The Niels Bohr International Academy

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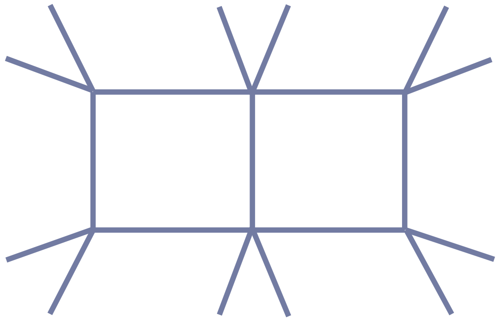
# Elliptic symbol bootstrap and Schubert analysis

based on arXiv:2212.09762

with Roger Morales, Matthias Wilhelm, 杨清霖 (Qinglin Yang) and 张驰 (Chi Zhang)

**Anne Spiering**
Niels Bohr Institute

# The 12-point double box

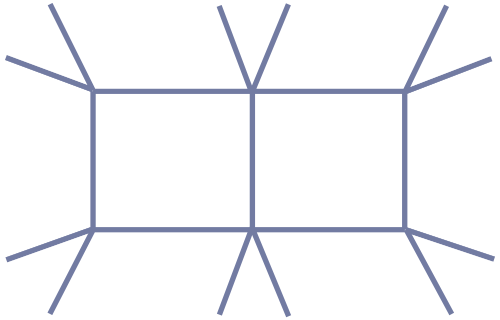


# The 12-point double box

**Special functions in QFT observables**

- 12pt double box evaluates to elliptic integrals
- elliptic – and more complicated – integrals ubiquitous in QFT (planar $\mathcal{N} = 4$ SYM, QCD,...) & gravitational-wave physics

[cf. talks by Broadhurst, Caola, Hidding, Nega, Plefka, Ruf]



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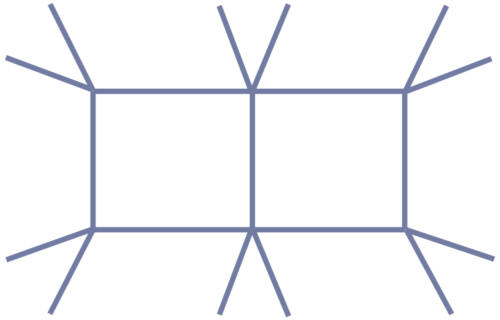

**Efficient analytic tools**

- 12pt double box is a "simple" testing ground for 4d elliptic Feynman diagrams
- generalisation of polylog tools to elliptic (and even CY-type) diagrams?

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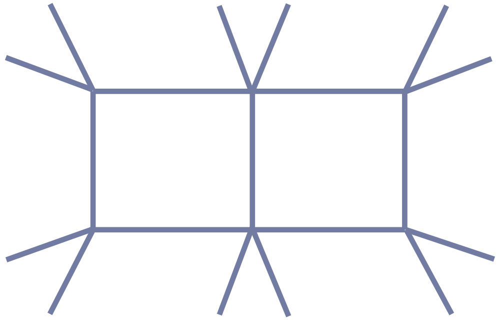
**Mathematical and physical structures appearing in scattering amplitudes**

- Which types of functions appear?
- unravelling singularity structure e.g. via coproduct and symbol map
  - natural kinematic variables? clusters, Landau, Schubert,…

[cf. talks by Britto, Drummond, Gardi, Hannesdottir, Vergu]



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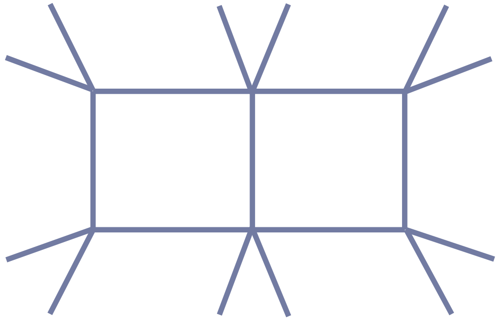

**Solving all 4d planar 2-loop amplitudes**

- 12pt double box is most general double box diagram appearing in massless planar scalar field theories
- and is one of the building blocks of general 2-loop diagrams

**Efficient analytic tools**

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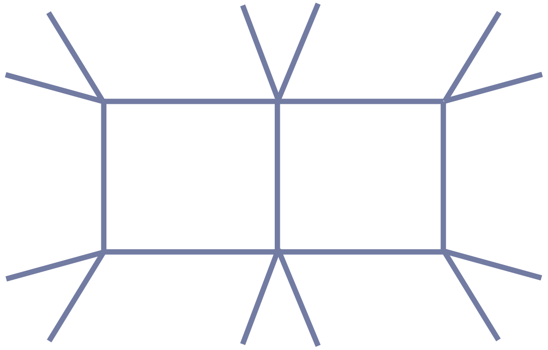
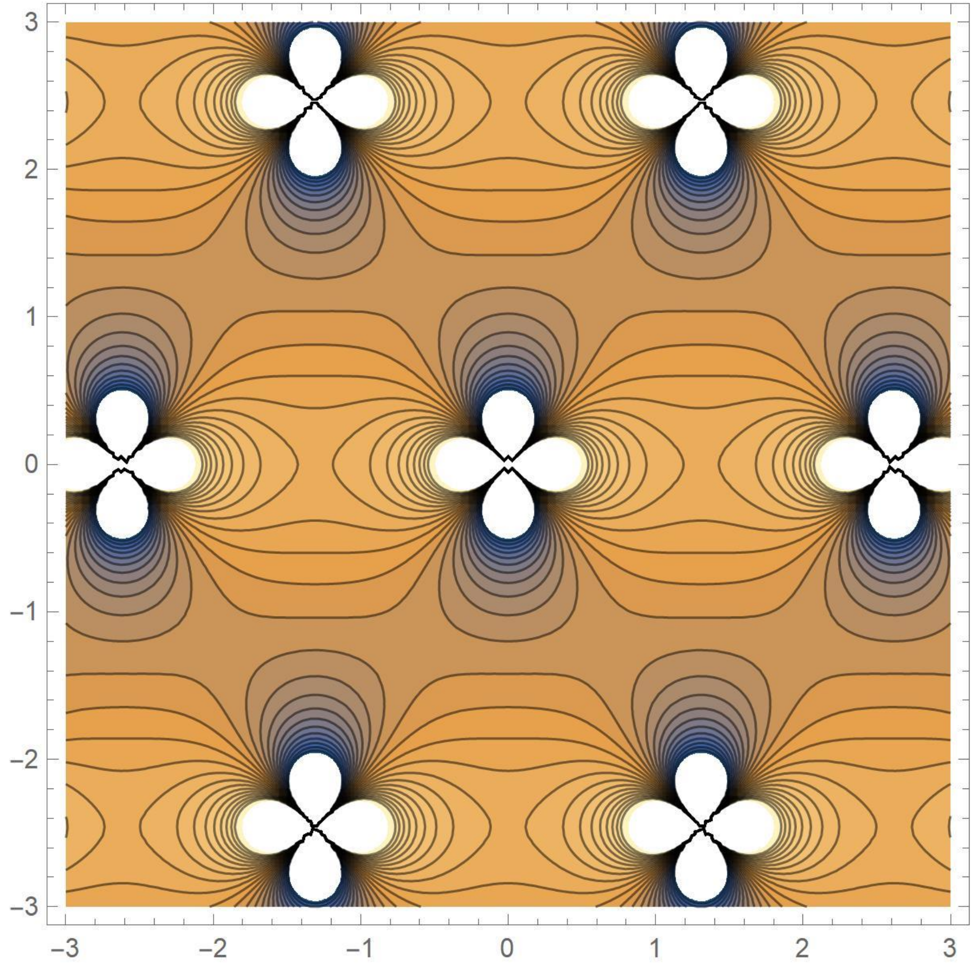
# Overview

1. **Motivation**
2. **Elliptic Feynman diagrams**
3. **Symbol techniques for amplitudes**
4. **Bootstrapping the elliptic double box**
5. **Summary & Outlook**

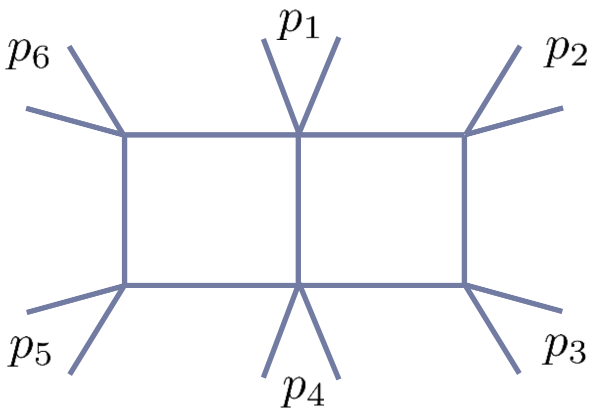


# Elliptic Feynman diagrams



# 2.1 Double box and hexagon

What makes the double box so hard to evaluate?



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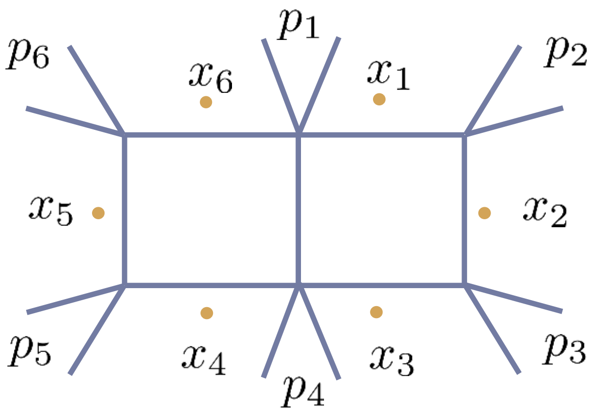


- introduce **dual momentum coordinates** via

$$p_i = x_i - x_{i-1}$$

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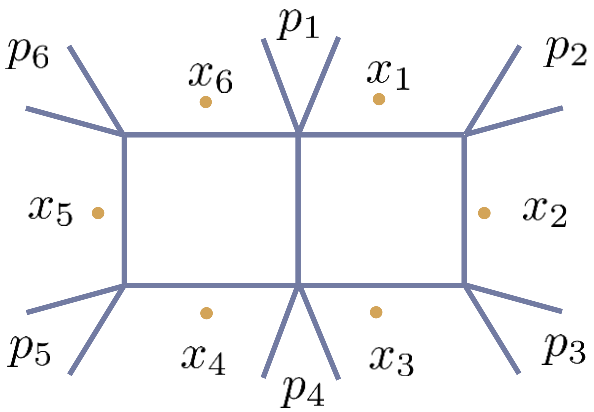


- introduce **dual momentum coordinates** via

$$p_i = x_i - x_{i-1}$$

- **nine** independent kinematic variables of the form

$$\frac{(x_i - x_j)^2 (x_k - x_l)^2}{(x_i - x_k)^2 (x_j - x_l)^2}$$

in particular introduce the cross ratio

$$u = \frac{(x_1 - x_3)^2 (x_4 - x_6)^2}{(x_1 - x_4)^2 (x_3 - x_6)^2}$$

- Maximal cut of 12pt double box evaluates to an elliptic integral
  [Caron-Huot, Larsen '12] [Vergu, Volk '20]

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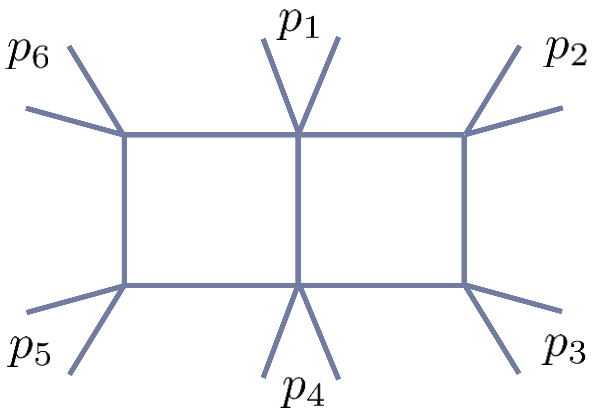
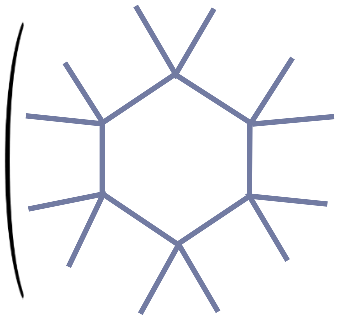



$$= \int_{-\infty}^{u} \frac{dz}{\sqrt{\Delta_6(z)}} \left( \vphantom{\int} \right)^{(d=6)}_{u \to z}$$



[Paulos, Spradlin, Volovich '12]

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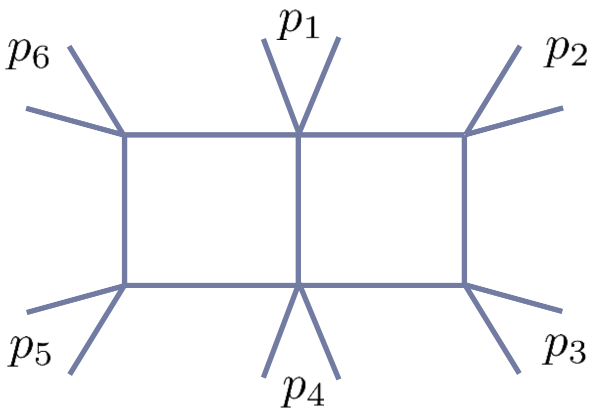


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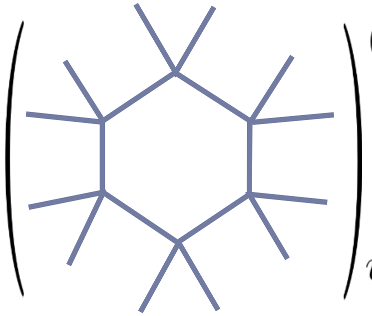

[Paulos, Spradlin, Volovich '12]

$$= \int_{-\infty}^{u} \frac{dz}{\sqrt{\Delta_6(z)}} \left( \mathrm{Li}_3(\ldots) + \ldots \right)$$

[Nandan, Paulos, Spradlin, Volovich '13]
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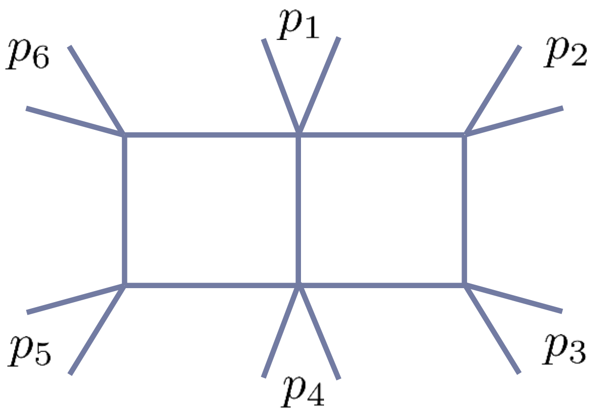


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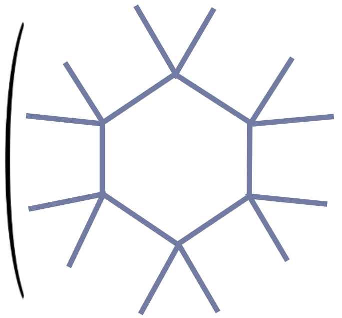

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$6 \times 6$ Gram determinant

cubic polynomial in $z$ with no repeated roots

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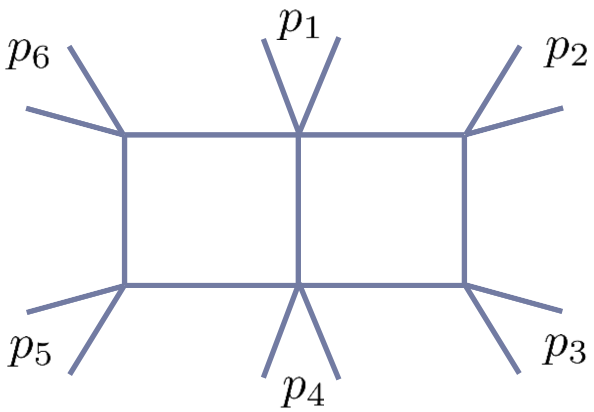


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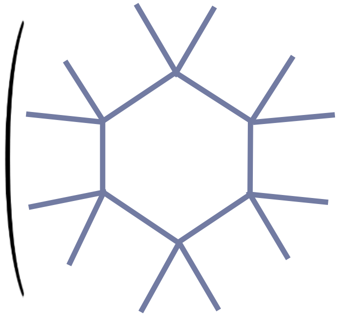

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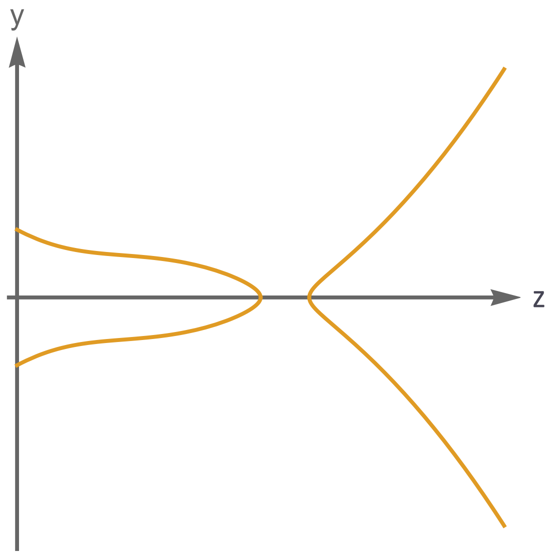
$6 \times 6$ Gram determinant
cubic polynomial in $z$ with no repeated roots

additional 9 square roots
depending on $z$

# 2.2 Elliptic curves and polylogs

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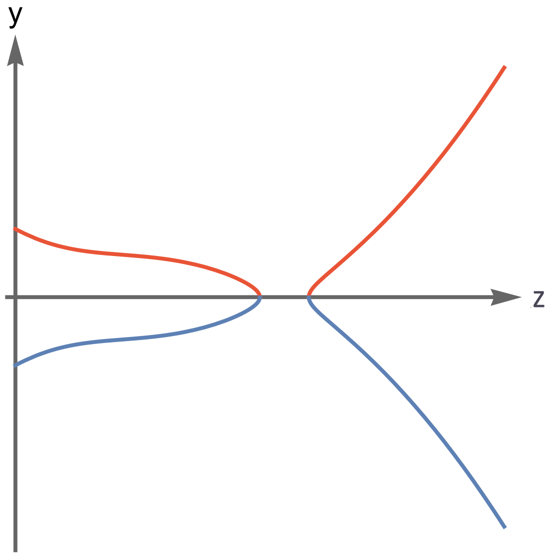


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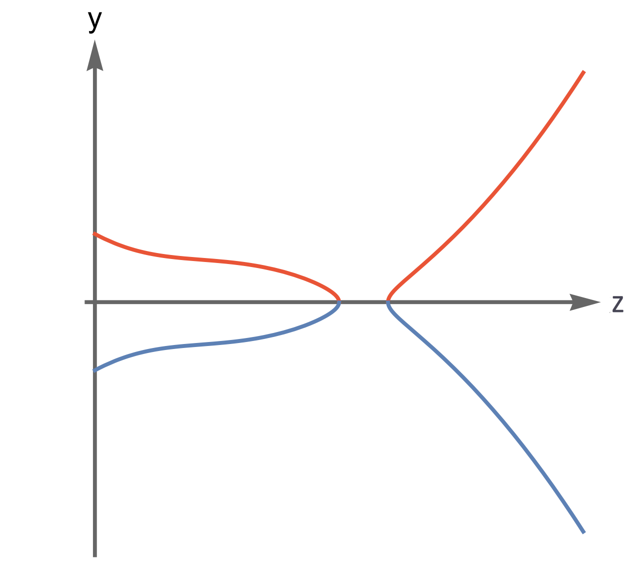


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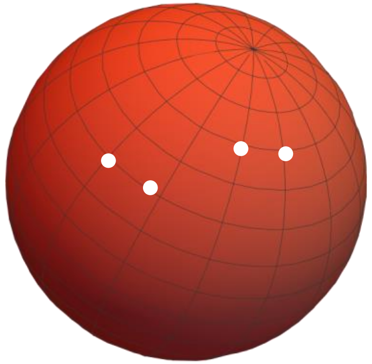

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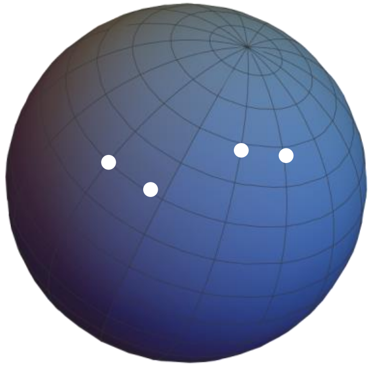

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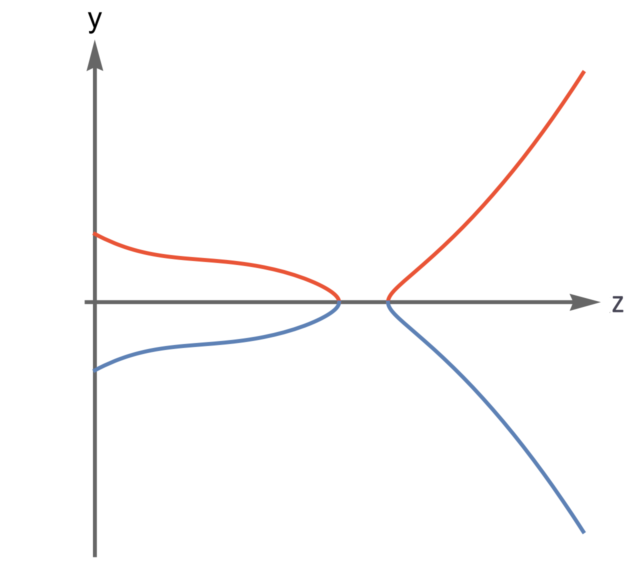


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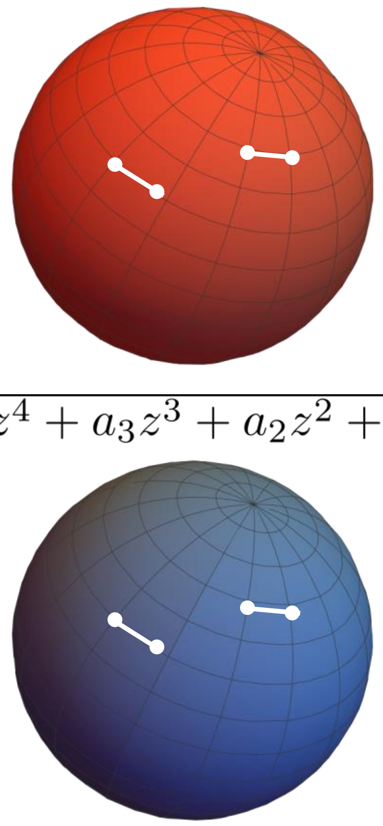

