Generalizing Polylogarithms to Riemann Surfaces of Arbitrary Genus

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Organization of the Talk

1. Introduction
2. Review of polylogarithms at genus zero and one
3. A brief overview of the geometry of higher-genus Riemann surfaces
4. Construction of higher-genus polylogarithms
5. Conclusion

Introduction

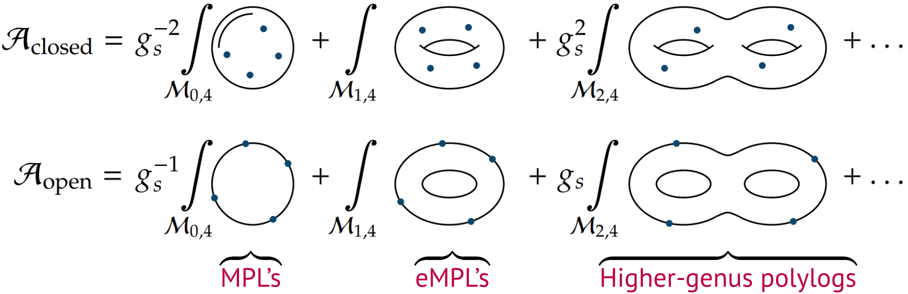
Introduction

- **Polylogarithms** play a significant role in scattering amplitudes for LHC processes, SYM theory, supergravity, and string theory.
- Suitable **generalizations of classical** polylogarithms are defined by considering **iterated integrals** on closed Riemann surfaces.
- Much of the literature on polylogarithms has focused on **genus zero and genus one** Riemann surfaces, with **higher-genus surfaces** less understood.
  - Proposals for higher-genus polylogarithm function spaces exist, but without explicit **formulas** for use in physics. [Enriquez, 1112.0864] [Enriquez, Zerbini, 2110.09341] [Enriquez, Zerbini, 2212.03119]
- Today, we will explore a **new** construction of **higher-genus polylogarithms**.
- Our method includes two key steps:
  - We create a new set of **integration kernels** using **convolutions** of certain functions defined on higher-genus Riemann surfaces.
  - With these kernels, we build a **generating function**, which helps define our **higher-genus polylogarithms** which are **closed under taking primitives**.

String amplitudes motivation

- String perturbation theory involves expanding in the **string coupling constant** g_s, which in turn is an expansion based on the **genus** of the string world-sheet.

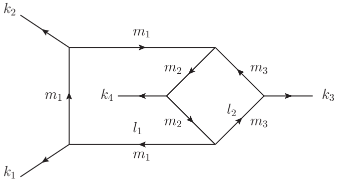
[Figure taken from PhD thesis of J. Gerken]



- Furthermore, typically we also expand in the **inverse string tension** α', which corresponds to low energy and weak coupling regimes.
- The resulting function space of these expansions is that of **polylogarithms**, (or single-valued combinations thereof.)

Higher genus curves in Feynman integrals

- The appearance of **hyperelliptic curves** in Feynman integrals has also been observed in a number of publications. See for example:
- *R. Huang and Y. Zhang, "On Genera of Curves from High-loop Generalized Unitarity Cuts," JHEP* ***04*** *(2013), 080 [arXiv:1302.1023 [hep-ph]].*
- *A. Georgoudis and Y. Zhang, "Two-loop Integral Reduction from Elliptic and Hyperelliptic Curves," JHEP* ***12*** *(2015), 086 [arXiv:1507.06310 [hep-th]].*



The maximal cut of this diagram yields a hyperelliptic curve. Figure taken from [1507.06310].

- *C. F. Doran, A. Harder, E. Pichon-Pharabod and P. Vanhove, "Motivic geometry of two-loop Feynman integrals," [arXiv:2302.14840 [math.AG]].*
- *R. Marzucca, A. J. McLeod, B. Page, S. Pögel, S. Weinzierl, "Genus Drop in Hyperelliptic Feynman Integrals," [arXiv:2307.11497 [hep-th]].*

Review of polylogarithms at genus zero and one

Building Polylogarithms as Iterated Integrals

- We want to construct **polylogarithms** in terms of iterated integrals on a **compact Riemann surface, Σ, with genus h**.
- The polylogarithms we construct should have these qualities:

1. **Homotopy Invariance**: The polylogarithms should retain their value when we smoothly change the path of integration, keeping the endpoints constant.
2. **Logarithmic Branch-Cuts**: The integration kernels should only have simple poles, meaning our integrals should show just logarithmic irregularities at branch points.
3. **Closed Under Integration**: Our function space should remain intact under integration, and form a basis for all iterated integrals on Σ.

Homotopy-Invariant Iterated Integrals on a Surface

- Let's consider the differential equation: $d\mathbf{\Gamma} = \mathcal{J}\mathbf{\Gamma}$.
- If we want the equation to be **integrable**, we need $d^2 = 0$. This leads us to the **Maurer-Cartan** equation for the connection $\mathcal{J}$:

$$d\mathcal{J} - \mathcal{J} \wedge \mathcal{J} = 0$$

- Such a connection is called **flat**. The solution $\mathbf{\Gamma}$ to our differential equation can be obtained by the path-ordered exponential (POE):

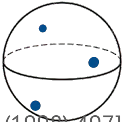
$$\mathbf{\Gamma}(\mathcal{C}) = \mathrm{P}\exp \int_{\mathcal{C}} \mathcal{J}(\cdot) = \mathrm{P}\exp \int_0^1 dt\, J(t)$$

- Let's denote $\mathcal{J} = J(t)dt$, following a path $\mathcal{C}$ where $t \in [0,1]$, $\mathcal{C}(0) = z_0$, and $\mathcal{C}(1) = z$. Using **physics conventions**, we position $J(t)$ to the **left** of $J(t')$ for $t > t'$:

$$\mathrm{P}\exp \int_{\mathcal{C}} \mathcal{J}(\cdot) = 1 + \int_0^1 dt\, J(t) + \int_0^1 dt \int_0^t dt'\, J(t)J(t') + \dots$$

- The flatness $\mathcal{J}$ leads to **homotopy-invariant** integrals over $\mathcal{C}$, (though results can differ for z_0 and z when the path circles around poles on Σ.)

Genus 0: MPLs and Generating Series



- Multiple polylogarithms (MPLs) are **iterated integrals** of rational forms $dz/(z-s)$ with $z, s \in \mathbb{C}$, on the Riemann sphere $\mathbb{CP}^1$.

  [A.B. Goncharov, Math. Res. Lett. 5 (1998) 497]

- They are **defined recursively** by: [A.B. Goncharov, math.AG/0103059]

$$G(s_1, s_2, \cdots, s_n; z) = \int_0^z \frac{dt}{t - s_1} G(s_2, \cdots, s_n; t)$$

  where we have the special case $G(\emptyset; z) = 1$. The integer $n \geq 0$ is referred to as the **transcendental weight**.

