The Nonperturbative S-matrix Bootstrap

EPFL

João Penedones

Based on work with: Correia, Elias-Miró, Guerrieri, Haring, Hebbar, Homrich, Karateev, Kuhn, Marucha, Meineri, Murali, Paulos, Sahoo, Toledo, van Rees, Vieira, Vuignier

Amplitudes 2023 - CERN - 11/08/2023

Outline

- Introduction

- Applications

  - $2\to 2$ Supergraviton Scattering in $d = 9, 10, 11$

    ['21 Guerrieri, JP, Vieira]
    ['22 Guerrieri, Murali, JP, Vieira]

  - a-anomaly bootstrap in 4d

    ['22 Karateev, Marucha, JP, Sahoo]

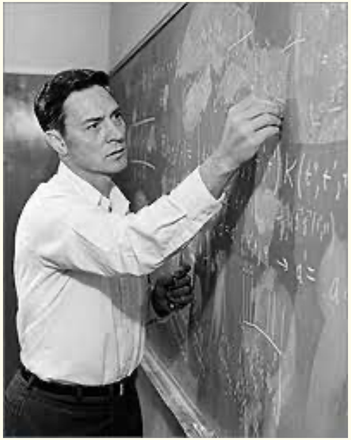
- Discussion

Introduction

The S-matrix program


Heisenberg

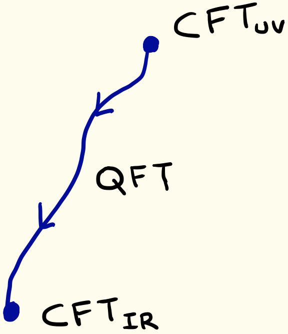
The "observable quantities" in the theory of elementary particles
1943


