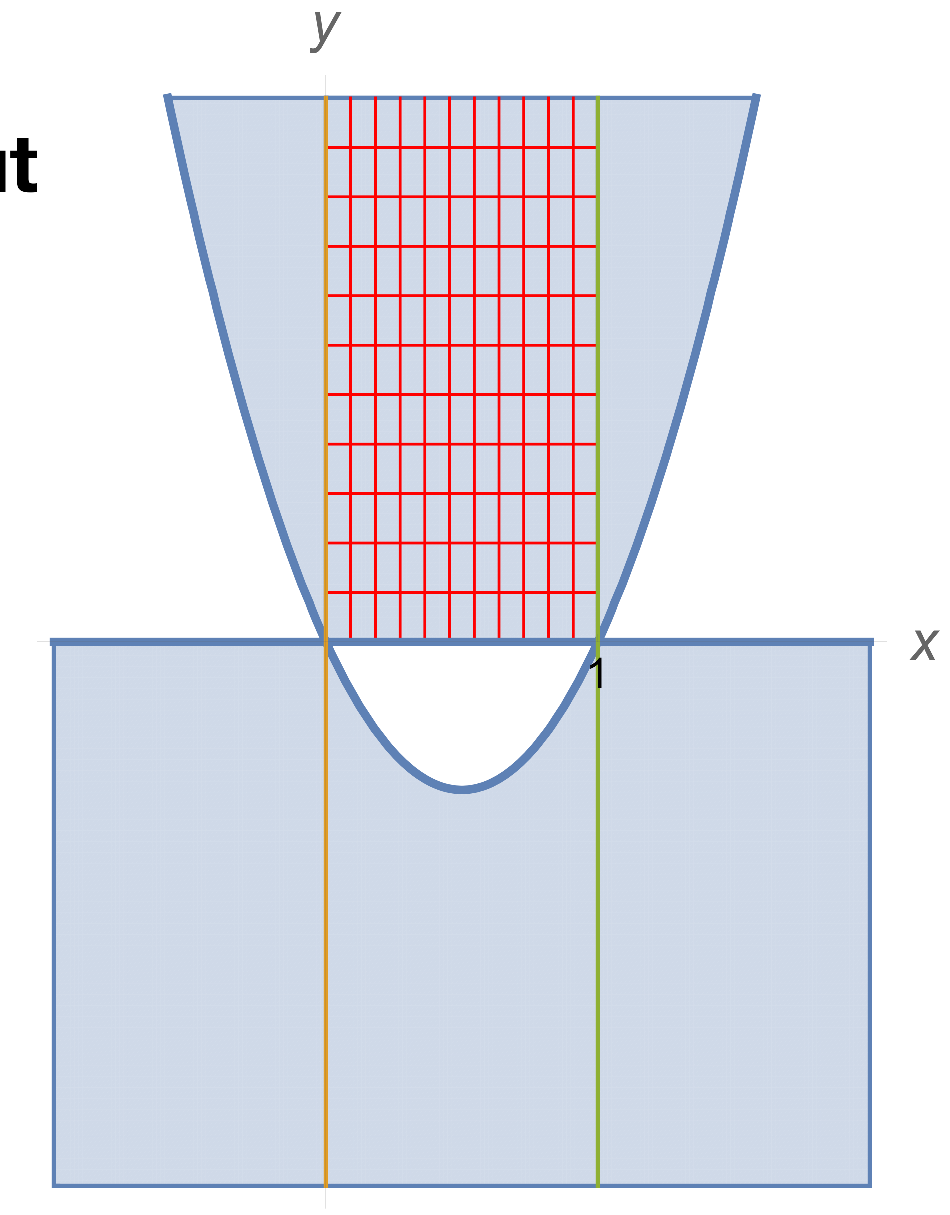
$$S = \{1,2\} : \ \alpha_1 = x, \ \alpha_2 = 1 - x, \ \alpha_3 = y$$

