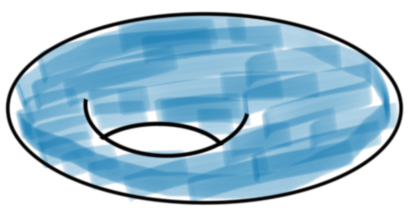
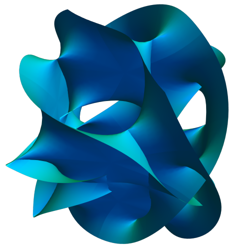
ϵ-Factorized Differential Equations Beyond Polylogarithms



Christoph Nega



Joint work with:

Lennard Görges, Lorenzo Tancredi & Fabian Wagner

Claude Duhr & Sara Maggio

"On a procedure to derive ϵ-factorised differential equations beyond polylogarithms" [1], current ongoing project [2]

Amplitudes 2023

CERN

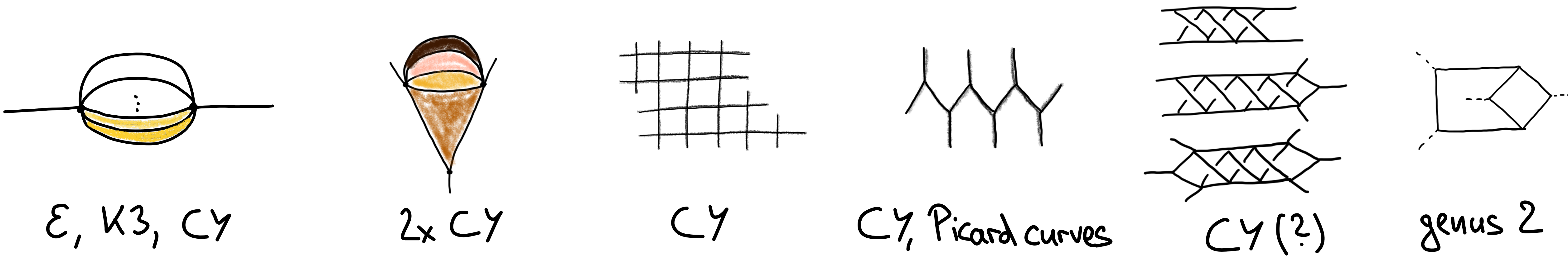
August 10, 2023

Motivation

- The precision of current high energy particle experiments require very accurate theoretical predictions.

- Usually these predictions are made in a perturbative QFT framework, where one has to perform **multi-loop Feynman integral** computations.

- It was observed that also functions beyond polylogarithms can appear. These functions are only properly defined on **non-trivial geometries**:



["Bestiary" collaboration] [Marzucca, McLeod, Page, Pögel, Weinzierl]

- For the computation of amplitudes or cross sections it is necessary to have these **function spaces** under control, e.g. relations between them, numerical evaluation, etc. .

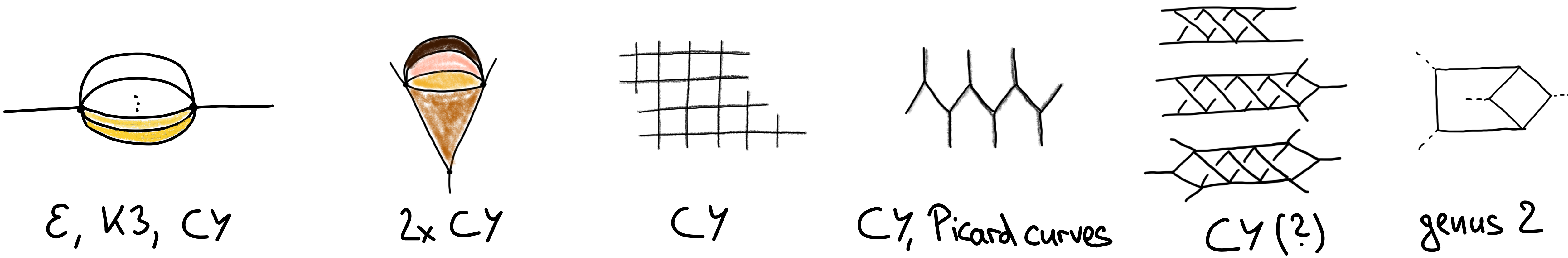
- So far, one of the bottlenecks in amplitude computations is a method to derive an ϵ-**factorized form** of the differential equations for these non-trivial functions such that their ϵ-**expansion** is under control.

Motivation

- The precision of current high energy particle experiments require very accurate theoretical predictions.