- Any integral of a **rational function** times a **multiple polylogarithm** (MPL) can be expressed in terms of MPLs.
- This is achieved by **partial fractioning** the rational function and/or using **integration by parts** (IBP) identities. For example:

$$\frac{1}{(x-s_1)(x-s_2)} = \frac{1}{(s_1 - s_2)} \left(\frac{1}{(x-s_1)} - \frac{1}{(x-s_2)} \right)$$

Generating Series

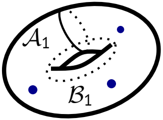
- A **generating series** for the polylogarithms can be constructed from the **Knizhnik-Zamolodchikov** (KZ) connection:

$$\mathcal{J}_{\mathrm{KZ}}(z) = \sum_{i=1}^{m} \frac{dz}{z - s_i} e_i$$

- The elements $e_1, \cdots, e_m$ are generators of a free Lie algebra $\mathcal{L}$ associated with the **marked points** $s_1, \cdots, s_m$.
- Choosing endpoints $z_0 = 0$ and $z_1 = z$, we can **organize** the expansion of the **path-ordered exponential** in terms of the **generators** $e_1, \cdots, e_m$:

$$\mathrm{P}\exp \int_0^z \mathcal{J}_{\mathrm{KZ}}(\cdot) = 1 + \sum_{i=1}^{m} e_i G(s_i; z) + \sum_{i=1}^{m}\sum_{j=1}^{m} e_i e_j G(s_i s_j; z)$$
$$+ \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} e_i e_j e_k G(s_i s_j s_k; z) + \cdots$$

Genus 1: Elliptic Multiple Polylogarithms



- Next, consider a compact **genus-one** surface, Σ, with modulus τ, denoted as a lattice by $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.
- For a surface with genus $h \geq 1$, there are two key options for constructing a connection: [Brown, Levin, arXiv:1110.6917]

  [Broedel, Mafra, Matthes, Schlotterer, arXiv:1412.5535]

  [Broedel, Duhr, Dulat, Tancredi, arXiv:1712.07089]

  1. A connection that is **single-valued** on Σ, but **non-meromorphic** (due to $\bar{z}$-dependence), with at most **simple poles**.
  2. A **meromorphic** connection that has at most **simple poles**, but is **not single-valued** (and lives on the universal cover of Σ). This can be obtained with a minor tweak of the first construction.
- The **Brown-Levin construction** opts for the first choice.
- Interestingly, the construction of elliptic multiple polylogarithms at genus 1 is quite different from the genus 0 case. Notably, there is an **infinite set of integration kernels** at genus one, even for **a single marked point** z.

The Brown-Levin Construction

- Brown and Levin pioneered a method of **homotopy-invariant iterated integrals** at genus one. [Brown, Levin, arXiv:1110.6917]
- The key element to their construction is the so-called **Kronecker-Eisenstein (KE-) series**:

$$\Omega(z,\alpha|\tau) = \exp\left(2\pi i\alpha \frac{\operatorname{Im} z}{\operatorname{Im}\tau}\right)\frac{\vartheta_1'(0|\tau)\vartheta_1(z+\alpha|\tau)}{\vartheta_1(z|\tau)\vartheta_1(\alpha|\tau)} = \sum_{n=0}^{\infty}\alpha^{n-1}f^{(n)}(z|\tau)$$

- The KE-series is **single-valued on the torus**, has a **simple pole at $z=0$** and satisfies the following **differential relation** (for $z \neq 0$):

$$\partial_{\bar z}\Omega(z,\alpha|\tau) = -\frac{\pi\,\alpha}{\operatorname{Im}\tau}\,\Omega(z,\alpha|\tau)$$

- They then constructed the **flat connection** $\mathcal{J}_{\mathrm{BL}}(z|\tau)$, which is valued in the Lie algebra $\mathcal{L}$, generated by elements a, b:

$$\mathcal{J}_{\mathrm{BL}}(z|\tau) = \frac{\pi}{\operatorname{Im}\tau}(dz - d\bar z)\,b + dz\,\mathrm{ad}_b\,\Omega\big(z,\mathrm{ad}_b|\tau\big)\,a$$

- Note that we have put $\alpha \to \mathrm{ad}_b = [b,\circ]$. **Flatness** can be proven using that $d_z = dz\partial_z + d\bar z\partial_{\bar z}$, and using the above differential equation.

Homotopy-Invariant Iterated Integrals

- We may write down **homotopy-invariant iterated integrals** on the torus by expanding the path-ordered exponential in terms of words in a, b:

$$\mathrm{P}\exp\int_0^z \mathcal{J}_{\mathrm{BL}}(\cdot|\tau) = 1 + a\,\Gamma(a;z|\tau) + b\,\Gamma(b;z|\tau) + ab\,\Gamma(ab;z|\tau) + ba\,\Gamma(ba;z|\tau) + \ldots$$

- The resulting coefficient functions $\Gamma(\mathfrak{w};z|\tau)$ are referred to as **elliptic polylogarithms**.
- While the connection is single-valued on the torus, the integrals are **not** and have monodromies along the $\mathfrak{A}$- and $\mathfrak{B}$-cycles.
- **Note**: In the physics literature we typically see the following functions:

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 & n_2 & \cdots & n_r \\ w_1 & w_2 & \cdots & w_r \end{smallmatrix};z|\tau\right) = \int_0^z dz_1\, g^{(n_1)}(z_1 - w_1|\tau)\,\tilde{\Gamma}\left(\begin{smallmatrix} n_2 & \cdots & n_r \\ w_2 & \cdots & w_r \end{smallmatrix};z_1|\tau\right)$$

which are a **meromorphic** variant of the elliptic polylogarithms that were constructed above. For example:

$$\Gamma(ab;z|\tau) = \int_0^z dt\left(2\pi i\,\frac{\mathrm{Im}\,t}{\mathrm{Im}\,\tau} - f^{(1)}(t|\tau)\right) = -\int_0^z dt\, g^{(1)}(t|\tau) = -\tilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix};z|\tau\right)$$

Closure under integration

- For the MPLs, we saw that partial fraction identities were essential for splitting up a product of integration kernels.
- We need similar identities for the **function space to close under integration** at genus one. For example, we might encounter an integral of the type:

$$\int_0^z \mathrm{d}t\, f^{(n_1)}(t-a_1)\, f^{(n_2)}(t-a_2)$$

[Broedel, Mafra, Matthes, Schlotterer, arXiv:1412.5535]

- The so-called **Fay identities** generalize the partial fraction relations. They are generated by:

$$\Omega(z_1,\alpha_1,\tau)\,\Omega(z_2,\alpha_2,\tau) = \Omega(z_1,\alpha_1+\alpha_2,\tau)\,\Omega(z_2-z_1,\alpha_2,\tau) + \Omega(z_2,\alpha_1+\alpha_2,\tau)\,\Omega(z_1-z_2,\alpha_1,\tau)$$

- For example we have that:

$$f^{(1)}(t-x)f^{(1)}(t) = f^{(1)}(t-x)f^{(1)}(x) - f^{(1)}(t)f^{(1)}(x) + f^{(2)}(t) + f^{(2)}(x) + f^{(2)}(t-x)$$

Alternative Construction via Convolutions

- An **alternative construction** of the functions $f^{(k)}(z|\tau)$ is in terms of the **scalar Green function** $g(z|\tau)$ on Σ. The Green function is defined by:

$$\partial_{\bar{z}}\partial_z g(z|\tau) = -\pi\delta(z) + \frac{\pi}{\operatorname{Im}\tau}, \quad \int_\Sigma d^2z\, g(z|\tau) = 0$$

- It can be expressed in terms of the Jacobi theta function ϑ_1 and the Dedekind eta-function η as follows:

$$g(z|\tau) = -\ln\left|\frac{\vartheta_1(z|\tau)}{\eta(\tau)}\right|^2 - \pi\frac{(z-\bar{z})^2}{2\operatorname{Im}\tau}$$

- We define the function $f^{(1)}(z|\tau)$ as the derivative of the Green's function:

$$f^{(1)}(z|\tau) = -\partial_z g(z|\tau)$$

- Subsequently, we can define **higher dimensional convolutions** of f recursively as follows:

$$f^{(k)}(z|\tau) = -\int_\Sigma \frac{d^2x}{\operatorname{Im}\tau}\,\partial_x g(x|\tau) f^{(k-1)}(x-z|\tau), \quad k \geq 2$$

- We will see in the following that **similar convolutions underlie** our higher-genus generalizations of these kernels.