$$\mathscr{U} = x(1-x) + y$$

$$\mathscr{F} = y\left[m^2 y + (m^2 - p^2)x(1-x)\right]$$

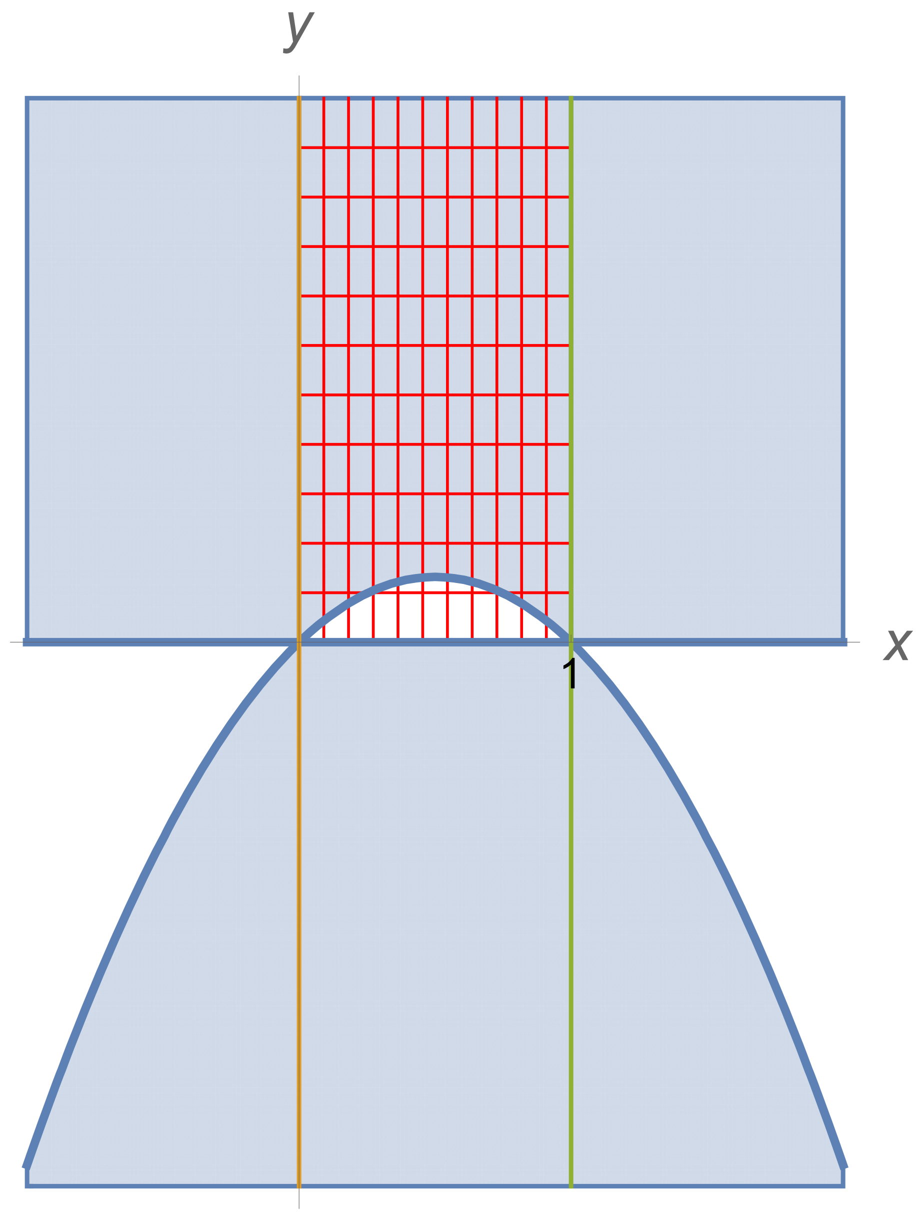
Here $\mathscr{F} = 0$ has two components: a variable parabola, and the coordinate hyperplane $y = 0$.