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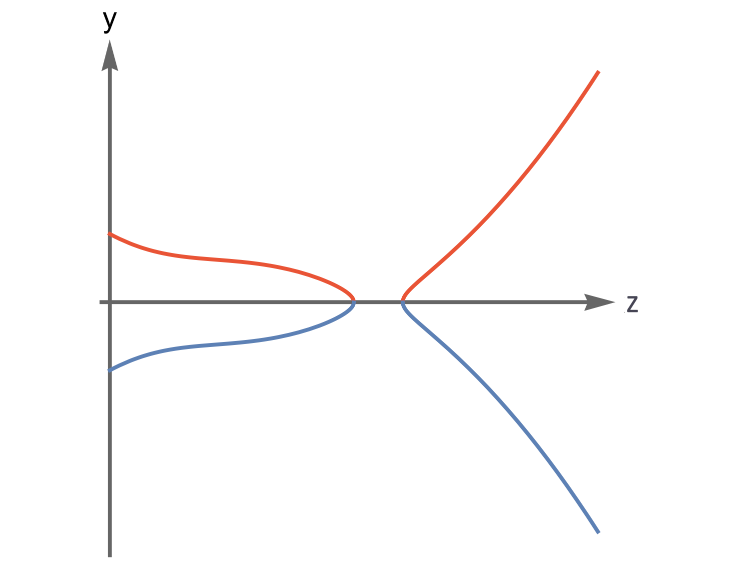


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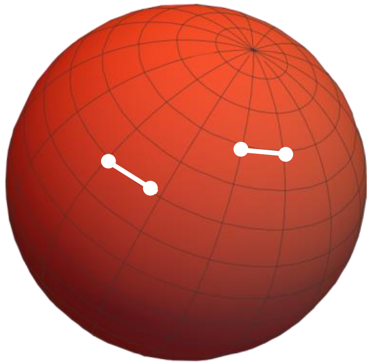

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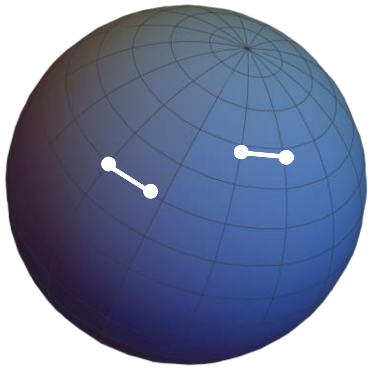

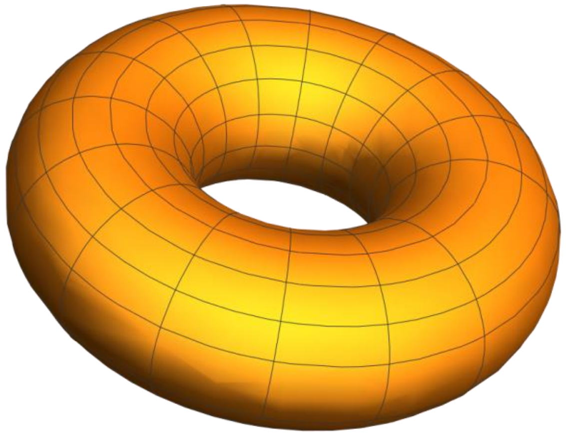

elliptic curve ~ torus

two periods $\omega_1$ & $\omega_2$, $\tau = \frac{\omega_2}{\omega_1}$

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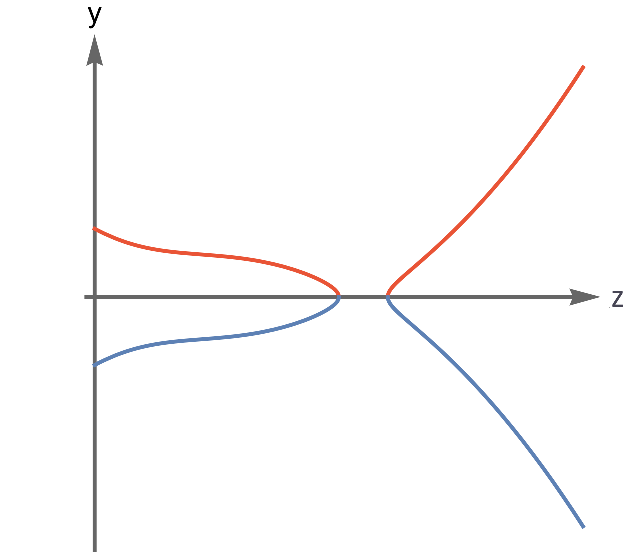


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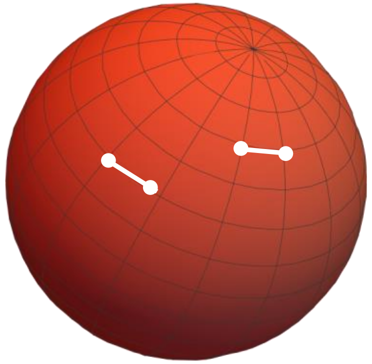

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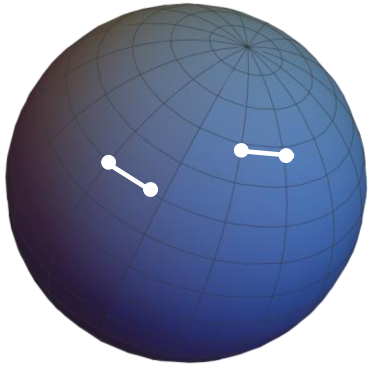

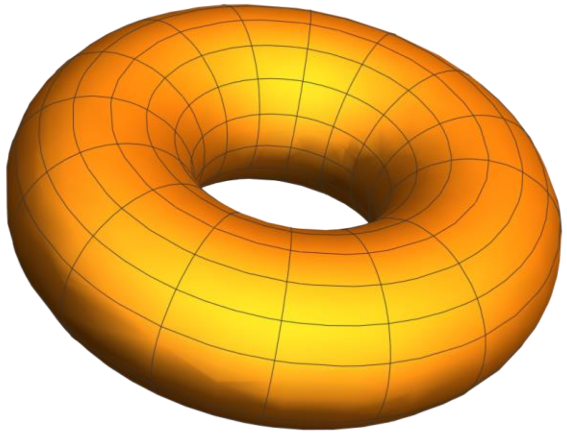

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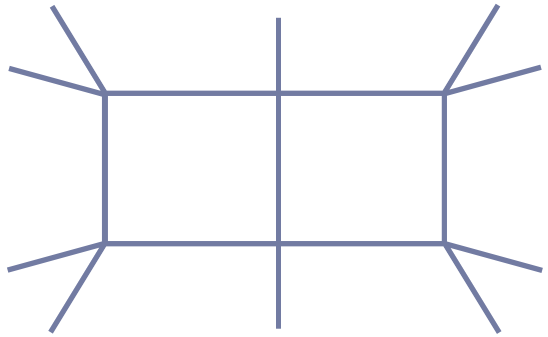
two periods $\omega_1$ & $\omega_2$ , $\tau = \frac{\omega_2}{\omega_1}$

[cf. talk by Hidding for generalisations]

- elliptic integrals: integrals over rational functions of $z$ and $y$, or alternatively, integrals on the (punctured) torus
- corresponding space of functions: elliptic polylogs [Brown, Levin '11][Brödel, Duhr, Dulat, Tancredi '17] [Bogner, Müller-Stach, Weinzierl '19] ...

# 2.3 10pt vs 12pt double box

**10pt double box**



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[Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm '17]
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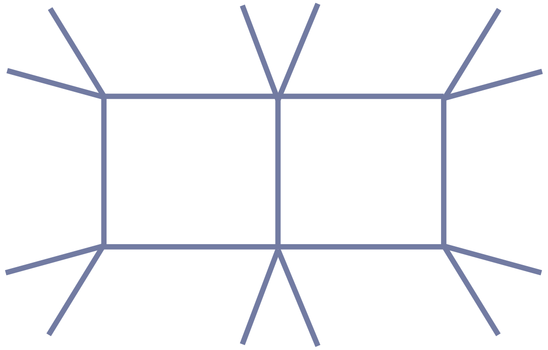

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Most generic elliptic double box in massless scalar theories

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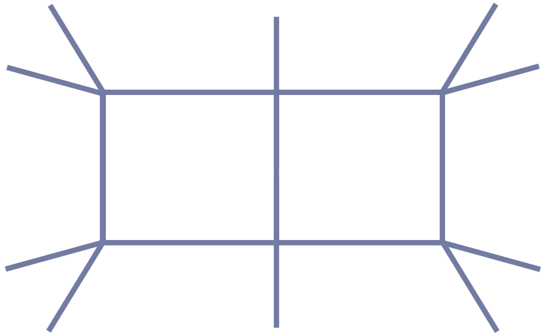

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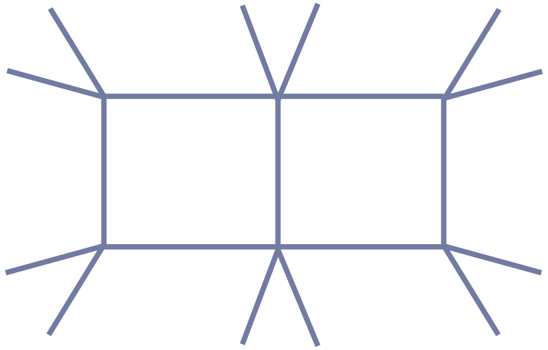

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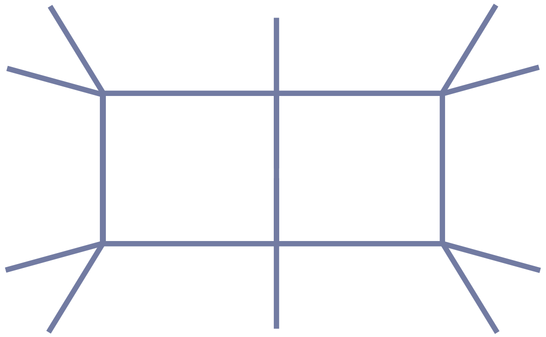

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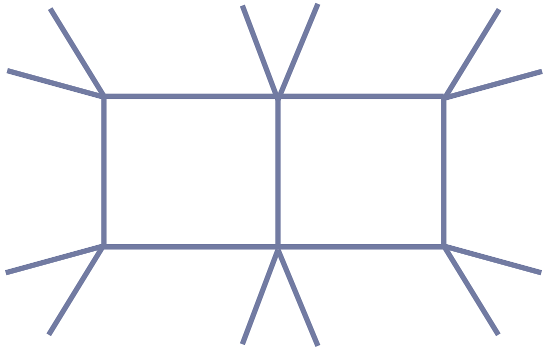

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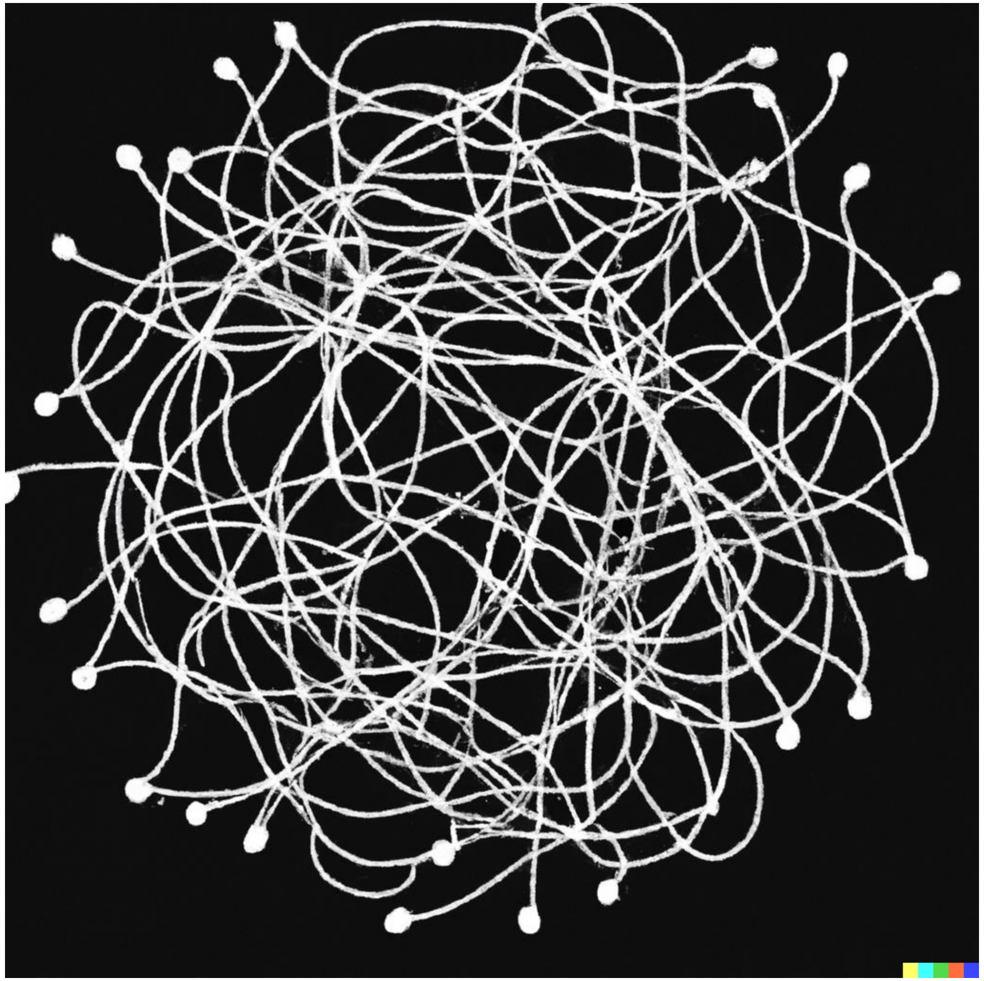
Most generic elliptic double box in massless scalar theories

Here: try to bootstrap* the integral!

*at the level of the symbol

# Symbol techniques for amplitudes



# 3.1 Symbol for polylogs

- **Goncharov polylogarithms:**

$$G_n(\alpha_1, ..., \alpha_n; x) = \int_0^x \frac{dz}{z - \alpha_n} G_{n-1}(\alpha_1, ..., \alpha_{n-1}; z) \qquad \text{with} \quad G_0(;x) = 1$$

- they satisfy the differential equation

$$dG_n(\alpha_1, ..., \alpha_n; \alpha_{n+1}) = \sum_{i=1}^{n} G_{n-1}(\alpha_1, ..., \hat{\alpha}_i, ..., \alpha_n; \alpha_{n+1}) d\log\left(\frac{\alpha_{i+1} - \alpha_i}{\alpha_{i-1} - \alpha_i}\right) \;, \quad \alpha_0 = 0$$

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- maximum recursion of this differential equation is the **symbol**
  [Chen '77] … [Goncharov, Spradlin, Vergu, Volovich '10]

$$\mathcal{S}(G_n) = \sum_i \mathcal{S}(G_{n-1}) \otimes \log\left(\frac{\alpha_{i+1} - \alpha_i}{\alpha_{i-1} - \alpha_i}\right)$$

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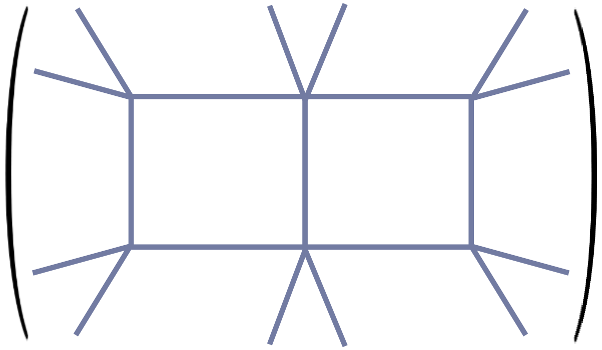
- maximum recursion of this differential equation is the **symbol**
  [Chen '77] ... [Goncharov, Spradlin, Vergu, Volovich '10]

$$\mathcal{S}(G_n) = \sum_i \mathcal{S}(G_{n-1}) \otimes \log\left(\frac{\alpha_{i+1} - \alpha_i}{\alpha_{i-1} - \alpha_i}\right)$$

- generalisation of **symbol to elliptic polylogs** [Broedel, Duhr, Dulat, Penante, Tancredi '18]

# 3.2 Symbol bootstrap

$\mathcal{S}$  $\overset{!}{=} \sum_i C_i \, \mathcal{S}(\text{elliptic polylogs})_i$

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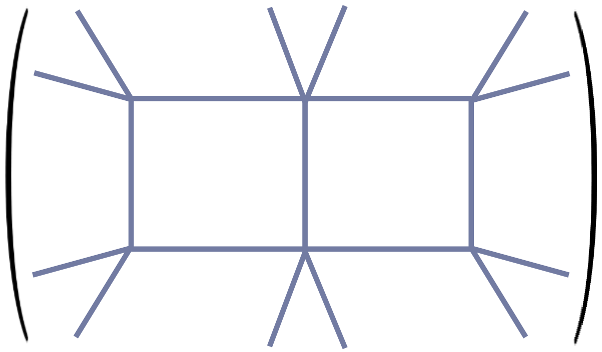

$$\mathcal{S}\left( \; \cdots \; \right) \overset{!}{=} \sum_i C_i \, \mathcal{S}(\text{elliptic polylogs})_i$$

**Symbol bootstrap**:

- circumvent complicated integration by writing an ansatz & afterwards imposing constraints to fix free coefficients
- polylog identities manifest
- physical constraints easier to impose (see later)

applications to many **purely polylog** Feynman integrals (and observables more generally)
[Dixon, Henn, von Hippel, McLeod, Almelid, Caron-Huot, Drummond, Duhr, Dulat, Foster, Gardi, Gürdoğan, Papathanasiou, Pennington, Spradlin, White, Wilhelm ...]
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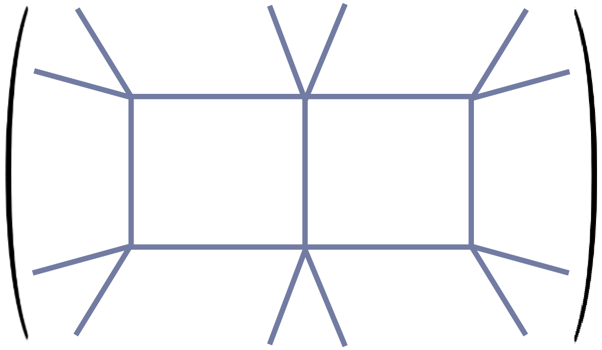
$\mathcal{S}$  $\overset{!}{=} \sum_i C_i \mathcal{S}(\text{elliptic polylogs})_i$

Here: **first bootstrap of an elliptic Feynman diagram**

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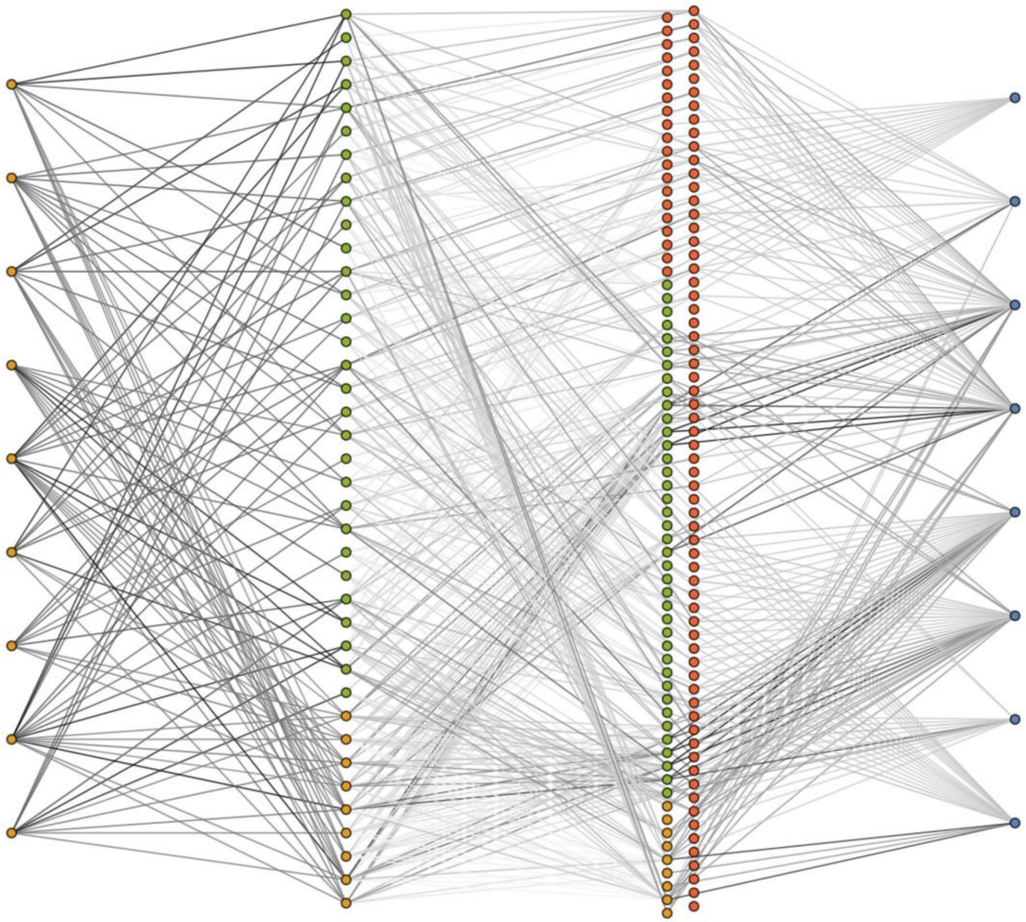
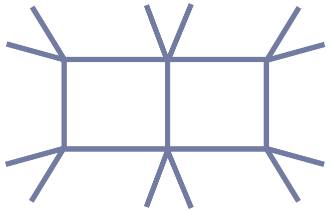
# Bootstrapping the elliptic double box



# 4.1 Bootstrap ansatz for 12pt double box

Based on the structure of the symbol of the 10pt double box [Kristensson, Wilhelm, Zhang '21]
we make the **bootstrapping ansatz for the 12pt double box**

$\mathcal{S}\Bigg($  $\Bigg) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

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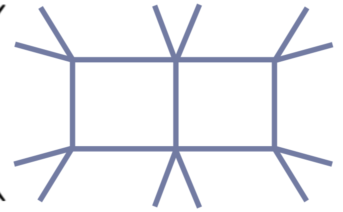
$\mathcal{S}\Bigg($  $\Bigg) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

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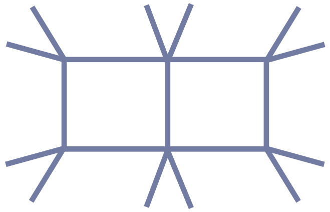
$\mathcal{S}\Bigg($  $\Bigg)$

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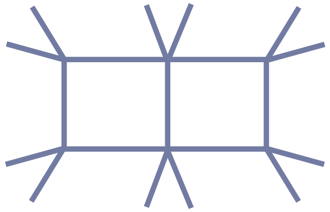
$\mathcal{S}\Bigg($  $\Bigg) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

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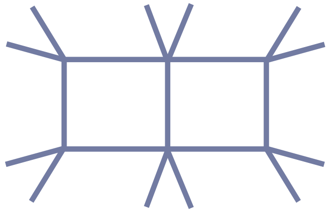
$\mathcal{S}\Bigg($  $\Bigg) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

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- What are the entries in the symbol ("**letters**")?
- What are the **(mathematical & physical) constraints** on the symbol?

# 4.2 Schubert approach to symbol letters

How to construct symbol letters of an amplitude?