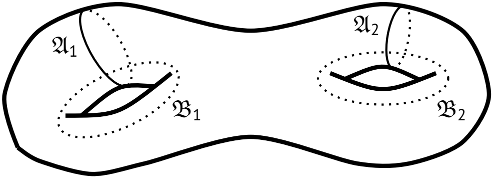
Summary: The Brown-Levin construction

Step	Brown-Levin construction	Higher-genus construction
1. Integration kernels	$f^{(k)}(z\|\tau) = -\int_\Sigma \frac{d^2x}{\mathrm{Im}\,\tau}\, \partial_x g(x\|\tau)\, f^{(k-1)}(x-z\|\tau)$	$\Phi^{I_1\cdots I_r}{}_J(x) = \int_\Sigma d^2z\, \mathcal{G}(x,z)\, \bar\omega^{I_1}(z)\, \partial_z \Phi^{I_2\cdots I_r}{}_J(z) \quad (r \geq 2)$ $\mathcal{G}^{I_1\cdots I_s}(x,y) = \int_\Sigma d^2z\, \mathcal{G}(x,z)\, \bar\omega^{I_1}(z)\, \partial_z \mathcal{G}^{I_2\cdots I_s}(z,y) \quad (s \geq 1)$
2. Generating series	$\alpha\Omega(z,\alpha\|\tau) = \sum_{n=0}^{\infty} \alpha^n f^{(n)}(z\|\tau)$	$\Psi_J(x,p;B) = \omega_J(x) + \left(\partial_x \Phi^{I_1}{}_J(x) - \partial_x \mathcal{G}(x,p)\delta^{I_1}{}_J\right) B_{I_1}$ $+\sum_{r=2}^{\infty} \left(\partial_x \Phi^{I_1 I_2\cdots I_r}{}_J(x) - \partial_x \mathcal{G}^{I_1 I_2\cdots I_{r-1}}(x,p)\delta^{I_r}{}_J\right)$ $\times B_{I_1} B_{I_2} \cdots B_{I_r}$
3. Flat connection $(d\mathcal{J} - \mathcal{J}\wedge\mathcal{J} = 0)$	$\mathcal{J}_{\mathrm{BL}}(x\|\tau) = -d\bar x\, b$ $+ \frac{\pi}{\mathrm{Im}\,\tau}\, dx\, b + dx\, \mathrm{ad}_b\, \Omega(x, \mathrm{ad}_b\|\tau)\, a$	$\mathcal{J}(x,p) = -\pi\, d\bar x\, \bar\omega^I(x)\, b_I$ $+ \pi\, dx\, \mathcal{H}^I(x;B)\, b_I + dx\, \Psi_I(x,p;B)\, a^I$
4. Path-ordered exponential	$\mathrm{P}\exp \int_0^x \mathcal{J}_{\mathrm{BL}}(\cdot\|\tau) =$ $1 + a\,\Gamma(a;x\|\tau) + b\,\Gamma(b;x\|\tau)$ $+ ab\,\Gamma(ab;x\|\tau) + ba\,\Gamma(ba;x\|\tau) + \ldots$	$\mathrm{P}\exp \int_y^x \mathcal{J}(t,p) =$ $1 + a^I\Gamma_I(x,y;p) + b_I\Gamma^I(x,y;p)$ $+ a^I a^J\Gamma_{IJ}(x,y;p) + b_I b_J \Gamma^{IJ}(x,y;p)$ $+ a^I b_J \Gamma_I{}^J(x,y;p) + b_I a^J \Gamma^I{}_J(x,y;p) + \cdots$
5. Polylogs	e.g. $\Gamma(ab;x\|\tau) =$ $\int_0^x dt \left(2\pi i \frac{\mathrm{Im}\,t}{\mathrm{Im}\,\tau} - f^{(1)}(t\|\tau)\right)$	e.g. $\Gamma^{IJ}(x,y;p) = \pi \int_y^x \Big(dt\, \left(\partial_t \Phi^I{}_K(t) Y^{KJ} - \partial_t \Phi^J{}_K(t) Y^{KI}\right)$ $+ \pi\left(\boldsymbol{\omega}^I(t) - \bar{\boldsymbol{\omega}}^I(t)\right) \int_y^t (\boldsymbol{\omega}^J - \bar{\boldsymbol{\omega}}^J)\Big)$

Brief overview of higher-genus Riemann surfaces

Topology of a Compact Riemann Surface Σ

- The **topology** of a **compact** Riemann surface Σ without boundary is specified by its **genus** h.
- The **homology group** $H_1(\Sigma, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2h}$ and supports an **anti-symmetric non-degenerate intersection pairing** denoted by $\mathfrak{J}$.



A choice of canonical homology basis on a compact **genus-two** Riemann surface Σ.

- A **canonical homology basis** of cycles $\mathfrak{A}_I$ and $\mathfrak{B}_J$ with $I, J = 1, \cdots, h$ has symplectic intersection matrix $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{B}_J) = -\mathfrak{J}(\mathfrak{B}_J, \mathfrak{A}_I) = \delta_{IJ}$, and $\mathfrak{J}(\mathfrak{A}_I, \mathfrak{A}_J) = \mathfrak{J}(\mathfrak{B}_I, \mathfrak{B}_J) = 0$.
- A **new canonical basis** $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{B}}$ is obtained by applying a **modular transformation** $M \in Sp(2h, \mathbb{Z})$, such that $M^t \mathfrak{J} M = \mathfrak{J}$.