- Usually these predictions are made in a perturbative QFT framework, where one has to perform **multi-loop Feynman integral** computations.

- It was observed that also functions beyond polylogarithms can appear. These functions are only properly defined on **non-trivial geometries**:



["Bestiary" collaboration]

[Marzucca, McLeod, Page, Pögel, Weinzierl]

- For the computation of amplitudes or cross sections it is necessary to have these **function spaces** under control, e.g. relations between them, numerical evaluation, etc. .

- So far, one of the bottlenecks in amplitude computations is a method to derive an ϵ-**factorized form** of the differential equations for these non-trivial functions such that their ϵ-**expansion** is under control.

Today I want to show you a method to derive an ϵ-form for non-trivial geometries.

Table of Content

Master Integrals and ϵ-factorized Differential Equations

- Using IBP and symmetry relations one finds a minimal set of integrals necessary for a given problem. These are called **master integrals**:

$$I = (I_1, I_2, \ldots, I_r)$$

- Usually Feynman integrals are **divergent**, so we have to regularize them. Mostly, one takes **dimensional regularization**:

$$d_0 \longrightarrow d = d_0 - 2\epsilon$$

- These master integrals can be computed using differential equations called the **Gauss-Manin system**:

$$\mathrm{d}I(z,\epsilon) = \mathbf{GM}(z,\epsilon) I(z,\epsilon)$$

Master Integrals and ϵ-factorized Differential Equations

- Using IBP and symmetry relations one finds a minimal set of integrals necessary for a given problem. These are called **master integrals**:

$$I = (I_1, I_2, \ldots, I_r)$$

- Usually Feynman integrals are **divergent**, so we have to regularize them. Mostly, one takes **dimensional regularization**:

$$d_0 \longrightarrow d = d_0 - 2\epsilon$$

- These master integrals can be computed using differential equations called the **Gauss-Manin system**:

$$\mathrm{d}I(z,\epsilon) = \mathbf{GM}(z,\epsilon)I(z,\epsilon)$$

- The ϵ-expansion can easily be solved if the GM system is ϵ**-factorized**: [Henn]

$$\tilde{I}(z,\epsilon) = T(z,\epsilon)I(z,\epsilon)$$ such that $$\mathrm{d}\tilde{I}(z,\epsilon) = \epsilon\,\tilde{\mathbf{GM}}(z)\tilde{I}(z,\epsilon)$$

$$\tilde{I}(z,\epsilon) = \mathbb{P}\exp\left(\epsilon\int_{z_0}^{z}\tilde{\mathbf{GM}}(z')\mathrm{d}z'\right)\tilde{I}(z_0,\epsilon)$$ and $\tilde{I}_k(z) =$ Iterated integrals over $\tilde{\mathbf{GM}}_{ij}(z)$

Get as many orders in ϵ-expansion as we want in a controlled way.

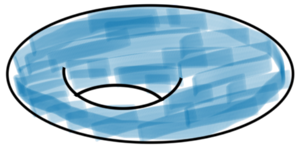
- So far, there is **no general method** known to get this form, in particular for geometries beyond polylogarithms. What "canonical-form" means is also only clear for polylogarithms.
  - Quite recent, algorithm for equal-mass banana graphs. [Pögel, Wang, Weinzierl]

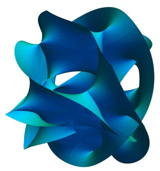
Review of Elliptic Curves and Calabi-Yau Manifolds

- **Calabi-Yau manifolds** are natural generalizations of elliptic curves:

  Calabi-Yaus are complex n-dim Kähler manifolds which have a unique holomorphic $(n,0)$-form.

  CYs are defined via polynomial constraints.





$$(\mathcal{E},\ \mathrm{d}x/y,\ \mathrm{d}x \wedge \mathrm{d}y) \qquad (X, \Omega, \omega)$$

$$\{Y^2 Z - 4X^3 + g_2(t) X Z^2 + g_3(t) Z^3 = 0\} \subset \mathbb{P}^2$$

$$\{\sum_{i=0}^{4} X_i^5 - \Psi X_0 \cdots X_4 = 0\} \subset \mathbb{P}^4$$

Review of Elliptic Curves and Calabi-Yau Manifolds

- **Calabi-Yau manifolds** are natural generalizations of elliptic curves:

  Calabi-Yaus are complex n-dim Kähler manifolds which have a unique holomorphic $(n,0)$-form.

$$(\mathcal{E},\, \mathrm{d}x/y,\, \mathrm{d}x \wedge \mathrm{d}y) \qquad (X, \Omega, \omega)$$