Chew

"Nuclear Democracy"
'60

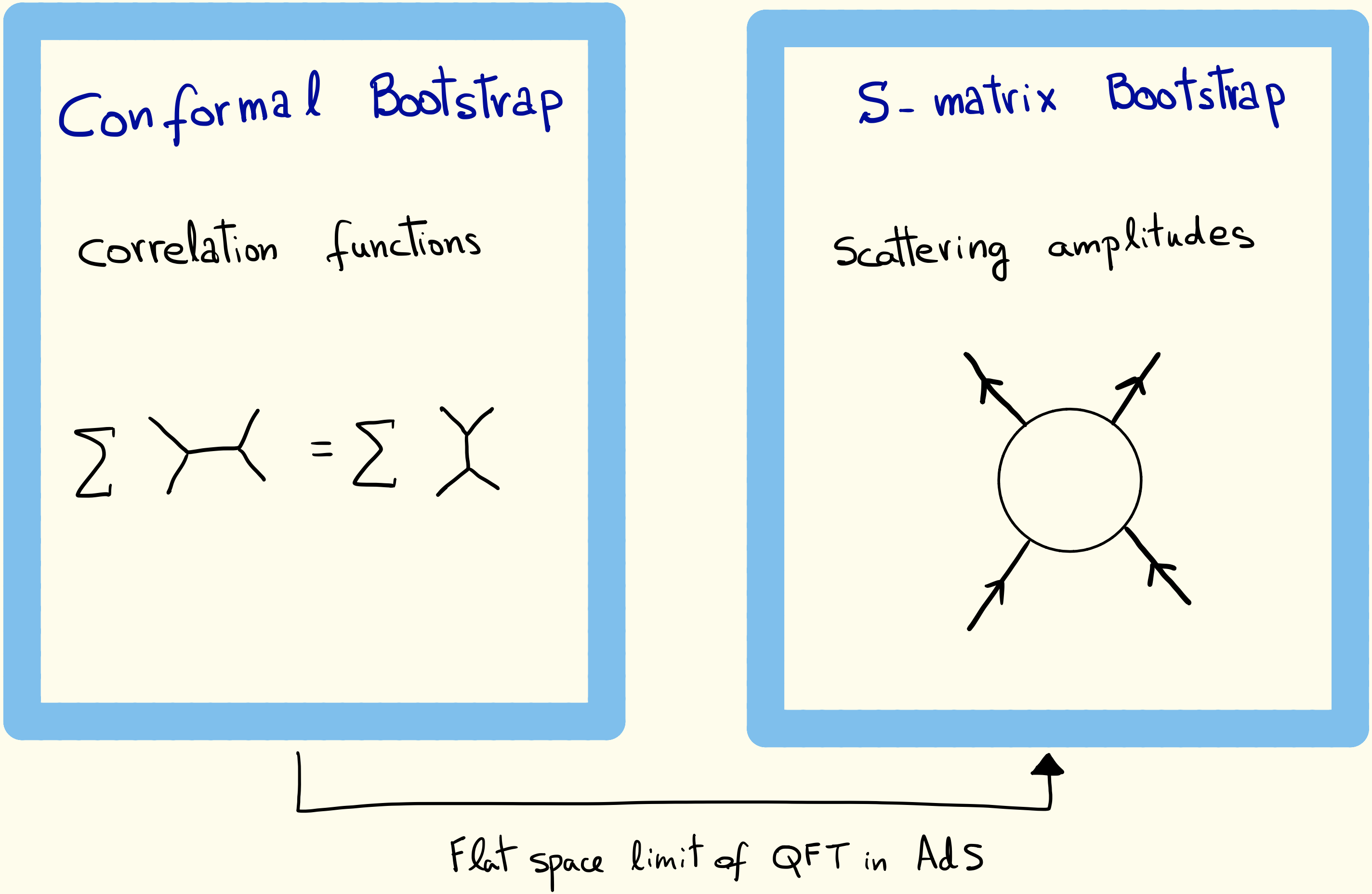
Modern QFT Bootstrap

Goal: map out the space of QFTs and develop nonperturbative methods to compute their observables.



Bootstrap approach: bound the space of theories by imposing consistency conditions on physical observables.

Strategy: extend recent success in CFT to QFT.


Conformal Bootstrap
correlation functions
S-matrix Bootstrap
Scattering amplitudes
Flat space limit of QFT in AdS

Applications

Some Applications

2D phonons with Elias-Miró, Guerrieri, Hebbar, Vieira

pions with Guerrieri, Vieira

TODAY → super-gravitons with Guerrieri, Murali, Vieira

photons with Meineri, Haring, Hebbar, Karateev

massive scalars with Homrich, Paulos, Toledo, van Rees, Vieira

massive fermions with Hebbar, Karateev

TODAY → 4d a-anomaly with Karateev, Marucha, Sahoo

2d central charge with Correia, Karateev, Kuhn, Vaignier

...

Goals: - bound the leading Wilson coefficients of the EFT.

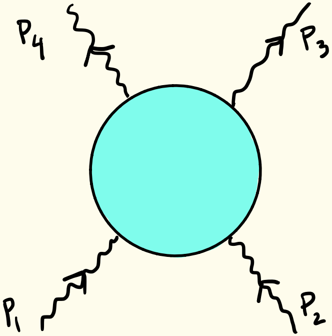
- bound the interaction strength (cubic & quartic couplings)

- bound a-anomaly of the UV CFT

...

Maximal Supergravity $d = 9, 10, 11$

$2 \to 2$ Supergraviton Scattering



$$\mathbb{A}_{2\to 2} = \mathbf{R}^4 A(s,t,u)\,.$$

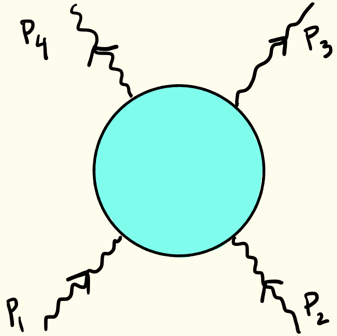
crossing symmetric function of

$$\begin{cases} s = (p_1+p_2)^2 \\ t = (p_1-p_3)^2 \\ u = (p_1-p_4)^2 \end{cases}$$

$$s+t+u = 0$$

pre-factor fixed by SUSY

$2 \to 2$ Supergraviton Scattering



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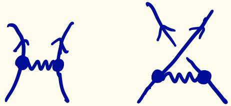