Schubert approach: symbol letters from cross ratios of intersection points of external lines and lines associated to the integral's maximal cut in $\mathbb{P}^3$

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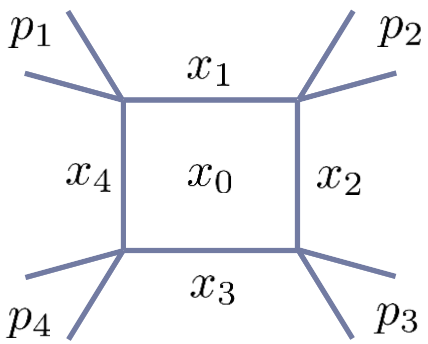

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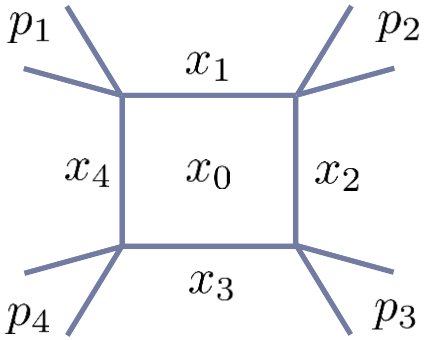
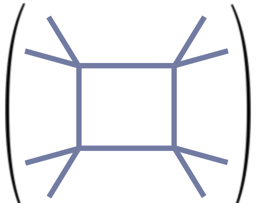



$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}$$

$$v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z})$$

Reminder: symbol of the single box

$\mathcal{S}\Big($  $\Big) \propto \log\left((1-z)(1-\bar{z})\right) \otimes \log\left(\frac{z}{\bar{z}}\right) - \log\left(z\bar{z}\right) \otimes \log\left(\frac{1-z}{1-\bar{z}}\right)$

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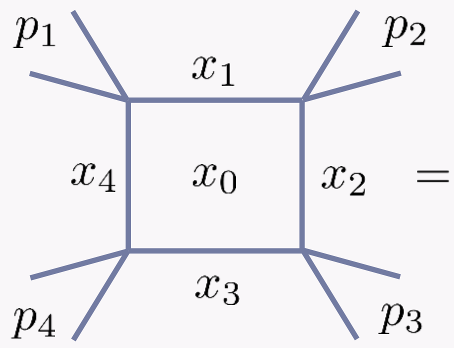


$$= \int \frac{d^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$$

Maximal cut:

$$x_{10}^2 = x_{20}^2 = x_{30}^2 = x_{40}^2 = 0$$

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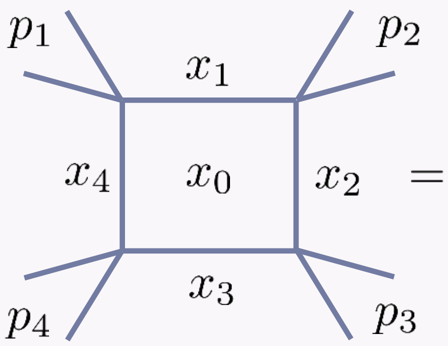


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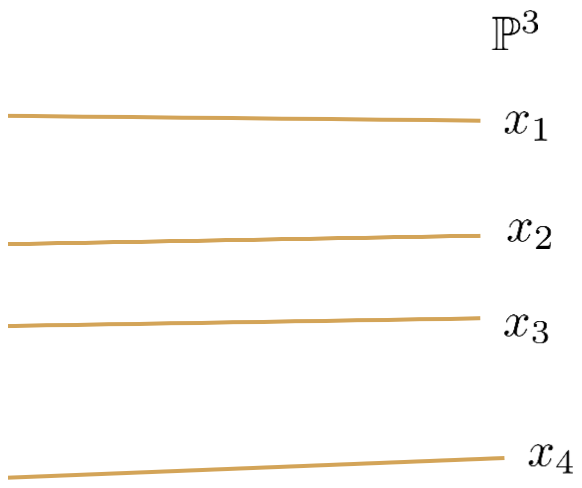

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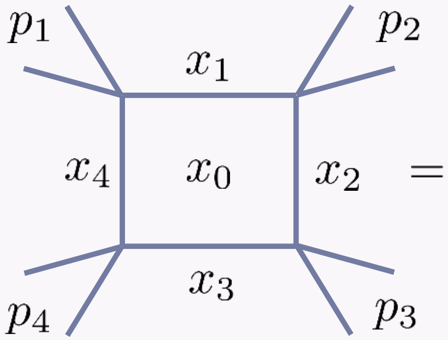


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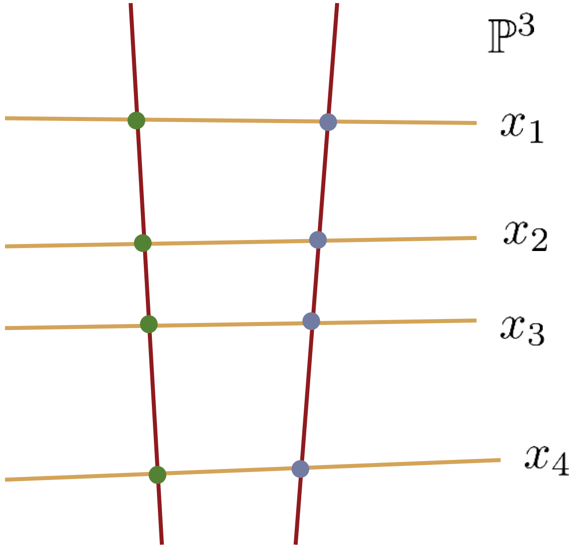


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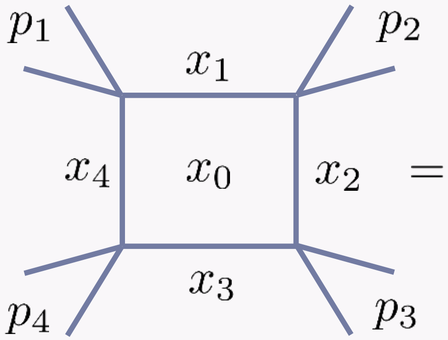


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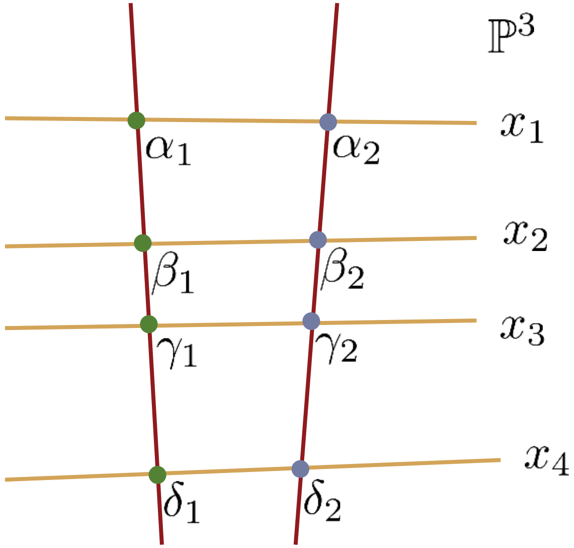


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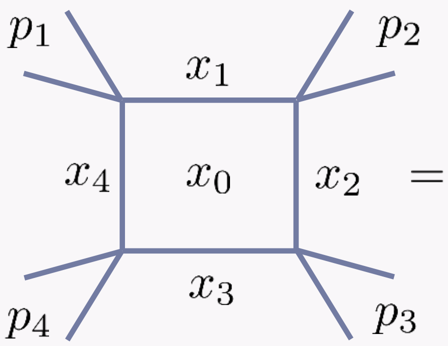


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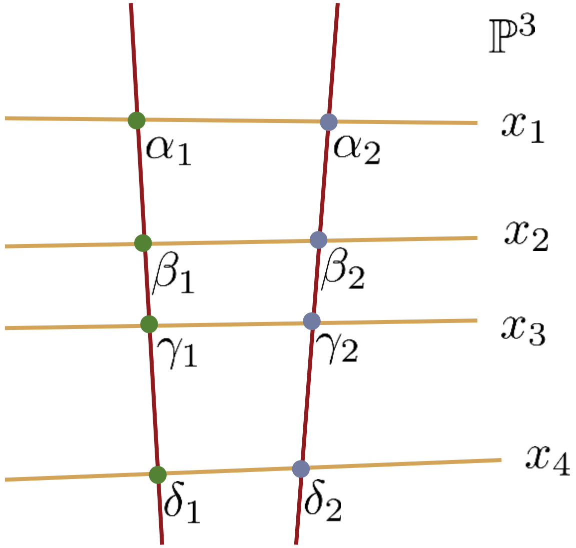


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form cross ratios along each loop line

$$\mathcal{U} = \frac{(\alpha_i - \beta_i)(\gamma_i - \delta_i)}{(\alpha_i - \gamma_i)(\beta_i - \delta_i)}$$

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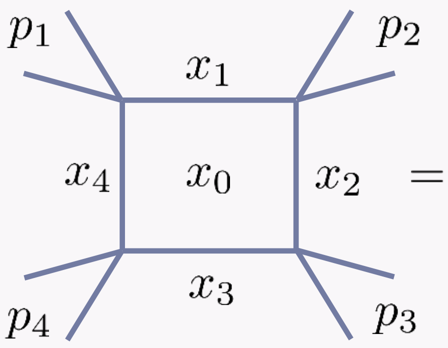


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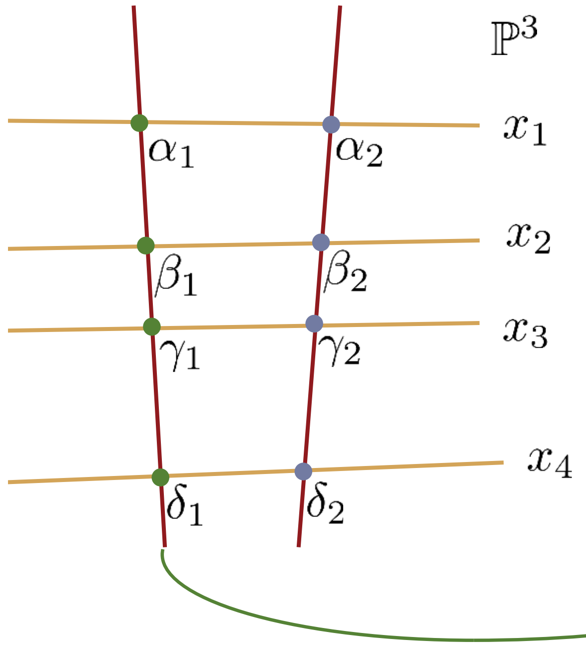


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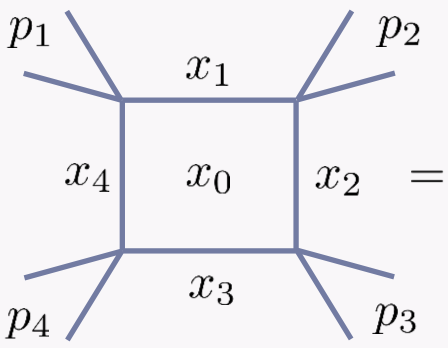


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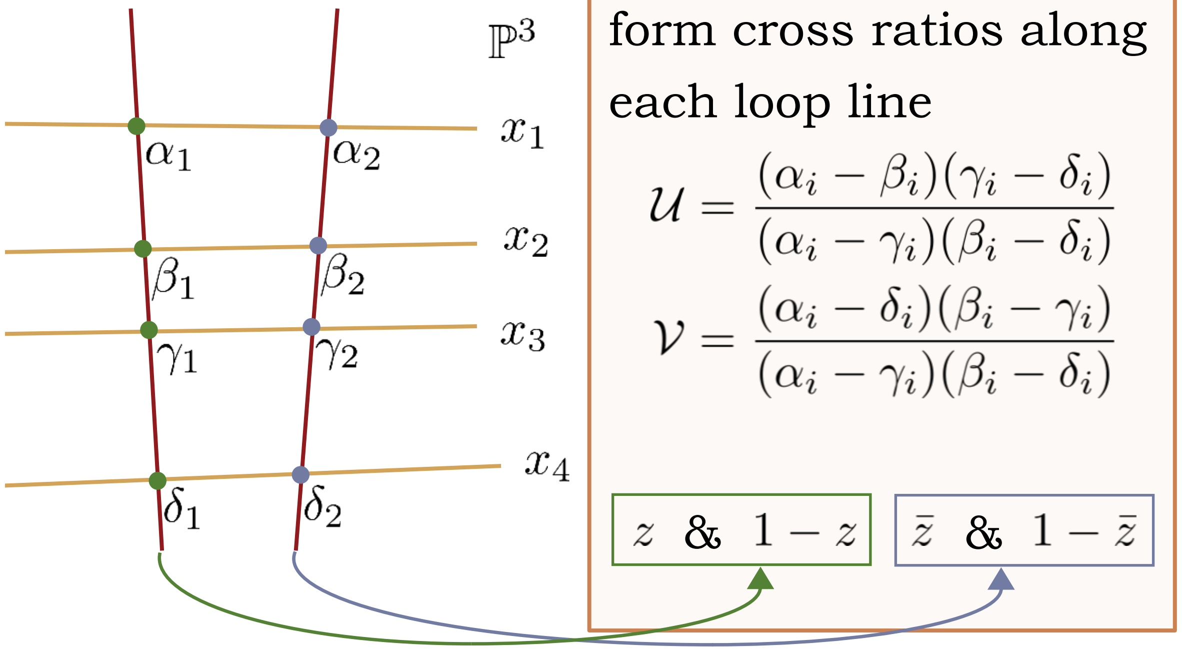


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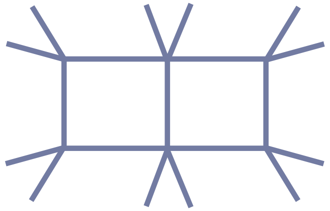
# 4.3 Symbol alphabet

$\mathcal{S}\Bigg($  $\Bigg) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

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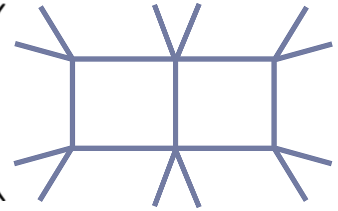
$\mathcal{S}\Bigg($  $\Bigg) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

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$\phi_i \in \{u_1, \ldots, u_9\}$

[Gaiotto, Sever, Maldacena, Vieira '10]

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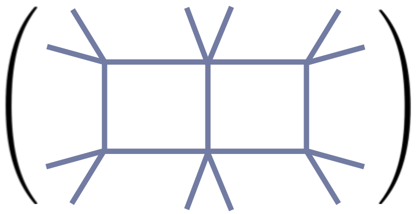
$\mathcal{S}$  $= \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

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**second entries from single-box subtopologies**

$$\phi_j \in \{u_1, \ldots, u_9\} \cup \{z_i/\bar{z}_i, 1 - z_i/1 - \bar{z}_i\}_{i=1}^{15}$$

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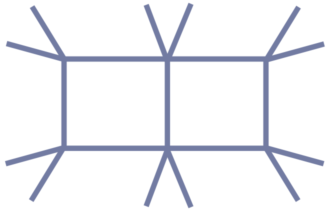
$\mathcal{S}\Bigg($  $\Bigg) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

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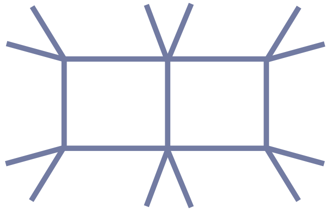
$\mathcal{S}\Bigg($  $\Bigg) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

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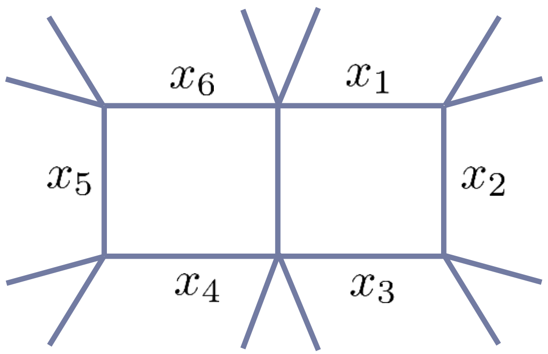
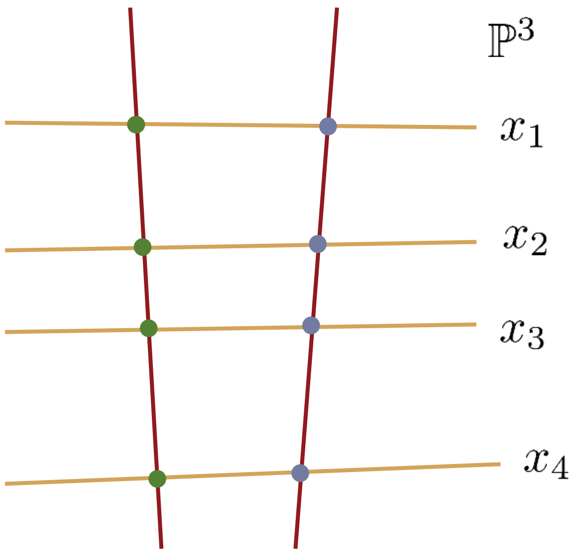
**last entries**: from an **elliptic generalisation of Schubert analysis**

[Morales, **AS**, Wilhelm, Yang, Zhang '22]

# 4.4 Schubert analysis for the double box



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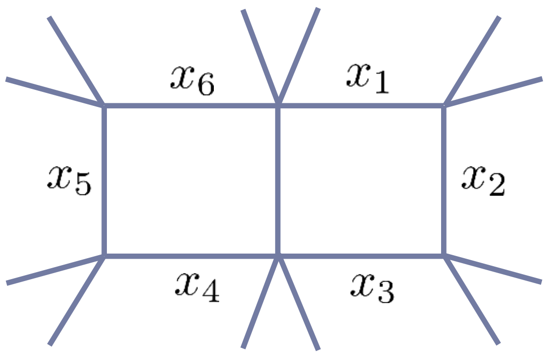
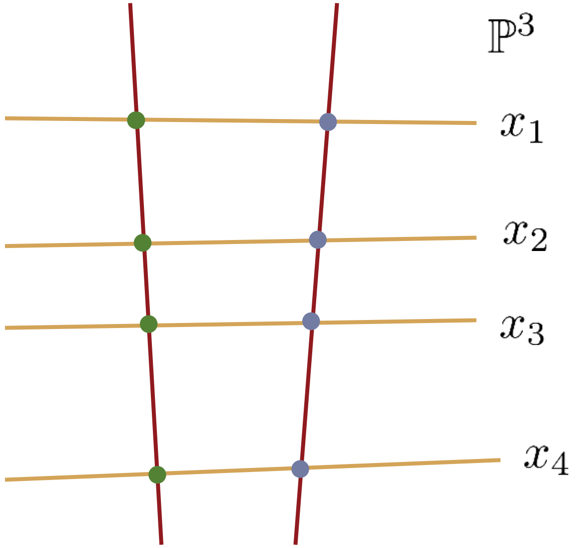



**First and second logarithmic entries** from Schubert problem of single-box subtopologies.



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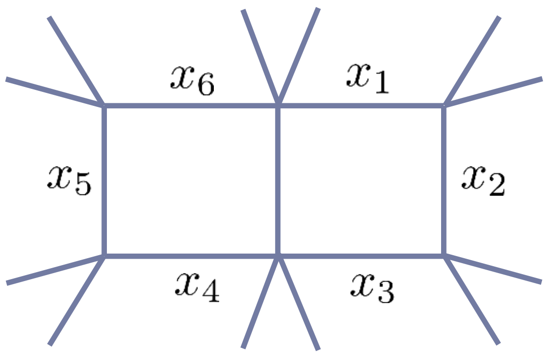
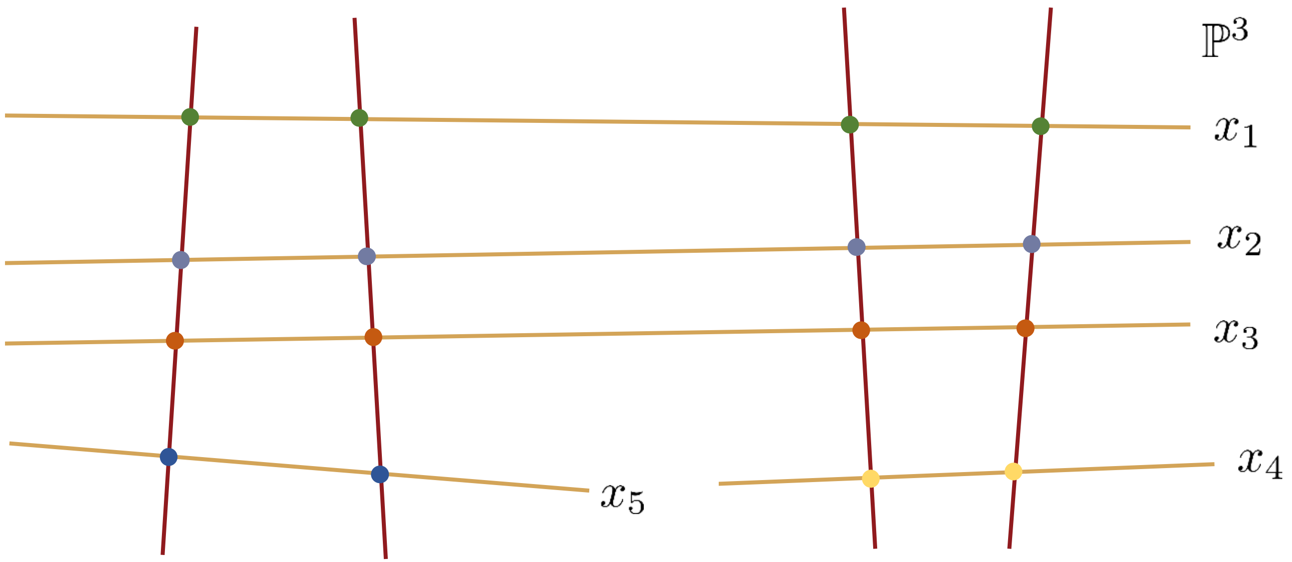
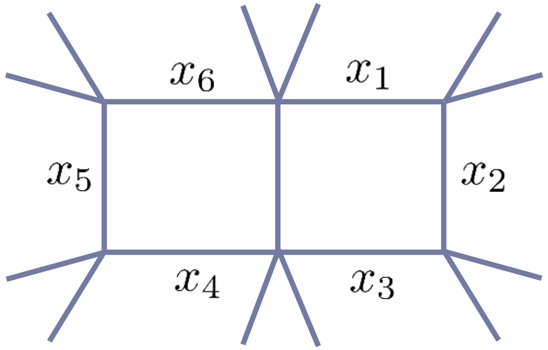


# 4.4 Schubert analysis for the double box

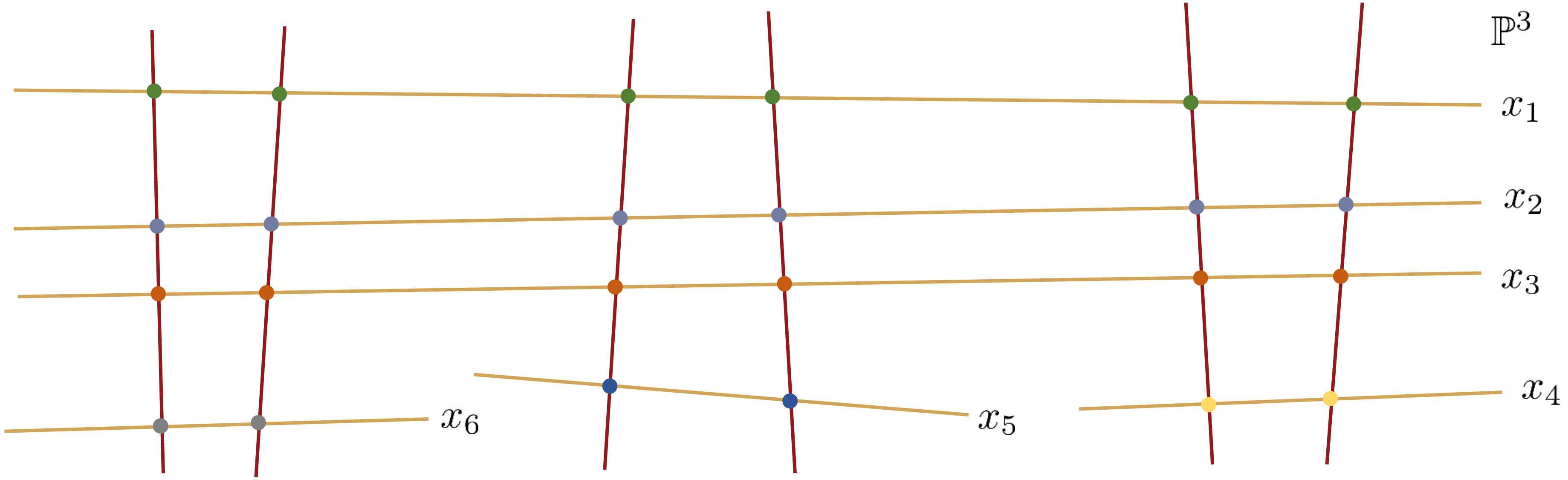
**Logarithmic third entries** from 3 overlapping single-box Schubert problems along external lines.