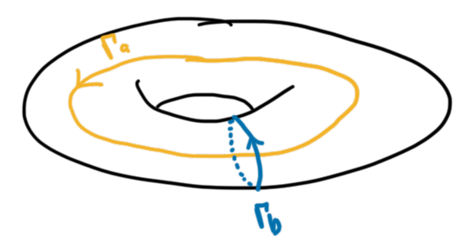
  CYs are defined via polynomial constraints.

$$\{Y^2 Z - 4X^3 + g_2(t) X Z^2 + g_3(t) Z^3 = 0\} \subset \mathbb{P}^2 \qquad \{\sum_{i=0}^{4} X_i^5 - \Psi X_0 \cdots X_4 = 0\} \subset \mathbb{P}^4$$

- **Period integrals** on Calabi-Yaus can be used to describe their shape and properties:

$$\Pi : \quad H_n(X) \times H^n_{\mathrm{dR}}(X) \longrightarrow \mathbb{C}$$
$$(\Gamma, \alpha) \longmapsto \int_\Gamma \alpha$$

,



$$\alpha = \frac{\mathrm{d}X}{Y} \qquad \beta = \frac{X\mathrm{d}X}{Y}$$

,

$$\Pi = \begin{pmatrix} \int_{\Gamma_a} \alpha & \int_{\Gamma_a} \beta \\ \int_{\Gamma_b} \alpha & \int_{\Gamma_b} \beta \end{pmatrix}$$

elliptic integrals
$K(\lambda), K(1-\lambda)$

Review of Elliptic Curves and Calabi-Yau Manifolds

- **Calabi-Yau manifolds** are natural generalizations of elliptic curves:

Calabi-Yaus are complex n-dim Kähler manifolds which have a unique holomorphic $(n,0)$-form.

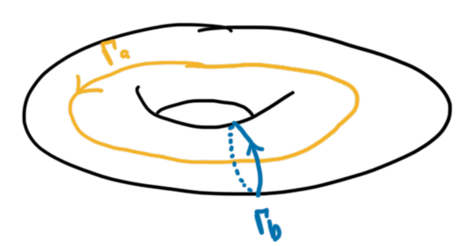
$(\mathcal{E},\, \mathrm{d}x/y,\, \mathrm{d}x \wedge \mathrm{d}y)$ $\quad\quad$ (X, Ω, ω)

CYs are defined via polynomial constraints.

$$\{Y^2 Z - 4X^3 + g_2(t) X Z^2 + g_3(t) Z^3 = 0\} \subset \mathbb{P}^2 \qquad \{\sum_{i=0}^{4} X_i^5 - \Psi X_0 \cdots X_4 = 0\} \subset \mathbb{P}^4$$

- **Period integrals** on Calabi-Yaus can be used to describe their shape and properties:

$$\begin{aligned} \Pi : \quad H_n(X) \times H^n_{\mathrm{dR}}(X) &\longrightarrow \mathbb{C} \\ (\Gamma, \alpha) &\longmapsto \int_\Gamma \alpha \end{aligned} \quad ,$$



$$\alpha = \frac{\mathrm{d}X}{Y} \qquad \beta = \frac{X\mathrm{d}X}{Y}$$

$$, \quad \Pi = \begin{pmatrix} \int_{\Gamma_a} \alpha & \int_{\Gamma_a} \beta \\ \int_{\Gamma_b} \alpha & \int_{\Gamma_b} \beta \end{pmatrix}$$

elliptic integrals $K(\lambda), K(1-\lambda)$

- Periods are governed by differential equations: **Picard-Fuchs equation** or **Gauss-Manin system**:

Point of maximal unipotent monodromy:

hierarchic logarithmic structure

$$\begin{aligned} \varpi_0 &= \text{power series in z} \\ \varpi_1 &= \varpi_0 \log(z) + \Sigma_1 \\ \varpi_2 &= \frac{1}{2}\varpi_0 \log(z)^2 + \Sigma_1 \log(z) + \Sigma_2 \\ &\vdots \end{aligned}$$

Review of Elliptic Curves and Calabi-Yau Manifolds

Using **Griffiths transversality** we can construct relations between the periods of a Calabi-Yau:

$$\begin{aligned}\Omega &\in H^{n,0}(X)\\ \partial_z\Omega &\in H^{n,0}(X)\oplus H^{n-1,1}(X)\\ &\vdots\\ \partial_z^n\Omega &\in H^{n,0}(X)\oplus\ldots\oplus H^{0,n}(X)\end{aligned}$$

- There are **quadratic relations** between periods:

$$\int_X \Omega\wedge\partial_z^k\Omega = \Pi^T\,\Sigma\,\partial_z^k\Pi = \begin{cases}0, & k<n\\ C_n, & k=n\end{cases}$$