Canonical Basis of Holomorphic Abelian Differentials

- A **canonical basis** of **holomorphic Abelian differentials** ω_I may be normalized on $\mathfrak{A}$-cycles:

$$\oint_{\mathfrak{A}_I} \omega_J = \delta_{IJ} \qquad \oint_{\mathfrak{B}_I} \omega_J = \Omega_{IJ}$$

- The complex variables Ω_{IJ} denote the components of the **period matrix** Ω of the surface Σ.
- By the **Riemann relations**, Ω is **symmetric**, and has **positive definite imaginary part**:

$$\Omega^t = \Omega \qquad Y = \operatorname{Im}\Omega > 0$$

- We will use the matrix $Y_{IJ} = \operatorname{Im}\Omega_{IJ}$ and its **inverse** $Y^{IJ} = \left((\operatorname{Im}\Omega)^{-1}\right)^{IJ}$ to **raise and lower** indices:

$$\omega^I = Y^{IJ}\omega_J \qquad \bar{\omega}^I = Y^{IJ}\bar{\omega}_J \qquad Y^{IK}Y_{KJ} = \delta^I_J$$

The Arakelov Green Function

- The **Arakelov Green function** $\mathcal{G}(x,y|\Omega)$ on $\Sigma \times \Sigma$ is a **single-valued** version of the Green function, defined by: [D'Hoker, Green, Pioline, arXiv:1712.06135] [G. Faltings, Ann. Math., 119(2), 1984]

$$\partial_{\bar{x}}\partial_x \mathcal{G}(x,y|\Omega) = -\pi\delta(x,y) + \pi\kappa(x)\,, \qquad \int_\Sigma \boldsymbol{\kappa}(x)\mathcal{G}(x,y|\Omega) = 0$$

where the **Kähler form** $\boldsymbol{\kappa}$ is given by:

$$\boldsymbol{\kappa} = \frac{i}{2h}\boldsymbol{\omega}_I \wedge \bar{\boldsymbol{\omega}}^I = \kappa(z)\,d^2z \qquad \int_\Sigma \boldsymbol{\kappa} = 1$$

- In what follows we will drop the explicit dependence on the moduli Ω.
- At genus one the (Arakelov) Green function only depends on a difference of points $\mathcal{G}(x,y)|_{h=1} = \mathcal{G}(x-y)|_{h=1}$.
- However, this **translation invariance** is **absent** on a Riemann surface Σ of genus $h > 1$.

The Interchange Lemma

- The tensor $\Phi^I{}_J(x)$, introduced by Kawazumi, compensates for the lack of translation invariance at higher genus: [Kawazumi, MCM2016] [Kawazumi, 2017]

$$\Phi^I{}_J(x) = \int_\Sigma d^2z\, \mathcal{G}(x,z)\, \bar{\omega}^I(z)\omega_J(z)$$

- Note that the **trace** of $\Phi^I{}_J(x)$ **vanishes** by the definition of the Arakelov Green function.
- In particular, the so-called **interchange lemma** provides a substitute for the absence of translation invariance:

$$\partial_x \mathcal{G}(x,y)\,\omega_J(y) + \partial_y \mathcal{G}(x,y)\,\omega_J(x) - \partial_x \Phi^I{}_J(x)\,\omega_I(y) - \partial_y \Phi^I{}_J(y)\,\omega_I(x) = 0$$

[E. D'Hoker et al., arXiv:2008.08687 [hep-th]]

Construction of higher-genus polylogarithms

Higher Convolution of the Arakelov Green Function

- Inspired by the alternative construction of the Kronecker-Eisenstein kernels through convolutions, we define the **tensors** $\Phi^{I_1\cdots I_r}{}_J(x)$ and $\mathcal{G}^{I_1\cdots I_s}(x,y)$:

$$\Phi^{I_1\cdots I_r}{}_J(x) = \int_\Sigma d^2z\,\mathcal{G}(x,z)\,\bar{\omega}^{I_1}(z)\,\partial_z\Phi^{I_2\cdots I_r}{}_J(z) \quad (r\geq 2)$$

$$\mathcal{G}^{I_1\cdots I_s}(x,y) = \int_\Sigma d^2z\,\mathcal{G}(x,z)\,\bar{\omega}^{I_1}(z)\,\partial_z\mathcal{G}^{I_2\cdots I_s}(z,y) \quad (s\geq 1)$$

- (We also encounter these tensors while decomposing cyclic products of Szegö kernels, see [D'Hoker, MH, Schlotterer, arXiv:2308.05044]).
- **At genus one**, the derivatives of the tensor $\mathcal{G}^{I_1\cdots I_s}$ for $I_1=\cdots=I_s=1$ equal the Kronecker-Eisenstein integration kernels $f^{(s+1)}$:

$$\partial_x\mathcal{G}^{I_1\cdots I_s}(x,y)\,\big|_{h=1} = -f^{(s+1)}(x-y|\tau)$$

- The trace $\Phi^{I_1\cdots I_r}{}_{I_r}=0$ for arbitrary genus implies that Φ-tensors for arbitrary $r\geq 1$ **vanish** identically for **genus one**.
- **In the next part**: we will construct **generating functions** of our kernels, and combine them into a flat connection.

Generating Functions

- Let us introduce a **non-commutative algebra freely generated by B_I** for $I = 1, \cdots, h$ (loosely inspired by the approach of Enriquez and Zerbini arXiv:2110.09341).
- Next, we fix an arbitrary **auxiliary marked point p** on the Riemann surface Σ and introduce the following **generating functions**:

$$\mathcal{H}(x,p;B) = \partial_x \mathcal{G}(x,p) + \sum_{r=1}^{\infty} \partial_x \mathcal{G}^{I_1 I_2 \cdots I_r}(x,p) B_{I_1} B_{I_2} \cdots B_{I_r}$$

$$\mathcal{H}_J(x;B) = \omega_J(x) + \sum_{r=1}^{\infty} \partial_x \Phi^{I_1 I_2 \cdots I_r}{}_J(x) B_{I_1} B_{I_2} \cdots B_{I_r}$$

- By forming the **combination** $\Psi_J(x,p;B) = \mathcal{H}_J(x;B) - \mathcal{H}(x,p;B)B_J$, we obtain a compact antiholomorphic derivative:

$$\partial_{\bar{x}} \Psi_J(x,p;B) = -\pi \bar{\omega}^I(x)\, B_I\, \Psi_J(x,p;B)$$

for $x \neq p$, which **generalizes the genus-one differential relation for Ω**.

The Flat Connection

- Next, we **extend** to a Lie algebra $\mathcal{L}$ **freely generated** by elements a^I and b_I for $I = 1, \cdots, h$ and set $B_I = \mathrm{ad}_{b_I} = [b_I, \cdot]$.

- Our **connection** $\mathcal{J}(x,p)$, on a **Riemann surface** Σ of arbitrary **genus** h with a **marked point** $p \in \Sigma$ and valued in the **Lie algebra** $\mathcal{L}$ is then given by:

$$\mathcal{J}(x,p) = -\pi\, d\bar{x}\, \bar{\omega}^I(x)\, b_I + \pi\, dx\, \mathcal{H}^I(x;B)\, b_I + dx\, \Psi_I(x,p;B)\, a^I$$

- Working out $d_x = dx\partial_x + d\bar{x}\partial_{\bar{x}}$, we may show that:

$$d_x\mathcal{J}(x,p) - \mathcal{J}(x,p) \wedge \mathcal{J}(x,p) = \pi d\bar{x} \wedge dx\, \delta(x,p)\, [b_I, a^I]$$

  proving that the connection is **flat** (away from $x = p$).