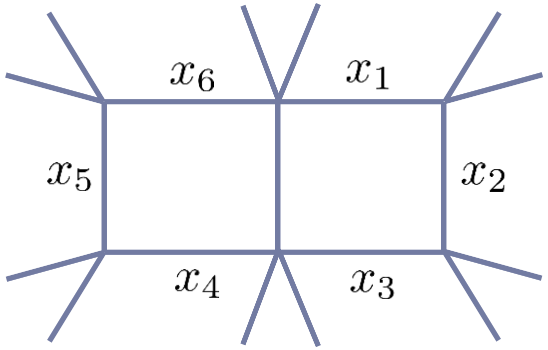

$$\left\{ \dots, \frac{\mathcal{G}_i}{\mathcal{N}_i}, \frac{\mathcal{G}_{ij}}{\mathcal{N}_{ij}}, \frac{\mathcal{G}^i_j - \sqrt{-\mathcal{G}_{ij}\mathcal{G}}}{\mathcal{G}^i_j + \sqrt{-\mathcal{G}_{ij}\mathcal{G}}}, \frac{\mathcal{G}^{ik}_{ij} - \sqrt{-\mathcal{G}_{ij}\mathcal{G}_{ik}}}{\mathcal{G}^{ik}_{ij} + \sqrt{-\mathcal{G}_{ij}\mathcal{G}_{ik}}} \right\}$$

$$\mathcal{G}^B_A := (-1)^{\sum_{c\in\{A,B\}} c} \det(x^2_{ab})_{\substack{a\in\{1,\dots,6\}\setminus\{A\}\\ b\in\{1,\dots,6\}\setminus\{B\}}}$$

$$\mathcal{G}_A := \mathcal{G}^A_A$$

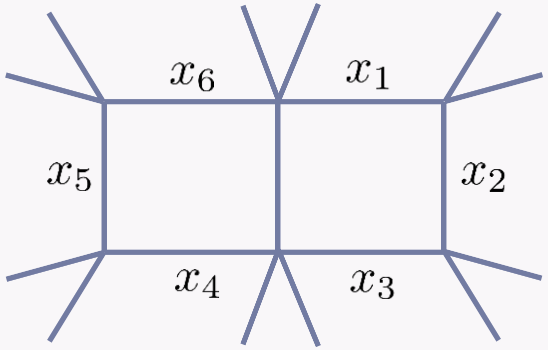


# 4.5 Schubert analysis for elliptic letters




$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

# 4.5 Schubert analysis for elliptic letters

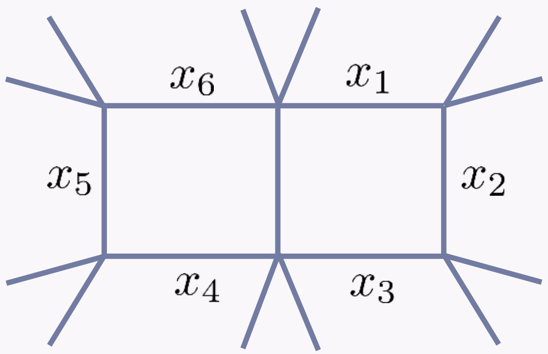



$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

Maximal cut: $x_{10}^2 = x_{20}^2 = x_{30}^2 = x_{00'}^2 = x_{40'}^2 = x_{50'}^2 = x_{60'}^2 = 0$

**Elliptic final entries** from two-loop generalisation of Schubert problem corresponding to leading singularity of double-box diagram.

# 4.5 Schubert analysis for elliptic letters




$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

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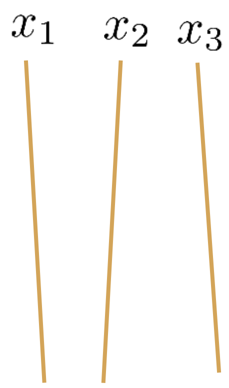
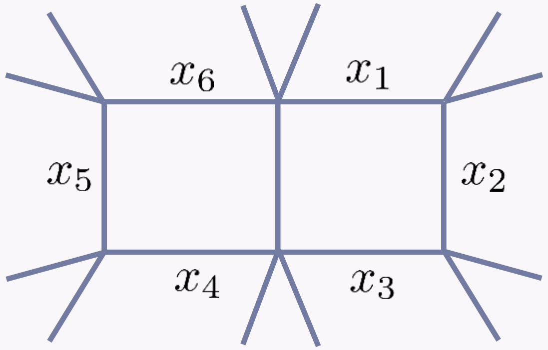


# 4.5 Schubert analysis for elliptic letters




$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

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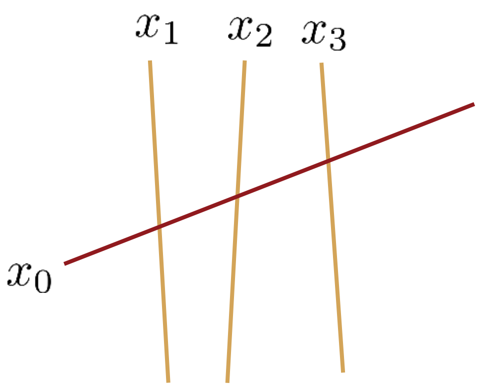
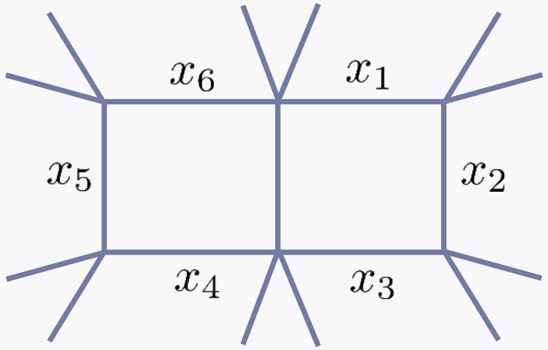
**Elliptic final entries** from two-loop generalisation of Schubert problem corresponding to leading singularity of double-box diagram.



# 4.5 Schubert analysis for elliptic letters




$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

Maximal cut: $x_{10}^2 = x_{20}^2 = x_{30}^2 = x_{00'}^2 = x_{40'}^2 = x_{50'}^2 = x_{60'}^2 = 0$

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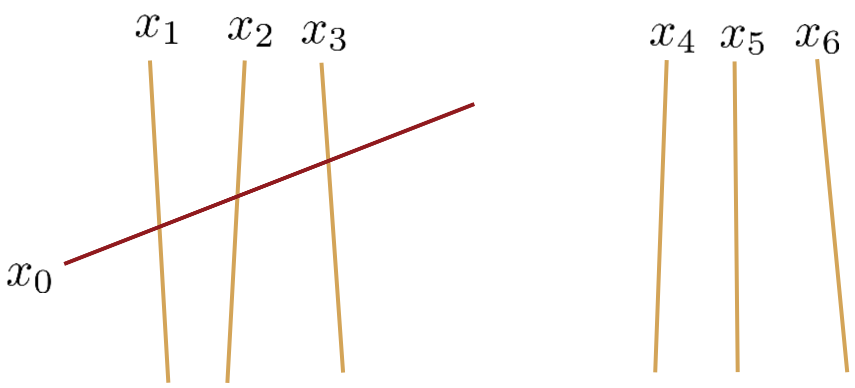
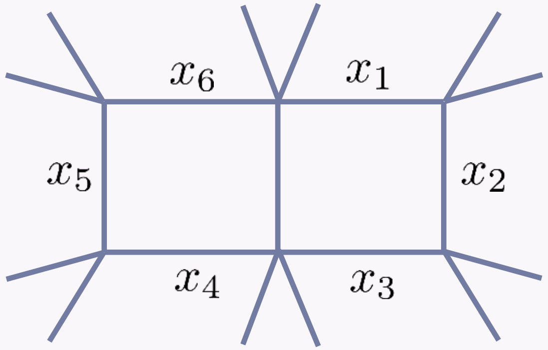


# 4.5 Schubert analysis for elliptic letters




$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

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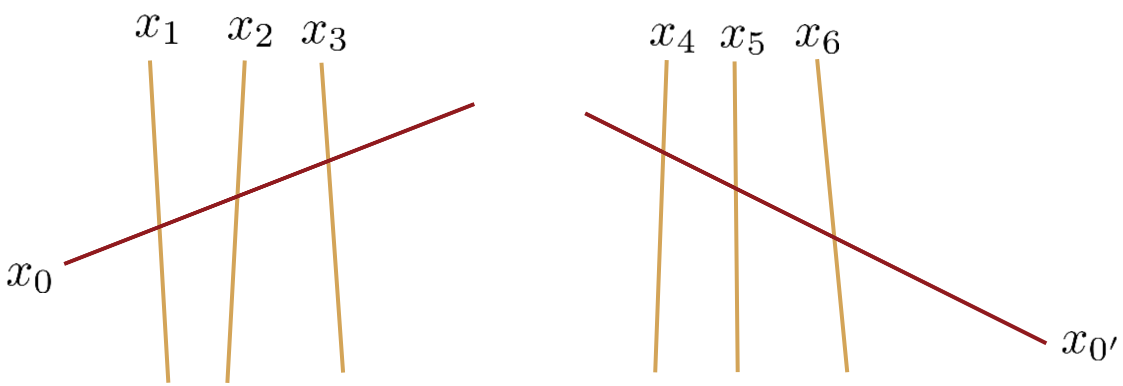
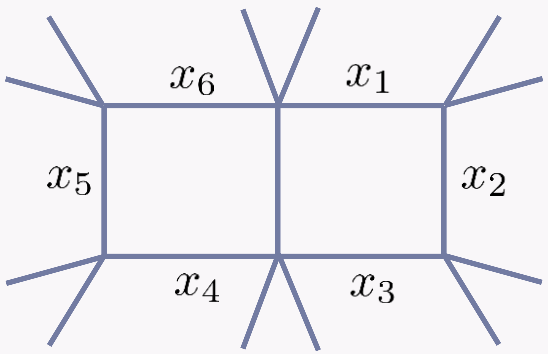


# 4.5 Schubert analysis for elliptic letters




$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

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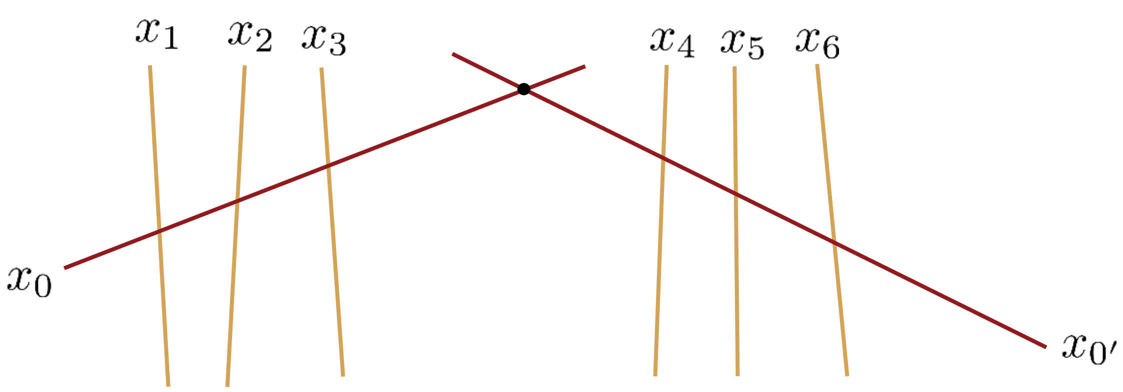
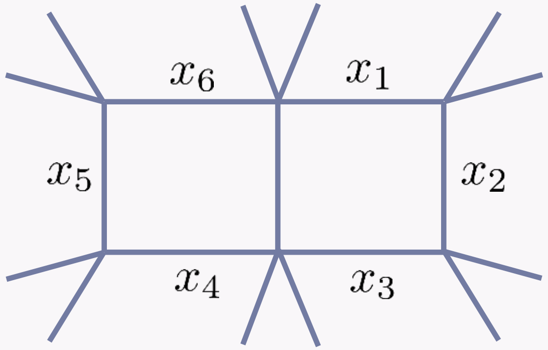


# 4.5 Schubert analysis for elliptic letters




$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

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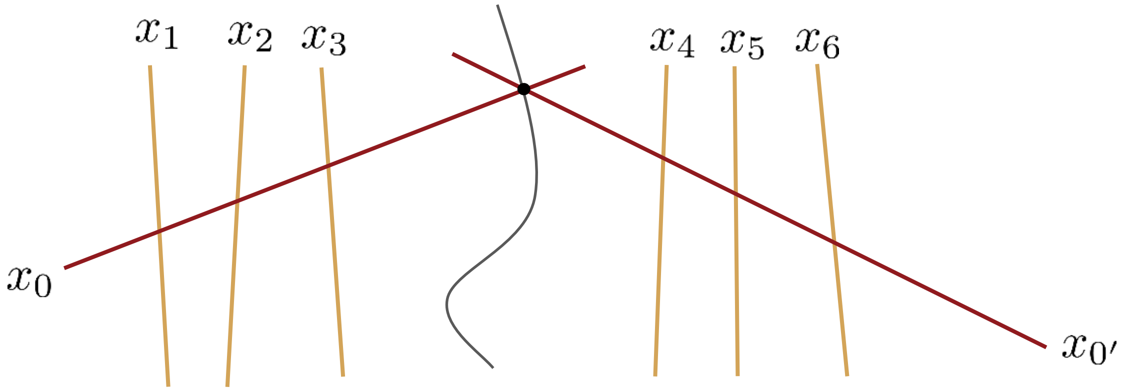
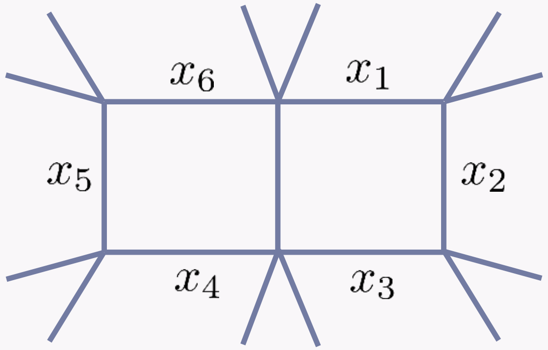


# 4.5 Schubert analysis for elliptic letters




$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

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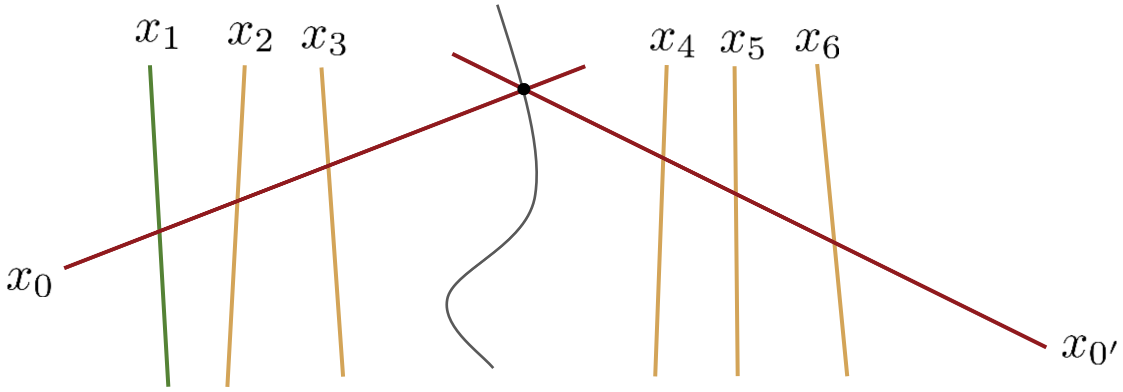
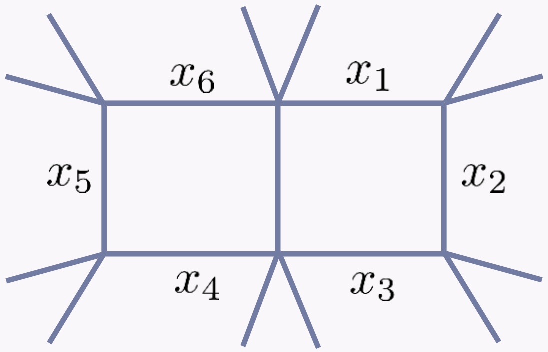
**Elliptic final entries** from two-loop generalisation of Schubert problem corresponding to leading singularity of double-box diagram.



# 4.5 Schubert analysis for elliptic letters




$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

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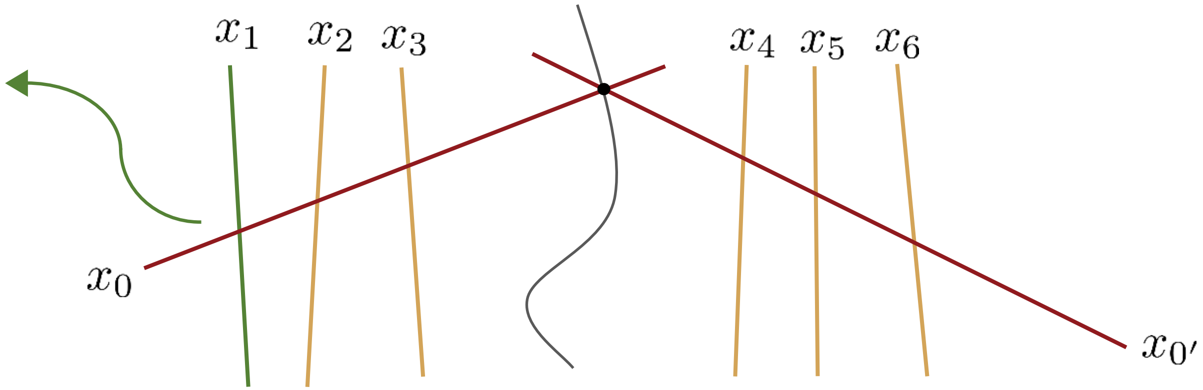
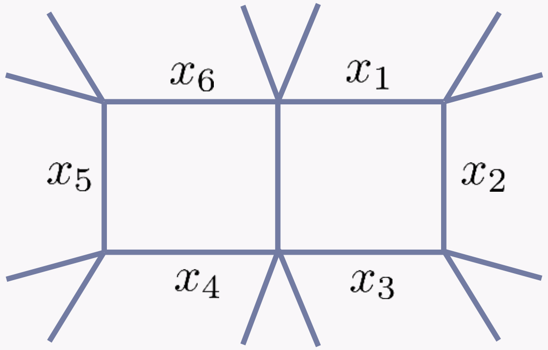
**Elliptic final entries** from two-loop generalisation of Schubert problem corresponding to leading singularity of double-box diagram.

$$\int \frac{dx}{y(x)}$$



# 4.5 Schubert analysis for elliptic letters

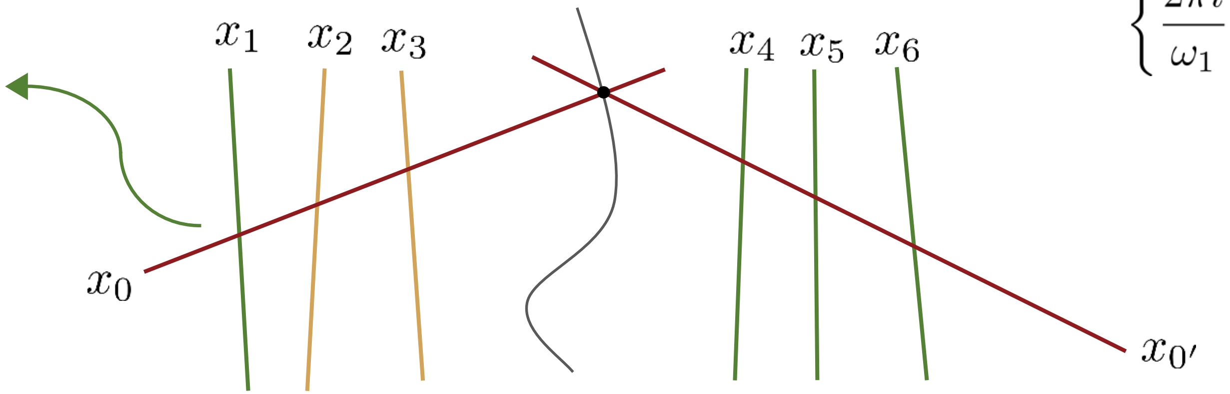



$$= \int \frac{d^4x_0 d^4x_{0'}}{x_{10}^2 x_{20}^2 x_{30}^2 x_{00'}^2 x_{40'}^2 x_{50'}^2 x_{60'}^2}$$

Maximal cut: $x_{10}^2 = x_{20}^2 = x_{30}^2 = x_{00'}^2 = x_{40'}^2 = x_{50'}^2 = x_{60'}^2 = 0$

**Elliptic final entries** from two-loop generalisation of Schubert problem corresponding to leading singularity of double-box diagram.