Uncut



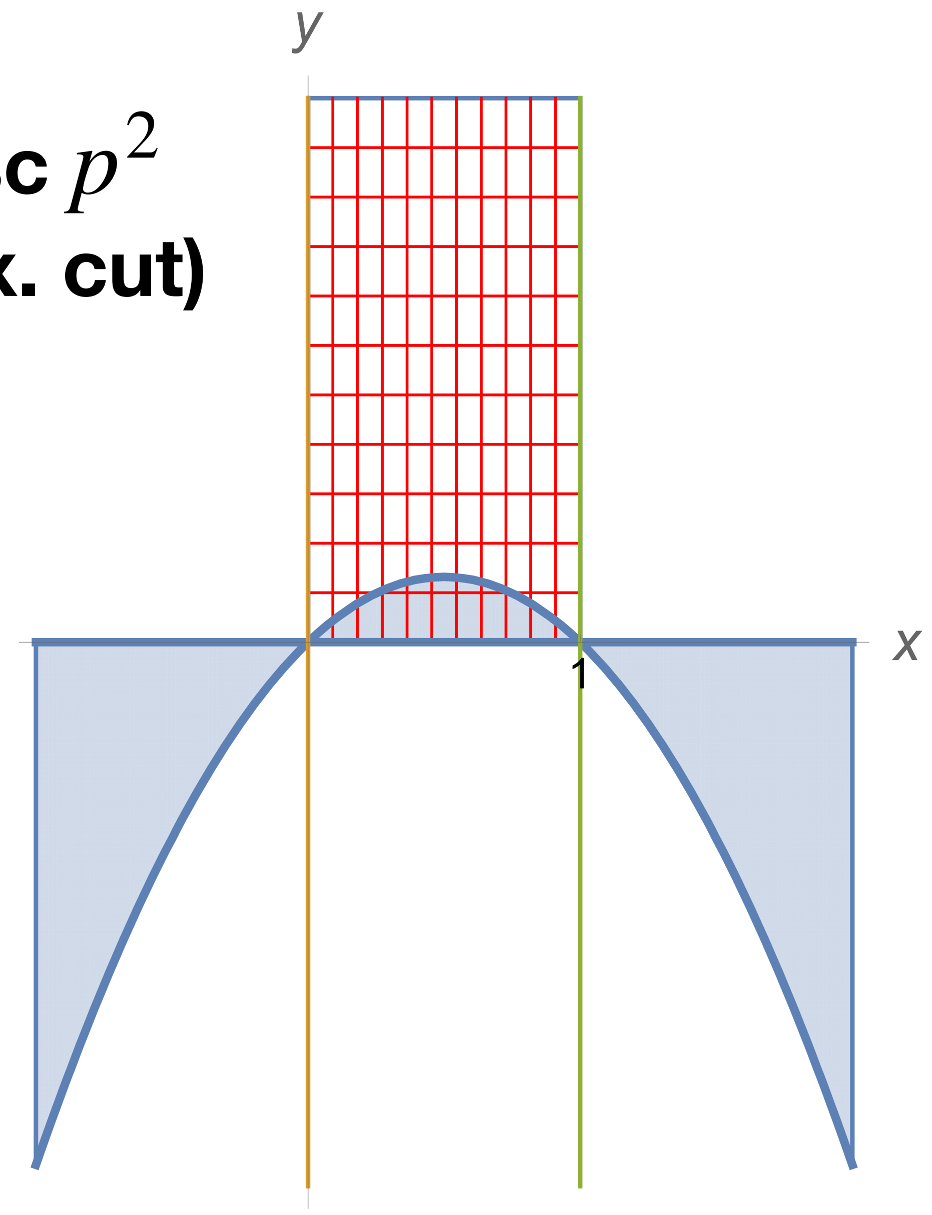
$\mathscr{F} > 0$ with $p^2 < 0 < m^2$

Disc m^2



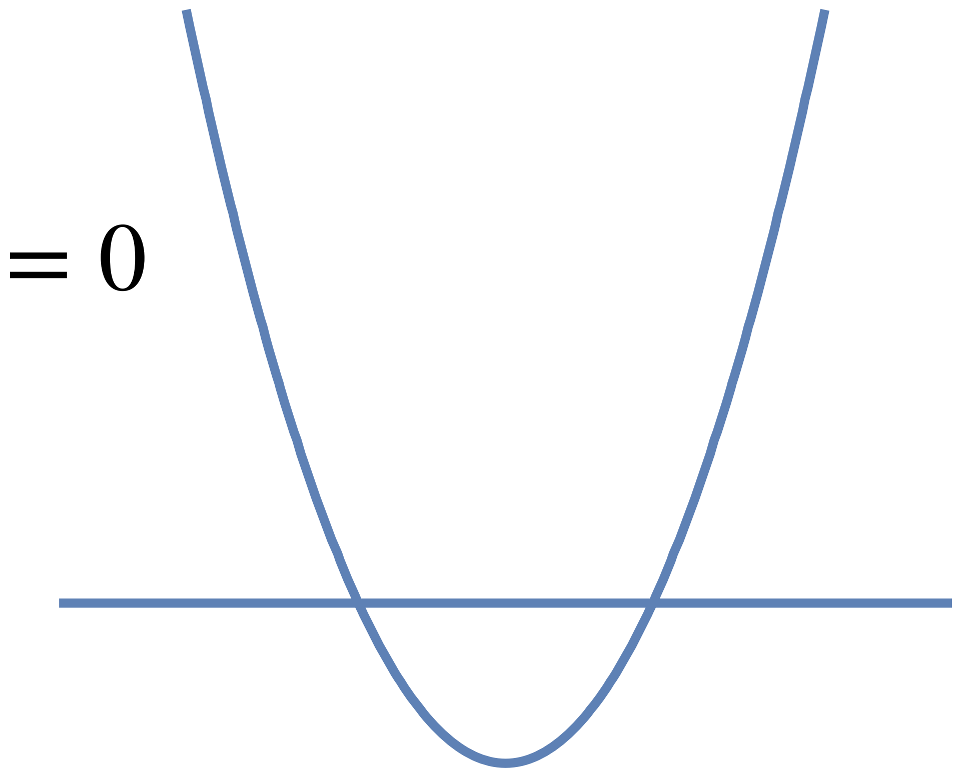