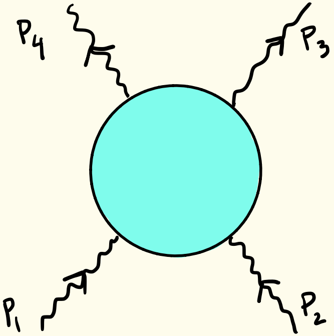
Focus on charged scalar :
[axi-dilaton in IIB]

$$T(s,t,u) \equiv s^4 A(s,t,u) = -8\pi G_N \left(\frac{s^2}{t} + \frac{s^2}{u}\right) + \ldots$$

$$\frac{T(s,t,u)}{8\pi G_N} = s^4 \left(\frac{1}{stu} + \alpha\,\ell_P^6 + \ldots\right) \equiv s^4 A(s,t,u)$$

$$16\pi G_N = (2\pi)^{d-3} \ell_P^{d-2}$$

$2 \to 2$ Supergraviton Scattering



crossing symmetric function of $\begin{cases} s = (p_1+p_2)^2 \\ t = (p_1-p_3)^2 \\ u = (p_1-p_4)^2 \end{cases}$

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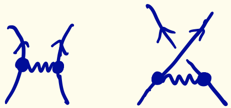

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$$16\pi G_N = (2\pi)^{d-3} \ell_P^{d-2}$$

What are the allowed values of α ?

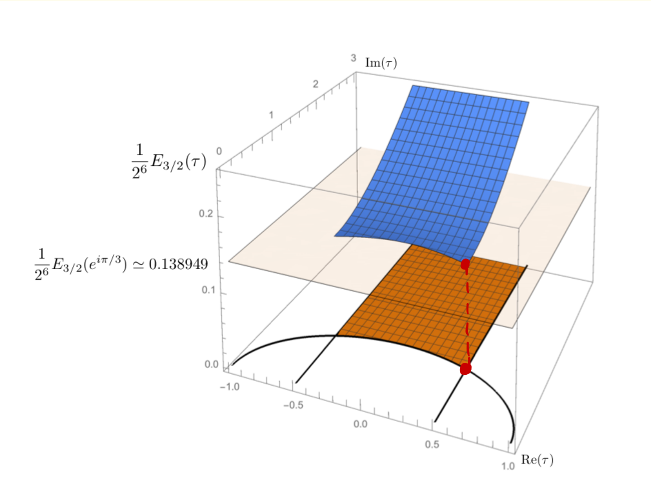
EFT: $S = \dfrac{1}{(2\pi)^7 \ell_p^8} \displaystyle\int d^{10}x \sqrt{-g} \left[R + \# \alpha\, \ell_p^6 R^4 + \dots \right]$

Superstrings $d=10$

IIB

$$\alpha^{\mathrm{IIB}} = \frac{1}{2^6} E_{\frac{3}{2}}(\tau,\bar\tau) \geq \frac{1}{2^6} E_{\frac{3}{2}}(e^{i\pi/3}, e^{-i\pi/3}) \approx 0.1389$$

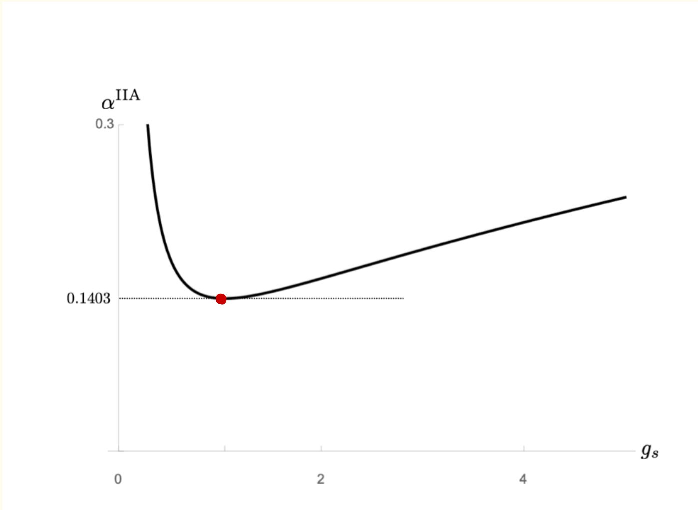
$\tau = x_s + i/g_s$



['97 Green, Gutperle]