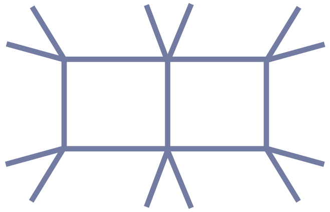
$$\int_{\alpha_1}^{\alpha_2} \frac{dx}{y(x)}$$




$$\left\{ \frac{2\pi i}{\omega_1} \int_{-\infty}^{0} \frac{dx}{y}, \frac{2\pi i}{\omega_1} \int_{-\infty}^{u} \frac{dx}{y}, \frac{2\pi i}{\omega_1} \int_{-\infty}^{c_k} \frac{dx}{y}, -2\pi i \tau \right\}$$

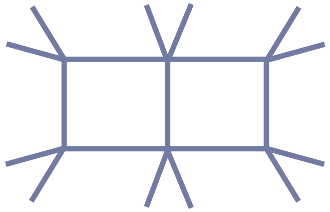
$$c_1 = u \frac{\mathcal{G}_2 x_{12}^4 + \mathcal{G}_3 x_{13}^4 + 2(\mathcal{G}_3^2 + \mathcal{G}_{23} x_{23}^2) x_{12}^2 x_{13}^2}{2\mathcal{G}_{23} x_{12}^2 x_{23}^2 x_{13}^2}$$

# 4.6 Imposing constraints on the symbol

$\mathcal{S}\Big($  $\Big) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

$|\{\phi_i\}| = 9$ $|\{\phi_j\}| = 39$ $|\{\phi_k\}| = 134$ $|\{c_l\}| = 8$

# 4.6 Imposing constraints on the symbol



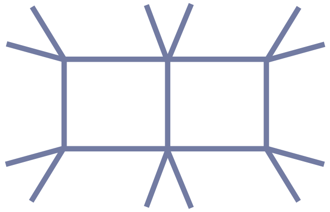
$$\mathcal{S}\left( \text{[double box]} \right) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$$

$|\{\phi_i\}| = 9$ $|\{\phi_j\}| = 39$ $|\{\phi_k\}| = 134$ $|\{c_l\}| = 8$

Not all tensor products of these building blocks correspond to the symbol of a function: **integrability constraints** [Chen '77]

$$\text{“} \left[ \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j} \right] \mathcal{S} = 0 \text{”}$$

## 4.6 Imposing constraints on the symbol

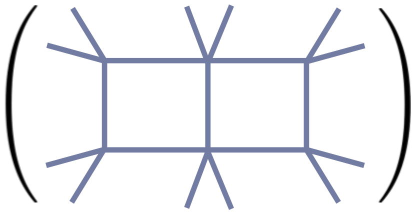
$\mathcal{S}\Bigg($  $\Bigg) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

$|\{\phi_i\}| = 9$ $\quad |\{\phi_j\}| = 39$ $\quad |\{\phi_k\}| = 134$ $\quad |\{c_l\}| = 8$

impose integrability constraints

$\underbrace{|\{\phi_i\}| \quad |\{\phi_j\}|}_{60}$

# 4.6 Imposing constraints on the symbol

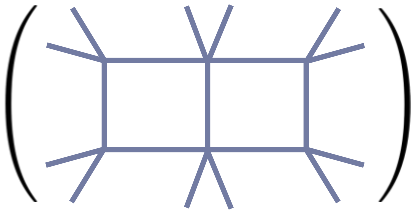
$\mathcal{S}\Big($  $\Big) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$

$|\{\phi_i\}| = 9$ $\quad$ $|\{\phi_j\}| = 39$ $\quad$ $|\{\phi_k\}| = 134$ $\quad$ $|\{c_l\}| = 8$

impose integrability constraints

60 $\quad$ 19

# 4.6 Imposing constraints on the symbol



$$\mathcal{S}\left(\text{[double box]}\right) = \sum_{ijkl} C_{ijkl} \log(\phi_i) \otimes \log(\phi_j) \otimes \log(\phi_k) \otimes \int_{-\infty}^{c_l} \frac{dx}{y} + F_\tau \otimes \tau$$

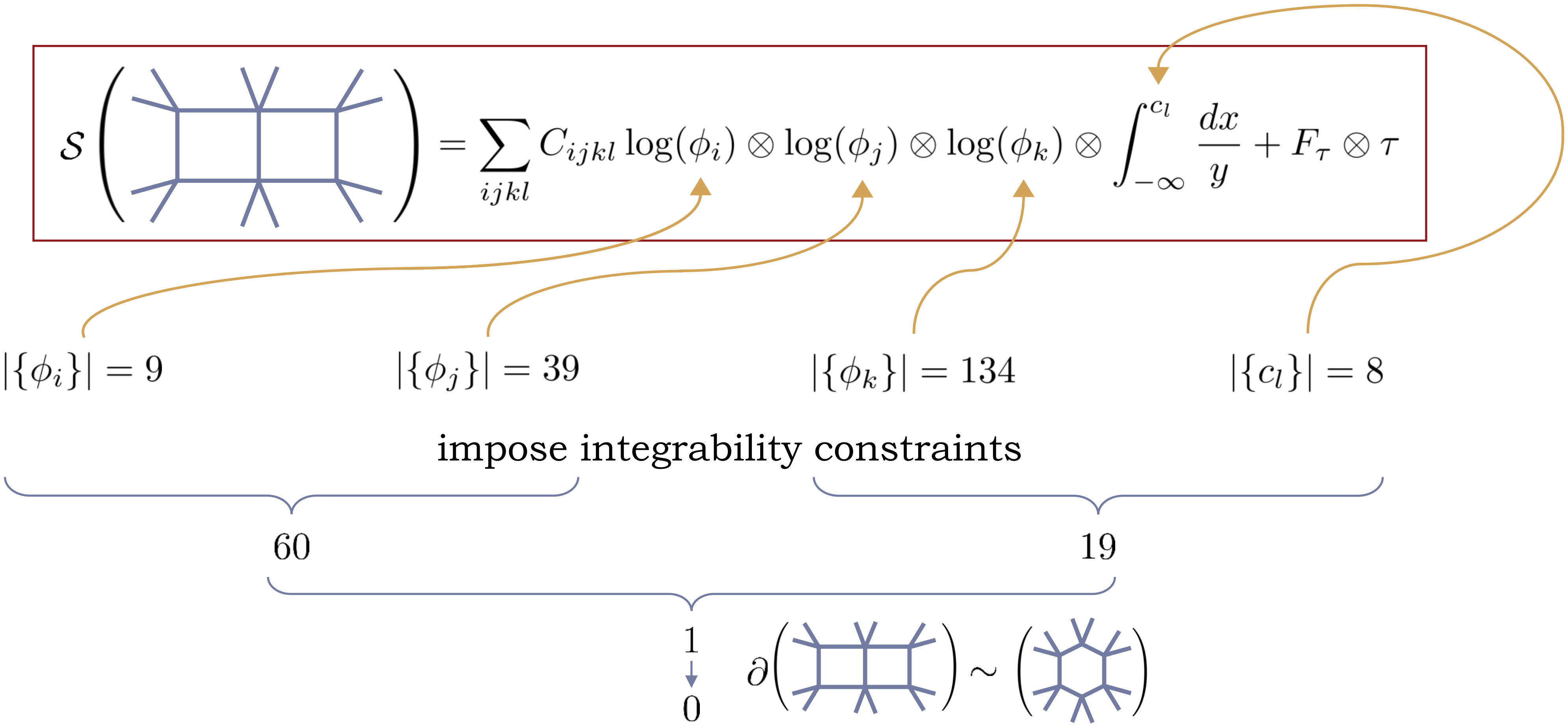
$|\{\phi_i\}| = 9$ $|\{\phi_j\}| = 39$ $|\{\phi_k\}| = 134$ $|\{c_l\}| = 8$

impose integrability constraints

60 19

1

# 4.6 Imposing constraints on the symbol




**Integrability constraints and the differential equation fully fix the ansatz!**

# 4.7 Result of bootstrap

$$-T[1,2,5,1]+T[1,2,12,3]-T[1,2,12,4]+T[1,2,14,3]+T[1,2,14,4]+T[1,2,37,5]+T[1,2,37,6]-T[1,2,39,5]+T[1,2,39,6]-2T[1,2,39,7]-\frac{1}{2}T[1,4,5,1]+\frac{1}{2}T[1,4,12,3]-\frac{1}{2}T[1,4,12,4]+\frac{1}{2}T[1,4,14,3]+\frac{1}{2}T[1,4,14,4]+\frac{1}{2}T[1,4,37,5]+\frac{1}{2}T[1,4,37,6]-\frac{1}{2}T[1,4,39,5]+\frac{1}{2}T[1,4,39,6]-$$

$$T[1,4,39,7]-\frac{1}{2}T[1,5,4,1]-T[1,5,10,2]+\frac{1}{2}T[1,5,10,3]+\frac{1}{2}T[1,5,10,4]-\frac{1}{2}T[1,5,12,3]+\frac{1}{2}T[1,5,12,4]-\frac{1}{2}T[1,5,36,5]-\frac{1}{2}T[1,5,36,6]+\frac{1}{2}T[1,5,38,5]-\frac{1}{2}T[1,5,38,6]+T[1,5,38,7]-T[1,7,4,1]-2T[1,7,10,2]+T[1,7,10,3]+T[1,7,10,4]-T[1,7,12,3]+$$

$$T[1,7,12,4]-T[1,7,36,5]-T[1,7,36,6]+T[1,7,38,5]-T[1,7,38,6]+2T[1,7,38,7]-T[1,9,5,1]+T[1,9,12,3]-T[1,9,12,4]+T[1,9,14,3]+T[1,9,14,4]+T[1,9,37,5]+T[1,9,37,6]-T[1,9,39,5]+T[1,9,39,6]-2T[1,9,39,7]-T[1,10,5,2]+\frac{1}{2}T[1,10,5,3]+$$

$$\frac{1}{2}T[1,10,5,4]-2T[1,10,7,2]+T[1,10,7,3]+T[1,10,7,4]-T[1,10,40,2]+\frac{1}{2}T[1,10,40,3]+\frac{1}{2}T[1,10,40,4]+T[1,10,41,2]-\frac{1}{2}T[1,10,41,3]-\frac{1}{2}T[1,10,41,4]+T[1,10,60,8]-\frac{1}{2}T[1,10,76,3]+\frac{1}{2}T[1,10,76,4]-\frac{1}{2}T[1,10,77,3]-\frac{1}{2}T[1,10,77,4]+\frac{1}{2}T[1,12,2,3]-$$

$$\frac{1}{2}T[1,12,2,4]-\frac{1}{2}T[1,12,5,3]+\frac{1}{2}T[1,12,5,4]-T[1,12,7,3]+T[1,12,7,4]+T[1,12,9,3]-T[1,12,9,4]+\frac{1}{2}T[1,12,41,3]-\frac{1}{2}T[1,12,41,4]-\frac{1}{2}T[1,12,45,3]+\frac{1}{2}T[1,12,45,4]+T[1,12,59,8]-T[1,12,76,2]+\frac{1}{2}T[1,12,76,3]+\frac{1}{2}T[1,12,76,4]+\frac{1}{2}T[1,12,84,3]+$$

$$\frac{1}{2}T[1,12,84,4]+\frac{1}{2}T[1,14,2,3]+\frac{1}{2}T[1,14,2,4]+T[1,14,9,3]+T[1,14,9,4]+\frac{1}{2}T[1,14,40,3]+\frac{1}{2}T[1,14,40,4]-\frac{1}{2}T[1,14,45,3]-\frac{1}{2}T[1,14,45,4]+T[1,14,55,8]+T[1,14,77,2]-\frac{1}{2}T[1,14,77,3]-\frac{1}{2}T[1,14,77,4]+\frac{1}{2}T[1,14,84,3]-\frac{1}{2}T[1,14,84,4]+T[1,31,30,1]-$$

$$\frac{1}{2}T[1,31,31,1]-T[1,31,56,8]-T[1,31,82,2]+\frac{1}{2}T[1,31,82,3]+\frac{1}{2}T[1,31,82,4]+\frac{1}{2}T[1,31,89,3]-\frac{1}{2}T[1,31,89,4]+\frac{1}{2}T[1,31,128,5]+\frac{1}{2}T[1,31,128,6]+\frac{1}{2}T[1,31,129,5]-\frac{1}{2}T[1,31,129,6]+T[1,31,129,7]-T[1,33,32,1]+\frac{1}{2}T[1,33,33,1]-T[1,33,52,8]+T[1,33,83,2]-$$

$$\frac{1}{2}T[1,33,83,3]-\frac{1}{2}T[1,33,83,4]+\frac{1}{2}T[1,33,95,3]+\frac{1}{2}T[1,33,95,4]-\frac{1}{2}T[1,33,131,5]-\frac{1}{2}T[1,33,131,6]-\frac{1}{2}T[1,33,132,5]+\frac{1}{2}T[1,33,132,6]-T[1,33,132,7]+T[1,35,34,1]-\frac{1}{2}T[1,35,35,1]-T[1,35,49,8]+\frac{1}{2}T[1,35,90,3]-\frac{1}{2}T[1,35,90,4]-\frac{1}{2}T[1,35,96,3]-$$

$$\frac{1}{2}T[1,35,96,4]+\frac{1}{2}T[1,35,133,5]+\frac{1}{2}T[1,35,133,6]+\frac{1}{2}T[1,35,134,5]-\frac{1}{2}T[1,35,134,6]+T[1,35,134,7]-T[2,1,5,1]+T[2,1,12,3]-T[2,1,12,4]+T[2,1,14,3]+T[2,1,14,4]+T[2,1,37,5]+T[2,1,37,6]-T[2,1,39,5]+T[2,1,39,6]-2T[2,1,39,7]-T[2,2,5,1]+$$

$$T[2,2,12,3]-T[2,2,12,4]+T[2,2,14,3]+T[2,2,14,4]+T[2,2,37,5]+T[2,2,37,6]-T[2,2,39,5]+T[2,2,39,6]-2T[2,2,39,7]+\frac{1}{2}T[2,5,2,1]+\frac{1}{2}T[2,5,5,1]-T[2,5,11,2]+\frac{1}{2}T[2,5,11,3]+\frac{1}{2}T[2,5,11,4]-\frac{1}{2}T[2,5,12,3]+\frac{1}{2}T[2,5,12,4]-\frac{1}{2}T[2,5,13,3]+\frac{1}{2}T[2,5,13,4]-$$

$$\frac{1}{2}T[2,5,14,3]-\frac{1}{2}T[2,5,14,4]-\frac{1}{2}T[2,5,28,5]+\frac{1}{2}T[2,5,28,6]-\frac{1}{2}T[2,5,36,5]-\frac{1}{2}T[2,5,36,6]-\frac{1}{2}T[2,5,37,5]-\frac{1}{2}T[2,5,37,6]+\frac{1}{2}T[2,5,39,5]-\frac{1}{2}T[2,5,39,6]+T[2,5,39,7]-\frac{1}{2}T[2,6,3,1]-\frac{1}{2}T[2,6,13,3]+\frac{1}{2}T[2,6,13,4]-\frac{1}{2}T[2,6,15,3]-\frac{1}{2}T[2,6,15,4]-$$

$$\frac{1}{2}T[2,6,29,5]+\frac{1}{2}T[2,6,29,6]-\frac{1}{2}T[2,6,37,5]-\frac{1}{2}T[2,6,37,6]-\frac{1}{2}T[2,7,4,1]-T[2,7,10,2]+\frac{1}{2}T[2,7,10,3]+\frac{1}{2}T[2,7,10,4]-\frac{1}{2}T[2,7,12,3]+\frac{1}{2}T[2,7,12,4]-\frac{1}{2}T[2,7,36,5]-\frac{1}{2}T[2,7,36,6]+\frac{1}{2}T[2,7,38,5]-\frac{1}{2}T[2,7,38,6]+T[2,7,38,7]-\frac{1}{2}T[2,8,5,1]+\frac{1}{2}T[2,8,12,3]-$$

$$\frac{1}{2}T[2,8,12,4]+\frac{1}{2}T[2,8,14,3]+\frac{1}{2}T[2,8,14,4]+\frac{1}{2}T[2,8,37,5]+\frac{1}{2}T[2,8,37,6]-\frac{1}{2}T[2,8,39,5]+\frac{1}{2}T[2,8,39,6]-T[2,8,39,7]-T[2,9,5,1]+T[2,9,12,3]-T[2,9,12,4]+T[2,9,14,3]+T[2,9,14,4]+T[2,9,37,5]+T[2,9,37,6]-T[2,9,39,5]+T[2,9,39,6]-2T[2,9,39,7]+$$

$$\frac{1}{2}T[2,12,2,3]-\frac{1}{2}T[2,12,2,4]-\frac{1}{2}T[2,12,5,3]+\frac{1}{2}T[2,12,5,4]-T[2,12,7,3]+T[2,12,7,4]+T[2,12,9,3]-T[2,12,9,4]+\frac{1}{2}T[2,12,41,3]-\frac{1}{2}T[2,12,41,4]-\frac{1}{2}T[2,12,45,3]+\frac{1}{2}T[2,12,45,4]+T[2,12,59,8]-T[2,12,76,2]+\frac{1}{2}T[2,12,76,3]+\frac{1}{2}T[2,12,76,4]+$$

$$\frac{1}{2}T[2,12,84,3]+\frac{1}{2}T[2,12,84,4]+\frac{1}{2}T[2,14,2,3]+\frac{1}{2}T[2,14,2,4]+T[2,14,9,3]+T[2,14,9,4]+\frac{1}{2}T[2,14,40,3]+\frac{1}{2}T[2,14,40,4]-\frac{1}{2}T[2,14,45,3]-\frac{1}{2}T[2,14,45,4]+T[2,14,55,8]+T[2,14,77,2]-\frac{1}{2}T[2,14,77,3]-\frac{1}{2}T[2,14,77,4]+\frac{1}{2}T[2,14,84,3]-\frac{1}{2}T[2,14,84,4]-$$

$$T[2,16,16,1]+\frac{1}{2}T[2,16,17,1]+T[2,16,58,8]+T[2,16,78,2]-\frac{1}{2}T[2,16,78,3]-\frac{1}{2}T[2,16,78,4]-\frac{1}{2}T[2,16,85,3]+\frac{1}{2}T[2,16,85,4]+\frac{1}{2}T[2,16,100,5]-\frac{1}{2}T[2,16,100,6]-\frac{1}{2}T[2,16,102,5]-\frac{1}{2}T[2,16,102,6]+T[2,18,18,1]-\frac{1}{2}T[2,18,19,1]+T[2,18,54,8]-T[2,18,79,2]+$$

$$\frac{1}{2}T[2,18,79,3]+\frac{1}{2}T[2,18,79,4]-\frac{1}{2}T[2,18,91,3]-\frac{1}{2}T[2,18,91,4]-\frac{1}{2}T[2,18,105,5]+\frac{1}{2}T[2,18,105,6]+\frac{1}{2}T[2,18,107,5]+\frac{1}{2}T[2,18,107,6]+T[2,22,22,1]-\frac{1}{2}T[2,22,23,1]+T[2,22,57,8]-T[2,22,80,2]+\frac{1}{2}T[2,22,80,3]+\frac{1}{2}T[2,22,80,4]+\frac{1}{2}T[2,22,87,3]-$$

$$\frac{1}{2}T[2,22,87,4]-\frac{1}{2}T[2,22,114,5]+\frac{1}{2}T[2,22,114,6]+\frac{1}{2}T[2,22,116,5]-\frac{1}{2}T[2,22,116,6]+T[2,22,116,7]-T[2,24,24,1]+\frac{1}{2}T[2,24,25,1]+T[2,24,53,8]+T[2,24,81,2]-\frac{1}{2}T[2,24,81,3]-\frac{1}{2}T[2,24,81,4]+\frac{1}{2}T[2,24,93,3]+\frac{1}{2}T[2,24,93,4]+\frac{1}{2}T[2,24,118,5]-$$

$$\frac{1}{2}T[2,24,118,6]-\frac{1}{2}T[2,24,120,5]+\frac{1}{2}T[2,24,120,6]-T[2,24,120,7]+T[2,35,34,1]-\frac{1}{2}T[2,35,35,1]-T[2,35,49,8]+\frac{1}{2}T[2,35,90,3]-\frac{1}{2}T[2,35,90,4]-\frac{1}{2}T[2,35,96,3]-\frac{1}{2}T[2,35,96,4]+\frac{1}{2}T[2,35,133,5]+\frac{1}{2}T[2,35,133,6]+\frac{1}{2}T[2,35,134,5]-\frac{1}{2}T[2,35,134,6]+$$

$$T[2,35,134,7]+T[2,37,1,5]+T[2,37,1,6]+\frac{1}{2}T[2,37,2,5]+\frac{1}{2}T[2,37,2,6]-\frac{1}{2}T[2,37,5,5]-\frac{1}{2}T[2,37,5,6]-T[2,37,6,5]-T[2,37,6,6]-\frac{1}{2}T[2,37,43,5]-\frac{1}{2}T[2,37,43,6]+\frac{1}{2}T[2,37,44,5]+\frac{1}{2}T[2,37,44,6]-T[2,37,47,8]-\frac{1}{2}T[2,37,124,5]+\frac{1}{2}T[2,37,124,6]-$$

$$\frac{1}{2}T[2,37,135,5]+\frac{1}{2}T[2,37,135,6]-T[2,37,135,7]-T[2,39,1,5]+T[2,39,1,6]-2T[2,39,1,7]-\frac{1}{2}T[2,39,2,5]+\frac{1}{2}T[2,39,2,6]-T[2,39,2,7]-\frac{1}{2}T[2,39,42,5]+\frac{1}{2}T[2,39,42,6]-T[2,39,42,7]+\frac{1}{2}T[2,39,43,5]-\frac{1}{2}T[2,39,43,6]+T[2,39,43,7]-T[2,39,46,8]+$$

$$\frac{1}{2}T[2,39,125,5]-\frac{1}{2}T[2,39,125,6]+\frac{1}{2}T[2,39,135,5]+\frac{1}{2}T[2,39,135,6]-\frac{1}{2}T[3,6,2,1]+T[3,6,11,2]-\frac{1}{2}T[3,6,11,3]-\frac{1}{2}T[3,6,11,4]+\frac{1}{2}T[3,6,13,3]-\frac{1}{2}T[3,6,13,4]+\frac{1}{2}T[3,6,28,5]-\frac{1}{2}T[3,6,28,6]+\frac{1}{2}T[3,6,36,5]+\frac{1}{2}T[3,6,36,6]+\frac{1}{2}T[3,8,4,1]+T[3,8,10,2]-$$

$$\frac{1}{2}T[3,8,10,3]-\frac{1}{2}T[3,8,10,4]+\frac{1}{2}T[3,8,12,3]-\frac{1}{2}T[3,8,12,4]+\frac{1}{2}T[3,8,36,5]+\frac{1}{2}T[3,8,36,6]-\frac{1}{2}T[3,8,38,5]+\frac{1}{2}T[3,8,38,6]-T[3,8,38,7]-\frac{1}{2}T[3,9,5,1]+\frac{1}{2}T[3,9,12,3]-\frac{1}{2}T[3,9,12,4]+\frac{1}{2}T[3,9,14,3]+\frac{1}{2}T[3,9,14,4]+\frac{1}{2}T[3,9,37,5]+\frac{1}{2}T[3,9,37,6]-$$