$\mathscr{F} < 0$ with $p^2 < m^2 < 0$

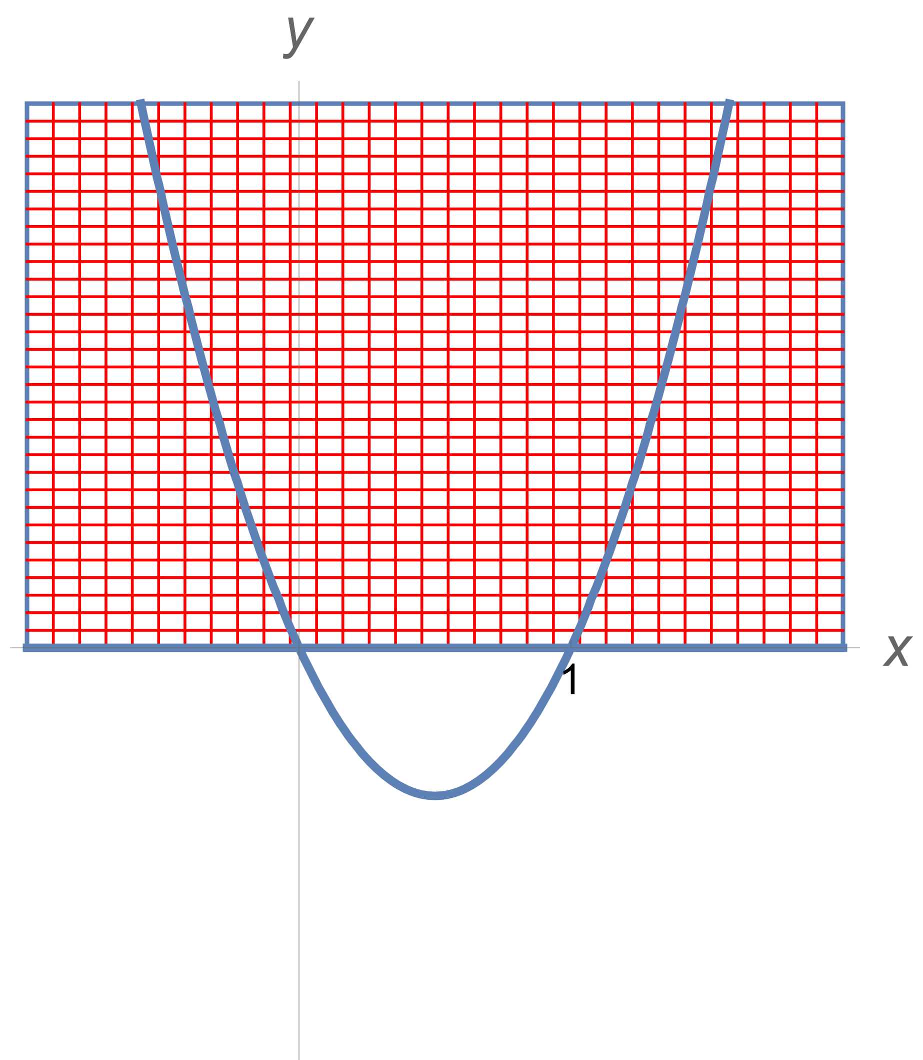
Disc p^2 (max. cut)



