Evaluating One-Loop String Amplitudes

Amplitudes 2023

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Based on work in collaboration with Sebastian Mizera
[2208.12233, 2302.12733 [hep-th]]

Surprisingly little is known about higher loop string amplitudes

Veneziano Amplitude

$$\mathscr{A}(s,t) = t_8 \times \mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}) \times \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)}$$

Polarization structure

Color structure

Veneziano '68

No explicit formulas for higher loops...

Veneziano Amplitude

$$\mathscr{A}(s,t) = t_8 \times \operatorname{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}) \times \boxed{\frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1-\alpha' s-\alpha' t)}}$$

Polarization structure

Color structure

Interesting part!

Veneziano '68

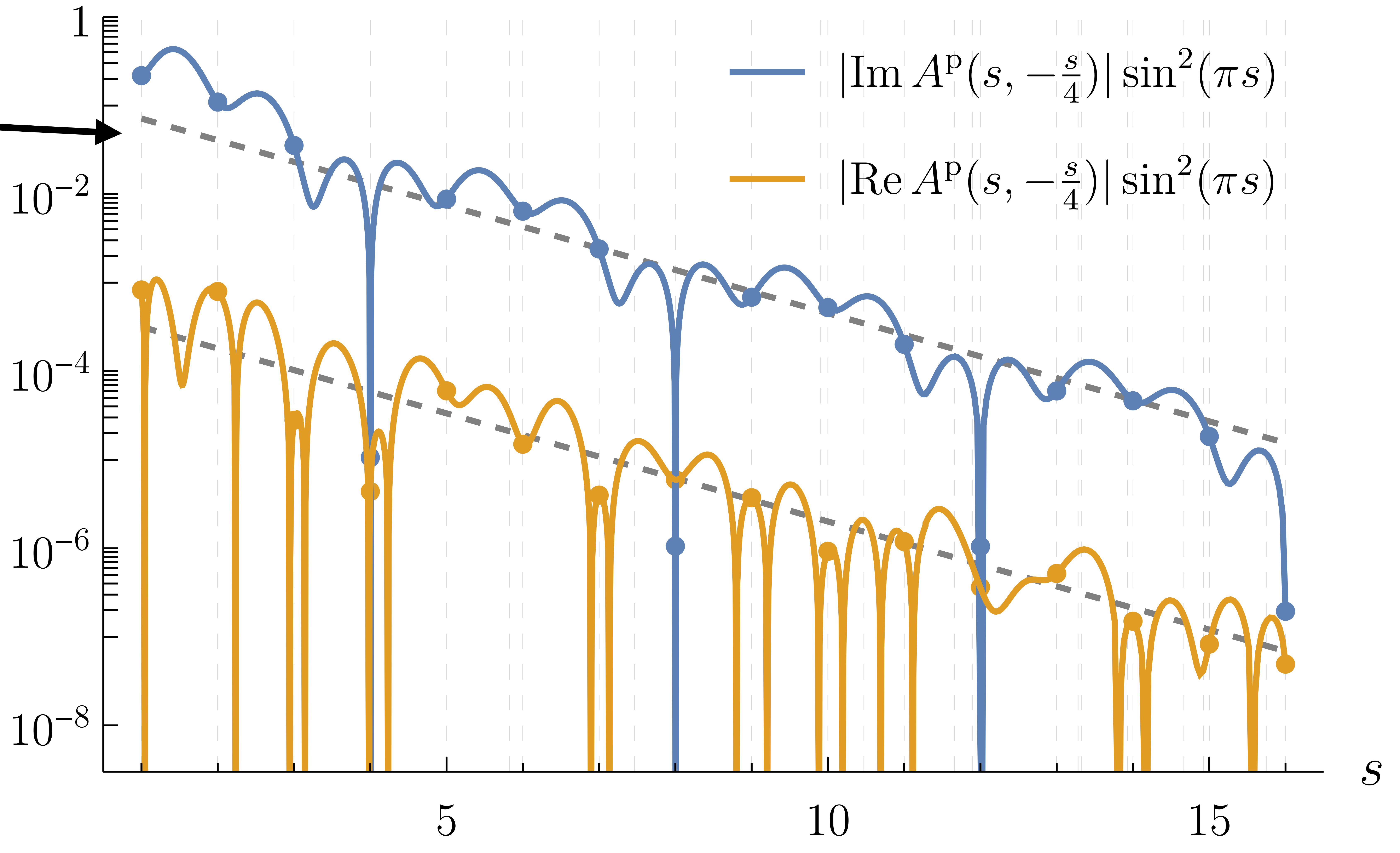
No explicit formulas for higher loops…

Type I one-loop amplitude in 10d @ 60 degrees

Field theory

+ α' corrections

[enormous literature: Green, Schwarz, Gross, Veneziano, Amati, Ciafaloni, Di Vecchia, Koba, Nielsen, D'Hoker, Phong, Martinec, Bern, Dixon, Polyakov, Kosower, Vanhove, Schlotterer, Mafra, Stieberger, Brown, Broedel, Hohenegger, Kleinschmidt, Gerken, Roiban, Lipstein, Mason, Monteiro, ...]



LE, Mizera '23

I. Defining the amplitude

Higher loop amplitudes

- Textbook definition:

$$A_{g,n}(p_1, \ldots, p_n) = \int_{\mathscr{M}_{g,n}} (\text{CFT correlation function})$$

- Not quite correct:
  - Supermoduli space or picture changing operators
  - Integral does not converge

Friedan, Martinec, Shenker '85-'86, Verlinde[2] '87, D'Hoker, Phong '86-'89, (Donagi), Witten '12, Sen, (Witten) '14 - '15, ...

A definition consistent with causality

- Problem: We treat the worldsheet as Euclidean, but spacetime is Lorentzian!

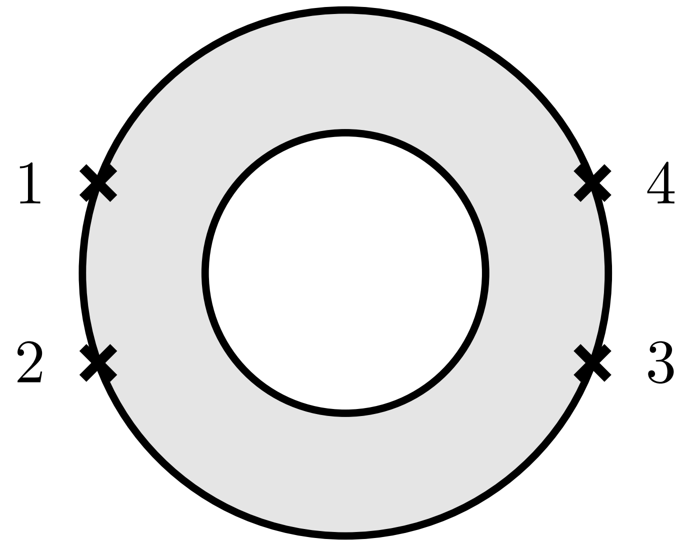
$$A_{g,n}(p_1, \ldots, p_n) = \int_{\Gamma \subset \mathscr{M}_{g,n}^{\mathbb{C}}} \text{(CFT correlation function)}$$

Contour consistent with the Lorentzian spacetime

Describes complex metrics on the worldsheet

Degenerations of the worldsheet

- Modify contour of integration near a degeneration of the worldsheet



Degenerations of the worldsheet

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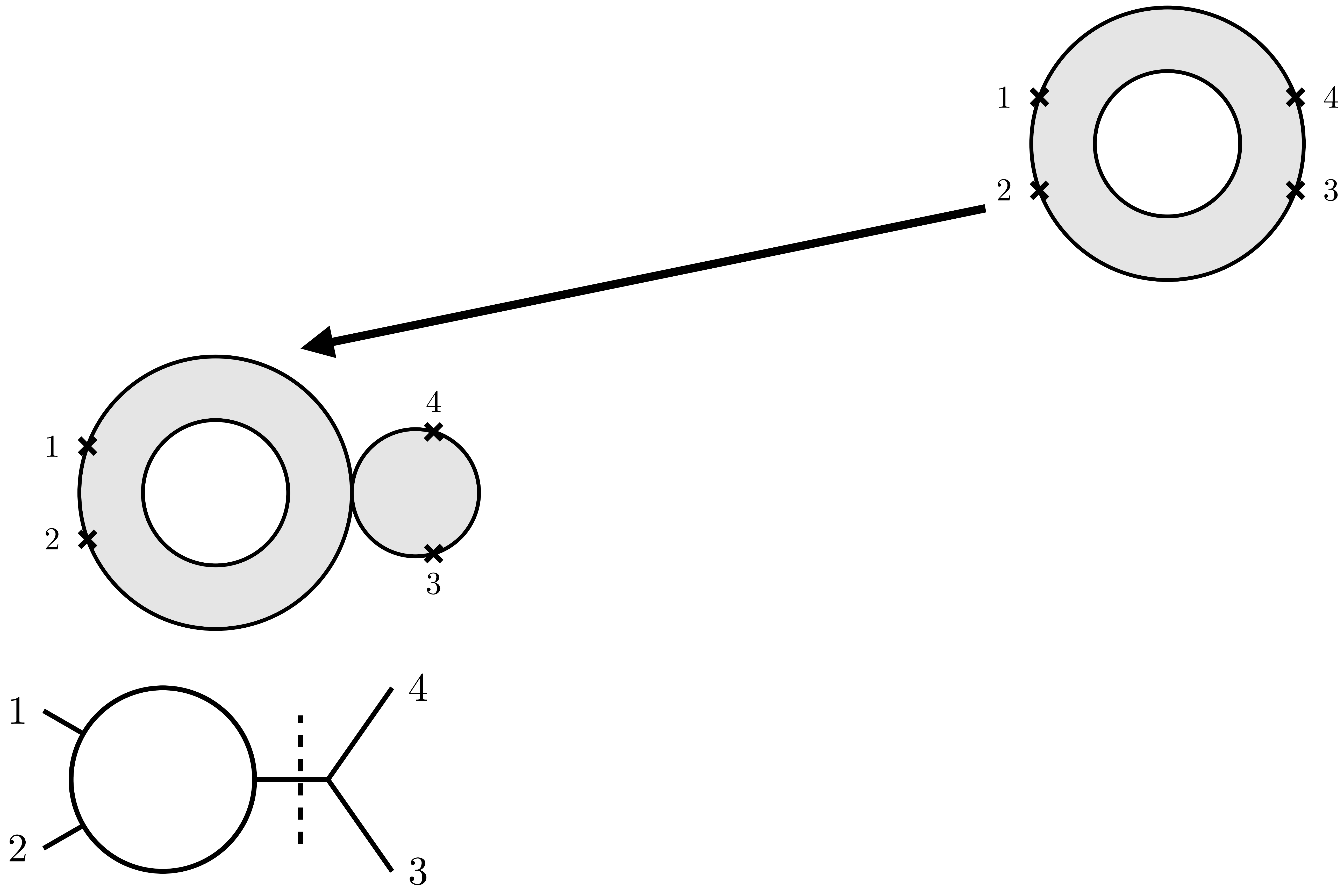

