Integrating Along the Cuts

Amplitudes 2023, CERN

C. Vergu

Main Questions

Nima Arkani-Hamed:

- "Half of the integrals are useless"
- "The integrals are *one-dimensional*"

Why is this true?

More precise formulation

- $$\int \frac{d^{dl}k}{\prod \text{prop}} = \pi^{\lceil \frac{dl}{2} \rceil} \times P \times T_{\lfloor \frac{dl}{2} \rfloor},$$
  with a prefactor P of weight zero and T_w a transcendental function of weight w and d is an *integer* dimension. We have $\lfloor x \rfloor + \lceil x \rceil = x$.
- $$T_w = \sum \int_{0<t_1 \cdots <t_w<1} f_1(t_1)dt_1 \cdots f_w(t_w)dt_w.$$

Why are half of the integrals "useless"?

Together with Matt Schwartz, Hofie Hannesdóttir and Andrew McLeod: Landau singularities would not have the required asymptotic behavior otherwise. Can be shown rigorously when the integrals are finite (no regularization) and the kinematics (including masses) is *generic*.

At non-generic kinematics more complicated behavior, such as weight drop, but weight can not increase.

The asymptotics of Feynman integrals was studied by Landau, Leray, Polkinghorne & Screaton (reviewed by Pham).

Integrate “useless” integrals

Can we do the “useless” integrals? Previous work (direct integration) by Brown, Panzer, Bogner, Schnetz. Some difficulties having to do with field extensions which sometimes result in high degree extensions which eventually cancel out. We have found extensions of degrees up to 16, containing nested square roots which eventually cancel out completely (Bourjaily, McLeod, CV, Volk, von Hippel, Wilhelm).
Algorithmic way which avoids un-necessary field extensions?

Cutkosky representation

$$I = \int_{a_1}^{b_1} \frac{dq_1^2}{q_1^2 - m_1^2} \int_{a_2}^{b_2} \frac{dq_2^2}{q_2^2 - m_2^2} \cdots \int_{a_n}^{b_n} \frac{dq_n^2}{q_n^2 - m_n^2} \int \frac{N d^{dl-n}\xi}{J}.$$

Two types of integrals:

- $\int_{a_i}^{b_i} \frac{dq_i^2}{q_i^2 - m_i^2}$
- $\int \frac{N d^{dl-n}\xi}{J}$ ("angular" integral in Cutkosky's terminology).

J is the Jacobian of the change of coordinates $(q_1^{\mu_1}, \ldots, q_n^{\mu_n}) \to (q_1^2, \ldots, q_n^2, \xi)$ and N a possible numerator.
The numerator N may vanish while J may factorize so sometimes we don't get a nowhere vanishing holomorphic top-form as in the Calabi-Yau case.
In lorentzian signature integral over $(-\infty, a_i] \cup [b_i, \infty)$ and in euclidean signature over $[a_i, b_i]$.

“Angular” integrals

In the polylogarithmic case the “angular” integral over ξ is either zero-dimensional or it can be computed in simple terms. For elliptic or Calabi-Yau cases the last integral is often the period integral of the holomorphic top-form along a distinguished real cycle. For Calabi-Yau varieties these cycle of integration is higher-dimensional.

No reason to expect a coproduct/coaction in general.

One way to make progress: keep the ξ integrals for last, but non-trivial since the moduli of the variety depend on the values of q_i^2. Can be done but, in the elliptic case transforms complete elliptic integrals to incomplete elliptic integrals.

Remarks on Cutkosky representation

- Works for non-planar, for Euclidean and Lorentzian signatures
- Canonical parameterization for the loop momenta (contrast to issues with loop momentum routing in non-planar integrals).
- $\frac{dq_i^2}{q_i^2 - m_i^2} = d\log(q_i^2 - m_i)$ (d-log forms)
- The integrals in q_i^2 are *one-dimensional*.
- Obvious choice of complexification and compactification[1] (necessary for technical reasons) where $(q_1^2, q_2^2, \dots, q_n^2) \in \mathbb{R}^n$ gets embedded in $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. The space of ξ is already compact.

[1] Does not respect dual conformal symmetry when present.

Main idea

The discontinuities across branch cuts corresponding to a Landau singularity involving loop momenta $q_1, \ldots, q_r$ are obtained by replacing the first r integrals in the Cutkosky representation by $\int dq_1^2(-2\pi i)\delta(q_1^2 - m_1^2) \cdots \int dq_r^2(-2\pi i)\delta(q_r^2 - m_r^2)$.
Since all the cuts are computable in terms of polylogarithms, we can do the integrals in one variable q_i^2 at a time without ever encountering an obstruction.
There are two types of of singularities: logarithms and square roots. We should be able to do the square root singularity integrals while introducing only a factor of π.