# 4.7 Result of bootstrap

$$\frac{i}{2}T[3, 9, 39, 5]+\frac{1}{2}T[3, 9, 39, 6]-T[3, 9, 39, 7]+\frac{i}{2}T[3, 14, 2, 3]+\frac{1}{2}T[3, 14, 2, 4]+T[3, 14, 9, 3]+T[3, 14, 9, 4]+\frac{1}{2}T[3, 14, 40, 3]+\frac{1}{2}T[3, 14, 40, 4]-\frac{i}{2}T[3, 14, 45, 3]-\frac{i}{2}T[3, 14, 45, 4]+T[3, 14, 55, 8]+T[3, 14, 77, 2]-\frac{1}{2}T[3, 14, 77, 3]-\frac{1}{2}T[3, 14, 77, 4]+\frac{1}{2}T[3, 14, 84, 3]-$$

$$\frac{1}{2}T[3, 14, 84, 4]+T[3, 18, 18, 1]-\frac{1}{2}T[3, 18, 19, 1]+T[3, 18, 54, 8]-T[3, 18, 79, 2]+\frac{1}{2}T[3, 18, 79, 3]+\frac{1}{2}T[3, 18, 79, 4]-\frac{1}{2}T[3, 18, 91, 3]-\frac{1}{2}T[3, 18, 91, 4]-\frac{1}{2}T[3, 18, 105, 5]+\frac{1}{2}T[3, 18, 105, 6]+\frac{1}{2}T[3, 18, 107, 5]+\frac{1}{2}T[3, 18, 107, 6]-T[3, 20, 20, 1]+\frac{1}{2}T[3, 20, 21, 1]+T[3, 20, 51, 8]-$$

$$\frac{1}{2}T[3, 20, 86, 3]+\frac{1}{2}T[3, 20, 86, 4]+\frac{1}{2}T[3, 20, 92, 3]+\frac{1}{2}T[3, 20, 92, 4]+\frac{1}{2}T[3, 20, 109, 5]-\frac{1}{2}T[3, 20, 109, 6]-\frac{1}{2}T[3, 20, 111, 5]-\frac{1}{2}T[3, 20, 111, 6]-T[3, 24, 24, 1]+\frac{1}{2}T[3, 24, 25, 1]+T[3, 24, 53, 8]+T[3, 24, 81, 2]-\frac{1}{2}T[3, 24, 81, 3]-\frac{1}{2}T[3, 24, 81, 4]+\frac{1}{2}T[3, 24, 93, 3]+$$

$$\frac{1}{2}T[3, 24, 93, 4]+\frac{1}{2}T[3, 24, 118, 5]-\frac{1}{2}T[3, 24, 118, 6]-\frac{1}{2}T[3, 24, 120, 5]+\frac{1}{2}T[3, 24, 120, 6]-T[3, 24, 120, 7]+T[3, 26, 26, 1]-\frac{1}{2}T[3, 26, 27, 1]+T[3, 26, 50, 8]+\frac{1}{2}T[3, 26, 88, 3]-\frac{1}{2}T[3, 26, 88, 4]-\frac{1}{2}T[3, 26, 94, 3]-\frac{1}{2}T[3, 26, 94, 4]-\frac{1}{2}T[3, 26, 121, 5]+\frac{1}{2}T[3, 26, 121, 6]+$$

$$\frac{1}{2}T[3, 26, 123, 5]-\frac{1}{2}T[3, 26, 123, 6]+T[3, 26, 123, 7]+\frac{1}{2}T[3, 28, 5, 5]-\frac{1}{2}T[3, 28, 5, 6]+T[3, 28, 6, 5]-T[3, 28, 6, 6]+\frac{1}{2}T[3, 28, 42, 5]-\frac{1}{2}T[3, 28, 42, 6]-\frac{1}{2}T[3, 28, 44, 5]+\frac{1}{2}T[3, 28, 44, 6]+T[3, 28, 48, 8]+\frac{1}{2}T[3, 28, 124, 5]+\frac{1}{2}T[3, 28, 124, 6]-\frac{1}{2}T[3, 28, 125, 5]+\frac{1}{2}T[3, 28, 125, 6]-$$

$$T[3, 28, 125, 7]-\frac{1}{2}T[4, 1, 5, 1]+\frac{1}{2}T[4, 1, 12, 3]-\frac{1}{2}T[4, 1, 12, 4]+\frac{1}{2}T[4, 1, 14, 3]+\frac{1}{2}T[4, 1, 14, 4]+\frac{1}{2}T[4, 1, 37, 5]+\frac{1}{2}T[4, 1, 37, 6]-\frac{1}{2}T[4, 1, 39, 5]+\frac{1}{2}T[4, 1, 39, 6]-T[4, 1, 39, 7]-\frac{1}{2}T[4, 7, 2, 1]+T[4, 7, 11, 2]-\frac{1}{2}T[4, 7, 11, 3]-\frac{1}{2}T[4, 7, 11, 4]+\frac{1}{2}T[4, 7, 13, 3]-\frac{1}{2}T[4, 7, 13, 4]+$$

$$\frac{1}{2}T[4, 7, 28, 5]-\frac{1}{2}T[4, 7, 28, 6]+\frac{1}{2}T[4, 7, 36, 5]+\frac{1}{2}T[4, 7, 36, 6]+\frac{1}{2}T[4, 8, 3, 1]+\frac{1}{2}T[4, 8, 13, 3]-\frac{1}{2}T[4, 8, 13, 4]+\frac{1}{2}T[4, 8, 15, 3]+\frac{1}{2}T[4, 8, 15, 4]+\frac{1}{2}T[4, 8, 29, 5]-\frac{1}{2}T[4, 8, 29, 6]+\frac{1}{2}T[4, 8, 37, 5]+\frac{1}{2}T[4, 8, 37, 6]+T[4, 11, 5, 2]-\frac{1}{2}T[4, 11, 5, 3]-\frac{1}{2}T[4, 11, 5, 4]+$$

$$2T[4, 11, 7, 2]-T[4, 11, 7, 3]-T[4, 11, 7, 4]+T[4, 11, 40, 2]-\frac{1}{2}T[4, 11, 40, 3]-\frac{1}{2}T[4, 11, 40, 4]-T[4, 11, 41, 2]+\frac{1}{2}T[4, 11, 41, 3]+\frac{1}{2}T[4, 11, 41, 4]-T[4, 11, 60, 8]+\frac{1}{2}T[4, 11, 76, 3]-\frac{1}{2}T[4, 11, 76, 4]+\frac{1}{2}T[4, 11, 77, 3]+\frac{1}{2}T[4, 11, 77, 4]+T[4, 22, 22, 1]-\frac{1}{2}T[4, 22, 23, 1]+$$

$$T[4, 22, 57, 8]-T[4, 22, 80, 2]+\frac{1}{2}T[4, 22, 80, 3]+\frac{1}{2}T[4, 22, 80, 4]+\frac{1}{2}T[4, 22, 87, 3]-\frac{1}{2}T[4, 22, 87, 4]-\frac{1}{2}T[4, 22, 114, 5]+\frac{1}{2}T[4, 22, 114, 6]+\frac{1}{2}T[4, 22, 116, 5]-\frac{1}{2}T[4, 22, 116, 6]+T[4, 22, 116, 7]-T[4, 24, 24, 1]+\frac{1}{2}T[4, 24, 25, 1]+T[4, 24, 53, 8]+T[4, 24, 81, 2]-\frac{1}{2}T[4, 24, 81, 3]-$$

$$\frac{1}{2}T[4, 24, 81, 4]+\frac{1}{2}T[4, 24, 93, 3]+\frac{1}{2}T[4, 24, 93, 4]+\frac{1}{2}T[4, 24, 118, 5]-\frac{1}{2}T[4, 24, 118, 6]-\frac{1}{2}T[4, 24, 120, 5]+\frac{1}{2}T[4, 24, 120, 6]-T[4, 24, 120, 7]-T[4, 30, 30, 1]+\frac{1}{2}T[4, 30, 31, 1]+T[4, 30, 56, 8]+T[4, 30, 82, 2]-\frac{1}{2}T[4, 30, 82, 3]-\frac{1}{2}T[4, 30, 82, 4]-\frac{1}{2}T[4, 30, 89, 3]+$$

$$\frac{1}{2}T[4, 30, 89, 4]-\frac{1}{2}T[4, 30, 128, 5]-\frac{1}{2}T[4, 30, 128, 6]-\frac{1}{2}T[4, 30, 129, 5]+\frac{1}{2}T[4, 30, 129, 6]-T[4, 30, 129, 7]+T[4, 32, 32, 1]-\frac{1}{2}T[4, 32, 33, 1]+T[4, 32, 52, 8]-T[4, 32, 83, 2]+\frac{1}{2}T[4, 32, 83, 3]+\frac{1}{2}T[4, 32, 83, 4]-\frac{1}{2}T[4, 32, 95, 3]-\frac{1}{2}T[4, 32, 95, 4]+\frac{1}{2}T[4, 32, 131, 5]+$$

$$\frac{1}{2}T[4, 32, 131, 6]+\frac{1}{2}T[4, 32, 132, 5]-\frac{1}{2}T[4, 32, 132, 6]+T[4, 32, 132, 7]-T[4, 39, 1, 5]+T[4, 39, 1, 6]-2T[4, 39, 1, 7]-\frac{1}{2}T[4, 39, 2, 5]+\frac{1}{2}T[4, 39, 2, 6]-T[4, 39, 2, 7]-\frac{1}{2}T[4, 39, 42, 5]+\frac{1}{2}T[4, 39, 42, 6]-T[4, 39, 42, 7]+\frac{1}{2}T[4, 39, 43, 5]-\frac{1}{2}T[4, 39, 43, 6]+T[4, 39, 43, 7]-$$

$$T[4, 39, 46, 8]+\frac{1}{2}T[4, 39, 125, 5]-\frac{1}{2}T[4, 39, 125, 6]+\frac{1}{2}T[4, 39, 135, 5]+\frac{1}{2}T[4, 39, 135, 6]-\frac{1}{2}T[5, 1, 4, 1]-T[5, 1, 10, 2]+\frac{1}{2}T[5, 1, 10, 3]+\frac{1}{2}T[5, 1, 10, 4]-\frac{1}{2}T[5, 1, 12, 3]+\frac{1}{2}T[5, 1, 12, 4]-\frac{1}{2}T[5, 1, 36, 5]-\frac{1}{2}T[5, 1, 36, 6]+\frac{1}{2}T[5, 1, 38, 5]-\frac{1}{2}T[5, 1, 38, 6]+T[5, 1, 38, 7]+$$

$$\frac{1}{2}T[5, 2, 2, 1]+\frac{1}{2}T[5, 2, 5, 1]-T[5, 2, 11, 2]+\frac{1}{2}T[5, 2, 11, 3]+\frac{1}{2}T[5, 2, 11, 4]-\frac{1}{2}T[5, 2, 12, 3]+\frac{1}{2}T[5, 2, 12, 4]-\frac{1}{2}T[5, 2, 13, 3]+\frac{1}{2}T[5, 2, 13, 4]-\frac{1}{2}T[5, 2, 14, 3]-\frac{1}{2}T[5, 2, 14, 4]-\frac{1}{2}T[5, 2, 28, 5]+\frac{1}{2}T[5, 2, 28, 6]-\frac{1}{2}T[5, 2, 36, 5]-\frac{1}{2}T[5, 2, 36, 6]-\frac{1}{2}T[5, 2, 37, 5]-$$

$$\frac{1}{2}T[5, 2, 37, 6]+\frac{1}{2}T[5, 2, 39, 5]-\frac{1}{2}T[5, 2, 39, 6]+T[5, 2, 39, 7]-T[5, 5, 2, 1]+2T[5, 5, 11, 2]-T[5, 5, 11, 3]-T[5, 5, 11, 4]+T[5, 5, 13, 3]-T[5, 5, 13, 4]+T[5, 5, 28, 5]-T[5, 5, 28, 6]+T[5, 5, 36, 5]+T[5, 5, 36, 6]-T[5, 6, 2, 1]+2T[5, 6, 11, 2]-T[5, 6, 11, 3]-T[5, 6, 11, 4]+$$

$$T[5, 6, 13, 3]-T[5, 6, 13, 4]+T[5, 6, 28, 5]-T[5, 6, 28, 6]+T[5, 6, 36, 5]+T[5, 6, 36, 6]-T[5, 7, 2, 1]+2T[5, 7, 11, 2]-T[5, 7, 11, 3]-T[5, 7, 11, 4]+T[5, 7, 13, 3]-T[5, 7, 13, 4]+T[5, 7, 28, 5]-T[5, 7, 28, 6]+T[5, 7, 36, 5]+T[5, 7, 36, 6]-\frac{1}{2}T[5, 8, 2, 1]+T[5, 8, 11, 2]-$$

$$\frac{1}{2}T[5, 8, 11, 3]-\frac{1}{2}T[5, 8, 11, 4]+\frac{1}{2}T[5, 8, 13, 3]-\frac{1}{2}T[5, 8, 13, 4]+\frac{1}{2}T[5, 8, 28, 5]-\frac{1}{2}T[5, 8, 28, 6]+\frac{1}{2}T[5, 8, 36, 5]+\frac{1}{2}T[5, 8, 36, 6]-\frac{1}{2}T[5, 9, 3, 1]-\frac{1}{2}T[5, 9, 13, 3]+\frac{1}{2}T[5, 9, 13, 4]-\frac{1}{2}T[5, 9, 15, 3]-\frac{1}{2}T[5, 9, 15, 4]-\frac{1}{2}T[5, 9, 29, 5]+\frac{1}{2}T[5, 9, 29, 6]-\frac{1}{2}T[5, 9, 37, 5]-$$

$$\frac{1}{2}T[5, 9, 37, 6]+T[5, 11, 5, 2]-\frac{1}{2}T[5, 11, 5, 3]-\frac{1}{2}T[5, 11, 5, 4]+2T[5, 11, 7, 2]-T[5, 11, 7, 3]-T[5, 11, 7, 4]+T[5, 11, 40, 2]-\frac{1}{2}T[5, 11, 40, 3]-\frac{1}{2}T[5, 11, 40, 4]-T[5, 11, 41, 2]+\frac{1}{2}T[5, 11, 41, 3]+\frac{1}{2}T[5, 11, 41, 4]-T[5, 11, 60, 8]+\frac{1}{2}T[5, 11, 76, 3]-\frac{1}{2}T[5, 11, 76, 4]+$$

$$\frac{1}{2}T[5, 11, 77, 3]+\frac{1}{2}T[5, 11, 77, 4]-\frac{1}{2}T[5, 13, 2, 3]+\frac{1}{2}T[5, 13, 2, 4]+\frac{1}{2}T[5, 13, 5, 3]-\frac{1}{2}T[5, 13, 5, 4]+T[5, 13, 7, 3]-T[5, 13, 7, 4]-T[5, 13, 9, 3]+T[5, 13, 9, 4]-\frac{1}{2}T[5, 13, 41, 3]+\frac{1}{2}T[5, 13, 41, 4]+\frac{1}{2}T[5, 13, 45, 3]-\frac{1}{2}T[5, 13, 45, 4]-T[5, 13, 59, 8]+T[5, 13, 76, 2]-$$

$$\frac{1}{2}T[5, 13, 76, 3]-\frac{1}{2}T[5, 13, 76, 4]-\frac{1}{2}T[5, 13, 84, 3]-\frac{1}{2}T[5, 13, 84, 4]+T[5, 17, 16, 1]-\frac{1}{2}T[5, 17, 17, 1]-T[5, 17, 58, 8]-T[5, 17, 78, 2]+\frac{1}{2}T[5, 17, 78, 3]+\frac{1}{2}T[5, 17, 78, 4]+\frac{1}{2}T[5, 17, 85, 3]-\frac{1}{2}T[5, 17, 85, 4]-\frac{1}{2}T[5, 17, 100, 5]+\frac{1}{2}T[5, 17, 100, 6]+\frac{1}{2}T[5, 17, 102, 5]+$$

$$\frac{1}{2}T[5, 17, 102, 6]-T[5, 24, 24, 1]+\frac{1}{2}T[5, 24, 25, 1]+T[5, 24, 53, 8]+T[5, 24, 81, 2]-\frac{1}{2}T[5, 24, 81, 3]-\frac{1}{2}T[5, 24, 81, 4]+\frac{1}{2}T[5, 24, 93, 3]+\frac{1}{2}T[5, 24, 93, 4]+\frac{1}{2}T[5, 24, 118, 5]-\frac{1}{2}T[5, 24, 118, 6]-\frac{1}{2}T[5, 24, 120, 5]+\frac{1}{2}T[5, 24, 120, 6]-T[5, 24, 120, 7]+T[5, 26, 26, 1]-$$

$$\frac{1}{2}T[5, 26, 27, 1]+T[5, 26, 50, 8]+\frac{1}{2}T[5, 26, 88, 3]-\frac{1}{2}T[5, 26, 88, 4]-\frac{1}{2}T[5, 26, 94, 3]-\frac{1}{2}T[5, 26, 94, 4]-\frac{1}{2}T[5, 26, 121, 5]+\frac{1}{2}T[5, 26, 121, 6]+\frac{1}{2}T[5, 26, 123, 5]-\frac{1}{2}T[5, 26, 123, 6]+T[5, 26, 123, 7]+\frac{1}{2}T[5, 28, 5, 5]-\frac{1}{2}T[5, 28, 5, 6]+T[5, 28, 6, 5]-T[5, 28, 6, 6]+\frac{1}{2}T[5, 28, 42, 5]-$$

$$\frac{1}{2}T[5, 28, 42, 6]-\frac{1}{2}T[5, 28, 44, 5]+\frac{1}{2}T[5, 28, 44, 6]+T[5, 28, 48, 8]+\frac{1}{2}T[5, 28, 124, 5]+\frac{1}{2}T[5, 28, 124, 6]-\frac{1}{2}T[5, 28, 125, 5]+\frac{1}{2}T[5, 28, 125, 6]-T[5, 28, 125, 7]+T[5, 32, 32, 1]-\frac{1}{2}T[5, 32, 33, 1]+T[5, 32, 52, 8]-T[5, 32, 83, 2]+\frac{1}{2}T[5, 32, 83, 3]+\frac{1}{2}T[5, 32, 83, 4]-$$

# 4.7 Result of bootstrap

$$\frac{1}{2} T[5, 32, 95, 3] - \frac{1}{2} T[5, 32, 95, 4] + \frac{1}{2} T[5, 32, 131, 5] + \frac{1}{2} T[5, 32, 131, 6] + \frac{1}{2} T[5, 32, 132, 5] - \frac{1}{2} T[5, 32, 132, 6] + T[5, 32, 132, 7] - T[5, 34, 34, 1] + \frac{1}{2} T[5, 34, 35, 1] + T[5, 34, 49, 8] - \frac{1}{2} T[5, 34, 90, 3] + \frac{1}{2} T[5, 34, 90, 4] + \frac{1}{2} T[5, 34, 96, 3] + \frac{1}{2} T[5, 34, 96, 4] - \frac{1}{2} T[5, 34, 133, 5] -$$

$$\frac{1}{2} T[5, 34, 133, 6] - \frac{1}{2} T[5, 34, 134, 5] + \frac{1}{2} T[5, 34, 134, 6] - T[5, 34, 134, 7] - T[5, 36, 1, 5] - T[5, 36, 1, 6] - \frac{1}{2} T[5, 36, 2, 5] - \frac{1}{2} T[5, 36, 2, 6] + \frac{1}{2} T[5, 36, 5, 5] + \frac{1}{2} T[5, 36, 5, 6] + T[5, 36, 6, 5] + T[5, 36, 6, 6] + \frac{1}{2} T[5, 36, 43, 5] + \frac{1}{2} T[5, 36, 43, 6] - \frac{1}{2} T[5, 36, 44, 5] - \frac{1}{2} T[5, 36, 44, 6] +$$

$$T[5, 36, 47, 8] + \frac{1}{2} T[5, 36, 124, 5] - \frac{1}{2} T[5, 36, 124, 6] + \frac{1}{2} T[5, 36, 135, 5] - \frac{1}{2} T[5, 36, 135, 6] + T[5, 36, 135, 7] - \frac{1}{2} T[6, 2, 3, 1] - \frac{1}{2} T[6, 2, 13, 3] + \frac{1}{2} T[6, 2, 13, 4] - \frac{1}{2} T[6, 2, 15, 3] - \frac{1}{2} T[6, 2, 15, 4] - \frac{1}{2} T[6, 2, 29, 5] + \frac{1}{2} T[6, 2, 29, 6] - \frac{1}{2} T[6, 2, 37, 5] - \frac{1}{2} T[6, 2, 37, 6] - \frac{1}{2} T[6, 3, 2, 1] +$$

$$T[6, 3, 11, 2] - \frac{1}{2} T[6, 3, 11, 3] - \frac{1}{2} T[6, 3, 11, 4] + \frac{1}{2} T[6, 3, 13, 3] - \frac{1}{2} T[6, 3, 13, 4] + \frac{1}{2} T[6, 3, 28, 5] - \frac{1}{2} T[6, 3, 28, 6] + \frac{1}{2} T[6, 3, 36, 5] + \frac{1}{2} T[6, 3, 36, 6] - T[6, 5, 2, 1] + 2 T[6, 5, 11, 2] - T[6, 5, 11, 3] - T[6, 5, 11, 4] + T[6, 5, 13, 3] - T[6, 5, 13, 4] + T[6, 5, 28, 5] - T[6, 5, 28, 6] +$$

$$T[6, 5, 36, 5] + T[6, 5, 36, 6] - T[6, 7, 2, 1] + 2 T[6, 7, 11, 2] - T[6, 7, 11, 3] - T[6, 7, 11, 4] + T[6, 7, 13, 3] - T[6, 7, 13, 4] + T[6, 7, 28, 5] - T[6, 7, 28, 6] + T[6, 7, 36, 5] + T[6, 7, 36, 6] - T[6, 9, 3, 1] - T[6, 9, 13, 3] + T[6, 9, 13, 4] - T[6, 9, 15, 3] - T[6, 9, 15, 4] - T[6, 9, 29, 5] +$$

$$T[6, 9, 29, 6] - T[6, 9, 37, 5] - T[6, 9, 37, 6] + T[6, 11, 5, 2] - \frac{1}{2} T[6, 11, 5, 3] - \frac{1}{2} T[6, 11, 5, 4] + 2 T[6, 11, 7, 2] - T[6, 11, 7, 3] - T[6, 11, 7, 4] + T[6, 11, 40, 2] - \frac{1}{2} T[6, 11, 40, 3] - \frac{1}{2} T[6, 11, 40, 4] - T[6, 11, 41, 2] + \frac{1}{2} T[6, 11, 41, 3] + \frac{1}{2} T[6, 11, 41, 4] - T[6, 11, 60, 8] + \frac{1}{2} T[6, 11, 76, 3] -$$

$$\frac{1}{2} T[6, 11, 76, 4] + \frac{1}{2} T[6, 11, 77, 3] + \frac{1}{2} T[6, 11, 77, 4] - \frac{1}{2} T[6, 13, 2, 3] + \frac{1}{2} T[6, 13, 2, 4] + \frac{1}{2} T[6, 13, 5, 3] - \frac{1}{2} T[6, 13, 5, 4] + T[6, 13, 7, 3] - T[6, 13, 7, 4] - T[6, 13, 9, 3] + T[6, 13, 9, 4] - \frac{1}{2} T[6, 13, 41, 3] + \frac{1}{2} T[6, 13, 41, 4] + \frac{1}{2} T[6, 13, 45, 3] - \frac{1}{2} T[6, 13, 45, 4] - T[6, 13, 59, 8] +$$

$$T[6, 13, 76, 2] - \frac{1}{2} T[6, 13, 76, 3] - \frac{1}{2} T[6, 13, 76, 4] - \frac{1}{2} T[6, 13, 84, 3] - \frac{1}{2} T[6, 13, 84, 4] - \frac{1}{2} T[6, 15, 2, 3] - \frac{1}{2} T[6, 15, 2, 4] - T[6, 15, 9, 3] - T[6, 15, 9, 4] - \frac{1}{2} T[6, 15, 40, 3] - \frac{1}{2} T[6, 15, 40, 4] + \frac{1}{2} T[6, 15, 45, 3] + \frac{1}{2} T[6, 15, 45, 4] - T[6, 15, 55, 8] - T[6, 15, 77, 2] + \frac{1}{2} T[6, 15, 77, 3] +$$

$$\frac{1}{2} T[6, 15, 77, 4] - \frac{1}{2} T[6, 15, 84, 3] + \frac{1}{2} T[6, 15, 84, 4] + T[6, 17, 16, 1] - \frac{1}{2} T[6, 17, 17, 1] - T[6, 17, 58, 8] - T[6, 17, 78, 2] + \frac{1}{2} T[6, 17, 78, 3] + \frac{1}{2} T[6, 17, 78, 4] + \frac{1}{2} T[6, 17, 85, 3] - \frac{1}{2} T[6, 17, 85, 4] - \frac{1}{2} T[6, 17, 100, 5] + \frac{1}{2} T[6, 17, 100, 6] + \frac{1}{2} T[6, 17, 102, 5] + \frac{1}{2} T[6, 17, 102, 6] -$$

$$T[6, 19, 18, 1] + \frac{1}{2} T[6, 19, 19, 1] - T[6, 19, 54, 8] + T[6, 19, 79, 2] - \frac{1}{2} T[6, 19, 79, 3] - \frac{1}{2} T[6, 19, 79, 4] + \frac{1}{2} T[6, 19, 91, 3] + \frac{1}{2} T[6, 19, 91, 4] + \frac{1}{2} T[6, 19, 105, 5] - \frac{1}{2} T[6, 19, 105, 6] - \frac{1}{2} T[6, 19, 107, 5] - \frac{1}{2} T[6, 19, 107, 6] + T[6, 21, 20, 1] - \frac{1}{2} T[6, 21, 21, 1] - T[6, 21, 51, 8] +$$