Massive poles

Degenerations of the worldsheet

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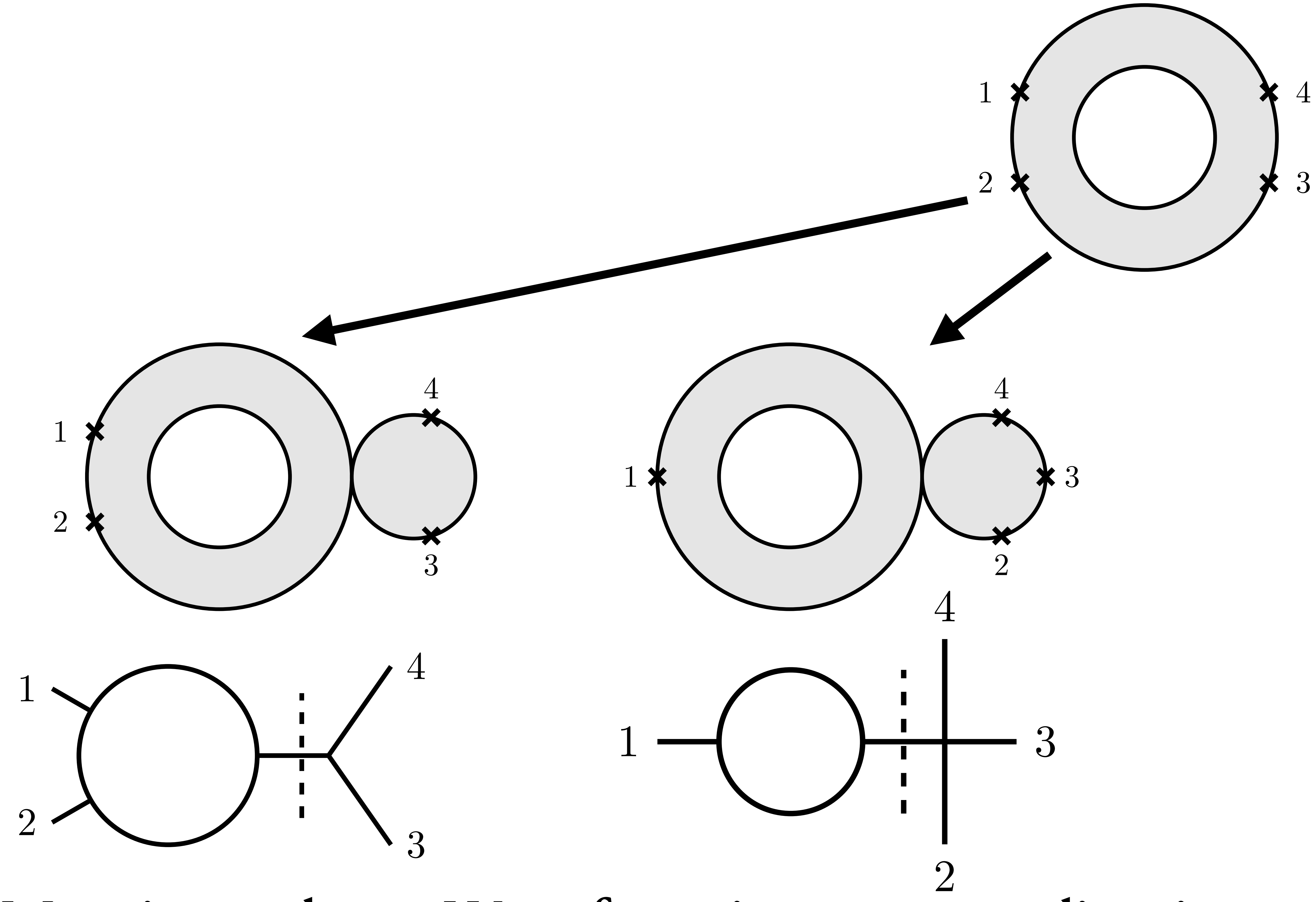

Massive poles   Wavefunction renormalization

Degenerations of the worldsheet

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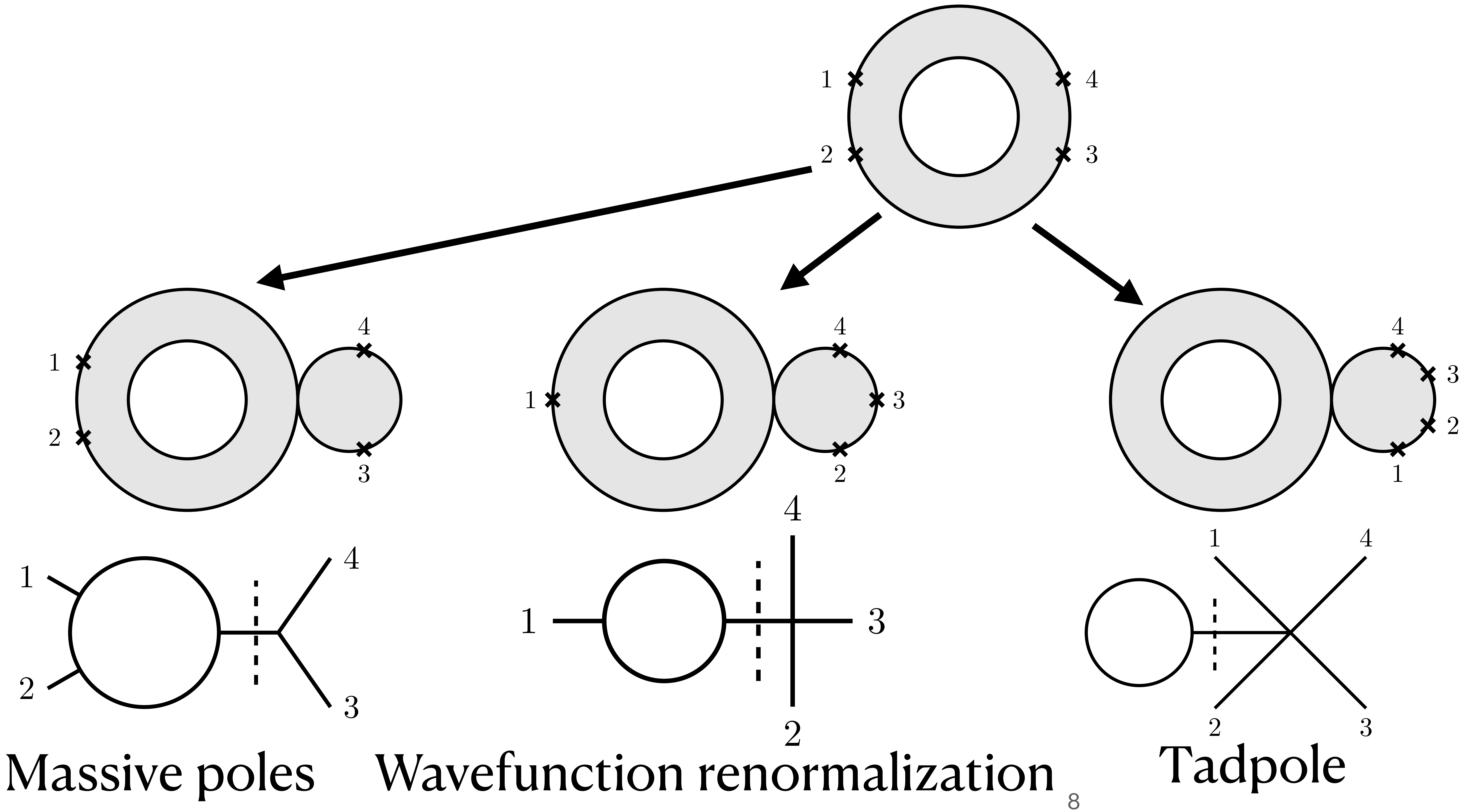

Degenerations of the worldsheet

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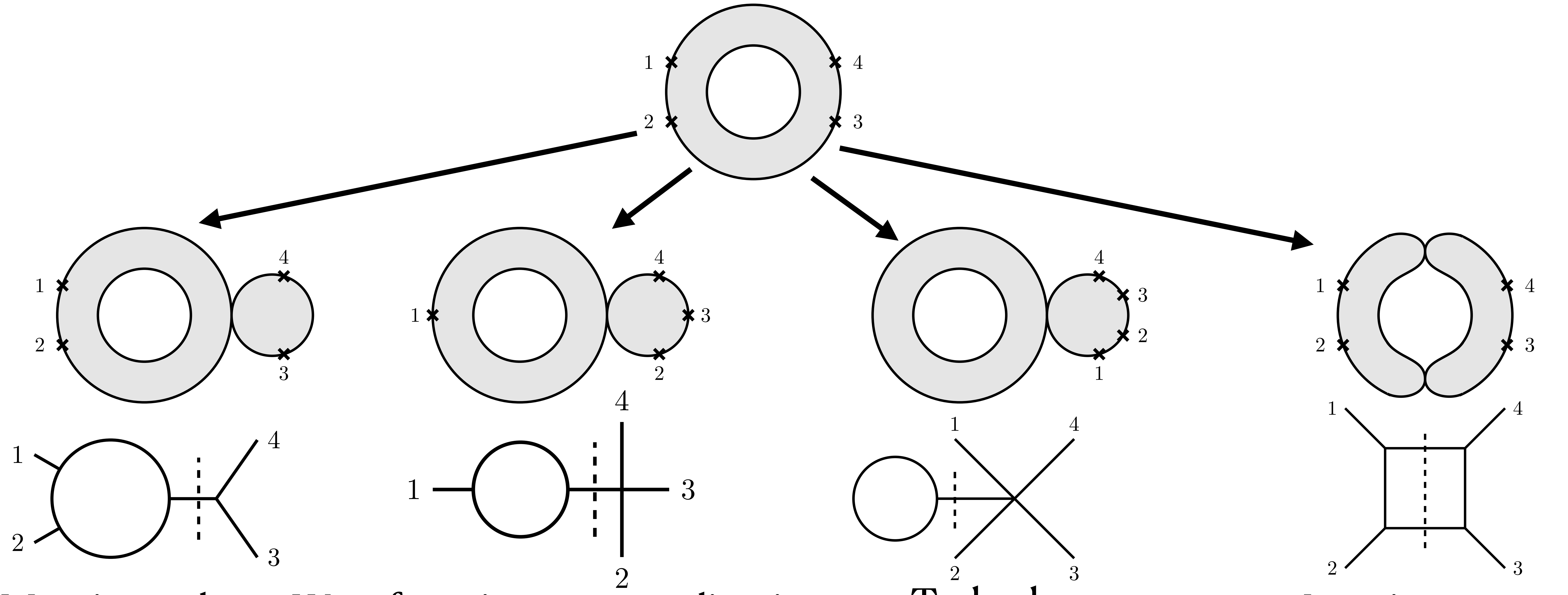
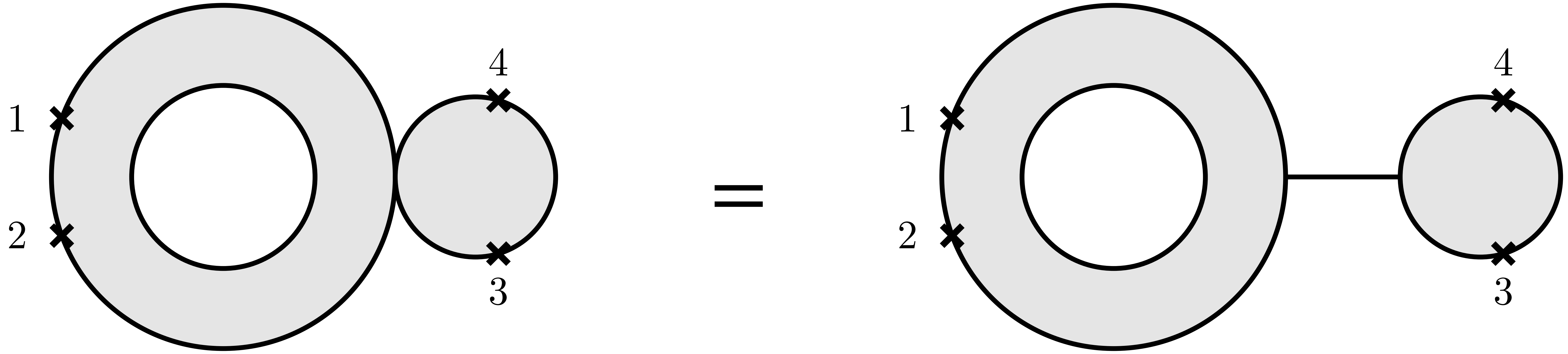