Example: massive bubble in 3D

$$\int \frac{d^3 q_1}{(q_1^2 + m_1^2)(q_2^2 + m_2^2)} = \int_0^\infty \frac{dq_1^2}{q_1^2 + m_1^2} \int_{a_2}^{b_2} \frac{dq_2^2}{q_2^2 + m_2^2} \int \frac{d^3 q_1}{dq_1^2 \wedge dq_2^2},$$

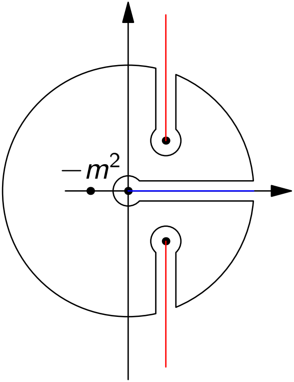
where $a_2 = (\|p\| - \|q_1\|)^2$ and $b_2 = (\|p\| - \|q_1\|)^2$. The third integral yields $-\frac{\pi}{2\|p\|}$ and can be pulled out of all integrals. We have

$$\int_{a_2}^{b_2} \frac{dq_2^2}{q_2^2 + m_2^2} = \log\left(\frac{(\|p\| + \sqrt{q_1^2})^2 + m_2^2}{(\|p\| - \sqrt{q_1^2})^2 + m_2^2}\right).$$

Put the branch cut along $\mathbb{R}_+$

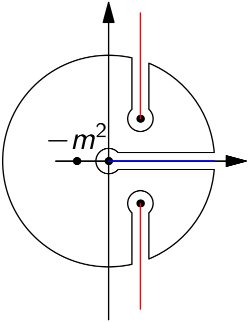
$$F_2(q_1^2) = \log\left(\frac{(\|p\| - i\sqrt{-q_1^2})^2 + m_2^2}{(\|p\| + i\sqrt{-q_1^2})^2 + m_2^2}\right).$$

Massive bubble in 3D, continued



$$\int_0^\infty \frac{dq_1^2}{q_1^2 + m_2^2} F_2(q_1^2),$$

with F_2 having a square root branch cut along $\mathbb{R}_+$.



$$\frac{1}{2}\lim_{\epsilon\to 0}\lim_{R\to\infty}\oint_{\gamma_1+\gamma_2+\gamma_3}\frac{dq_1^2}{q_1^2+m_1^2}F_2(q_1^2).$$

Several contributions:

- The pole at $q_1^2 = -m_1^2$, has the effect of setting $q_1^2 = -m_1^2$
- The integral along the branch cut, doubles the original integral
- Integral along the logarithmic branch cuts. *The discontinuity is $2\pi i$.*
- Integrals along the large and small circles vanish.

Massive bubble in 3D, final answer

$$\int_0^\infty \frac{dq_1^2}{q_1^2+m_1^2} F_2(q_1^2) = \pi i \left(\log \frac{(\|p\| - i m_1)^2 + m_2^2}{(\|p\| + i m_1)^2 + m_2^2} + (m_1 \leftrightarrow m_2) \right).$$

Symmetry arises non-trivially: one of the terms arises from the residue at $q_1^2 = -m_1^2$ while the other from the integration along the logarithmic cuts.

The final answer can be rewritten as

$$\frac{\pi^2}{i\,\|p\|} \log \frac{m_1 + m_2 + i\,\|p\|}{m_1 + m_2 - i\,\|p\|} = \frac{2\pi^2}{\|p\|} \arctan \frac{\|p\|}{m_1 + m_2}.$$

These forms assume $m_1, m_2 > 0$.

Triangle in 3D

$$\int \frac{d^3 q_1}{(q_1^2+m_1^2)(q_2^2+m_2^2)(q_3^2+m_3^2)} = \int_0^\infty \frac{dq_1^2}{q_1^2+m_1^2} \int_{a_2}^{b_2} \frac{dq_2^2}{q_2^2+m_2^2} \int_{a_3}^{b_3} \frac{dq_3^2}{q_3^2+m_3^2} \frac{d^3 q_1}{dq_1^2 \wedge dq_2^2 \wedge dq_3^2}. \quad (1)$$

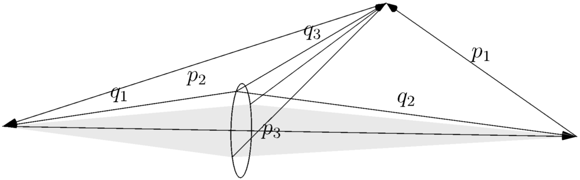
We have

$$\frac{d^3 q_1}{dq_1^2 \wedge dq_2^2 \wedge dq_3^2} = \frac{1}{8\sqrt{\det(q_i \cdot q_j)_{i,j=1,2,3}}}.$$

Denote $\det(q_i \cdot q_j)_{i,j=1,2,3} = P_3(q_3^2) = \alpha_3 (q_3^2)^2 + \beta_3 q_3^2 + \gamma_3$. We have $P_3(a_3) = P_3(b_3) = 0$. Then,

$$\int_{a_3}^{b_3} \frac{dq_3^2}{q_3^2+m_3^2} \frac{1}{\sqrt{P_3(q_3^2)}} = \frac{\pi}{\sqrt{-\alpha_3 m_3^4 + \beta_3 m_3^2 - \gamma_3}}.$$