Review of Elliptic Curves and Calabi-Yau Manifolds

> Using **Griffiths transversality** we can construct relations between the periods of a Calabi-Yau:

$$\begin{aligned} \Omega &\in H^{n,0}(X) \\ \partial_z \Omega &\in H^{n,0}(X) \oplus H^{n-1,1}(X) \\ &\vdots \\ \partial_z^n \Omega &\in H^{n,0}(X) \oplus \ldots \oplus H^{0,n}(X) \end{aligned}$$

- There are **quadratic relations** between periods:

$$\int_X \Omega \wedge \partial_z^k \Omega = \Pi^T \, \Sigma \, \partial_z^k \Pi = \begin{cases} 0, & k < n \\ C_n, & k = n \end{cases}$$

- One can **express** or **eliminate** some periods (or their derivatives) through the others:

$\mathcal{E}$ — Legendre relations

$$\begin{vmatrix} \varpi_0 & \varpi_1 \\ \varpi_0' & \varpi_1' \end{vmatrix} \sim \frac{1}{\Delta(z)}$$

K3

$$\varpi_0 \varpi_2 \sim \varpi_1^2$$

$$\varpi_0'' \sim R_1(z)\varpi_0 + R_2(z)\varpi_0' + R_3(z)\frac{\varpi_0'^2}{\varpi_0}$$

CY 3-fold

...

Review of Elliptic Curves and Calabi-Yau Manifolds

Using **Griffiths transversality** we can construct relations between the periods of a Calabi-Yau:

$$\begin{aligned} \Omega &\in H^{n,0}(X) \\ \partial_z \Omega &\in H^{n,0}(X) \oplus H^{n-1,1}(X) \\ &\vdots \\ \partial_z^n \Omega &\in H^{n,0}(X) \oplus \ldots \oplus H^{0,n}(X) \end{aligned}$$

- There are **quadratic relations** between periods:

$$\int_X \Omega \wedge \partial_z^k \Omega = \Pi^T \, \Sigma \, \partial_z^k \Pi = \begin{cases} 0, & k < n \\ C_n, & k = n \end{cases}$$

- One can **express** or **eliminate** some periods (or their derivatives) through the others:

$\mathcal{E}$ — Legendre relations

$$\begin{vmatrix} \varpi_0 & \varpi_1 \\ \varpi_0' & \varpi_1' \end{vmatrix} \sim \frac{1}{\Delta(z)}$$

K3

$$\varpi_0 \varpi_2 \sim \varpi_1^2$$

$$\varpi_0'' \sim R_1(z)\varpi_0 + R_2(z)\varpi_0' + R_3(z)\frac{\varpi_0'^2}{\varpi_0}$$

CY 3-fold

...

- We can simplify the **inverse Wronskian:**

$$\mathbf{W}(z)_{i,j} = \left\{\partial_z^i \varpi_j\right\}, \qquad \mathbf{W}(z)^{-1} = \Sigma \mathbf{W}(z)^T \mathbf{Z}(z)$$

- Can write the solutions for $\epsilon = 0$ in terms of **iterated period integrals**:

$$\mathcal{L}_{\text{CY}} I(z) = \text{Inhom}(z) \quad \longrightarrow$$

$$\begin{aligned} I &\sim \Pi(z)^T \int_0^z \mathrm{d}z' \, \mathbf{W}(z')^{-1} \text{Inhom}(z') + \text{periods} \\ &\sim \Pi(z)^T \mathbf{\Sigma} \int_0^z \mathrm{d}z' \, \mathbf{W}(z') \widehat{\text{Inhom}}(z') + \text{periods} \end{aligned}$$

Our Procedure

[1]

- We can work bottom up and sector-by sector.
  First, maximal cuts of a sector.
  Then continue with the subtopologies.

→

Pick your start sector.
Sectors below are already in ϵ-form.

Our Procedure

[1]

- We can work bottom up and sector-by sector. First, maximal cuts of a sector. Then continue with the subtopologies.

  → Pick your start sector. Sectors below are already in ϵ-form.

- Our procedure has basically **4 different steps**:

1) Choose a "good" **initial basis** s.t. the different geometries are visible in $d = d_0$.

   (c.f. polylogs: no higher poles)

2) Split Wronskian into a semi-simple and unipotent part. Rotate with the **inverse semi-simple** part.

   (c.f. polylogs: unit leading singularities)

3) Clean up your Sector:

   a) Perform ϵ-rescalings to achieve upper triangular ϵ-form.

   b) Remove total derivatives.

   c) Introduce **new functions**, if necessary, for full ϵ-form. These turn out to be iterated integrals of the functions introduced in the steps before.