Massive poles Wavefunction renormalization Tadpole Imaginary part

The contour Γ

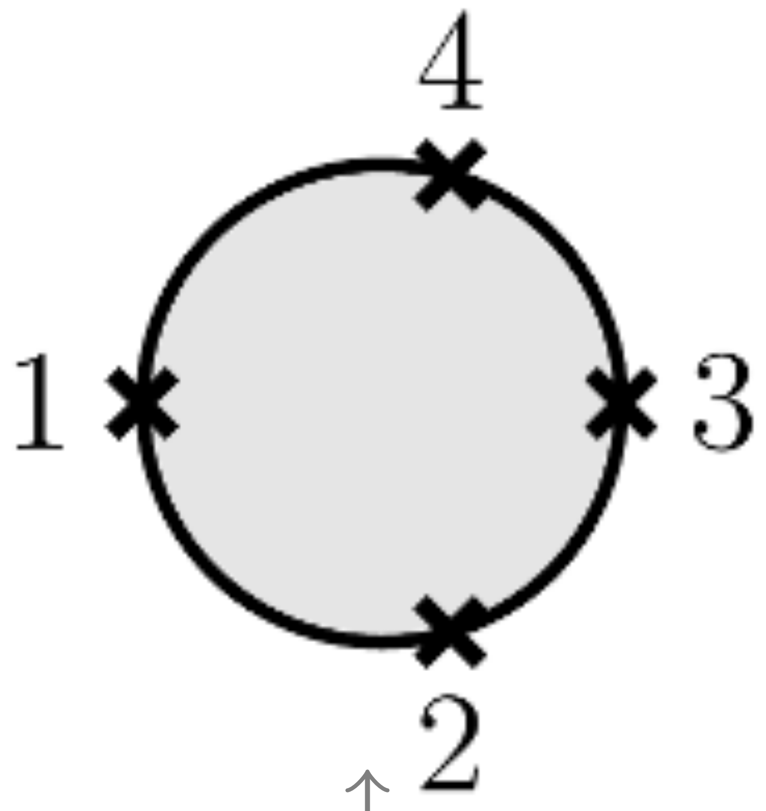
- Near a degeneration, the worldsheet can be approximated by a worldline



- **Euclidean** worldline Schwinger parameter = one of the modulus in $\mathscr{M}_{1,4}^{\text{open}}$
- To define Γ, Wick rotate to a Lorentzian Schwinger parameter and use field theory $i\varepsilon$

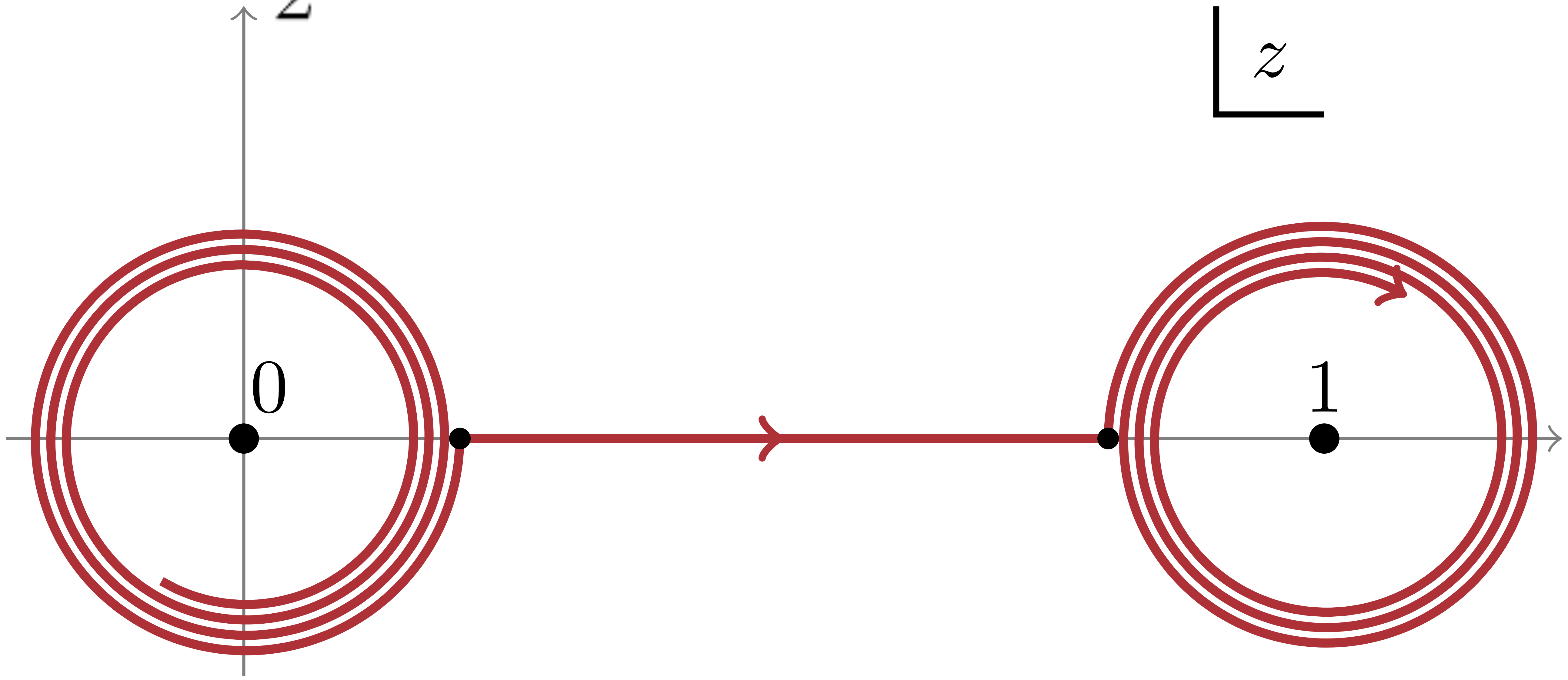
Witten '13

The tree-level amplitude



$$= \frac{1}{t}\int_\Gamma \mathrm{d}z\; z^{-s-1}(1-z)^{-t}$$

$\alpha' = 1$

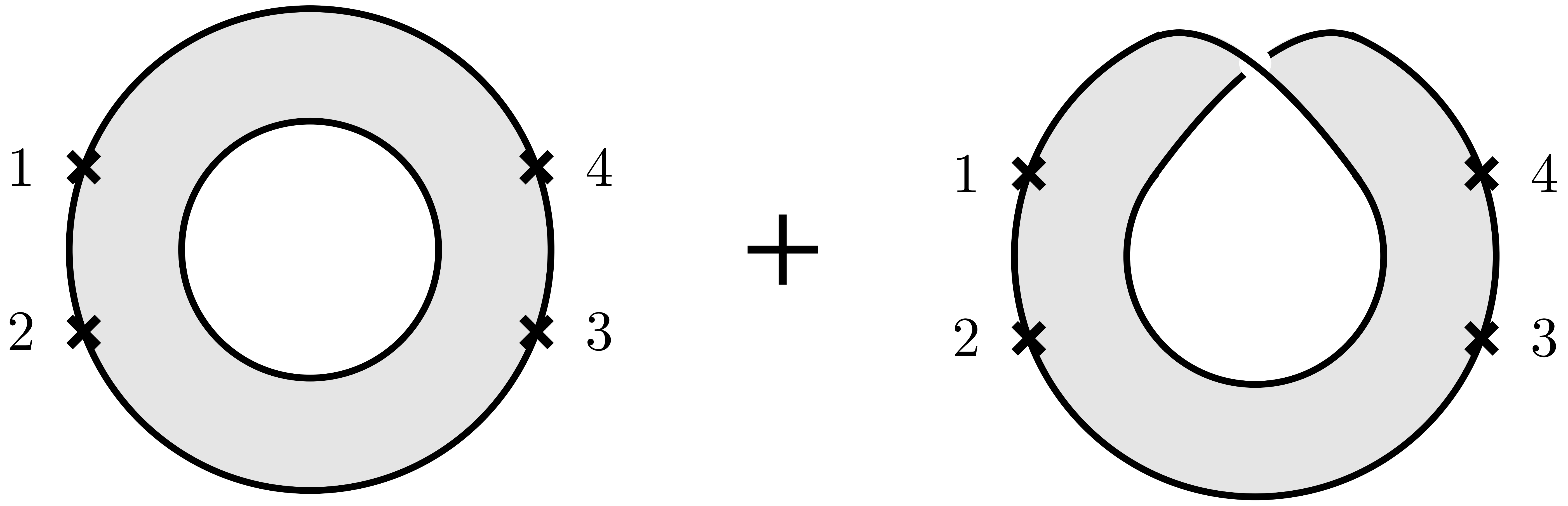


$z = \mathrm{e}^{-\tau}$

Euclidean
Schwinger
parameter

The one-loop amplitude

- Focus on planar color ordering:



$$A(s,t) = \frac{-iN}{32}\int_{\Gamma\subset\mathcal{M}_{1,4}^{\mathrm{open},\mathbb{C}}} \mathrm{d}\tau\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3 \left(\frac{\vartheta_1(z_{21},\tau)\vartheta_1(z_{43},\tau)}{\vartheta_1(z_{31},\tau)\vartheta_1(z_{42},\tau)}\right)^{-s}\left(\frac{\vartheta_1(z_{21},\tau)\vartheta_1(z_{41},\tau)}{\vartheta_1(z_{32},\tau)\vartheta_1(z_{42},\tau)}\right)^{-t}$$