- At **genus one**, $\mathcal{J}(x,p)$ **reduces** to the **Brown-Levin connection**, upon relabeling $a^1 = a$ and $b_1 = b$. In particular:

$$\Psi_1(x,p;B)\Big|_{h=1} = \mathrm{ad}_b\, \Omega(x-p, \mathrm{ad}_b|\tau)$$

Expansion of the Connection

- The connection $\mathcal{J}$ may be **expanded in words** in the basis (a^I, b_I):

$$\mathcal{J}(x,p) = \pi(dx\,\omega^I(x) - d\bar{x}\,\bar{\omega}^I(x))b_I + \pi\,dx \sum_{r=1}^{\infty} \partial_x \Phi^{I_1\cdots I_r}{}_J(x)\,Y^{JK}\,B_{I_1}\cdots B_{I_r}\,b_K$$
$$+\ dx \sum_{r=1}^{\infty} \left(\partial_x \Phi^{I_1\cdots I_r}{}_J(x) - \partial_x \mathcal{G}^{I_1\cdots I_{r-1}}(x,p)\delta_J^{I_r}\right) B_{I_1}\cdots B_{I_r}\,a^J$$

- Like before, the flat connection $\mathcal{J}(x,p)$ **integrates** to a homotopy-invariant path-ordered exponential $\mathbf{\Gamma}(x,y;p)$:

$$\mathbf{\Gamma}(x,y;p) = \mathrm{P}\exp\int_y^x \mathcal{J}(t,p)$$

- For example, for words with at most two letters in the basis (a^I, b_I):

$$\begin{aligned}\mathbf{\Gamma}(x,y;p) = 1 &+ a^I\Gamma_I(x,y;p) + b_I\Gamma^I(x,y;p)\\ &+ a^I a^J\Gamma_{IJ}(x,y;p) + b_I b_J\Gamma^{IJ}(x,y;p)\\ &+ a^I b_J\Gamma_I{}^J(x,y;p) + b_I a^J\Gamma^I{}_J(x,y;p) + \cdots\end{aligned}$$

Summary: Construction of higher-genus polylogs

Step	Brown-Levin construction	Higher-genus construction
1. Integration kernels	$f^{(k)}(z\|\tau) = -\int_\Sigma \frac{d^2x}{\text{Im}\,\tau}\, \partial_x g(x\|\tau)\, f^{(k-1)}(x-z\|\tau)$	$\Phi^{I_1\cdots I_r}{}_J(x) = \int_\Sigma d^2z\, \mathcal{G}(x,z)\, \bar\omega^{I_1}(z)\, \partial_z \Phi^{I_2\cdots I_r}{}_J(z) \quad (r \geq 2)$ $\mathcal{G}^{I_1\cdots I_s}(x,y) = \int_\Sigma d^2z\, \mathcal{G}(x,z)\, \bar\omega^{I_1}(z)\, \partial_z \mathcal{G}^{I_2\cdots I_s}(z,y) \quad (s \geq 1)$
2. Generating series	$\alpha\Omega(z,\alpha\|\tau) = \sum_{n=0}^{\infty} \alpha^n f^{(n)}(z\|\tau)$	$\Psi_J(x,p;B) = \omega_J(x) + \left(\partial_x \Phi^{I_1}{}_J(x) - \partial_x \mathcal{G}(x,p)\delta^{I_1}{}_J\right) B_{I_1} + \sum_{r=2}^{\infty} \left(\partial_x \Phi^{I_1 I_2\cdots I_r}{}_J(x) - \partial_x \mathcal{G}^{I_1 I_2 \cdots I_{r-1}}(x,p)\delta^{I_r}{}_J\right) \times B_{I_1} B_{I_2} \cdots B_{I_r}$
3. Flat connection	$\mathcal{J}_{\text{BL}}(x\|\tau) = -d\bar{x}\, b + \frac{\pi}{\text{Im}\,\tau}\, dx\, b + dx\, \text{ad}_b\, \Omega(x, \text{ad}_b\|\tau)\, a$	$\mathcal{J}(x,p) = -\pi\, d\bar{x}\, \bar\omega^I(x)\, b_I + \pi\, dx\, \mathcal{H}^I(x;B)\, b_I + dx\, \Psi_I(x,p;B)\, a^I$
4. Path-ordered exponential	$\text{P}\exp \int_0^x \mathcal{J}_{\text{BL}}(\cdot\|\tau) = 1 + a\,\Gamma(a;x\|\tau) + b\,\Gamma(b;x\|\tau) + ab\,\Gamma(ab;x\|\tau) + ba\,\Gamma(ba;x\|\tau) + \ldots$	$\text{P}\exp \int_y^x \mathcal{J}(t,p) = 1 + a^I\Gamma_I(x,y;p) + b_I\Gamma^I(x,y;p) + a^I a^J \Gamma_{IJ}(x,y;p) + b_I b_J \Gamma^{IJ}(x,y;p) + a^I b_J \Gamma_I{}^J(x,y;p) + b_I a^J \Gamma^I{}_J(x,y;p) + \cdots$
5. Polylogs	e.g. $\Gamma(ab;x\|\tau) = \int_0^x dt \left(2\pi i\, \frac{\text{Im}\, t}{\text{Im}\,\tau} - f^{(1)}(t\|\tau)\right)$	e.g. $\Gamma^{IJ}(x,y;p) = \pi \int_y^x \Big(dt\, \left(\partial_t \Phi^I{}_K(t) Y^{KJ} - \partial_t \Phi^J{}_K(t) Y^{KI}\right) + \pi\left(\boldsymbol{\omega}^I(t) - \bar{\boldsymbol{\omega}}^I(t)\right) \int_y^t \left(\boldsymbol{\omega}^J - \bar{\boldsymbol{\omega}}^J\right)\Big)$

Polylogarithms for Words without b_l

- The polylogarithms associated with words $\mathfrak{w}$ that do not involve any of the letters b_l are given by the following simple formula:

$$\Gamma_{l_1 l_2 \ldots l_r}(x,y;p) = \int_y^x \omega_{l_1}(t_1) \int_y^{t_1} \omega_{l_2}(t_2) \cdots \int_y^{t_{r-1}} \omega_{l_r}(t_r)$$

which we'll refer to as iterated Abelian integrals.

- These polylogarithms are **independent of the marked point p**.
- They obey the differential equations:

$$\partial_x \Gamma_{l_1 l_2 \ldots l_r}(x,y;p) = \omega_{l_1}(x) \Gamma_{l_2 \ldots l_r}(x,y;p)$$

- For the case $h=1$, we simply obtain:

$$\Gamma_{\underbrace{11\ldots1}_{r}}(x,y;z)\Big|_{h=1} = \frac{1}{r!}(x-y)^r$$

Low Letter Count Polylogarithms

- Next let us consider some cases involving the letters b_I. For the **single-letter word** b_I, we obtain:

$$\Gamma^I(x,y;p) = \pi \int_y^x (\omega^I - \bar{\varpi}^I)$$

- For **double-letter words** with **at least one letter** b_I, we obtain:

$$\Gamma^{IJ}(x,y;p) = \pi \int_y^x \left(dt \left(\partial_t \Phi^I{}_K(t) Y^{KJ} - \partial_t \Phi^J{}_K(t) Y^{KI} \right) + \pi \left(\omega^I(t) - \bar{\varpi}^I(t) \right) \int_y^t (\omega^J - \bar{\varpi}^J) \right)$$

$$\Gamma^J{}_I(x,y;p) = \int_y^x \left(dt\, \partial_t \Phi^J{}_I(t) - dt\, \partial_t \mathcal{G}(t,p) \delta^J_I + \pi \left(\omega^J(t) - \bar{\varpi}^J(t) \right) \int_y^t \omega_I \right)$$

$$\Gamma_I{}^J(x,y;p) = \int_y^x \left(-dt\, \partial_t \Phi^J{}_I(t) + dt\, \partial_t \mathcal{G}(t,p) \delta^J_I + \pi \omega_I(t) \int_y^t (\omega^J - \bar{\varpi}^J) \right)$$