$$\frac{1}{2} T[6, 21, 86, 3] - \frac{1}{2} T[6, 21, 86, 4] - \frac{1}{2} T[6, 21, 92, 3] - \frac{1}{2} T[6, 21, 92, 4] - \frac{1}{2} T[6, 21, 109, 5] + \frac{1}{2} T[6, 21, 109, 6] + \frac{1}{2} T[6, 21, 111, 5] + \frac{1}{2} T[6, 21, 111, 6] - T[7, 1, 4, 1] - 2 T[7, 1, 10, 2] + T[7, 1, 10, 3] + T[7, 1, 10, 4] - T[7, 1, 12, 3] + T[7, 1, 12, 4] - T[7, 1, 36, 5] - T[7, 1, 36, 6] +$$

$$T[7, 1, 38, 5] - T[7, 1, 38, 6] + 2 T[7, 1, 38, 7] - \frac{1}{2} T[7, 2, 4, 1] - T[7, 2, 10, 2] + \frac{1}{2} T[7, 2, 10, 3] + \frac{1}{2} T[7, 2, 10, 4] - \frac{1}{2} T[7, 2, 12, 3] + \frac{1}{2} T[7, 2, 12, 4] - \frac{1}{2} T[7, 2, 36, 5] - \frac{1}{2} T[7, 2, 36, 6] + \frac{1}{2} T[7, 2, 38, 5] - \frac{1}{2} T[7, 2, 38, 6] + T[7, 2, 38, 7] - \frac{1}{2} T[7, 4, 2, 1] + T[7, 4, 11, 2] - \frac{1}{2} T[7, 4, 11, 3] -$$

$$\frac{1}{2} T[7, 4, 11, 4] + \frac{1}{2} T[7, 4, 13, 3] - \frac{1}{2} T[7, 4, 13, 4] + \frac{1}{2} T[7, 4, 28, 5] - \frac{1}{2} T[7, 4, 28, 6] + \frac{1}{2} T[7, 4, 36, 5] + \frac{1}{2} T[7, 4, 36, 6] - T[7, 5, 2, 1] + 2 T[7, 5, 11, 2] - T[7, 5, 11, 3] - T[7, 5, 11, 4] + T[7, 5, 13, 3] - T[7, 5, 13, 4] + T[7, 5, 28, 5] - T[7, 5, 28, 6] + T[7, 5, 36, 5] + T[7, 5, 36, 6] -$$

$$T[7, 6, 2, 1] + 2 T[7, 6, 11, 2] - T[7, 6, 11, 3] - T[7, 6, 11, 4] + T[7, 6, 13, 3] - T[7, 6, 13, 4] + T[7, 6, 28, 5] - T[7, 6, 28, 6] + T[7, 6, 36, 5] + T[7, 6, 36, 6] + T[7, 17, 16, 1] - \frac{1}{2} T[7, 17, 17, 1] - T[7, 17, 58, 8] - T[7, 17, 78, 2] + \frac{1}{2} T[7, 17, 78, 3] + \frac{1}{2} T[7, 17, 78, 4] + \frac{1}{2} T[7, 17, 85, 3] -$$

$$\frac{1}{2} T[7, 17, 85, 4] - \frac{1}{2} T[7, 17, 100, 5] + \frac{1}{2} T[7, 17, 100, 6] + \frac{1}{2} T[7, 17, 102, 5] + \frac{1}{2} T[7, 17, 102, 6] - T[7, 23, 22, 1] + \frac{1}{2} T[7, 23, 23, 1] - T[7, 23, 57, 8] + T[7, 23, 80, 2] - \frac{1}{2} T[7, 23, 80, 3] - \frac{1}{2} T[7, 23, 80, 4] - \frac{1}{2} T[7, 23, 87, 3] + \frac{1}{2} T[7, 23, 87, 4] + \frac{1}{2} T[7, 23, 114, 5] - \frac{1}{2} T[7, 23, 114, 6] -$$

$$\frac{1}{2} T[7, 23, 116, 5] + \frac{1}{2} T[7, 23, 116, 6] - T[7, 23, 116, 7] + \frac{1}{2} T[7, 28, 5, 5] - \frac{1}{2} T[7, 28, 5, 6] + T[7, 28, 6, 5] - T[7, 28, 6, 6] + \frac{1}{2} T[7, 28, 42, 5] - \frac{1}{2} T[7, 28, 42, 6] - \frac{1}{2} T[7, 28, 44, 5] + \frac{1}{2} T[7, 28, 44, 6] + T[7, 28, 48, 8] + \frac{1}{2} T[7, 28, 124, 5] + \frac{1}{2} T[7, 28, 124, 6] - \frac{1}{2} T[7, 28, 125, 5] + \frac{1}{2} T[7, 28, 125, 6] -$$

$$T[7, 28, 125, 7] + T[7, 31, 30, 1] - \frac{1}{2} T[7, 31, 31, 1] - T[7, 31, 56, 8] - T[7, 31, 82, 2] + \frac{1}{2} T[7, 31, 82, 3] + \frac{1}{2} T[7, 31, 82, 4] + \frac{1}{2} T[7, 31, 89, 3] - \frac{1}{2} T[7, 31, 89, 4] + \frac{1}{2} T[7, 31, 128, 5] + \frac{1}{2} T[7, 31, 128, 6] + \frac{1}{2} T[7, 31, 129, 5] - \frac{1}{2} T[7, 31, 129, 6] + T[7, 31, 129, 7] - T[7, 36, 1, 5] - T[7, 36, 1, 6] -$$

$$\frac{1}{2} T[7, 36, 2, 5] - \frac{1}{2} T[7, 36, 2, 6] + \frac{1}{2} T[7, 36, 5, 5] + \frac{1}{2} T[7, 36, 5, 6] + T[7, 36, 6, 5] + T[7, 36, 6, 6] + \frac{1}{2} T[7, 36, 43, 5] + \frac{1}{2} T[7, 36, 43, 6] - \frac{1}{2} T[7, 36, 44, 5] - \frac{1}{2} T[7, 36, 44, 6] + T[7, 36, 47, 8] + \frac{1}{2} T[7, 36, 124, 5] - \frac{1}{2} T[7, 36, 124, 6] + \frac{1}{2} T[7, 36, 135, 5] - \frac{1}{2} T[7, 36, 135, 6] + T[7, 36, 135, 7] +$$

$$T[7, 38, 1, 5] - T[7, 38, 1, 6] + 2 T[7, 38, 1, 7] + \frac{1}{2} T[7, 38, 2, 5] - \frac{1}{2} T[7, 38, 2, 6] + T[7, 38, 2, 7] + \frac{1}{2} T[7, 38, 42, 5] - \frac{1}{2} T[7, 38, 42, 6] + T[7, 38, 42, 7] - \frac{1}{2} T[7, 38, 43, 5] + \frac{1}{2} T[7, 38, 43, 6] - T[7, 38, 43, 7] + T[7, 38, 46, 8] - \frac{1}{2} T[7, 38, 125, 5] + \frac{1}{2} T[7, 38, 125, 6] - \frac{1}{2} T[7, 38, 135, 5] -$$

$$\frac{1}{2} T[7, 38, 135, 6] - \frac{1}{2} T[8, 2, 5, 1] + \frac{1}{2} T[8, 2, 12, 3] - \frac{1}{2} T[8, 2, 12, 4] + \frac{1}{2} T[8, 2, 14, 3] + \frac{1}{2} T[8, 2, 14, 4] + \frac{1}{2} T[8, 2, 37, 5] + \frac{1}{2} T[8, 2, 37, 6] - \frac{1}{2} T[8, 2, 39, 5] + \frac{1}{2} T[8, 2, 39, 6] - T[8, 2, 39, 7] + \frac{1}{2} T[8, 3, 4, 1] + T[8, 3, 10, 2] - \frac{1}{2} T[8, 3, 10, 3] - \frac{1}{2} T[8, 3, 10, 4] + \frac{1}{2} T[8, 3, 12, 3] -$$

$$\frac{1}{2} T[8, 3, 12, 4] + \frac{1}{2} T[8, 3, 36, 5] + \frac{1}{2} T[8, 3, 36, 6] - \frac{1}{2} T[8, 3, 38, 5] + \frac{1}{2} T[8, 3, 38, 6] - T[8, 3, 38, 7] + \frac{1}{2} T[8, 4, 3, 1] + \frac{1}{2} T[8, 4, 13, 3] - \frac{1}{2} T[8, 4, 13, 4] + \frac{1}{2} T[8, 4, 15, 3] + \frac{1}{2} T[8, 4, 15, 4] + \frac{1}{2} T[8, 4, 29, 5] - \frac{1}{2} T[8, 4, 29, 6] + \frac{1}{2} T[8, 4, 37, 5] + \frac{1}{2} T[8, 4, 37, 6] - \frac{1}{2} T[8, 5, 2, 1] +$$

$$T[8, 5, 11, 2] - \frac{1}{2} T[8, 5, 11, 3] - \frac{1}{2} T[8, 5, 11, 4] + \frac{1}{2} T[8, 5, 13, 3] - \frac{1}{2} T[8, 5, 13, 4] + \frac{1}{2} T[8, 5, 28, 5] - \frac{1}{2} T[8, 5, 28, 6] + \frac{1}{2} T[8, 5, 36, 5] + \frac{1}{2} T[8, 5, 36, 6] + T[8, 17, 16, 1] - \frac{1}{2} T[8, 17, 17, 1] - T[8, 17, 58, 8] - T[8, 17, 78, 2] + \frac{1}{2} T[8, 17, 78, 3] + \frac{1}{2} T[8, 17, 78, 4] + \frac{1}{2} T[8, 17, 85, 3] -$$

$$\frac{1}{2} T[8, 17, 85, 4] - \frac{1}{2} T[8, 17, 100, 5] + \frac{1}{2} T[8, 17, 100, 6] + \frac{1}{2} T[8, 17, 102, 5] + \frac{1}{2} T[8, 17, 102, 6] - T[8, 19, 18, 1] + \frac{1}{2} T[8, 19, 19, 1] - T[8, 19, 54, 8] + T[8, 19, 79, 2] - \frac{1}{2} T[8, 19, 79, 3] - \frac{1}{2} T[8, 19, 79, 4] + \frac{1}{2} T[8, 19, 91, 3] + \frac{1}{2} T[8, 19, 91, 4] + \frac{1}{2} T[8, 19, 105, 5] - \frac{1}{2} T[8, 19, 105, 6] -$$

$$\frac{1}{2} T[8, 19, 107, 5] - \frac{1}{2} T[8, 19, 107, 6] + T[8, 21, 20, 1] - \frac{1}{2} T[8, 21, 21, 1] - T[8, 21, 51, 8] + \frac{1}{2} T[8, 21, 86, 3] - \frac{1}{2} T[8, 21, 86, 4] - \frac{1}{2} T[8, 21, 92, 3] - \frac{1}{2} T[8, 21, 92, 4] - \frac{1}{2} T[8, 21, 109, 5] + \frac{1}{2} T[8, 21, 109, 6] + \frac{1}{2} T[8, 21, 111, 5] + \frac{1}{2} T[8, 21, 111, 6] - T[8, 23, 22, 1] + \frac{1}{2} T[8, 23, 23, 1] -$$

# 4.7 Result of bootstrap

$$T[8,23,57,8]+T[8,23,80,2]-\frac{1}{2}T[8,23,80,3]-\frac{1}{2}T[8,23,80,4]-\frac{1}{2}T[8,23,87,3]+\frac{1}{2}T[8,23,87,4]+\frac{1}{2}T[8,23,114,5]-\frac{1}{2}T[8,23,114,6]-\frac{1}{2}T[8,23,116,5]+\frac{1}{2}T[8,23,116,6]-T[8,23,116,7]+T[8,25,24,1]-\frac{1}{2}T[8,25,25,1]-T[8,25,53,8]-T[8,25,81,2]+\frac{1}{2}T[8,25,81,3]+$$

$$\frac{1}{2}T[8,25,81,4]-\frac{1}{2}T[8,25,93,3]-\frac{1}{2}T[8,25,93,4]-\frac{1}{2}T[8,25,118,5]+\frac{1}{2}T[8,25,118,6]+\frac{1}{2}T[8,25,120,5]-\frac{1}{2}T[8,25,120,6]+T[8,25,120,7]-T[8,27,26,1]+\frac{1}{2}T[8,27,27,1]-T[8,27,50,8]-\frac{1}{2}T[8,27,88,3]+\frac{1}{2}T[8,27,88,4]+\frac{1}{2}T[8,27,94,3]+\frac{1}{2}T[8,27,94,4]+$$

$$\frac{1}{2}T[8,27,121,5]-\frac{1}{2}T[8,27,121,6]-\frac{1}{2}T[8,27,123,5]+\frac{1}{2}T[8,27,123,6]-T[8,27,123,7]+T[8,31,30,1]-\frac{1}{2}T[8,31,31,1]-T[8,31,56,8]-T[8,31,82,2]+\frac{1}{2}T[8,31,82,3]+\frac{1}{2}T[8,31,82,4]+\frac{1}{2}T[8,31,89,3]-\frac{1}{2}T[8,31,89,4]+\frac{1}{2}T[8,31,128,5]+\frac{1}{2}T[8,31,128,6]+$$

$$\frac{1}{2}T[8,31,129,5]-\frac{1}{2}T[8,31,129,6]+T[8,31,129,7]-T[8,33,32,1]+\frac{1}{2}T[8,33,33,1]-T[8,33,52,8]+T[8,33,83,2]-\frac{1}{2}T[8,33,83,3]-\frac{1}{2}T[8,33,83,4]+\frac{1}{2}T[8,33,95,3]+\frac{1}{2}T[8,33,95,4]-\frac{1}{2}T[8,33,131,5]-\frac{1}{2}T[8,33,131,6]-\frac{1}{2}T[8,33,132,5]+\frac{1}{2}T[8,33,132,6]-$$

$$T[8,33,132,7]+T[8,35,34,1]-\frac{1}{2}T[8,35,35,1]-T[8,35,49,8]+\frac{1}{2}T[8,35,90,3]-\frac{1}{2}T[8,35,90,4]-\frac{1}{2}T[8,35,96,3]-\frac{1}{2}T[8,35,96,4]+\frac{1}{2}T[8,35,133,5]+\frac{1}{2}T[8,35,133,6]+\frac{1}{2}T[8,35,134,5]-\frac{1}{2}T[8,35,134,6]+T[8,35,134,7]-T[9,1,5,1]+T[9,1,12,3]-T[9,1,12,4]+$$

$$T[9,1,14,3]+T[9,1,14,4]+T[9,1,37,5]+T[9,1,37,6]-T[9,1,39,5]+T[9,1,39,6]-2T[9,1,39,7]-T[9,2,5,1]+T[9,2,12,3]-T[9,2,12,4]+T[9,2,14,3]+T[9,2,14,4]+T[9,2,37,5]+T[9,2,37,6]-T[9,2,39,5]+T[9,2,39,6]-2T[9,2,39,7]-\frac{1}{2}T[9,3,5,1]+$$

$$\frac{1}{2}T[9,3,12,3]-\frac{1}{2}T[9,3,12,4]+\frac{1}{2}T[9,3,14,3]+\frac{1}{2}T[9,3,14,4]+\frac{1}{2}T[9,3,37,5]+\frac{1}{2}T[9,3,37,6]-\frac{1}{2}T[9,3,39,5]+\frac{1}{2}T[9,3,39,6]-T[9,3,39,7]-\frac{1}{2}T[9,5,3,1]-\frac{1}{2}T[9,5,13,3]+\frac{1}{2}T[9,5,13,4]-\frac{1}{2}T[9,5,15,3]-\frac{1}{2}T[9,5,15,4]-\frac{1}{2}T[9,5,29,5]+\frac{1}{2}T[9,5,29,6]-$$

$$\frac{1}{2}T[9,5,37,5]-\frac{1}{2}T[9,5,37,6]-T[9,6,3,1]-T[9,6,13,3]+T[9,6,13,4]-T[9,6,15,3]-T[9,6,15,4]-T[9,6,29,5]+T[9,6,29,6]-T[9,6,37,5]-T[9,6,37,6]+T[9,21,20,1]-\frac{1}{2}T[9,21,21,1]-T[9,21,51,8]+\frac{1}{2}T[9,21,86,3]-\frac{1}{2}T[9,21,86,4]-\frac{1}{2}T[9,21,92,3]-$$

$$\frac{1}{2}T[9,21,92,4]-\frac{1}{2}T[9,21,109,5]+\frac{1}{2}T[9,21,109,6]+\frac{1}{2}T[9,21,111,5]+\frac{1}{2}T[9,21,111,6]-T[9,27,26,1]+\frac{1}{2}T[9,27,27,1]-T[9,27,50,8]-\frac{1}{2}T[9,27,88,3]+\frac{1}{2}T[9,27,88,4]+\frac{1}{2}T[9,27,94,3]+\frac{1}{2}T[9,27,94,4]+\frac{1}{2}T[9,27,121,5]-\frac{1}{2}T[9,27,121,6]-\frac{1}{2}T[9,27,123,5]+$$

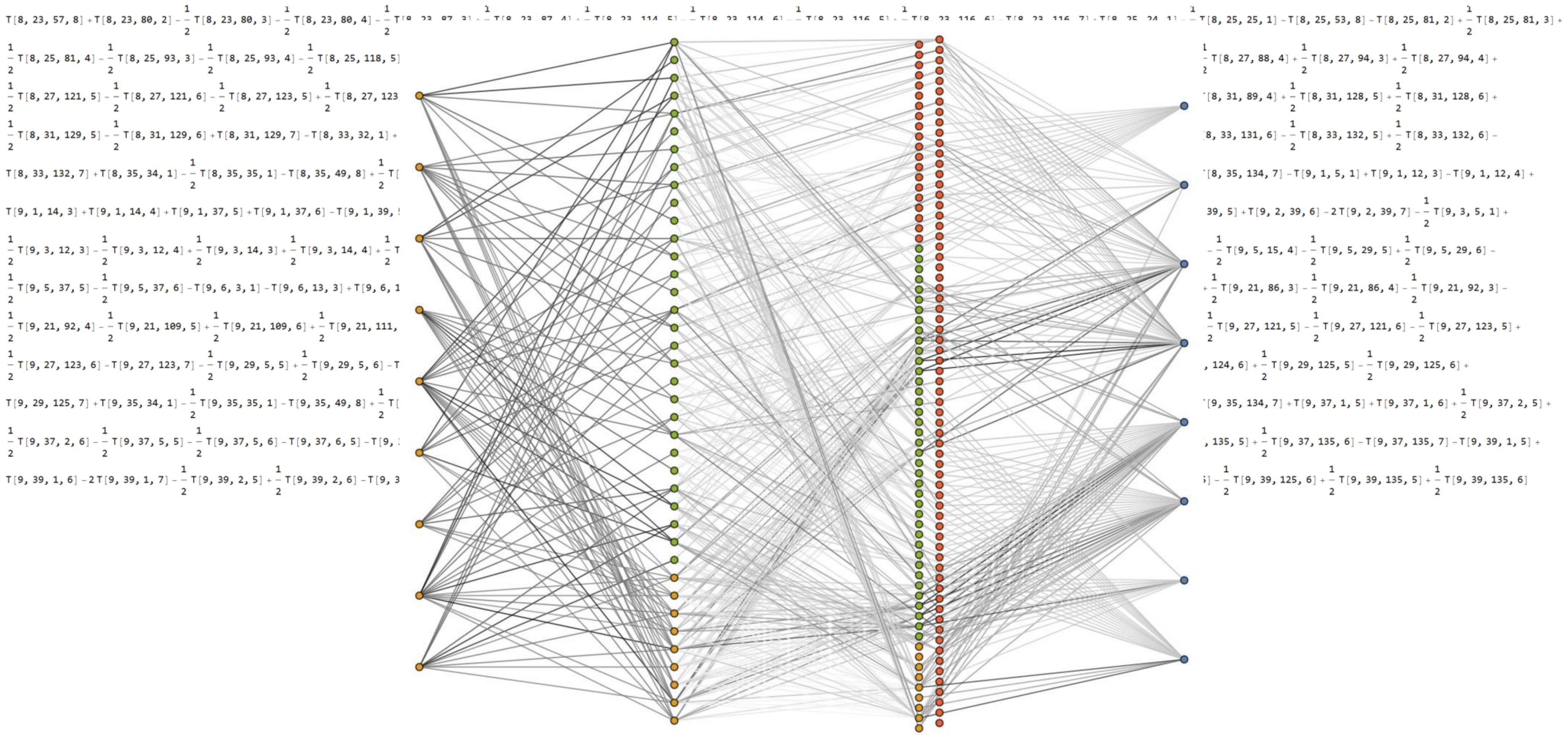
$$\frac{1}{2}T[9,27,123,6]-T[9,27,123,7]-\frac{1}{2}T[9,29,5,5]+\frac{1}{2}T[9,29,5,6]-T[9,29,6,5]+T[9,29,6,6]-\frac{1}{2}T[9,29,42,5]+\frac{1}{2}T[9,29,42,6]+\frac{1}{2}T[9,29,44,5]-\frac{1}{2}T[9,29,44,6]-T[9,29,48,8]-\frac{1}{2}T[9,29,124,5]-\frac{1}{2}T[9,29,124,6]+\frac{1}{2}T[9,29,125,5]-\frac{1}{2}T[9,29,125,6]+$$

$$T[9,29,125,7]+T[9,35,34,1]-\frac{1}{2}T[9,35,35,1]-T[9,35,49,8]+\frac{1}{2}T[9,35,90,3]-\frac{1}{2}T[9,35,90,4]-\frac{1}{2}T[9,35,96,3]-\frac{1}{2}T[9,35,96,4]+\frac{1}{2}T[9,35,133,5]+\frac{1}{2}T[9,35,133,6]+\frac{1}{2}T[9,35,134,5]-\frac{1}{2}T[9,35,134,6]+T[9,35,134,7]+T[9,37,1,5]+T[9,37,1,6]+\frac{1}{2}T[9,37,2,5]+$$

$$\frac{1}{2}T[9,37,2,6]-\frac{1}{2}T[9,37,5,5]-\frac{1}{2}T[9,37,5,6]-T[9,37,6,5]-T[9,37,6,6]-\frac{1}{2}T[9,37,43,5]-\frac{1}{2}T[9,37,43,6]+\frac{1}{2}T[9,37,44,5]+\frac{1}{2}T[9,37,44,6]-T[9,37,47,8]-\frac{1}{2}T[9,37,124,5]+\frac{1}{2}T[9,37,124,6]-\frac{1}{2}T[9,37,135,5]+\frac{1}{2}T[9,37,135,6]-T[9,37,135,7]-T[9,39,1,5]+$$

$$T[9,39,1,6]-2T[9,39,1,7]-\frac{1}{2}T[9,39,2,5]+\frac{1}{2}T[9,39,2,6]-T[9,39,2,7]-\frac{1}{2}T[9,39,42,5]+\frac{1}{2}T[9,39,42,6]-T[9,39,42,7]+\frac{1}{2}T[9,39,43,5]-\frac{1}{2}T[9,39,43,6]+T[9,39,43,7]-T[9,39,46,8]+\frac{1}{2}T[9,39,125,5]-\frac{1}{2}T[9,39,125,6]+\frac{1}{2}T[9,39,135,5]+\frac{1}{2}T[9,39,135,6]$$