Green, Schwarz '82

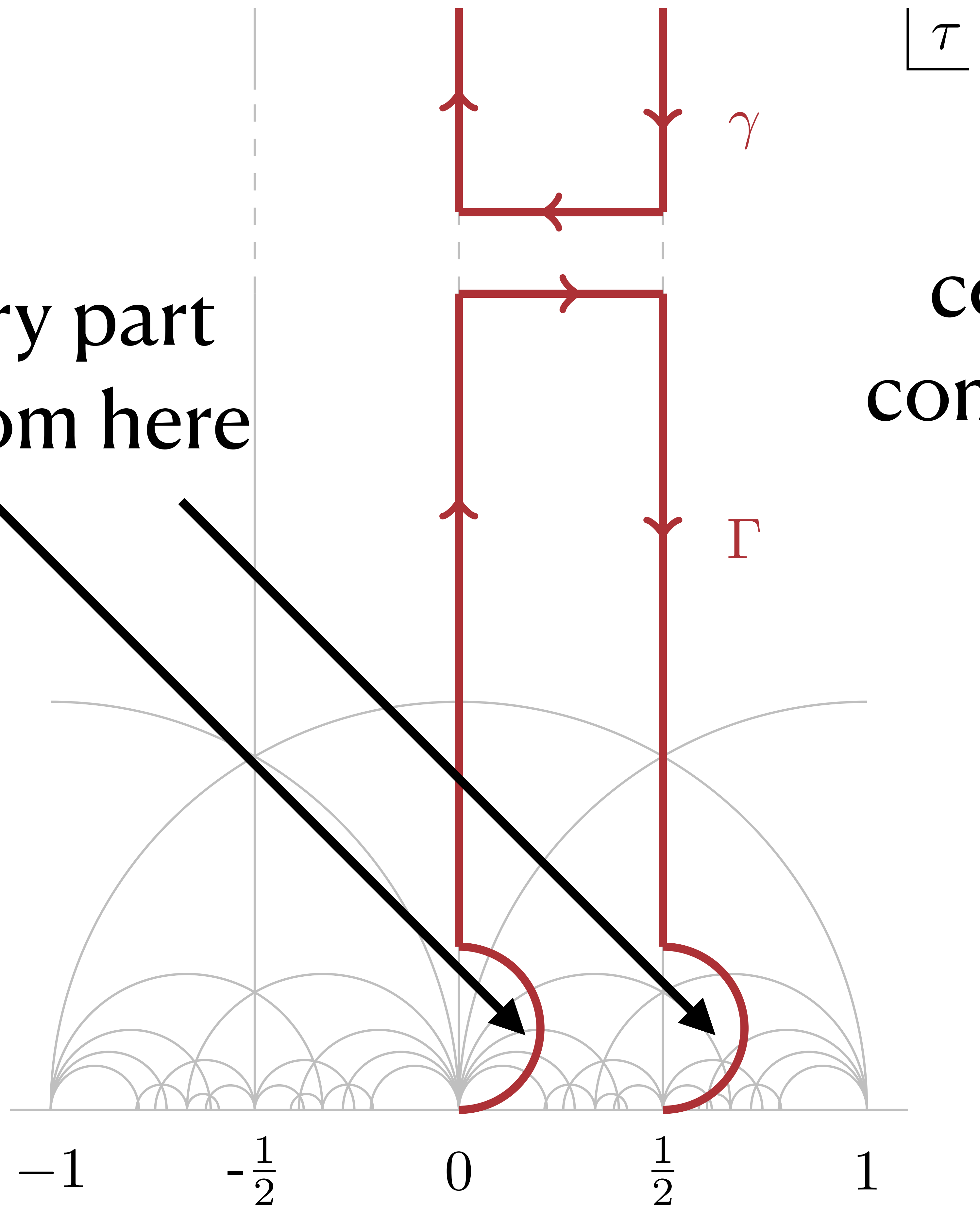
Rank of gauge group

Jacobi theta function

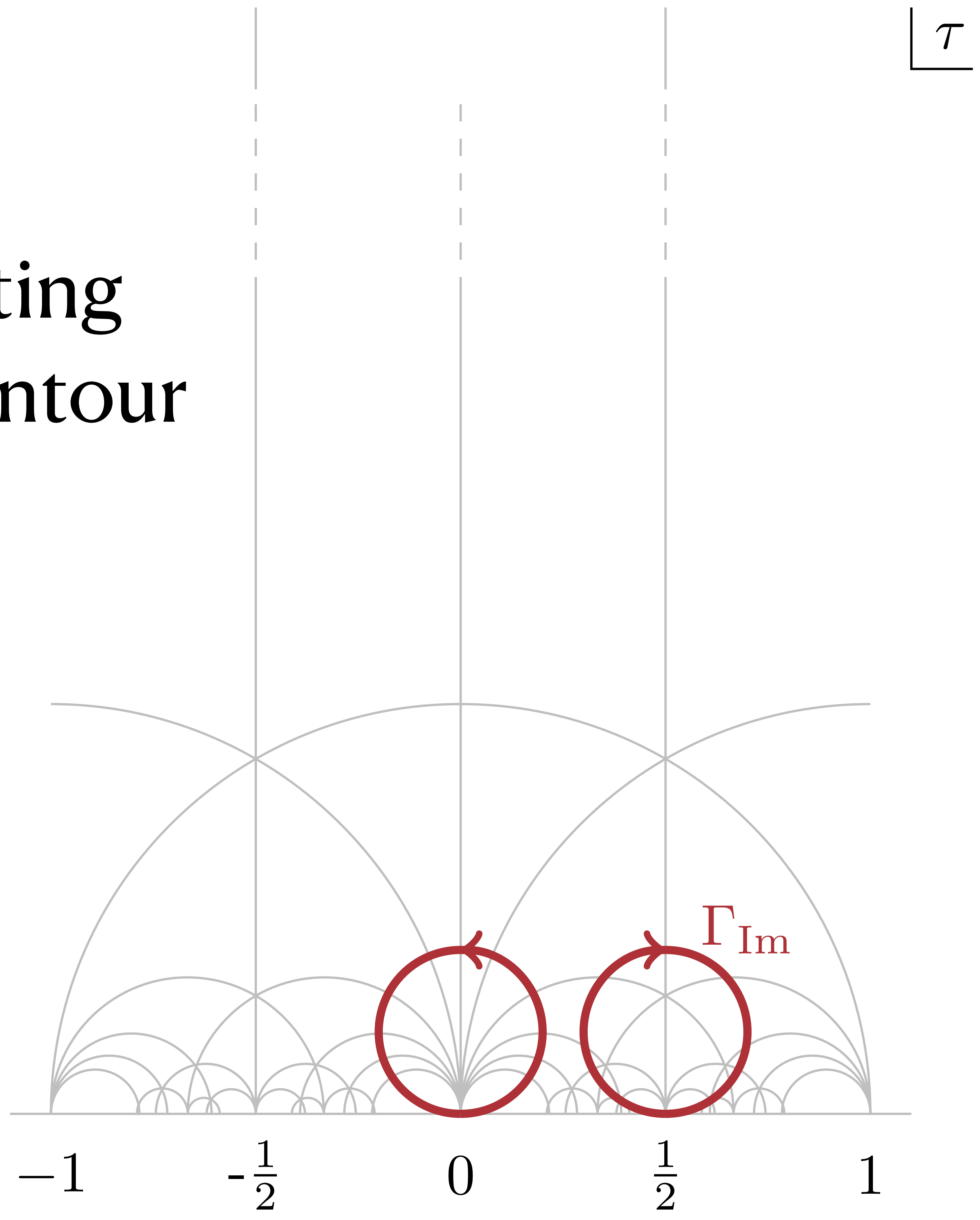
The contour Γ

The contour for Im A

Imaginary part
comes from here



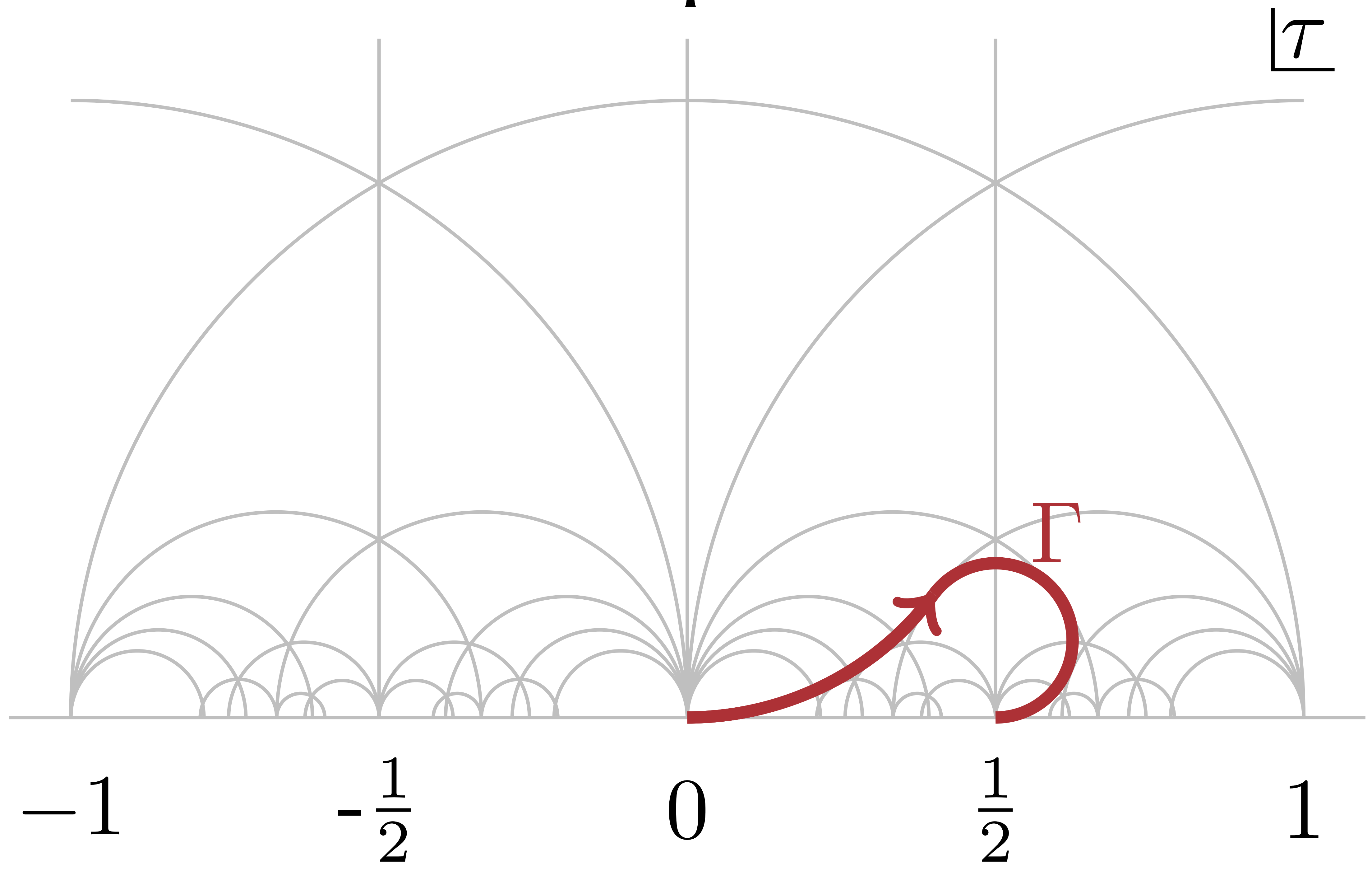
Imaginary part
computed by subtracting
complex conjugated contour