Meromorphic Variants of Polylogarithms

- Lastly, let's explore an instance showcasing where the **meromorphic variants** of polylogarithms live in our function space.
- Consider again the following higher-genus polylogarithm:

$$\Gamma_I{}^J(x,y;p) = \int_y^x dt\,\Big(-\partial_t \Phi^J{}_I(t) + \delta_I^J \partial_t \mathcal{G}(t,p) + \pi\omega_I(t) Y^{JK}\left(\Gamma_K(t,y;p) - \overline{\Gamma_K(t,y;p)}\right)\Big)$$

- Upon specializing to genus $h = 1$ and setting $p = y = 0$, this reproduces the Brown-Levin polylogarithm $\Gamma(ab;p|\tau) = -\tilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix};p|\tau\right)$.
- The integrand with respect to t in the equation above can be viewed as a **higher-genus uplift** of the Kronecker-Eisenstein kernel $g^{(1)}(t|\tau)$:

$$g^J{}_I(t,y;p) = \partial_t \Phi^J{}_I(t) - \delta_I^J \partial_t \mathcal{G}(t,p) - 2\pi i \omega_I(t) Y^{JK}\,\mathrm{Im}\int_y^t \boldsymbol{\omega}_K$$

- One may verify that indeed (for $t \neq p$):

$$\partial_{\bar{t}} g^J{}_I(t,y;p) = 0$$

Conclusion

Conclusion

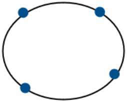
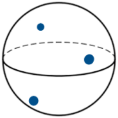
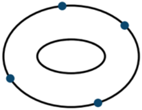
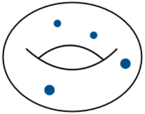
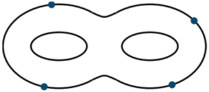
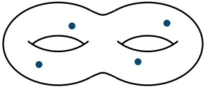
- We have presented an explicit construction of **polylogarithms** on **higher-genus** compact Riemann surfaces.
- Our construction relies on a **flat connection** whose **path-ordered exponential** plays the role of a **generating series** for **higher-genus polylogarithms**.
- The flat connection takes values in the **freely-generated Lie algebra generated by elements a^I and b_I** for $I = 1, \cdots, h$, introduced by Enriquez and Zerbini.
- Although we have strong evidence the function space of our polylogarithms is closed under integration, we have not yet proven this conjecture.
- Our construction provides the first **explicit proposal** for a **complete** set of **integration kernels beyond genus one**.

Thank you for listening!

Backup Slides

String amplitudes and special functions

- Different types of special functions emerge depending on whether we are considering **open/closed** strings, and depending on the **genus**:

	Open string	Closed string
g = 0	 (MPL’s)	 (sv. MPL’s)
g = 1	 (eMPL’s)	 eMGF’s (≈ sv. eMPL’s)
g = 2, g >= 2	 **Higher-genus polylogs** **(this talk)**	 Single-valued analogues: To be explored

Closure of MPLs Under Integration

- Any integral of a **rational function** times a **multiple polylogarithm** (MPL) can be expressed in terms of MPLs.
- This is achieved by **partial fractioning** the rational function and/or using **integration by parts** (IBP) identities. For example:

$$\frac{1}{(x-s_1)(x-s_2)} = \frac{1}{(s_1-s_2)}\left(\frac{1}{(x-s_1)} - \frac{1}{(x-s_2)}\right)$$

- After partial fractioning, we distinguish the following cases:

$$\int_0^z dt\,\frac{1}{(t-b)^k} G\left(\vec{s};t\right), \qquad \int_0^z dt\,G\left(\vec{s};t\right), \qquad \int_0^z dt\,t^k G\left(\vec{s};t\right)$$

where $0 < k \neq 1$. We then use **IBP identities** to **iteratively reduce** the value of k. For example:

$$\int_0^z dt\,\frac{1}{(t+1)^2} G(0;t) = \frac{z}{1+z} G(0;z) - G(-1;z)$$

Shuffle Algebra for Multiple Polylogarithms

- Multiple polylogarithms satisfy a **shuffle algebra**, which is expressed as:

$$G(s_1, s_2, \dots, s_k; z) \cdot G(s_{k+1}, \dots, s_r; z) = \sum_{\text{shuffles } \sigma} G(s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(r)}; z),$$

  where the sum runs over all permutations σ which are **shuffles** of $(1, \dots, k)$ and $(k+1, \dots, r)$, **preserving the relative order** of $1, 2, \dots, k$ and of $k+1, \dots, r$.

- A **simple example** of the shuffle product of two multiple polylogarithms is:

$$G(s_1; z) \cdot G(s_2; z) = G(s_1, s_2; z) + G(s_2, s_1; z).$$

- The proof of the shuffle product formula relies on the integral representation of multiple polylogarithms. In fact, a shuffle algebra structure holds for **all the homotopy-invariant iterated integrals** which we consider.

Removing Trailing Zeros

- Multiple polylogarithms with **trailing zeroes** do **not** have a Taylor expansion in z around $z = 0$, but **logarithmic singularities** at $z = 0$.
- We can use the shuffle product to **remove trailing zeros**, **separating** these logarithmic terms, such that the rest has a regular expansion around $z = 0$.
- For example, for $G(s_1, 0; z)$ with $s_1 \neq 0$, we have:

$$G(s_1, 0; z) = G(0; z)\, G(s_1; z) - G(0, s_1; z)\,.$$

- Both $G(s_1; z)$ and $G(0, s_1; z)$ are **free** of trailing zeros. We then define the **special cases**:

$$G(0; z) = \log(z) \qquad\qquad G\left(\vec{0}_n; z\right) = \frac{1}{n!}\log(z)^n,$$

where $\vec{0}_n$ denotes a sequence of n zeros. These definitions follow the **tangential basepoint prescription**:

$$\int_{0+\varepsilon}^{x} \frac{dt}{t} = \log(x) - \log(\epsilon) \to \log(x)$$

for a prescribed tangent vector (in $\mathbb{C}$) with $|\varepsilon| \ll 1$.