4) Clean up your Subsectors for full ϵ-form.

Our Procedure

[1]

- We can work bottom up and sector-by sector.
  First, maximal cuts of a sector.
  Then continue with the subtopologies.

  → Pick your start sector.
  Sectors below are already in ϵ-form.

- Our procedure has basically **4 different steps**:

1) Choose a "good" **initial basis** s.t. the different geometries are visible in $d = d_0$.
   (c.f. polylogs: no higher poles)

2) Split Wronskian into a semi-simple and unipotent part. Rotate with the **inverse semi-simple** part.
   (c.f. polylogs: unit leading singularities)

3) Clean up your Sector:
   a) Perform ϵ-rescalings to achieve upper triangular ϵ-form.
   b) Remove total derivatives.
   c) Introduce **new functions**, if necessary, for full ϵ-form. These turn out to be iterated integrals of the functions introduced in the steps before.

4) Clean up your Subsectors for full ϵ-form.

$$\mathbf{T}(z,\epsilon) = \mathbf{T}_{\text{clean2}}(z,\epsilon) \times \mathbf{T}_{\text{clean1}}(z,\epsilon) \times \mathbf{T}_{\text{ss}}^{-1}(z,\epsilon) \times \mathbf{T}_{\text{initial}}(z,\epsilon)$$

Initial Basis

[1]

- Either trivial or quite complicated. Not unique. But a good initial basis simplifies subsequent steps drastically.
- No power-like UV or IR divergent integrals in $d = d_0$. No non-trivial ϵ-dependencies in denominators.
- Search for minimally coupled systems in your sector, e.g. check factorizations of the Picard-Fuchs operator.

  ⟶ Gives us information about the **non-trivial geometries** in a sector.

Initial Basis

[1]

- Either trivial or quite complicated. Not unique. But a good initial basis simplifies subsequent steps drastically.

- No power-like UV or IR divergent integrals in $d = d_0$. No non-trivial ϵ-dependencies in denominators.

- Search for minimally coupled systems in your sector, e.g. check factorizations of the Picard-Fuchs operator.

  $\longrightarrow$ Gives us information about the **non-trivial geometries** in a sector.

- For each **minimally coupled system** "good" integrals might be:

  - Start with the integral having as max. cut the **standard period integral** of the geometry.

    $$\underline{\mathcal{E}|} \qquad \cdots \int \mathrm{d}\log \int \frac{\mathrm{d}x}{y} \int \mathrm{d}\log \cdots \qquad\qquad \underline{\mathrm{CY}|} \qquad \cdots \int \mathrm{d}\log \int \mathrm{d}\Omega \int \mathrm{d}\log \cdots$$

    Integral of the first kind — Integral of (n,0)-form

  - Take further integrals with max. cut related to the other cohomology elements.

    "In practice, this means one can take dots on massive propagators if no UV or IR divergencies are introduced."

  - For practical reasons, one can go later to a **derivative basis**.

Initial Basis

[1]

- Either trivial or quite complicated. Not unique. But a good initial basis simplifies subsequent steps drastically.
- No power-like UV or IR divergent integrals in $d = d_0$. No non-trivial ϵ-dependencies in denominators.
- Search for minimally coupled systems in your sector, e.g. check factorizations of the Picard-Fuchs operator.

  → Gives us information about the **non-trivial geometries** in a sector.

- For each **minimally coupled system** "good" integrals might be:
  - Start with the integral having as max. cut the **standard period integral** of the geometry.

    $\mathcal{E}\rfloor \quad \cdots \int \mathrm{d}\log \int \frac{\mathrm{d}x}{y} \int \mathrm{d}\log \cdots$ — Integral of the first kind

    $\mathrm{CY}\rfloor \quad \cdots \int \mathrm{d}\log \int \mathrm{d}\Omega \int \mathrm{d}\log \cdots$ — Integral of (n,0)-form

  - Take further integrals with max. cut related to the other cohomology elements.

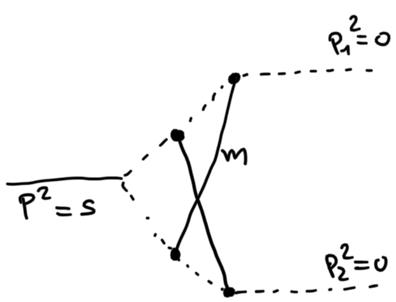
    "In practice, this means one can take dots on massive propagators if no UV or IR divergencies are introduced."