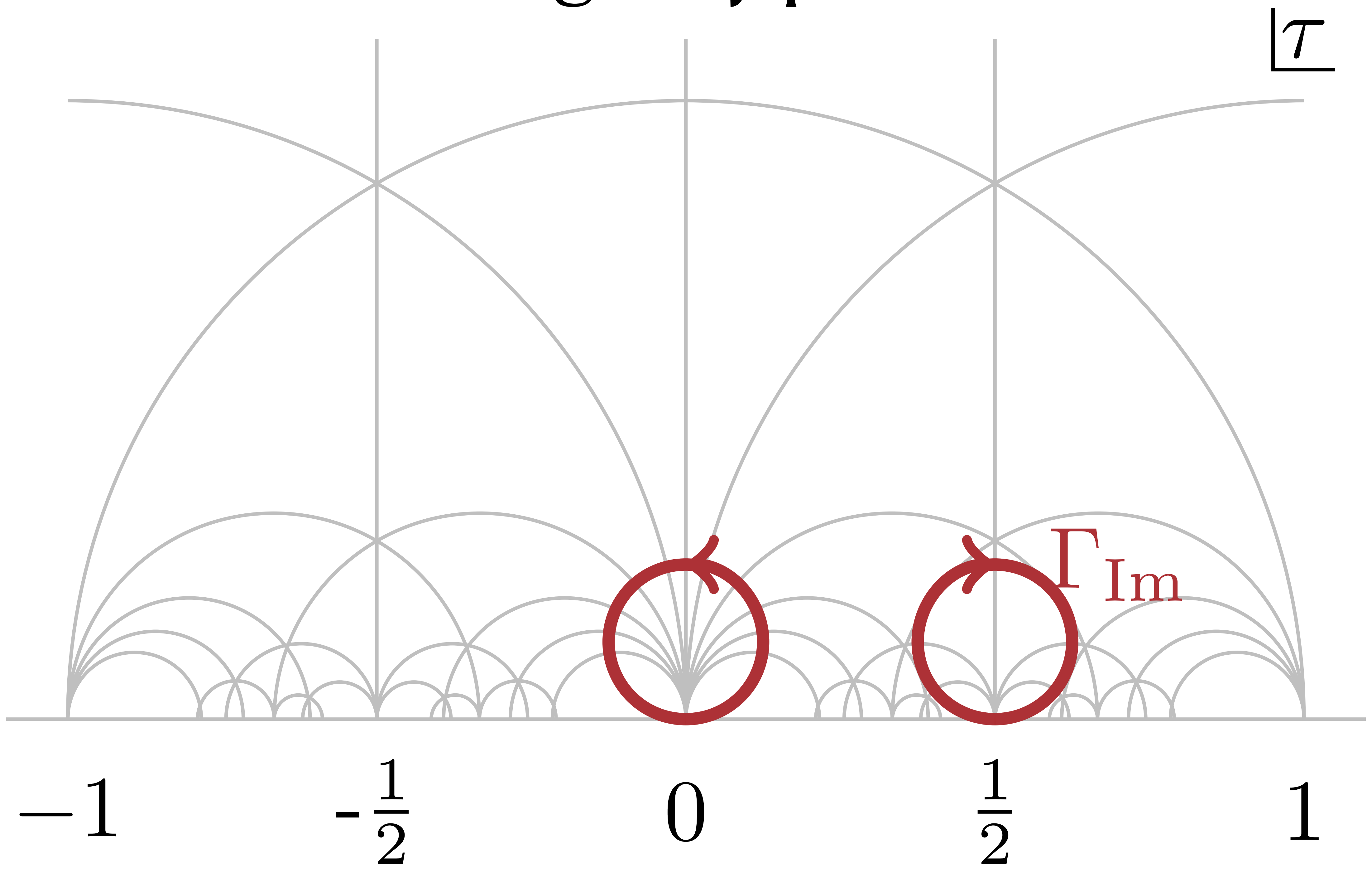
Summary of contours

$$A(s,t) = -i \int_{\Gamma \subset \mathcal{M}_{1,4}^{\text{open},\mathbb{C}}} \mathrm{d}\tau\, \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3 \left(\frac{\vartheta_1(z_{21},\tau)\vartheta_1(z_{43},\tau)}{\vartheta_1(z_{31},\tau)\vartheta_1(z_{42},\tau)} \right)^{-s} \left(\frac{\vartheta_1(z_{21},\tau)\vartheta_1(z_{41},\tau)}{\vartheta_1(z_{32},\tau)\vartheta_1(z_{42},\tau)} \right)^{-t}$$

Full amplitude



Imaginary part



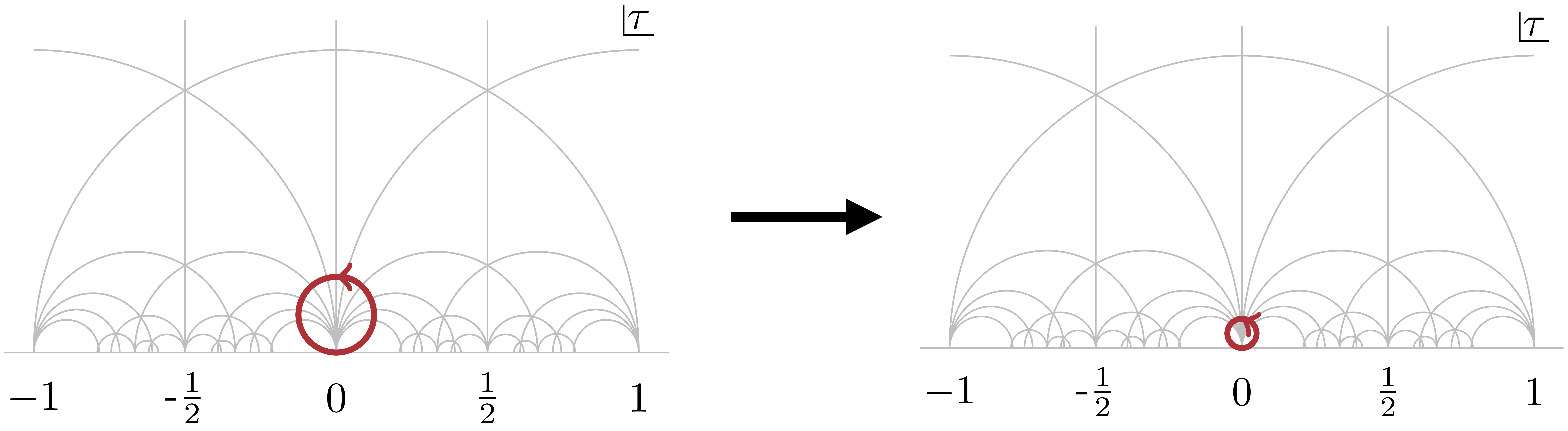
II. Evaluating the amplitude

Contour still not very useful

- Have convergent representation of the amplitude, but still very bad for e.g. numerical integration
- Integrand highly oscillatory
- Use contour deformations to proceed!

Imaginary part

- Shrink contour:



- Only need to know the local behavior of the integrand near the cusp

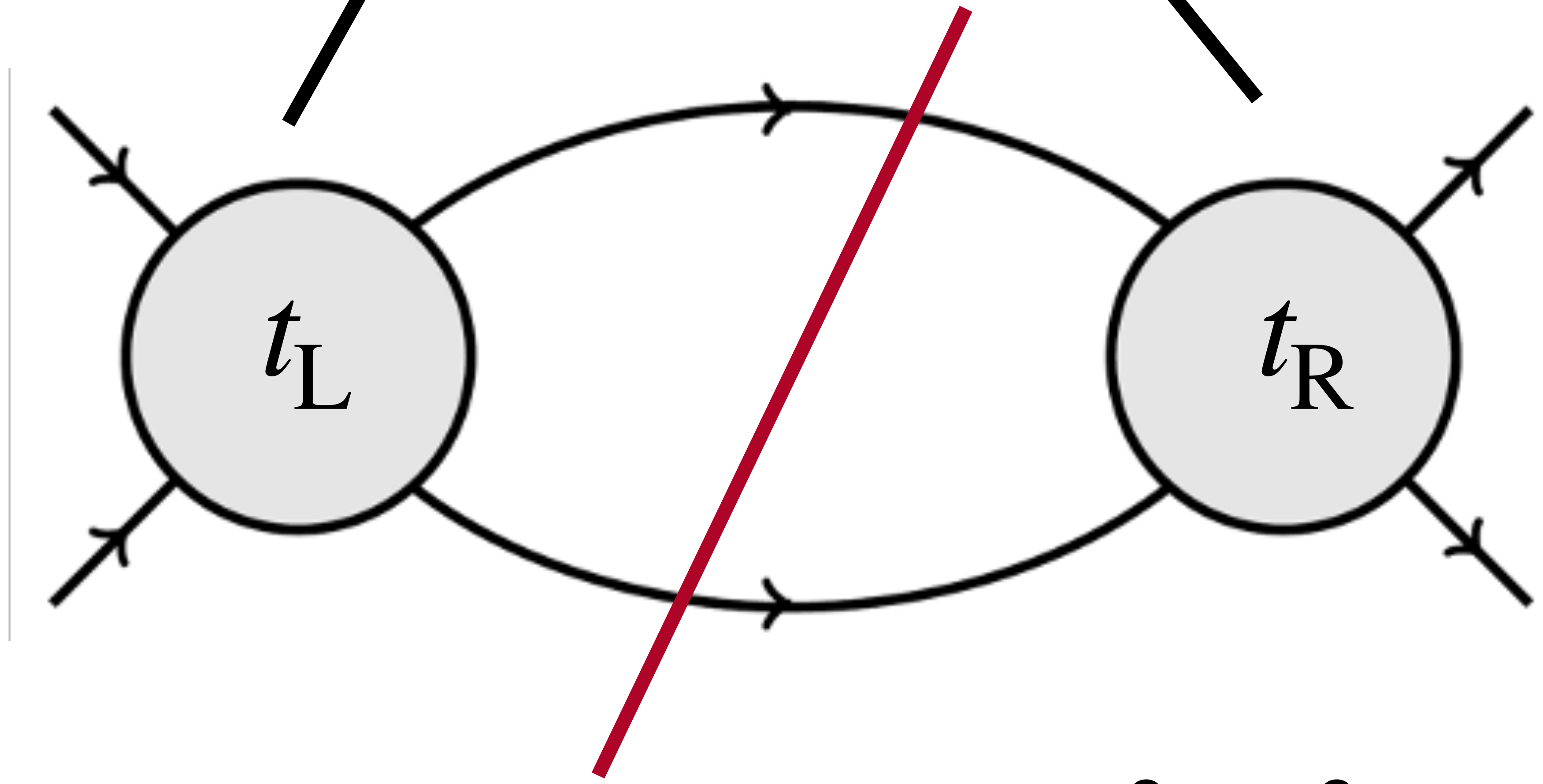
Baikov representation

LE, Mizera '22

- The integral for the imaginary part can then be computed:

$$\mathrm{Im}A\,|_{s\leq 1} = -\frac{16\pi i}{15\sqrt{stu}}\int_{P>0} \mathrm{d}t_{\mathrm{L}}\,\mathrm{d}t_{\mathrm{R}}\; P(s,t,t_{\mathrm{L}},t_{\mathrm{R}})^{\frac{5}{2}} \;\times\; A_{\mathrm{L}}(s,t_{\mathrm{L}}) \;\times\; A_{\mathrm{R}}(s,t_{\mathrm{R}})$$