IIA

$$\alpha^{\mathrm{IIA}} = \frac{\zeta(3)}{32 g_s^{3/2}} + g_s^{1/2}\frac{\pi^2}{96} \geq \frac{\pi^{3/2}(\zeta(3))^{1/4}}{24\sqrt{3}} \approx 0.1403$$



$$\alpha^{\mathrm{IIB}} = \alpha^{\mathrm{IIA}} + O\left(e^{-\frac{2\pi}{g_s}}\right) .$$

M-theory $d = 11$

$$\alpha = \frac{(2\pi)^2}{3 \times 2^7} \simeq 0.1028$$

String theory $d = 9$

$$\alpha = \frac{1}{2^6}\left[\mathcal{V}^{-3/7} E_{3/2}(\tau, \bar{\tau}) + \frac{2\pi^2}{3} \mathcal{V}^{4/7} \right] \gtrsim 0.2417$$

related to compactification radius

See ['23 Bossard, Loty] for lower bounds in $d = 6, 7, 8$.

Numerical S-matrix Bootstrap

PRIMAL ALGORITHM

['17 Paulos, JP, Toledo, Van Rees, Vieira]

1. Write amplitude ansatz obeying: $\begin{cases} \text{Lorentz invariance (+ SUSY)} \\ \text{Crossing symmetry} \\ \text{Analyticity} \\ \text{Low energy from EFT} \end{cases}$

2. Impose unitarity of each partial wave

3. Minimize a linear observable of the amplitude

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   # of parameters $\sim N^2$

2. Impose unitarity of each partial wave $\longrightarrow$ spin $\ell \leq L$

3. Minimize a linear observable of the amplitude

4. Extrapolate $L \to \infty$ and $N \to \infty$

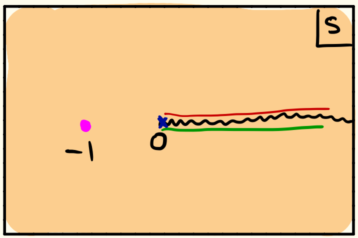
1. Analytic and crossing symmetric ansatz:

free parameters

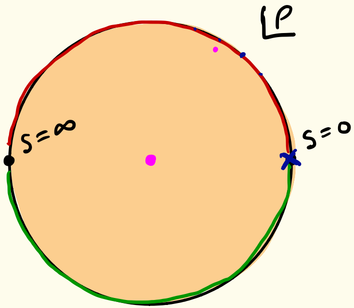
$$T = 8\pi G_N \, s^4 \left[\underbrace{\frac{1}{s\,t\,u}}_{\text{SUGRA}} + \underbrace{\sum_{a+b+c\le N}{}' \beta_{(abc)}\, \rho(s)^a \rho(t)^b \rho(u)^c}_{\text{"UV completion"}} \right]$$

$s+t+u=0$

$[\ell_p = 1]$

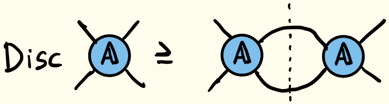


$$\rho(s) = \frac{1-\sqrt{-s}}{1+\sqrt{-s}}$$



2. Unitarity constraints

['98 Bern et al.]



$\Rightarrow$

$$\mathrm{Disc} A \geq s^4 \int d\mathrm{LIPS}\, A \times A\,.$$

$$\sum_{\text{two pt}} \mathbf{R}^4_{12\to\text{two pt}} \mathbf{R}^4_{\text{two pt}\to 34} = \mathbf{R}^4_{12\to 34} \times s^4$$

Unitarity of $\mathbb{A}_{2\to 2} = \mathbf{R}^4 A(s,t,u)\,.$ $\Leftrightarrow$ Unitarity of $T = s^4 A(s,t,u)$

$\mathrm{Disc}\, T \geq \int d\mathrm{LIPS}\; T \times T$

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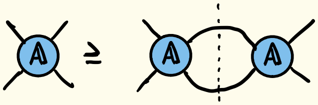
['98 Bern et al.]