  - For practical reasons, one can go later to a **derivative basis**.
- Try to **separate** the different **minimally coupled systems**:
  - Search for integrals which localize on non-trivial geometries and having residues. Normalize these residues, i.e. integrals of third kind for elliptic curves.
  - Take integrals vanishing in $d = d_0$.

Initial Basis: Examples

- Banana graphs: Trivial, top sector equals CY $(l-1)$-fold, take dots or derivative basis. [1,2]

- Triangle graph: Elliptic top sector with additional residue



[Jiang, Wang, Yang, Zhao]

$$I_{1,1,1,1,1,1,0} \sim \int \frac{\mathrm{d}x}{y}$$

$$I_{1,1,1,1,2,1,0} \sim \int \partial_s \frac{\mathrm{d}x}{y}$$

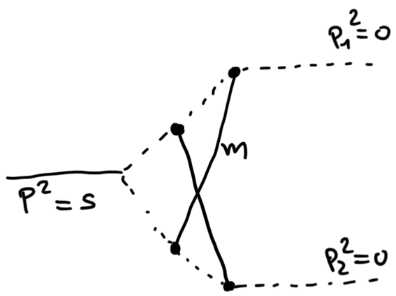
$$I_{1,1,1,1,1,1,-1} \sim \int \frac{\mathrm{d}x}{y(x-x_0)}$$

$$\mathbf{GM}^{d=4}_{\text{tri,top}} = \begin{pmatrix} -\frac{1}{z} & -\frac{2}{z} & 0 \\ \frac{4-z}{z(1-z)(8+z)} & \frac{8+z^2}{(1-z)z(8+z)} & 0 \\ -\frac{2}{3z} & -\frac{4(1-z)}{3z} & 0 \end{pmatrix}$$

Initial Basis: Examples

- **Banana graphs:** Trivial, top sector equals CY $(l-1)$-fold, take dots or derivative basis. [1,2]

- **Triangle graph:** Elliptic top sector with additional residue

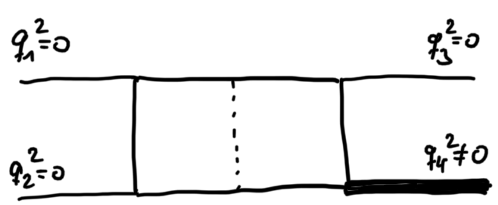


[Jiang, Wang, Yang, Zhao]

$$I_{1,1,1,1,1,1,0} \sim \int \frac{\mathrm{d}x}{y}$$
$$I_{1,1,1,1,2,1,0} \sim \int \partial_s \frac{\mathrm{d}x}{y}$$
$$I_{1,1,1,1,1,1,-1} \sim \int \frac{\mathrm{d}x}{y(x-x_0)}$$

$$\mathbf{GM}^{d=4}_{\text{tri,top}} = \begin{pmatrix} -\frac{1}{z} & -\frac{2}{z} & 0 \\ \frac{4-z}{z(1-z)(8+z)} & \frac{8+z^2}{(1-z)z(8+z)} & 0 \\ -\frac{2}{3z} & -\frac{4(1-z)}{3z} & 0 \end{pmatrix}$$

- **Double box:** Contains a 4x4 elliptic block in the sector below the top sector.



[Bonciani et al. Primo, Tancredi]

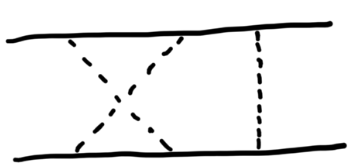
I_1, I_2 are standard elliptic integrals.

I_3 is an integral of third kind.

I_4 is chosen s.t. it vanishes in $d=4$.

$$\mathbf{GM}^{d=4}_{\text{box},s} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \mathrm{F1}(s,t,M) & \mathrm{F2}(s,t,M) & 0 & 0 \\ \mathrm{R1}(s,t,M) & 0 & 0 & 0 \\ \mathrm{R2}(s,t,M) & 0 & 0 & 0 \end{pmatrix}$$

- **Non-planar graph:** Contains a 6x6 elliptic block in the sector below the top sector.



I_1, I_2 are standard elliptic integrals.

I_3 from factorization of PF operator.

I_4, I_5, I_6 are residues with unit leading singularity.

$$\mathbf{GM}^{d=4}_{\text{np},t} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \mathrm{F1}(s,t) & \mathrm{F2}(s,t) & 0 & 0 & 0 & 0 \\ \mathrm{R1}(s,t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{R2}(s,t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Inverse Semi-Simple Part

[1]

"afterwards we have pure functions"

- This step generalizes the normalization with leading singularities.
- Take homogeneous diff. eq. in every minimally coupled block in $d = d_0$.
  The corresponding Wronskian satisfies the same system.