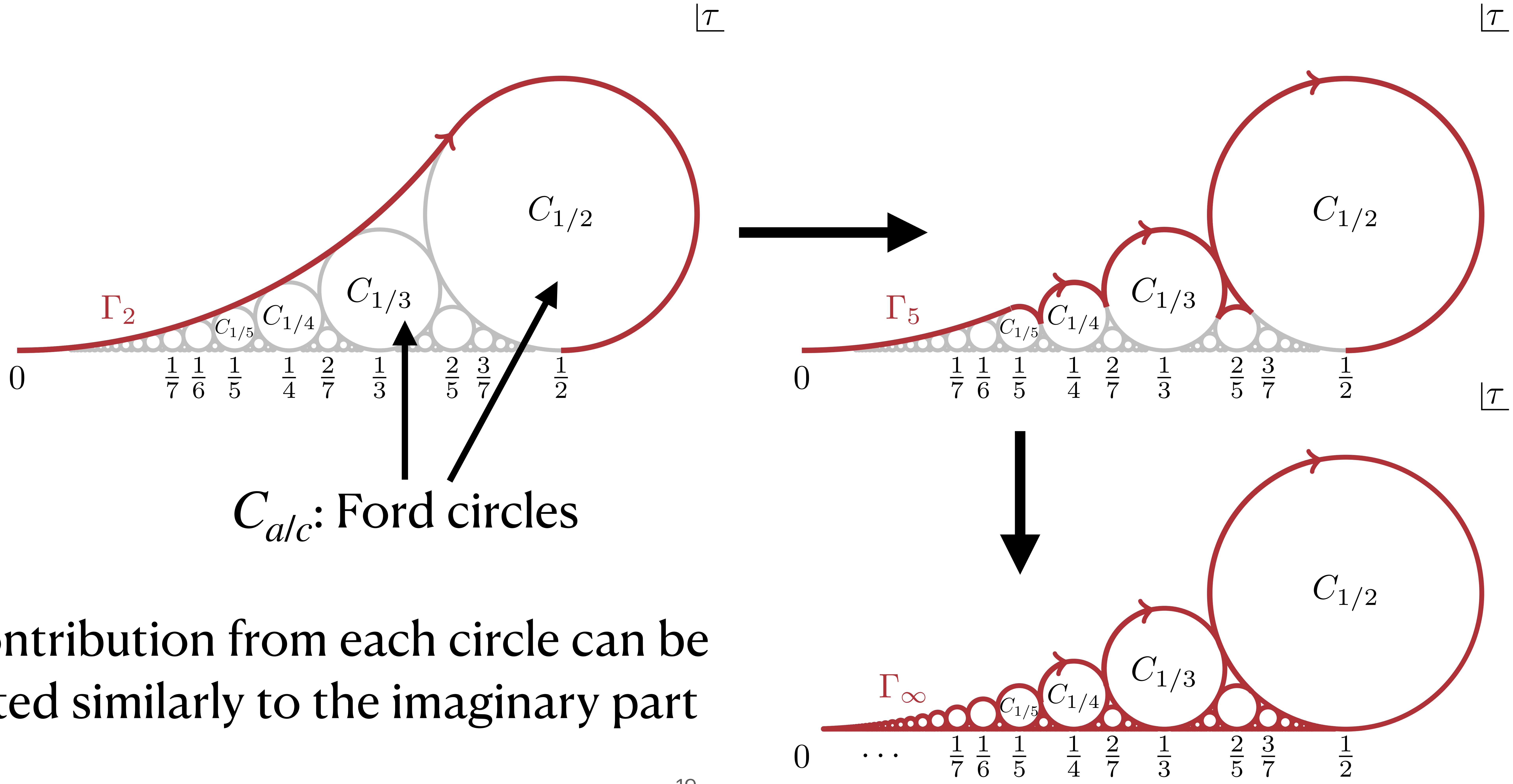
Integrate over phase space



- For higher s, we also need to sum over the internal mass squares $m_1^2, m_2^2 \in \mathbb{N}$, degeneracies and polarizations with $(m_1+m_2)^2 \leq s$

Rademacher contour

- Deform contour for the full amplitude



- The contribution from each circle can be evaluated similarly to the imaginary part

Infinite sum representation

- This gives rise to an infinite sum representation of the amplitude

$$A(s,t) = \Delta A(s,t) + \sum_{\substack{\text{irreducible} \\ \text{fractions} \\ 0 < \frac{a}{c} \le \frac{1}{2}}} \sum_{\substack{\text{windings} \\ n_\mathrm{L}, n_\mathrm{D}, n_\mathrm{R}, n_\mathrm{U} \ge 0 \\ n_\mathrm{L} + n_\mathrm{D} + n_\mathrm{R} + n_\mathrm{U} = c - 1}} A_{a/c}^{n_\mathrm{L}, n_\mathrm{D}, n_\mathrm{R}, n_\mathrm{U}}(s,t)$$

Simple correction from closing the contour at the cusp

Sum over Ford circles

Additional sum because of z Integration

Each contribution has a Baikov-like representation with additional phases

Convergence

- This gives a convergent representation of the amplitude
- The representation is practical for