Disc 

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$\Rightarrow$

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Unitarity of $\mathbb{A}_{2\to 2} = \mathbf{R}^4 A(s,t,u)\,.$ $\Longleftrightarrow$ Unitarity of $T = s^4 A(s,t,u)$

$$\Rightarrow \quad \boxed{|S_\ell(s)|^2 \le 1} \quad \text{for} \quad \begin{cases} \ell = 0, 2, 4, \dots, L \\ s > 0 \quad \text{[grid]} \end{cases}$$

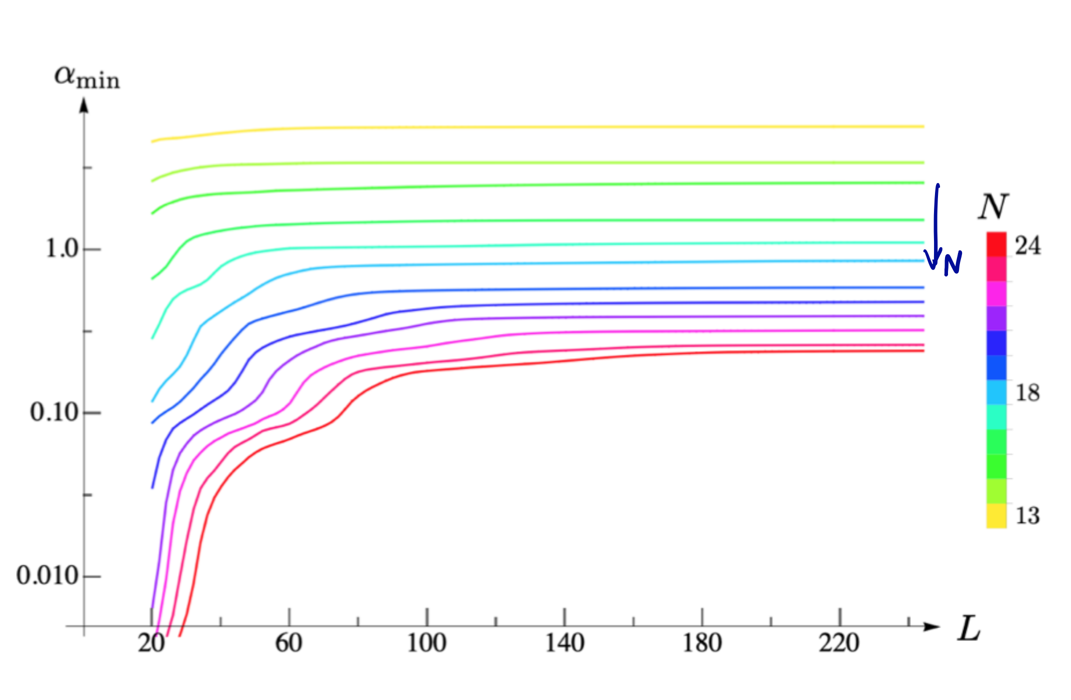
Gegenbauer polynomial

$$S_\ell(s) = 1 + i N_d\, s^{\frac{d}{2}-2} \int_{-1}^{1} dz\, (1-z^2)^{\frac{d-4}{2}}\, C_\ell^{\frac{d-3}{2}}(z)\, T(s,z) \qquad \begin{cases} t = -\frac{s}{2}(1-z) \\ u = -\frac{s}{2}(1+z) \end{cases}$$