$\mathscr{F} < 0$ with $0 < m^2 < p^2$

$\mathscr{F} = 0$





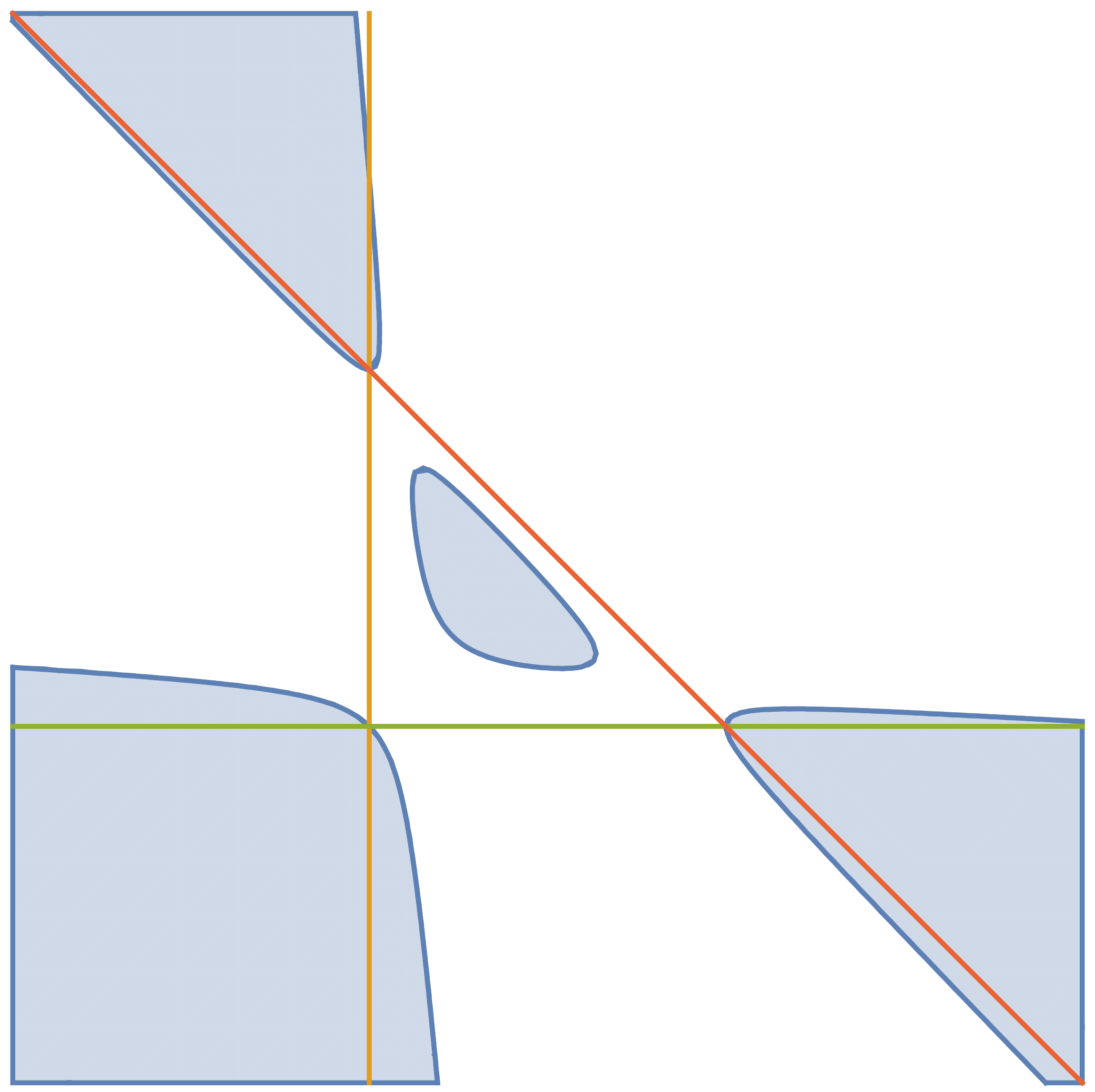
Second maximal cut

$$\mathscr{F} = y\left[m^2 y + (m^2 - p^2)x(1-x)\right]$$

Generic sunrise

$$\mathscr{F} = -\, p^2\alpha_1\alpha_2\alpha_3 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)(m_1^2\alpha_1 + m_2^2\alpha_2 + m_3^2\alpha_3)\,.$$

$$\alpha_1 = x,\ \alpha_2 = y,\ \alpha_3 = 1 - x - y\,.$$



4 independent regions bounded by $\mathscr{F}$.

Summary

- Proposal: cuts of Feynman integrals are obtained by integrating the parametric integrand over a region with certain boundaries. The boundaries are $\mathscr{F} = 0$, along with the subset of the coordinate hyperplanes complementary to the cut propagators.
- Consistent with discontinuities and Landau conditions.
- Evidence from simpler integrals. Agreement with some known results and with discontinuities.
- Relations among cuts are visible at 1-loop. Some 2-loop cases also.

Future explorations

- Can we generate/predict linear relations among multiloop cut integrals?
- Cuts as hypergeometric functions and in coactions; cf. period matrix $P_{ij} = \int_{\Gamma_j} \omega_i$.
- Exploit graphical properties of $\mathscr{U}$ and $\mathscr{F}$ in multiloop applications.
- Useful for (numerical) parametric computation?
- Cuts in related parametrizations: Schwinger, Lee-Pomeransky, Baikov, etc. with $\mathscr{F} = 0$ as a boundary.
- What about full amplitudes? Cuts as volumes in certain geometries?