$$1 \lesssim s \lesssim 20$$

Convergence gets worse for small s

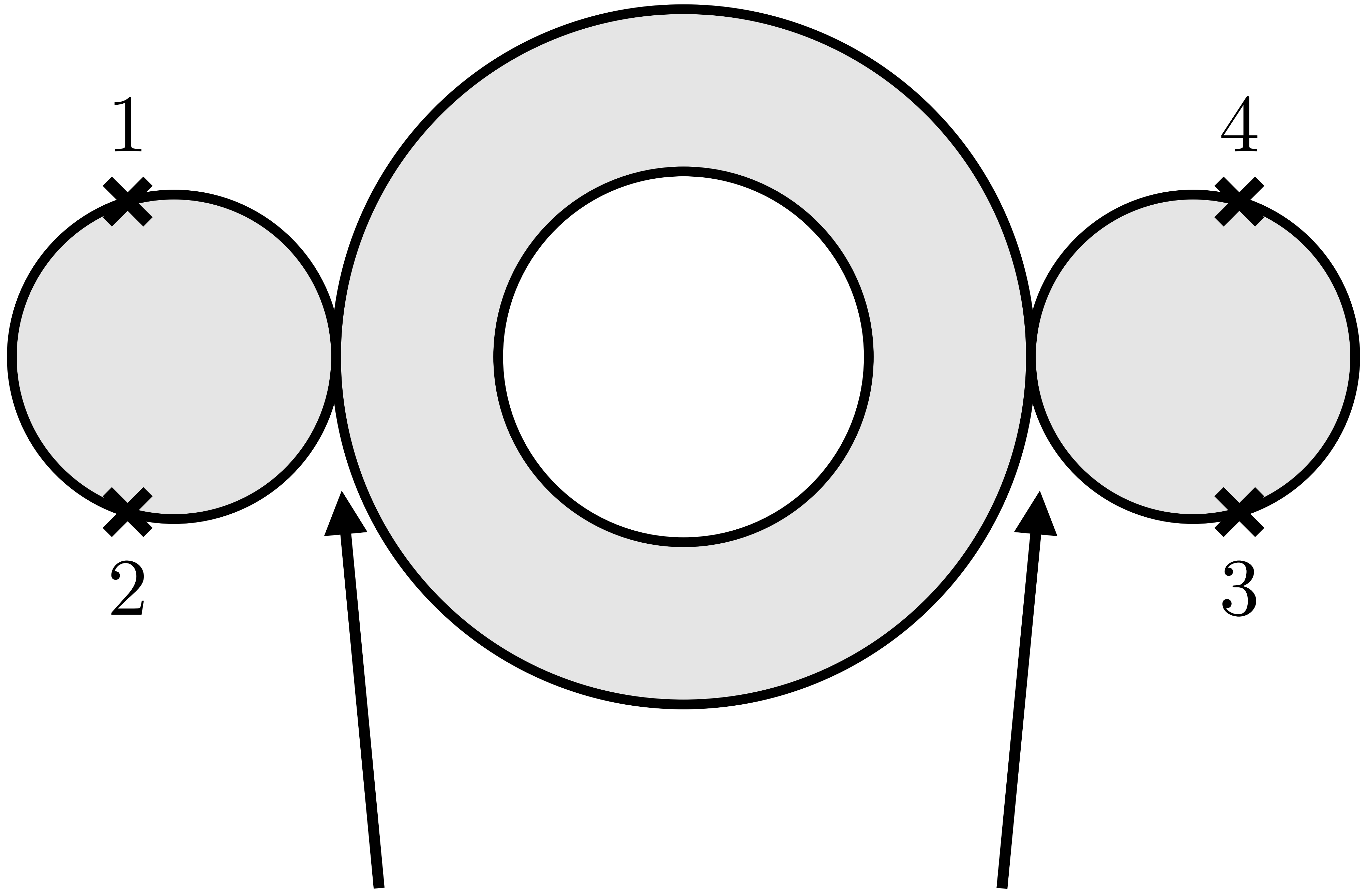
Summing over polarizations becomes hard

- But for small s direct numerical integration becomes good

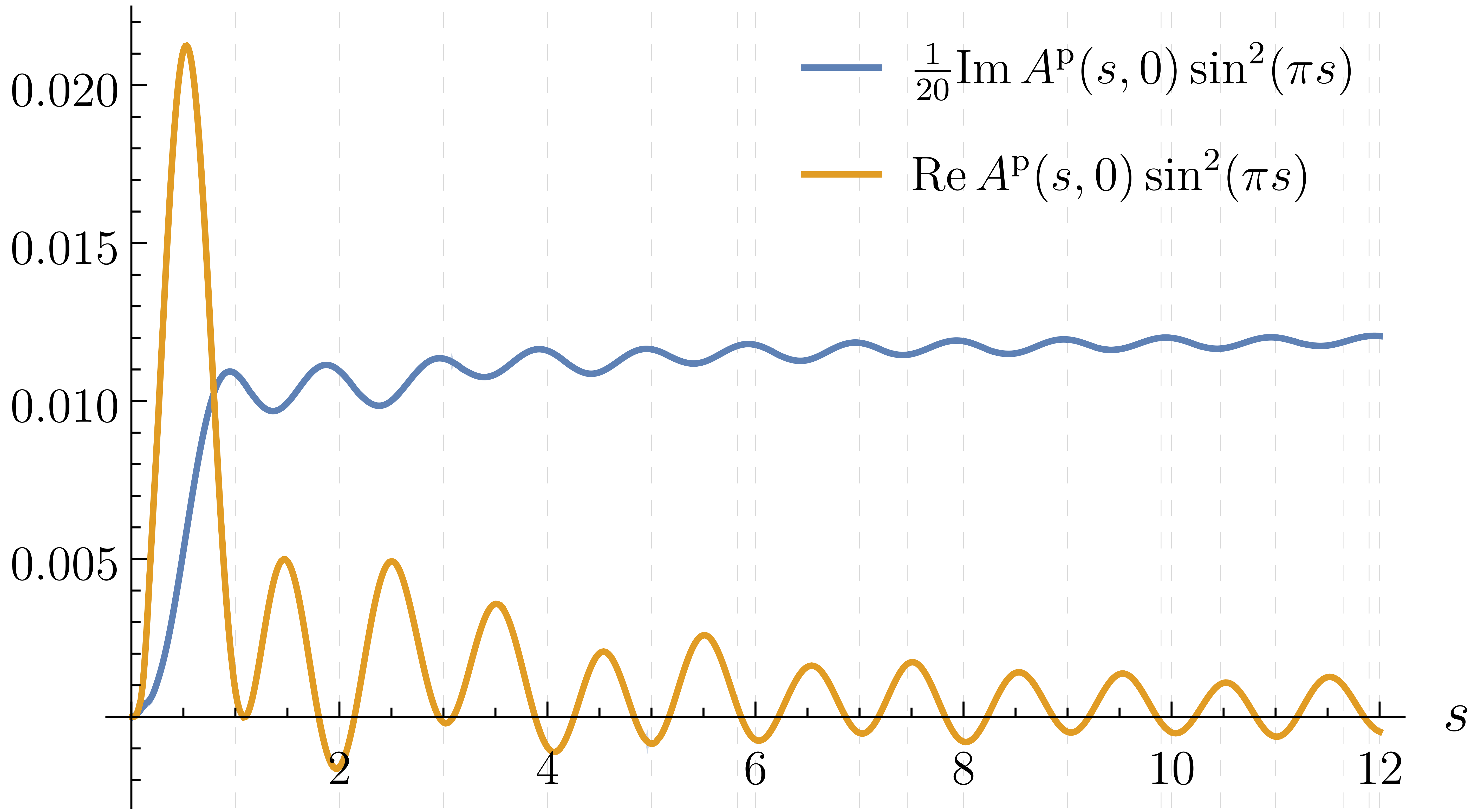
III. Results

Remove double poles

- Multiply the amplitude by $\sin(\pi s)^2$ to remove double poles coming from



Forward limit



- Amplitude mostly imaginary

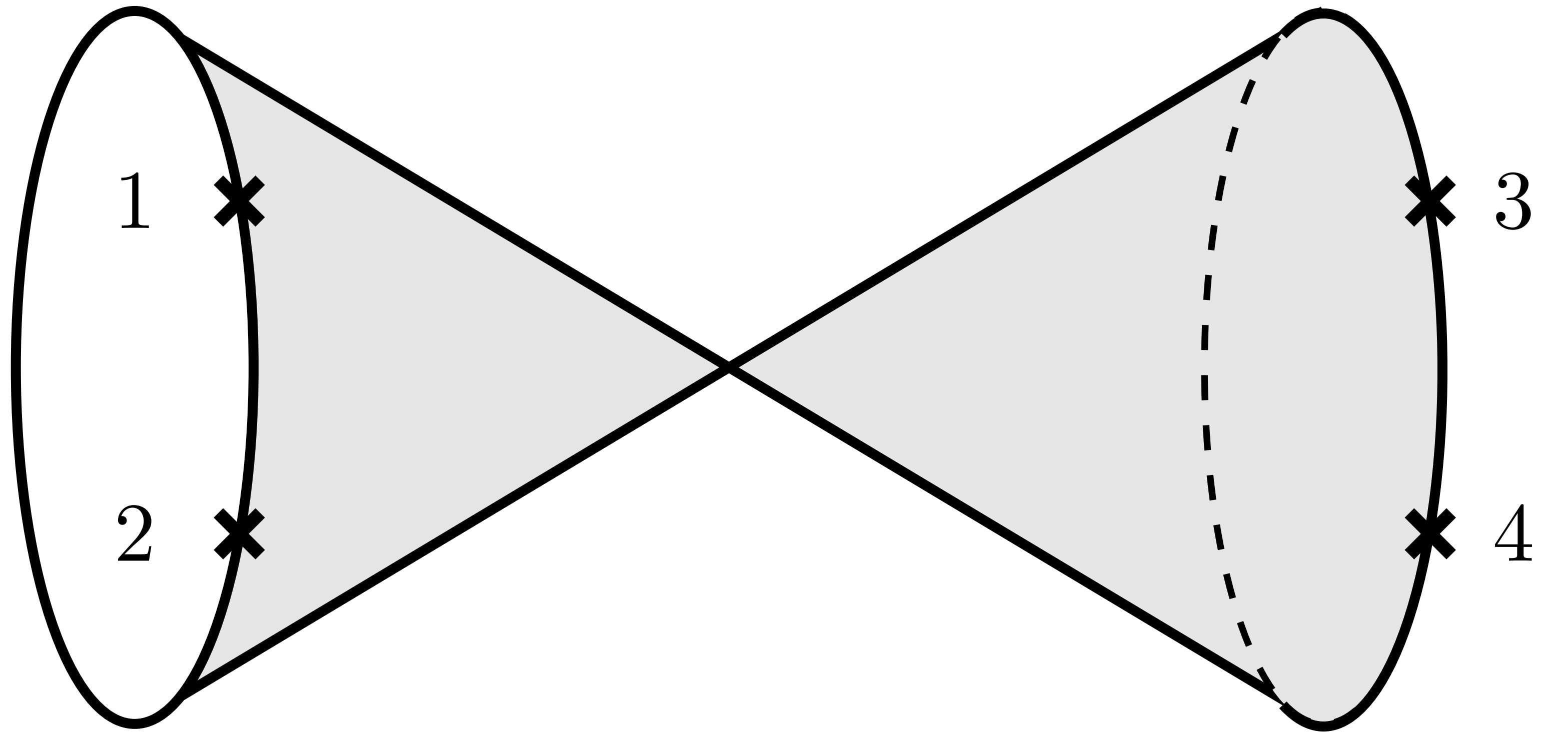
Regge growth

- The amplitude grows like

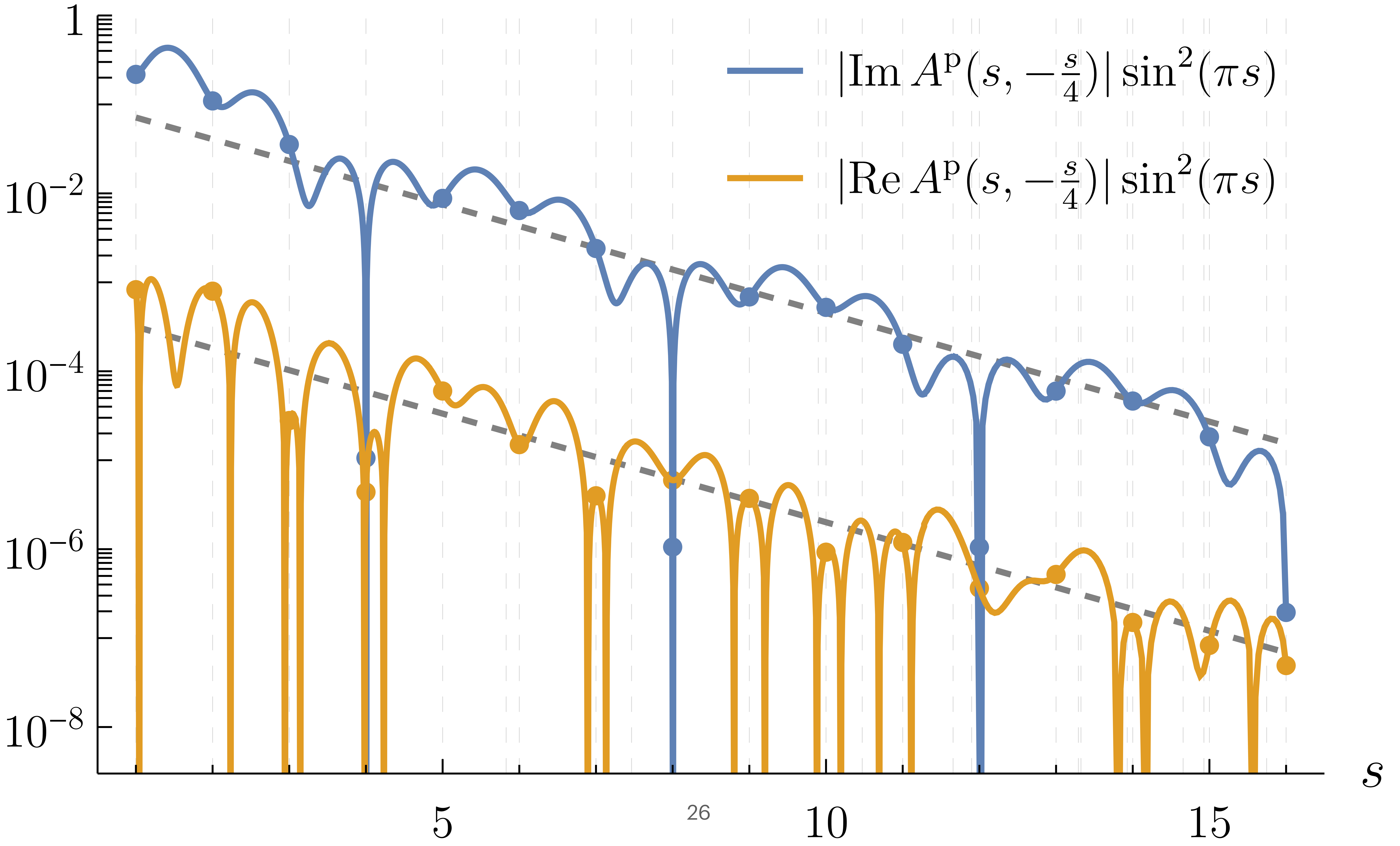
$$\mathscr{A}(s,0) = \underset{\substack{\uparrow \\ \mathcal{O}(s^2)}}{t_8} \times \text{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}) \times \underset{\substack{\uparrow \\ \mathcal{O}(1)}}{A(s,0)} = \mathcal{O}(s^2)$$

- For fixed non-zero t: $\mathscr{A}(s,t) = \mathcal{O}(s^{2+t})$
- Looks like the tree-level growth for a gravity theory
- Different as field theory!