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} I_{1,\max} \\ \vdots \\ I_{r,\max} \end{pmatrix} = \mathbf{GM}(z)\begin{pmatrix} I_{1,\max} \\ \vdots \\ I_{r,\max} \end{pmatrix} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}z}\mathbf{W} = \mathbf{GM}(z)\mathbf{W}$$

Inverse Semi-Simple Part

[1]

- This step generalizes the normalization with leading singularities. "afterwards we have pure functions"

- Take homogeneous diff. eq. in every minimally coupled block in $d = d_0$.
  The corresponding Wronskian satisfies the same system.

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} I_{1,\max} \\ \vdots \\ I_{r,\max} \end{pmatrix} = \mathbf{GM}(z)\begin{pmatrix} I_{1,\max} \\ \vdots \\ I_{r,\max} \end{pmatrix} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}z}\mathbf{W} = \mathbf{GM}(z)\mathbf{W}$$

- Split the Wronskian into a **semi-simple** (leading singularities) and **unipotent** (logs) **part**:

$$\mathbf{W} = \mathbf{W}_{\mathrm{ss}}\mathbf{W}_{\mathrm{u}} \quad \text{with} \quad \frac{\mathrm{d}}{\mathrm{d}z}\mathbf{W}_{\mathrm{u}} = \widehat{\mathbf{GM}}(z)\mathbf{W}_{\mathrm{u}} \quad \text{s.t.} \quad \widehat{\mathbf{GM}}^{k}(z) = 0 \quad \text{(nilpotent)}$$

  - Splitting is not unique. Normalize diagonal of unipotent part to be unity.

  - For Calabi-Yau manifolds the nilpotent matrix is known:

$$\widehat{\mathbf{GM}}_{\mathrm{CY}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Y_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

  - Use **relations** from **Griffiths transversality** to simplify the semi-simple part.
    For example for the three-loop banana (K3) we can remove ϖ_0'' in $\mathbf{W}_{\mathrm{ss}}$.

Clean Up

[1]

- There are **two clean ups** to do. One inside a sector and one between sectors and subsectors:

  a) Perform ϵ-rescalings to achieve upper triangular ϵ-form.

  b) Remove total derivatives.

  c) Introduce **new functions**, if necessary, for full ϵ-form. These turn out to be **iterated integrals** of the functions introduced in the steps before.

Clean Up

[1]

- There are **two clean ups** to do. One inside a sector and one between sectors and subsectors:

  a) Perform ϵ-rescalings to achieve upper triangular ϵ-form.

  b) Remove total derivatives.

  c) Introduce **new functions**, if necessary, for full ϵ-form. These turn out to be **iterated integrals** of the functions introduced in the steps before.

    (i) **Inside** a minimally coupled block:

K3

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & G_2(z) & 1 & 0 \\ 0 & -\frac{1}{2}G_2(z)^2 & -G_2(z) & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{G_1(z)}{\epsilon} & 0 & 1 \end{pmatrix}$$

$$G_1(z) = \int_0^z \mathrm{d}z' \, \frac{\left(2(1-8z')(1+8z')^3\right)}{z'^2(1-4z')^2(1-16z')^2} \, \varpi_0^2(z')$$

$$G_2(z) = \int_0^z \mathrm{d}z' \, \frac{G_1(z')}{\sqrt{(1-4z')(1-16z')}\, \varpi_0(z')}$$

    (ii) **Between** minimally coupled blocks:

Non-planar

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ G_1(s,t) & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2G_2(s,t) & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ G_2(s,t)^2 - \frac{G_1(s,t)^2}{4} & -\frac{G_1(s,t)}{2} & 0 & G_2(s,t) & 0 & 1 \end{pmatrix}$$