# 4.7 Result of bootstrap



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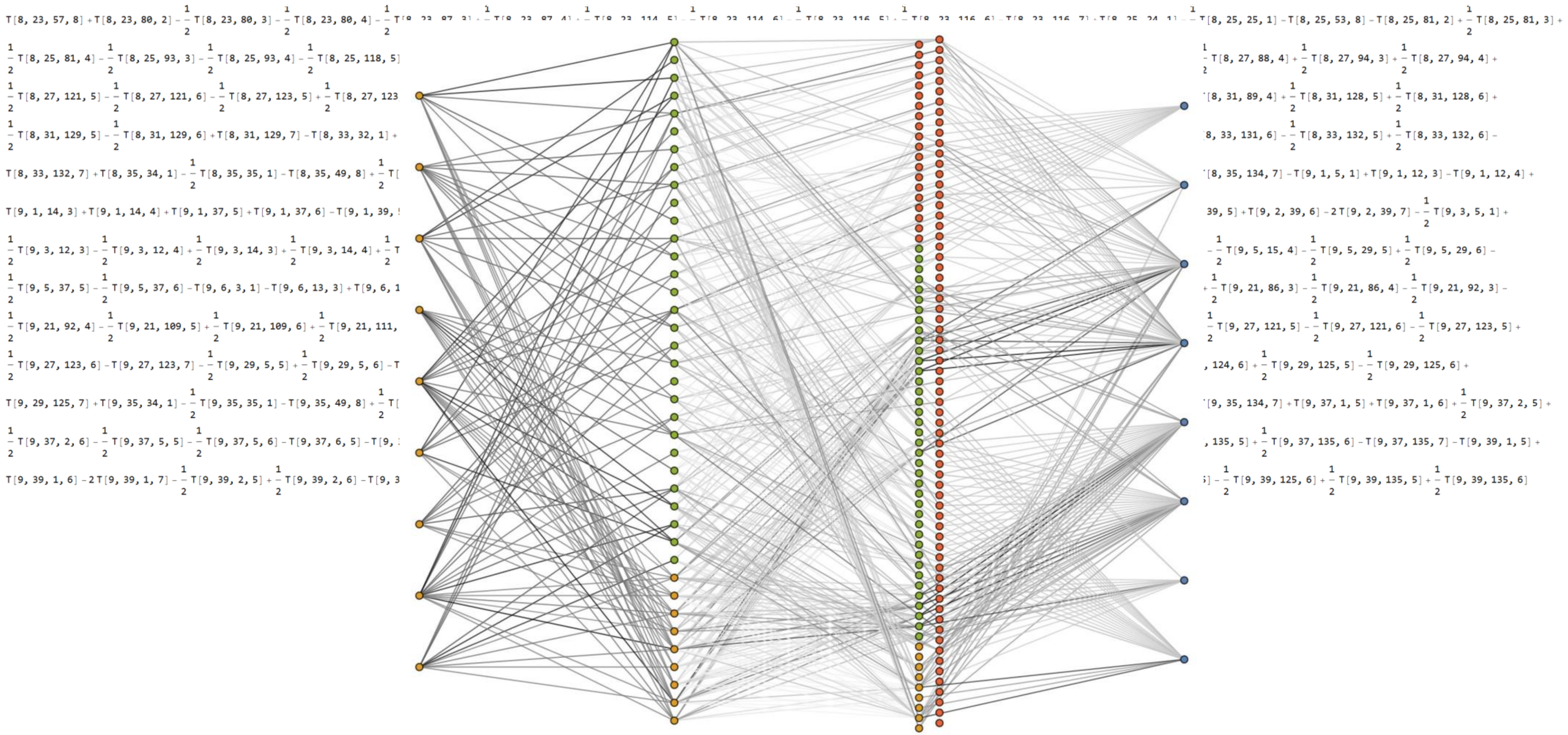
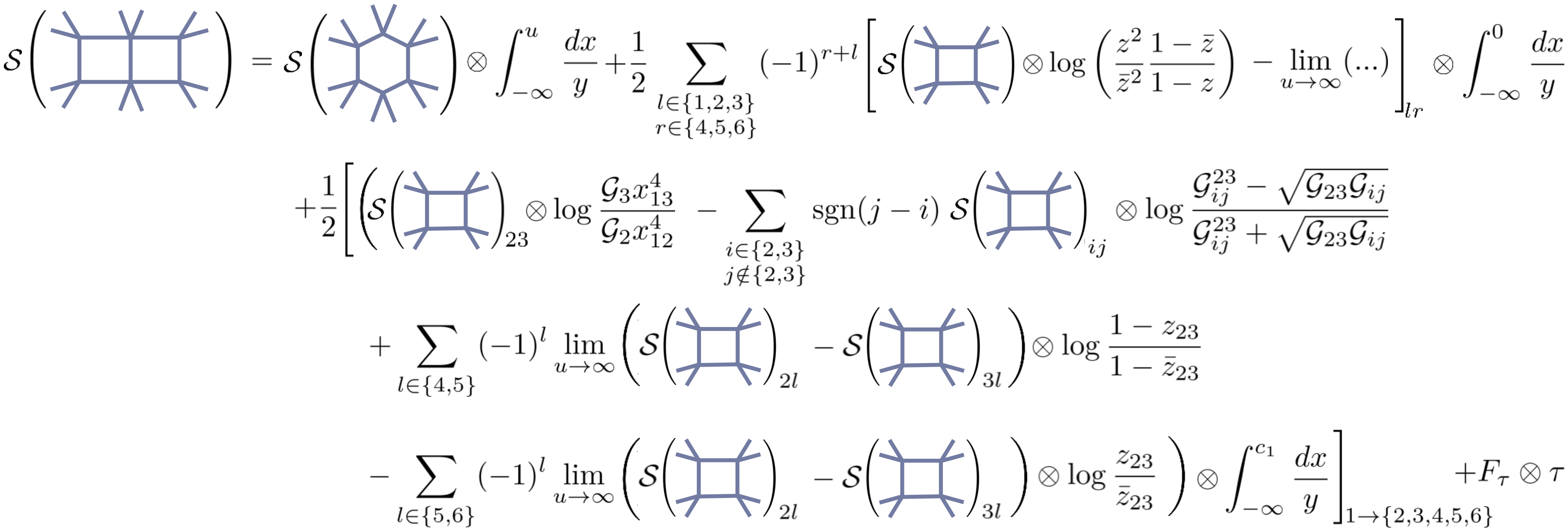



**Can one simplify the resulting expression?**

# 4.8 Simplifying the result



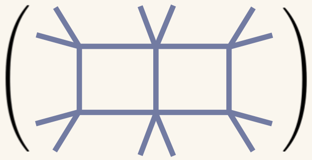
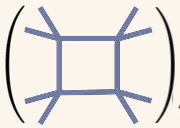
$$\mathcal{S}(\cdot) = \mathcal{S}(\cdot) \otimes \int_{-\infty}^{u} \frac{dx}{y} + \frac{1}{2} \sum_{\substack{l\in\{1,2,3\}\\ r\in\{4,5,6\}}} (-1)^{r+l} \left[ \mathcal{S}(\cdot) \otimes \log\left(\frac{z^2}{\bar{z}^2}\frac{1-\bar{z}}{1-z}\right) - \lim_{u\to\infty}(\ldots)\right]_{lr} \otimes \int_{-\infty}^{0} \frac{dx}{y}$$

$$+\frac{1}{2}\Bigg[ \Big( \mathcal{S}(\cdot)_{23} \otimes \log \frac{\mathcal{G}_3 x_{13}^4}{\mathcal{G}_2 x_{12}^4} - \sum_{\substack{i\in\{2,3\}\\ j\notin\{2,3\}}} \operatorname{sgn}(j-i)\, \mathcal{S}(\cdot)_{ij} \otimes \log \frac{\mathcal{G}_{ij}^{23} - \sqrt{\mathcal{G}_{23}\mathcal{G}_{ij}}}{\mathcal{G}_{ij}^{23} + \sqrt{\mathcal{G}_{23}\mathcal{G}_{ij}}}$$

$$+ \sum_{l\in\{4,5\}} (-1)^l \lim_{u\to\infty} \Big( \mathcal{S}(\cdot)_{2l} - \mathcal{S}(\cdot)_{3l} \Big) \otimes \log \frac{1-z_{23}}{1-\bar{z}_{23}}$$

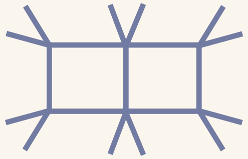
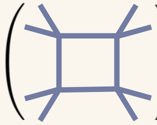
$$- \sum_{l\in\{5,6\}} (-1)^l \lim_{u\to\infty} \Big( \mathcal{S}(\cdot)_{2l} - \mathcal{S}(\cdot)_{3l} \Big) \otimes \log \frac{z_{23}}{\bar{z}_{23}} \Big) \otimes \int_{-\infty}^{c_1} \frac{dx}{y} \Bigg]_{1\to\{2,3,4,5,6\}} + F_\tau \otimes \tau$$

# 4.9 Simplifying further

$\Delta_{2,2}$  $= \sum_{1 \leq i < j \leq 6}$  $_{ij} \otimes \frac{2\pi i}{\omega_1} \int_{-\infty}^{u} \frac{dx}{y(x)} \log \frac{\mathcal{G}^i_j - \sqrt{-\mathcal{G}_{ij}\mathcal{G}}}{\mathcal{G}^i_j + \sqrt{-\mathcal{G}_{ij}\mathcal{G}}}(x) - \lim_{u \to \infty}(\ldots)$

[Morales, **AS**, Wilhelm, Yang, Zhang '22]

# 4.9 Simplifying further

$$\Delta_{2,2}\Bigg($$

$$\Bigg) = \sum_{1\leq i<j\leq 6} \Bigg($$

$$\Bigg)_{ij} \otimes \frac{2\pi i}{\omega_1} \int_{-\infty}^{u} \frac{dx}{y(x)} \log \frac{\mathcal{G}^i_j - \sqrt{-\mathcal{G}_{ij}\mathcal{G}}}{\mathcal{G}^i_j + \sqrt{-\mathcal{G}_{ij}\mathcal{G}}}(x) - \lim_{u\to\infty}(\ldots)$$

[Morales, **AS**, Wilhelm, Yang, Zhang '22]

**Cross checks**

- symmetry structure of integral manifest
- elliptic 10pt result recovered in appropriate limit
- differential equation with 1-loop hexagon, conformal Ward identity and $2^{\text{nd}}$-order differential equations
  [Chicherin, Sokatchev '17] [Drummond, Henn, Smirnov '06] [Drummond, Henn, Trnka '10]

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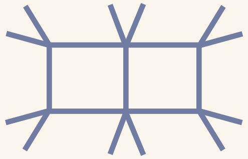
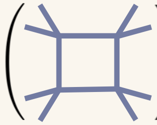
$\Delta_{2,2}\Bigg($  $\Bigg) = \sum_{1\le i<j\le 6} \Bigg($  $\Bigg)_{ij} \otimes \frac{2\pi i}{\omega_1}\int_{-\infty}^{u}\frac{dx}{y(x)}\log\frac{\mathcal{G}^i_j - \sqrt{-\mathcal{G}_{ij}\mathcal{G}}}{\mathcal{G}^i_j + \sqrt{-\mathcal{G}_{ij}\mathcal{G}}}(x) - \lim_{u\to\infty}(\ldots)$

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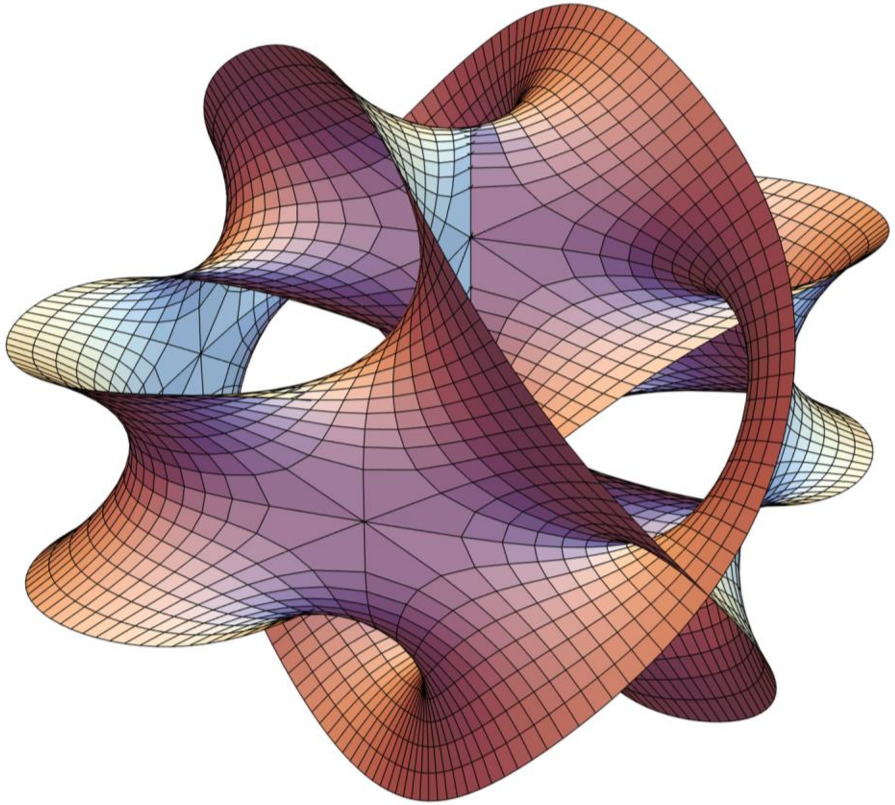
**Comments**

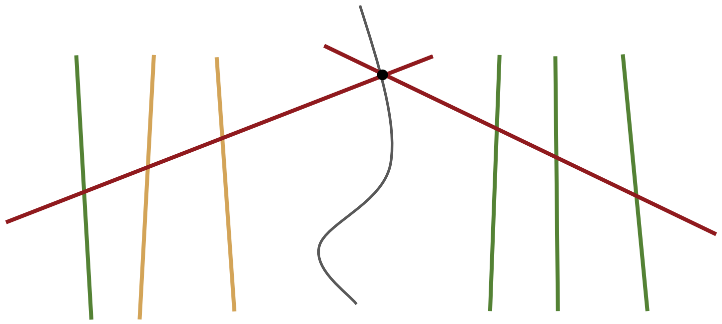
- (extended) Steinmann condition satisfied in logarithmic entries
  [Steinmann '60] [Caron-Huot, Dixon, Dulat, von Hippel, McLeod, Papathanasiou '19]
- missing $\tau$-dependence encoded in missing definition of integration contour

# Summary & Outlook



# 5 Summary & Outlook

Generalisation of the **Schubert problem** to elliptic cases: **134** logarithmic & **8+1** elliptic symbol entries for the double box



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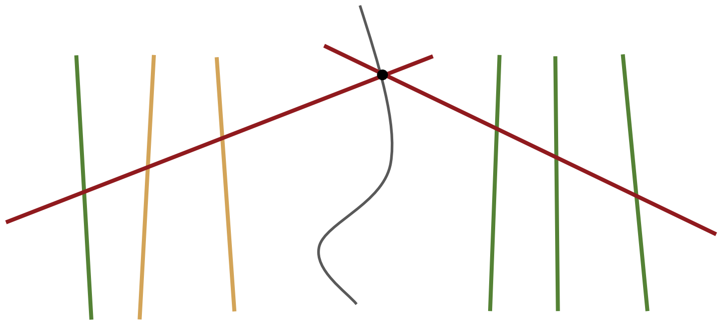
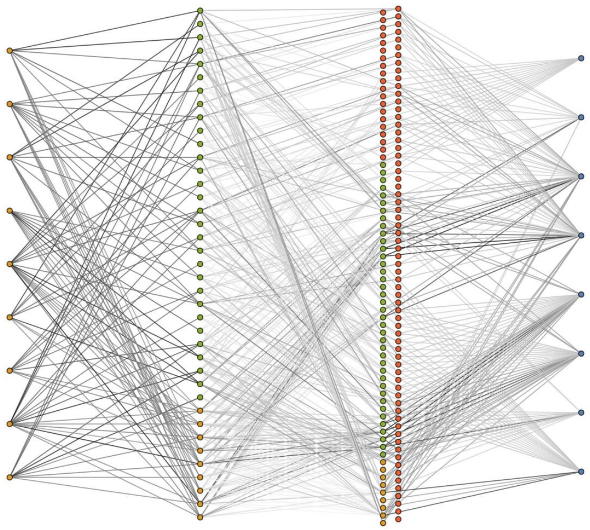


Generalisation of **symbol-bootstrapping techniques** to elliptic cases



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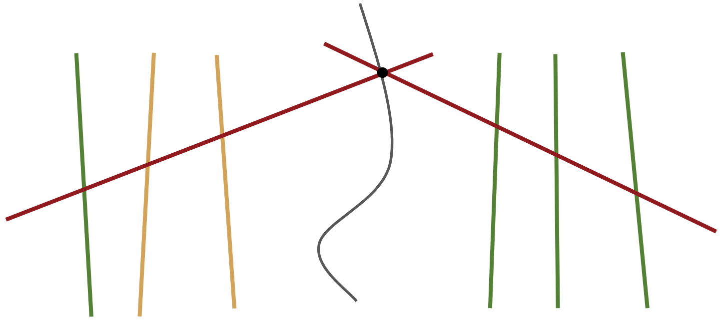

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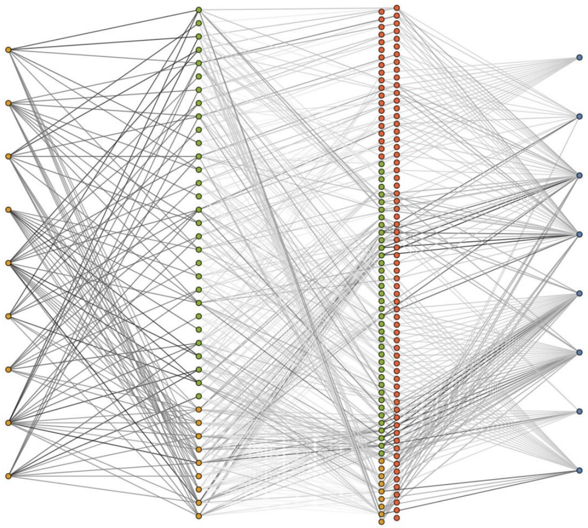
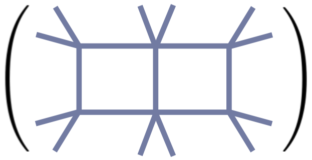
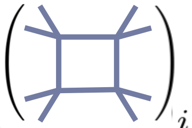


**Result:** 12pt double-box symbol in a compact analytic form

$\Delta_{2,2}$  $= \sum_{1\leq i<j\leq 6}$  $_{ij} \otimes \frac{2\pi i}{\omega_1}\int_{-\infty}^{u}\frac{dx}{y(x)}\log\frac{\mathcal{G}^i_j-\sqrt{-\mathcal{G}_{ij}\mathcal{G}}}{\mathcal{G}^i_j+\sqrt{-\mathcal{G}_{ij}\mathcal{G}}}(x)-\lim_{u\to\infty}(\ldots)$

[Morales, **AS**, Wilhelm, Yang, Zhang '22]

# 5 Summary & Outlook

Uplift of symbol to function, boundary conditions etc

Bootstrap of other elliptic Feynman diagrams, scattering amplitudes, form factors…
Generalisation of amplitudes techniques to elliptic cases, e.g. elliptic symbol-level integration [He, Tang '23]

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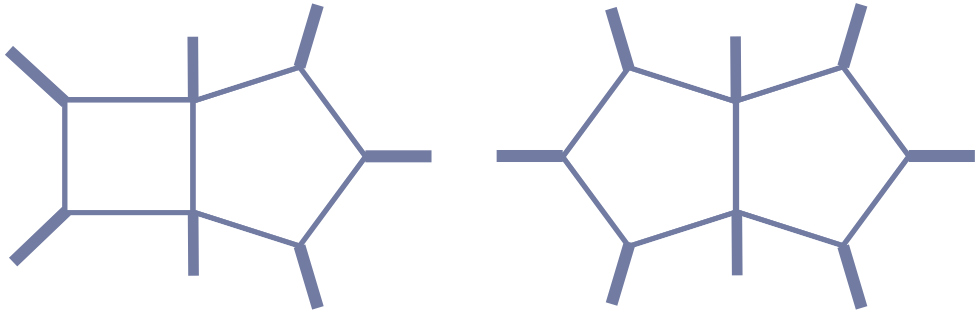

[**AS**, Wilhelm, Zhang arXiv:23xx.xxxxx]

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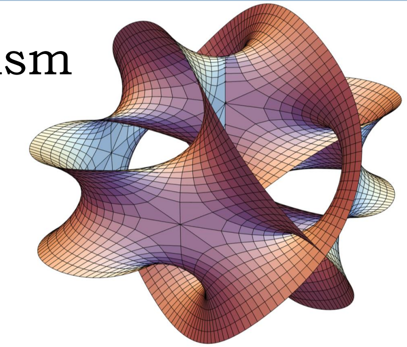

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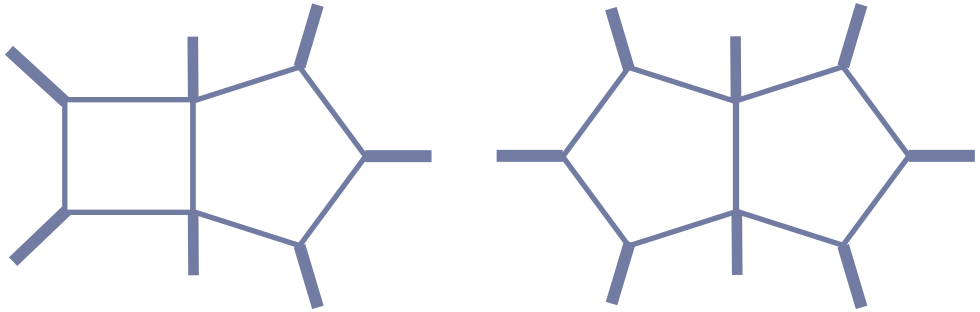

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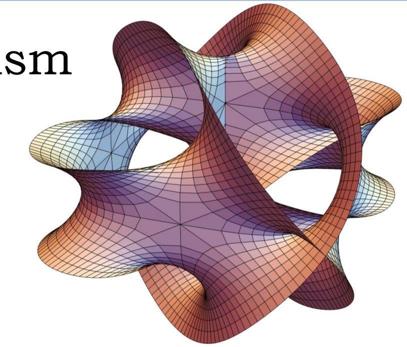

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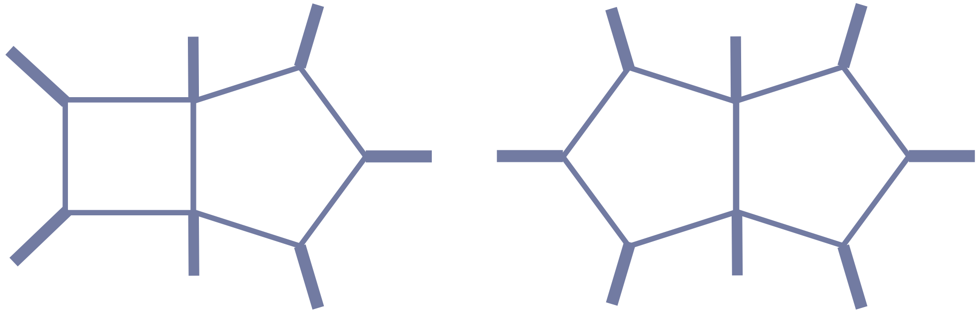

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