For non-planar amplitude:

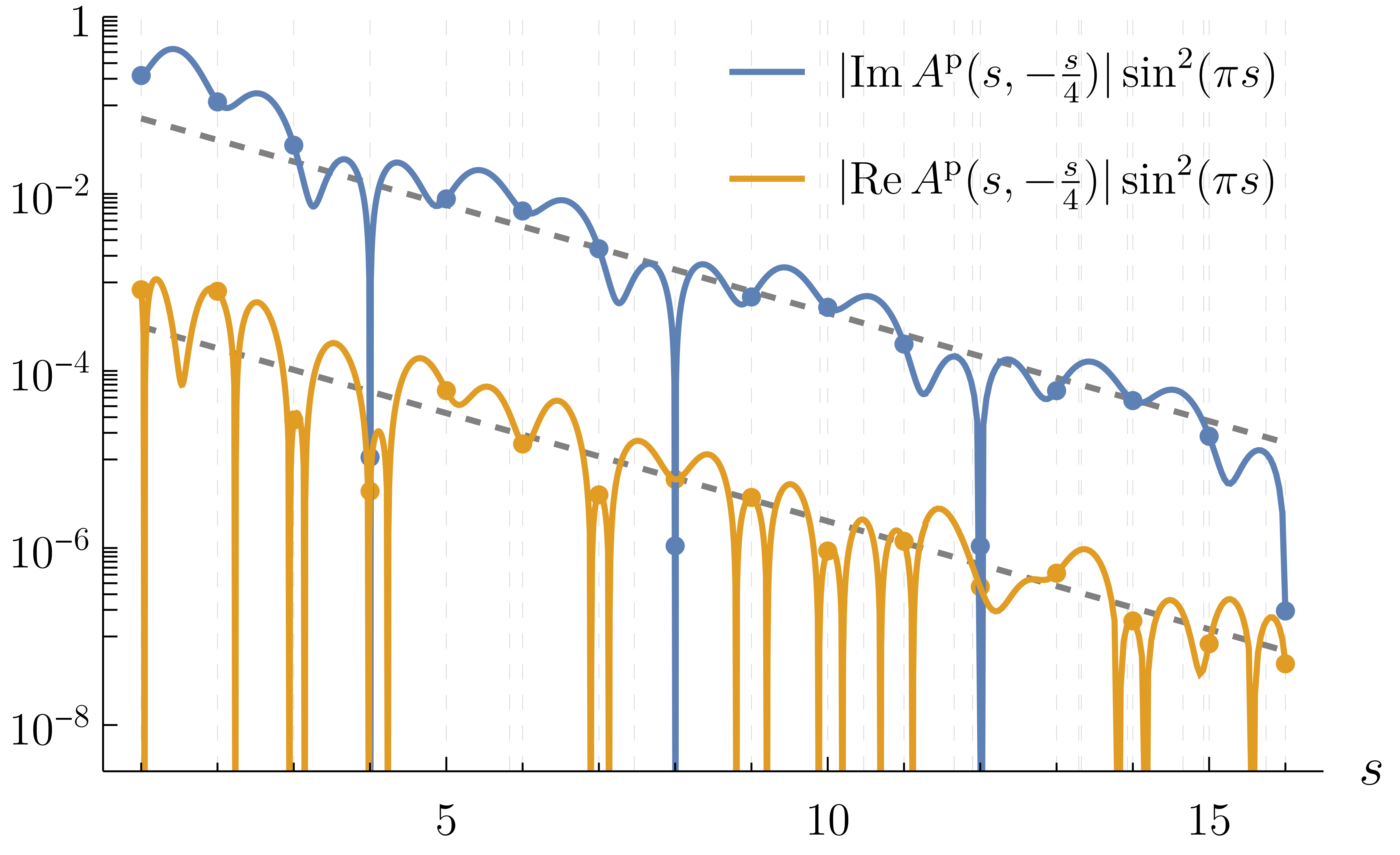


Fixed angle scattering @ 60 degrees



Properties

- Amplitude mostly imaginary
- Amplitude decays exponentially at large energies
- Predicted by Gross, (Mañes) Mende ’89, but never demonstrated explicitly
- Complicated oscillations on top of exponential decay

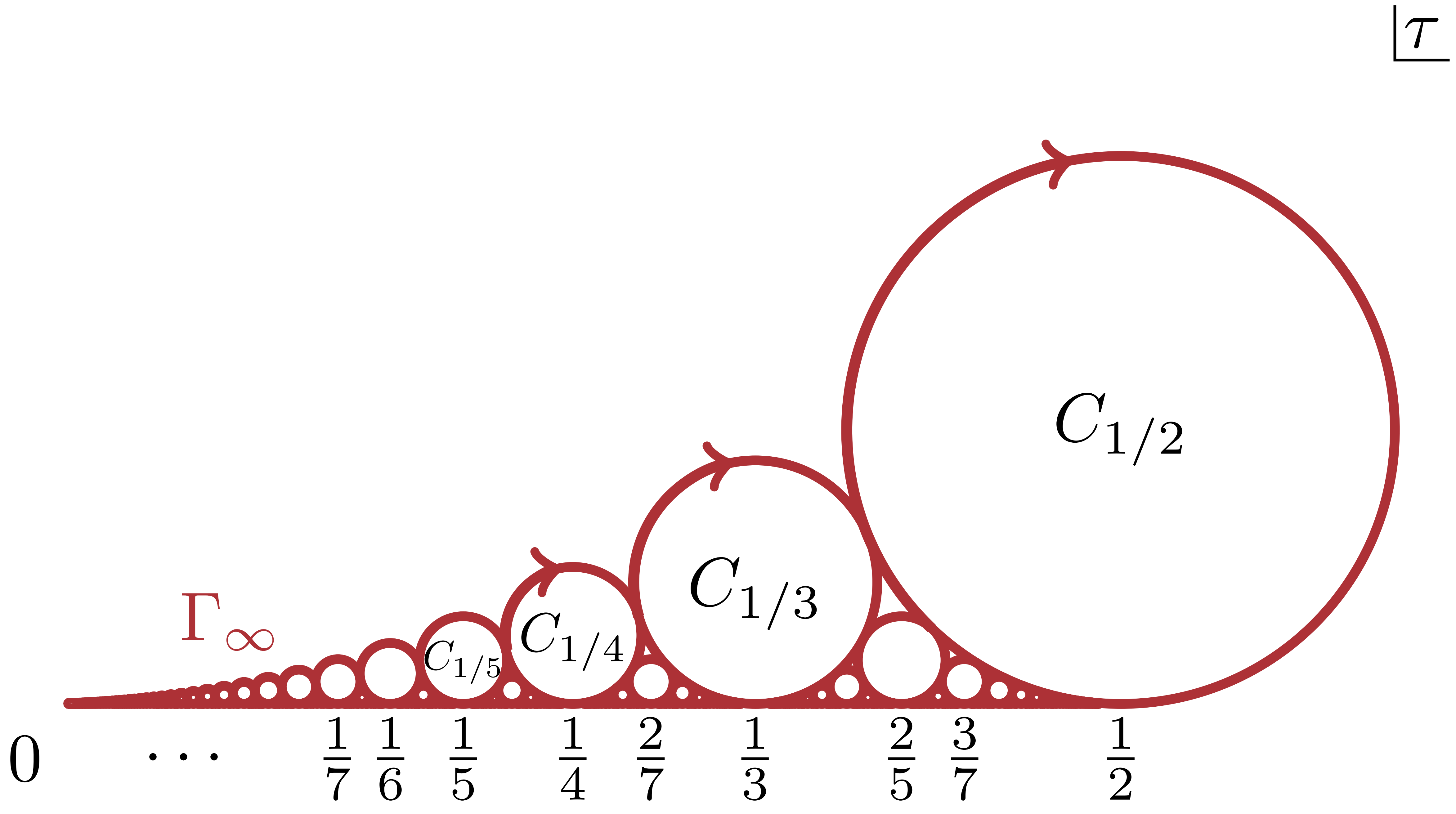


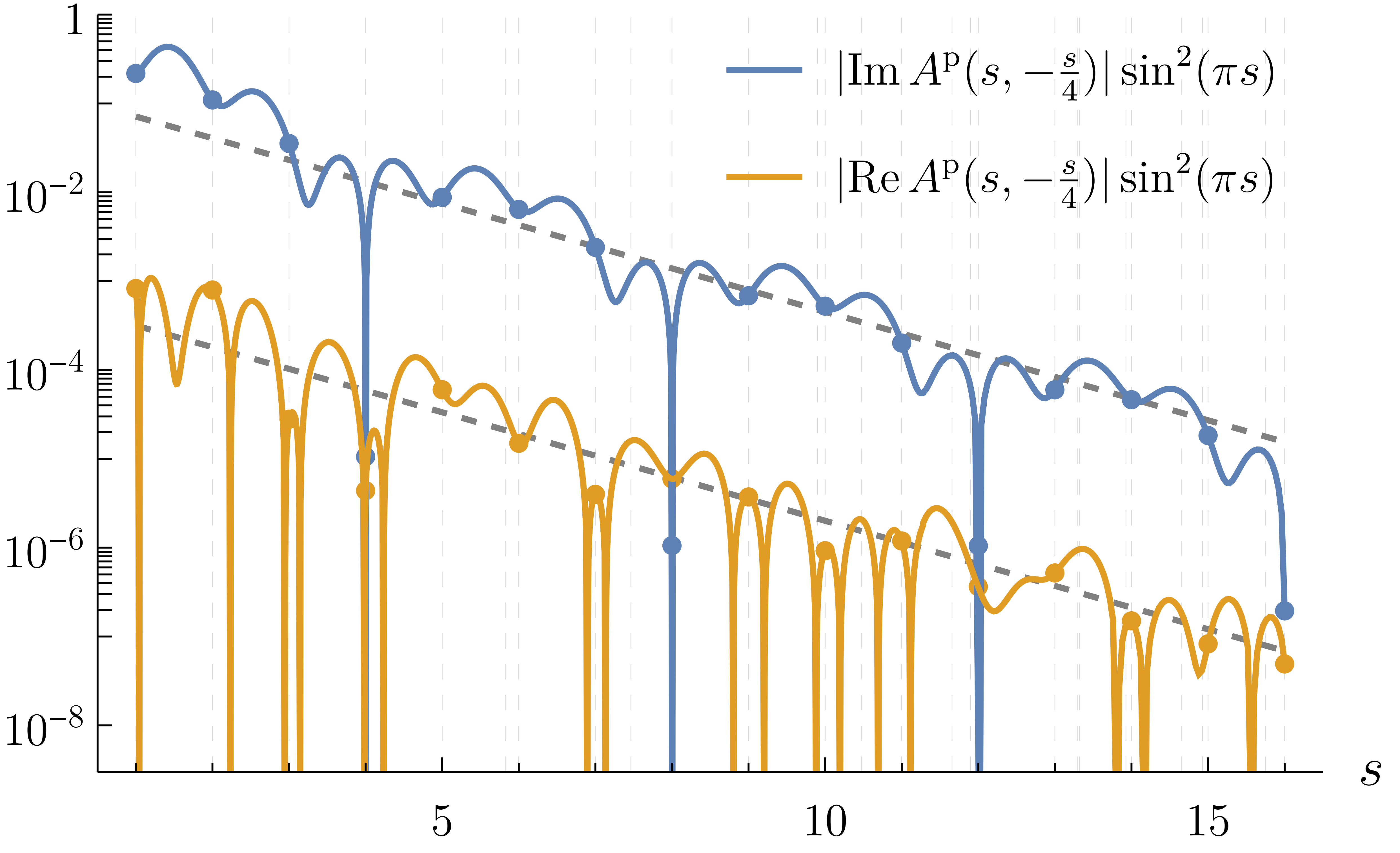
Summary

Deform worldsheet integration contour to Rademacher contour

→

Explicit formula for the one-loop string amplitude amenable to numerical evaluation





Thank you!