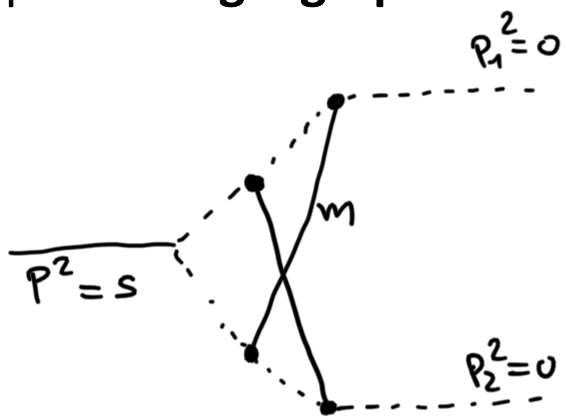
related to integrals of third kind

$$G_1(s,t) = -\int_0^t \mathrm{d}t' \, R1(s,t) \, \varpi_0(s,t')$$

$$G_2(s,t) = -\int_0^t \mathrm{d}t' \, R2(s,t) \, \varpi_0(s,t')$$

Examples

- Example: **Triangle graph**



$d = 4 - 2\epsilon$

$z = s/m^2$

18 master integrals

elliptic top sector
with residue

$$T(z,\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_{\text{new objects}}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3}(z+1)\varpi_0(z) & 1 & 0 \\ \frac{1}{24}\left(5z^2-44z-76\right)\varpi_0(z)^2 & 0 & 1 \end{pmatrix} \qquad T_{\text{tot. deri.}}$$

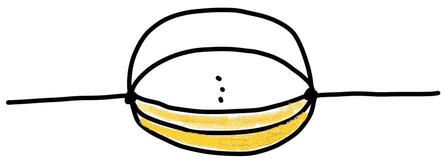
$$\cdot \begin{pmatrix} \epsilon^4 & 0 & 0 \\ 0 & 0 & \epsilon^4 \\ 0 & \epsilon^3 & 0 \end{pmatrix} \qquad T_{\epsilon-\text{scalings}}$$

$$\cdot \begin{pmatrix} \frac{1}{\varpi_0(z)} & 0 & 0 \\ \frac{1}{8}(z-8)(z+1)\left(\varpi_0(z)+z\varpi_0'(z)\right) & \frac{1}{4}(z-8)(z+1)\varpi_0(z) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_{\text{ss}}^{-1}$$

$$\cdot \begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix} \qquad T_{\text{initial}}$$

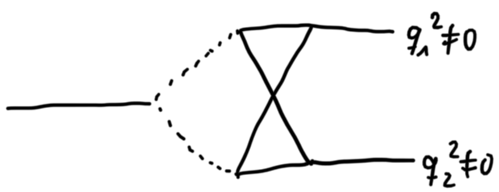
Examples

- So far, we found the ϵ-form for many graphs having **different geometries** and **number of parameters**: [1,2]

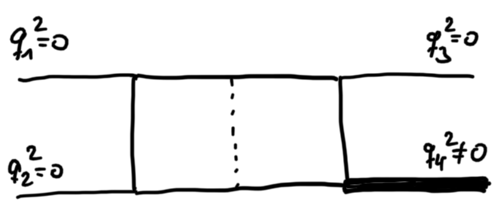


achieved ϵ-form up to five loops (CY four-fold), same form as [Pögel, Wang, Weinzierl]
also two- and three-mass configuration for elliptic case

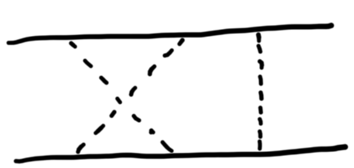
up to five loops, top sector contains two CYs



analyzed many triangles with different mass configurations and multi-parameter



double box including all subsectors and topsector



non-planar double box including all subsectors and topsector

- More examples are coming!

Conclusion

- To understand the ϵ-structure of Feynman integrals a "good" ϵ-form of the GM system is essential.
- Understanding the geometries appearing in a Feynman graph is important to achieve ϵ-form.
- We think, that if the splitting of the Wronskian into semi-simple and unipotent part for the relevant geometries is under control, one can derive an ϵ-form following our procedure.
- Our method is nearly algorithmic. In particular, the steps for a given class of geometries is similar independent of the explicit geometry, i.e. for all elliptic curves, K3, etc. .

Conclusion

- To understand the ϵ-structure of Feynman integrals a "good" ϵ-form of the GM system is essential.
- Understanding the geometries appearing in a Feynman graph is important to achieve ϵ-form.
- We think, that if the splitting of the Wronskian into semi-simple and unipotent part for the relevant geometries is under control, one can derive an ϵ-form following our procedure.
- Our method is nearly algorithmic. In particular, the steps for a given class of geometries is similar independent of the explicit geometry, i.e. for all elliptic curves, K3, etc. .

- Nevertheless, there are some parts we want to understand better:
    - Better understanding of the initial basis. How to find enough possible candidates?
    - What are the properties of the new functions G? Can we predict how many we need?

      They are iterated integrals. In elliptic case they are related to integrals of the third kind.

      For CYs they have integer expansions (magnetic modular forms). Pole structure?
    - Limitations of our procedure? Can we rigorously prove why our method works?
- Outlook: Use our bases to try to understand the analytic structure of amplitudes associated to non-trivial geometries.

Thank you for your attention