Triangle in 3D, geometry



To determine the range of integration for $q_3^2 \in [a_3(q_1^2, q_2^2), b_3(q_1^2, q_2^2)]$, use a constrained extremization approach (Lagrange multipliers). Define

$$F(q_3^2, \alpha_1, \alpha_2) = q_3^2 - \alpha_1(q_1^2 - z_1) - \alpha_2(q_2^2 - z_2).$$

Then $\partial_{q_3} F = 2(q_3 - \alpha_1 q_1 - \alpha_2 q_2) = 0$ is similar to a Landau loop equation (with one non-vanishing α and modified on-shell conditions $q_2^2 = z_2$ and $q_3^2 = z_3$).
To determine the nature of these extrema, use the bordered Hessian. In Lorentzian signature the integration domain is rather

Denote $-P_3(-m_3^2) = P_2(q_2^2) = \alpha_2(q_2^2)^2 + \beta_2 q_2^2 + \gamma_2$. Then

$$\int_{a_2}^{b_2} \frac{dq_2^2}{q_2^2+m_2^2}\frac{1}{\sqrt{P_2(q_2^2)}} = \frac{1}{\sqrt{P_2(-m_2^2)}}\Bigg[\log\left(\frac{-b_2-m_2^2+\sqrt{P_2(b_2)}-\sqrt{P_2(-m_2^2)}}{-b_2-m_2^2+\sqrt{P_2(b_2)}+\sqrt{P_2(-m_2^2)}}\right) - (b_2 \to a_2)\Bigg] \quad (2)$$

There are unwanted potential square root singularities in $\|q_1\|$ at $P_2(b_2)=0$ and $P_2(a_2)=0$. But $P_2(b_2)$ and $P_2(a_2)$ are actually *perfect squares*. Indeed,

$$P_2(a_2) = \frac{1}{64p_3^2}\times \left(\sqrt{-4m_3^2p_3^2+(p_1^2)^2-2p_1^2p_2^2-2p_1^2p_3^2+(p_2^2)^2-2p_2^2p_3^2+(p_3^2)^2}-p_1^2+p_2^2+p_3^2-2\|p_3\|\|q_1\|\right)^2 \left(\sqrt{-4m_3^2p_3^2+(p_1^2)^2-2p_1^2p_2^2-2p_1^2p_3^2+(p_2^2)^2-2p_2^2p_3^2+(p_3^2)^2}+p_1^2-p_2^2-p_3^2+2\|p_3\|\|q_1\|\right)^2. \quad (3)$$

Square root singularities

The integrals which yield square root singularities can be done while producing only factors of π. For the contour trick to work, the only square root singularity should be along the integration contour. There are other square roots in the formulas, but they *do not* produce square root branch points. For example, quantities such as

$$\frac{1}{\sqrt{P}} \log \frac{Q + \sqrt{P}}{Q - \sqrt{P}}$$

do not have a square root singularity at $P = 0$ (on the main sheet where we do not cross the logarithmic branch cut starting at $Q^2 - P = 0$).

Indeed, under $P \to e^{2\pi i} P$ we have $\sqrt{P} \to -\sqrt{P}$ so the sign of the prefactor cancels the sign from the logarithm $\log \frac{Q+\sqrt{P}}{Q-\sqrt{P}} \to \log \frac{Q-\sqrt{P}}{Q+\sqrt{P}} = -\log \frac{Q+\sqrt{P}}{Q-\sqrt{P}}$.

Including the prefactor makes the function nicer!

Weight vs depth

The number of integrals in an iterated integral can not be reduced if the integrand is algebraic. It *can* be reduced for more complicated integrands

$$\mathrm{Li}_n(z) = \frac{1}{(n-1)!} \int_0^\infty \frac{u^{n-1} du}{z^{-1} e^u - 1},$$

obtained by change of variable $t_i = e^{u_i}$ so $\frac{dt_i}{t_i} = du_i$ then most of the u-integrals can be done.
Depth is more important than the weight if we are willing to reduce the number of integrals at the cost of an integrand which itself contains transcendental functions (exponentials).

Goncharov depth conjecture

Theorem (Goncharov 2001, Rudenko 2020)

Any multiple polylog of weight $n \geq 2$ can be expressed as a linear combination of multiple polylogs of depth at most $\lfloor \frac{n}{2} \rfloor$ and products of polylogs of lower weight.

Rudenko's approach is algorithmic and involves *quadrangular polylogarithms* (see talk by Anastasia). *It is possible to write all polylogarithmic l-loop integrals in $d = 4$ as l-fold integrals!* Open question: find optimal (fewest terms) depth-reduced expression.

Summary

- No reason to expect a coproduct/coaction in general, once we encounter Calabi-Yau on-shell spaces.
- Two types of singularities, square roots and logarithmic (true even in the Calabi-Yau case). In the polylogarithmic case can do the "useless" square root integrals explicitly.
- The prefactors make the function nicer by eliminating some branch cuts in the product.
- Direct integration feasible in Cutkosky representation. Built-in checks: cancellation of spurious square roots, symmetry of the integral.
- Explains the transcendental weight and the "one-dimensional" nature of integrals.
- In principle can apply the same strategy in Baikov representation (work in progress).
- If you want the minimum number of integrations, use depth reduction.

Thank You!