Meromorphic Variant

- We can define a **meromorphic counterpart** of the doubly-periodic Kronecker-Eisenstein series and its expansion coefficients $g^{(n)}(z|\tau)$:

$$\frac{\vartheta_1'(0|\tau)\vartheta_1(z+\alpha|\tau)}{\vartheta_1(z|\tau)\vartheta_1(\alpha|\tau)} = \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z|\tau)$$

- The meromorphic integration kernels $g^{(n)}(z|\tau)$ are **multiple-valued on the torus**, and actually **live on the universal covering space**, which is $\mathbb{C}$.
- Brown-Levin polylogarithms associated with **words** $\mathfrak{w} \to ab\cdots b$ **reduce to a single integral over the meromorphic kernels**. For example:

$$\Gamma(ab;z|\tau) = \int_0^z dt \left(2\pi i \frac{\operatorname{Im} t}{\operatorname{Im} \tau} - f^{(1)}(t|\tau)\right) = -\int_0^z dt\, g^{(1)}(t|\tau) = -\tilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix}; z|\tau\right)$$

- **More generally**, $\Gamma(ab\cdots b;z|\tau)$ can be expressed as:

$$\Gamma(a\underbrace{b\cdots b}_{n};z|\tau) = (-1)^n \int_0^z dt\, g^{(n)}(t|\tau) = (-1)^n \tilde{\Gamma}\left(\begin{smallmatrix}n\\0\end{smallmatrix}; z|\tau\right)$$

Modular Transformations

- A **new canonical basis** $\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{B}}$ is obtained by applying a **modular transformation** $M \in Sp(2h, \mathbb{Z})$, such that $M^t \mathfrak{J} M = \mathfrak{J}$.
- Under a modular transformation, we have:

$$\tilde{\omega} = \omega \, (C\Omega + D)^{-1}, \quad \tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1}$$
$$\tilde{Y} = \left(\bar{\Omega} C^t + D^t\right)^{-1} \, Y \, (C\Omega + D)^{-1}$$

- The **moduli space of compact Riemann surfaces of genus** h will be denoted by $\mathcal{M}_h$.
- The moduli space $\mathcal{M}_h$ for $h = 1, 2, 3$ may be identified with $\mathcal{H}_h / Sp(2h, \mathbb{Z})$ provided we remove from the **Siegel upper half space** $\mathcal{H}_h$ for $h = 2, 3$ all elements which correspond to disconnected surfaces, and take into account the effect of automorphisms including the involution on the hyper-elliptic locus for $h = 3$.
- For $h \geq 4$, the moduli space $\mathcal{M}_h$ is a complex co-dimension $\frac{1}{2}(h-2)(h-3)$ subspace of $\mathcal{H}_h / Sp(2h, \mathbb{Z})$ known as the **Schottky locus**.

Definition of Modular Tensors

- **Modular tensors**, defined on **Torelli space**, employ a unique homology basis for $\mathfrak{A}$ and $\mathfrak{B}$ cycles.
- They **generalize modular forms**, replacing $(C\tau + D)$ of $SL(2,\mathbb{Z})$ with Q and $R = Q^{-1}$.

$$Q = Q(M,\Omega) = C\Omega + D$$
$$R = R(M,\Omega) = (C\Omega + D)^{-1}$$

- The tensors ω_I, ω^I, Y_{IJ}, and its inverse Y^{IJ} transform as follows under a modular transformation:

$$\tilde{\omega}_I = \omega_{I'} R^{I'}{}_I \qquad \tilde{Y}_{IJ} = Y_{I'J'} \bar{R}^{I'}{}_I R^{J'}{}_J$$
$$\tilde{\omega}^J = \bar{Q}^J{}_{J'} \omega^{J'} \qquad \tilde{Y}^{IJ} = Q^I{}_{I'} \bar{Q}^J{}_{J'} Y^{I'J'}$$

- A modular tensor $\mathcal{T}$ of arbitrary rank **transforms** as follows:

$$\tilde{\mathcal{T}}^{I_1,\cdots,I_n;J_1,\cdots,J_{\bar{n}}}(\tilde{\Omega}) = Q^{I_1}{}_{I'_1} \cdots Q^{I_n}{}_{I'_n} \bar{Q}^{J_1}{}_{J'_1} \cdots \bar{Q}^{J_{\bar{n}}}{}_{J'_{\bar{n}}} \mathcal{T}^{I'_1,\cdots,I'_n;J'_1,\cdots,J'_{\bar{n}}}(\Omega)$$

- The tensors Y_{IJ} and Y^{IJ} may be used to lower and raise indices, respectively, and can be made to compensate any anti-holomorphic automorphy factor.

Modular Properties of the Brown-Levin Construction

- Lastly, let us consider the **modular properties** of the Brown-Levin construction. Consider a modular transformation on the modulus τ, z, and α given by:
$$\tau \to \tilde{\tau} = \frac{A\tau + B}{C\tau + D}, \quad z \to \tilde{z} = \frac{z}{C\tau + D}, \quad \alpha \to \tilde{\alpha} = \frac{\alpha}{C\tau + D}$$
where $A, B, C, D \in \mathbb{Z}$ with $AD - BC = 1$.
- The Kronecker-Eisenstein series Ω and the functions $f^{(n)}$ transform as **modular forms of weight $(1, 0)$ and $(n, 0)$**, respectively:
$$\Omega(\tilde{z}, \tilde{\alpha}|\tilde{\tau}) = (C\tau + D)\Omega(z, \alpha|\tau),$$
$$f^{(n)}(\tilde{z}|\tilde{\tau}) = (C\tau + D)^n f^{(n)}(z|\tau)$$
- These transformation properties can be established by using the transformation properties of the **Jacobi θ-function**:
$$\theta_1(\tilde{z}, \tilde{\alpha}|\tilde{\tau}) = \epsilon(C\tau + D)^{\frac{1}{2}} e^{i\pi C z^2/(C\tau+D)}\theta_1(z|\tau), \quad \epsilon^8 = 1$$
- Or the **modular invariance of the functions $g_n(z|\tau)$** along with the relation
$$f^{(n)}(z|\tau) = -\partial_z^n g_n(z|\tau)$$

Modular Properties of the Brown-Levin Construction

- The **modular properties** of the Brown-Levin connection and polylogarithms are most transparent by assigning the following **transformation law** to the generators a, b:

$$a \to \tilde{a} = (C\tau + D)a + 2\pi i C b, \quad b \to \tilde{b} = \frac{b}{C\tau + D}$$

- This choice renders the flat connection $\mathcal{J}_{\rm BL}$ **modular invariant** under the transformation.
- The **extra contribution** $2\pi i C b$ to $\tilde{a}$ is engineered to compensate the transformation of the first term in the expression for the connection:

$$\frac{\pi \, d\tilde{z}}{\operatorname{Im} \tilde{\tau}} \, \tilde{b} = \frac{C\bar{\tau} + D}{C\tau + D} \frac{\pi \, dz}{\operatorname{Im} \tau} \, b$$

Modular Invariance of the Connection

- More generally, at higher-genus we may define an **alternative basis** $(\hat{a}^I, b_I)$ of generators of the Lie algebra $\mathcal{L}$:

$$\hat{a}^I = a^I + \pi Y^{IJ} b_J$$

- In this basis, the connection $\mathcal{J}(x,p)$ takes on a **simplified form**:

$$\mathcal{J}(x,p) = -\pi\, d\bar{x}\, \bar{\omega}^I(x)\, b_I + dx\, \Psi_I(x,p;B)\, \hat{a}^I$$

- A **modular transformation** $M \in Sp(2h,\mathbb{Z})$, acts on $\bar{\omega}^I$, B_I, $\mathcal{H}_I$, and Ψ_I, and on the Lie algebra generators a^I and b_I by:

$$a^I \to \tilde{a}^I = Q^I{}_J\, a^J + 2\pi i\, C^{IJ}\, b_J$$
$$b_I \to \tilde{b}_I = b_J\, R^J{}_I$$

- Then also

$$\hat{a}^I \to \tilde{\hat{a}}^I = Q^I_J \hat{a}^J$$

- The connection $\mathcal{J}(x,p)$ is seen to be **manifestly invariant** under $Sp(2h,\mathbb{Z})$.

Polylogarithms In The Hatted Basis

- In the basis $(\hat{a}^I, b_I)$, the expansion is given by:

$$\begin{aligned}\boldsymbol{\Gamma}(x,y;p) = 1 &+ \hat{a}^I \hat{\Gamma}_I(x,y;p) + b_I \hat{\Gamma}^I(x,y;p) \\ &+ \hat{a}^I \hat{a}^J \hat{\Gamma}_{IJ}(x,y;p) + b_I b_J \hat{\Gamma}^{IJ}(x,y;p) \\ &+ \hat{a}^I b_J \hat{\Gamma}_I{}^J(x,y;p) + b_I \hat{a}^J \hat{\Gamma}^I{}_J(x,y;p) + \cdots\end{aligned}$$

- Identifying term by term in both expansions gives the relations $\Gamma_I = \hat{\Gamma}_I$ and $\Gamma_{IJ} = \hat{\Gamma}_{IJ}$, as well as the following relations:

$$\begin{aligned}\hat{\Gamma}^I &= \Gamma^I - \pi Y^{IJ} \Gamma_J \\ \hat{\Gamma}^I{}_J &= \Gamma^I{}_J - \pi Y^{IK} \Gamma_{KJ} \\ \hat{\Gamma}_I{}^J &= \Gamma_I{}^J - \pi \Gamma_{IK} Y^{KJ} \\ \hat{\Gamma}^{IJ} &= \Gamma^{IJ} - \pi Y^{IK} \Gamma_K{}^J - \pi \Gamma^I{}_K Y^{KJ} + \pi^2 Y^{IK} \Gamma_{KL} Y^{LJ}\end{aligned}$$

- The polylogarithms $\hat{\Gamma}(x,y;p)$ in the basis $(\hat{a}^I, b_I)$ are **modular tensors** by the *$Sp(2h,\mathbb{Z})$* **invariance** of the connection $\mathcal{J}(x,p)$.

The Arakelov Green Function

- The **Arakelov Green function** $\mathcal{G}(x,y|\Omega)$ on $\Sigma \times \Sigma$ is a **single-valued version** of the Green function, defined by: [D'Hoker, Green, Pioline, arXiv:1712.06135] [G. Faltings, Ann. Math., 119(2), 1984]

$$\partial_{\bar{x}}\partial_x \mathcal{G}(x,y|\Omega) = -\pi\delta(x,y) + \pi\kappa(x)\,, \qquad \int_\Sigma \boldsymbol{\kappa}(x)\mathcal{G}(x,y|\Omega) = 0$$

- The **string Green function** is given in terms of the **prime form** $E(x,y)$ by:

$$G(x,y) = -\log|E(x,y)|^2 + 2\pi\left(\mathrm{Im}\int_y^x \boldsymbol{\omega}_I\right)\left(\mathrm{Im}\int_y^x \boldsymbol{\omega}^I\right)$$

- The prime form $E(x,y)$ is a unique form that is **holomorphic** in x and y and **vanishes linearly** as x approaches y.
- An explicit formula for $\mathcal{G}(x,y)$ may then be given in terms of the non-conformally invariant string Green function $G(x,y)$:

$$\mathcal{G}(x,y) = G(x,y) - \gamma(x) - \gamma(y) + \gamma_0$$

The Arakelov Green Function

- The functions $\gamma(x)$ and γ_0 are given by:

$$\gamma(x) = \int_\Sigma \boldsymbol{\kappa}(z) G(x,z) \qquad \gamma_0 = \int_\Sigma \boldsymbol{\kappa}\gamma$$

- The **Kähler form $\boldsymbol{\kappa}$** is given by the pull-back to Σ under the Abel map of the unique translation invariant Kähler form on the Jacobian variety $J(\Sigma) = \mathbb{C}^h/(\mathbb{Z}^h + \Omega\mathbb{Z}^h)$, normalized to unit volume:

$$\boldsymbol{\kappa} = \frac{i}{2h} \boldsymbol{\omega}_I \wedge \bar{\boldsymbol{\omega}}^I = \kappa(z)\, d^2 z \qquad \int_\Sigma \boldsymbol{\kappa} = 1$$

- Both $\boldsymbol{\kappa}$ and $\mathcal{G}(x,y)$ are **conformally invariant**.
- The Arakelov Green function also obeys the following derivatives:

$$\partial_x \partial_y \mathcal{G}(x,y) = -\partial_x \partial_y \ln E(x,y) + \pi\, \omega_I(x)\, \omega^I(y)$$
$$\partial_x \partial_{\bar{y}} \mathcal{G}(x,y) = \pi\, \delta(x,y) - \pi\, \omega_I(x)\, \bar{\omega}^I(y)$$

Simplified Representations

- The polylogarithms with **upper indices** admit **simplified representations** in terms of the **iterated abelian integrals**, their **complex conjugates** and **contractions with Y^{IJ}**.
- For words with a **single letter b_I** we have:

$$\Gamma^I(x,y;p) = \pi Y^{IJ}\left(\Gamma_J(x,y;p) - \overline{\Gamma_J(x,y;p)}\right)$$

- For **two-letter words that contain at least one b_I**, we have:

$$\Gamma_I{}^J(x,y;p) = \pi Y^{JK}\Gamma_{IK}(x,y;p) + \int_y^x dt\left(-\partial_t\Phi^J{}_I(t) + \delta_I^J\partial_t\mathcal{G}(t,p) - \pi\omega_I(t)Y^{JK}\overline{\Gamma_K(t,y;p)}\right)$$

$$\begin{aligned}\Gamma^I{}_J(x,y;p) &= \pi Y^{IK}\left(\Gamma_{KJ}(x,y;p) - \Gamma_J(x,y;p)\overline{\Gamma_K(x,y;p)}\right)\\ &\quad+ \int_y^x dt\left(\partial_t\Phi^I{}_J(t) - \delta_J^I\partial_t\mathcal{G}(t,p) + \pi\omega_J(t)Y^{IK}\overline{\Gamma_K(t,y;p)}\right)\end{aligned}$$

$$\begin{aligned}\Gamma^{IJ}(x,y;p) &= \pi^2 Y^{IK}Y^{JL}\left(\Gamma_{KL}(x,y;p) + \overline{\Gamma_{KL}(x,y;p)} - \overline{\Gamma_K(x,y;p)}\Gamma_L(x,y;p)\right)\\ &\quad+ \pi\int_y^x dt\left(\partial_t\Phi^I{}_K(t)Y^{KJ} - \partial_t\Phi^J{}_K(t)Y^{KI}\right.\\ &\quad\left.+ \pi\omega^J(t)Y^{IK}\overline{\Gamma_K(t,y;p)} - \pi\omega^I(t)Y^{JK}\overline{\Gamma_K(t,y;p)}\right)\end{aligned}$$