$\cos\theta$ ← scattering angle

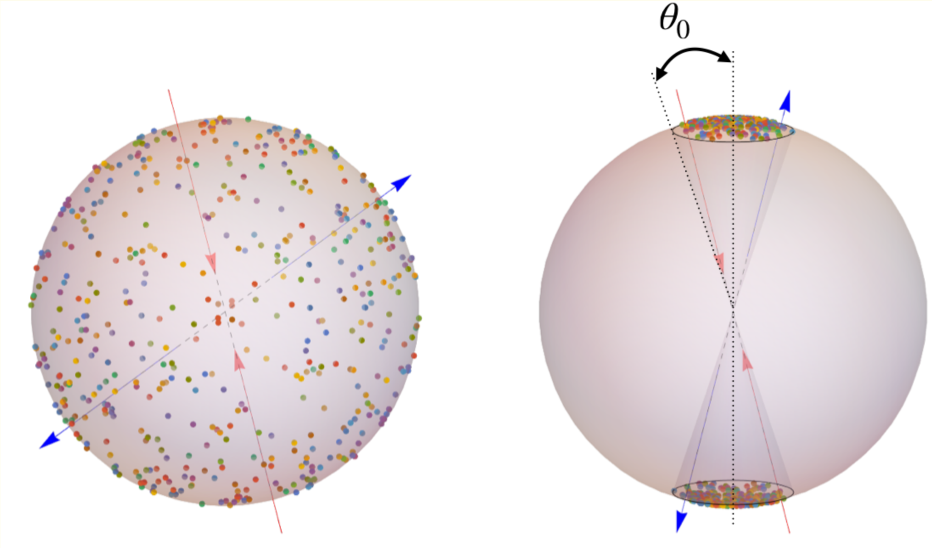
3. Minimize α for each N and L.

$\rightarrow$ Semi-definite Programming SDPB ['15 Simmons-Duffin]



$d = 10$

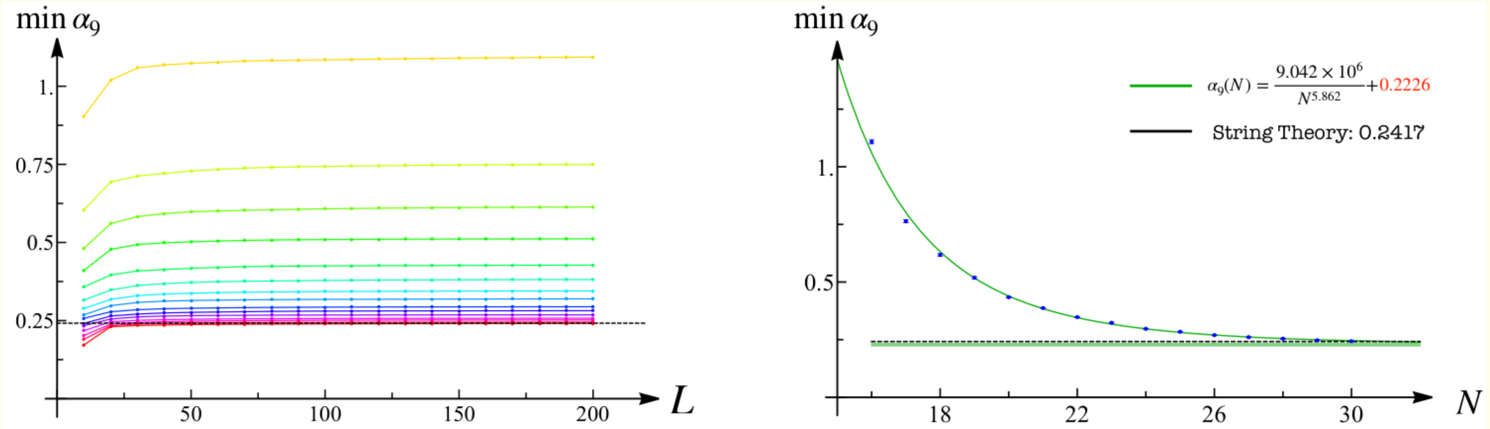
Positivity in the Sky



$$\boxed{(\mathrm{Im}\,T)_{ij} \equiv \mathrm{Im}\,T\left(s, t=-\frac{s}{2}(1-\cos\theta_{ij}), u=-\frac{s}{2}(1+\cos\theta_{ij})\right) \geq 0}$$

This is not an independent condition but it accelerates convergence in spin ($L \to \infty$).

4. Extrapolate $L \to \infty$ and $N \to \infty$



Minimal values of α

Dimension	Bootstrap	String/M-Theory
9	0.223 ± 0.002	0.241752
10	0.124 ± 0.003	0.138949
11	0.101 ± 0.005	0.102808

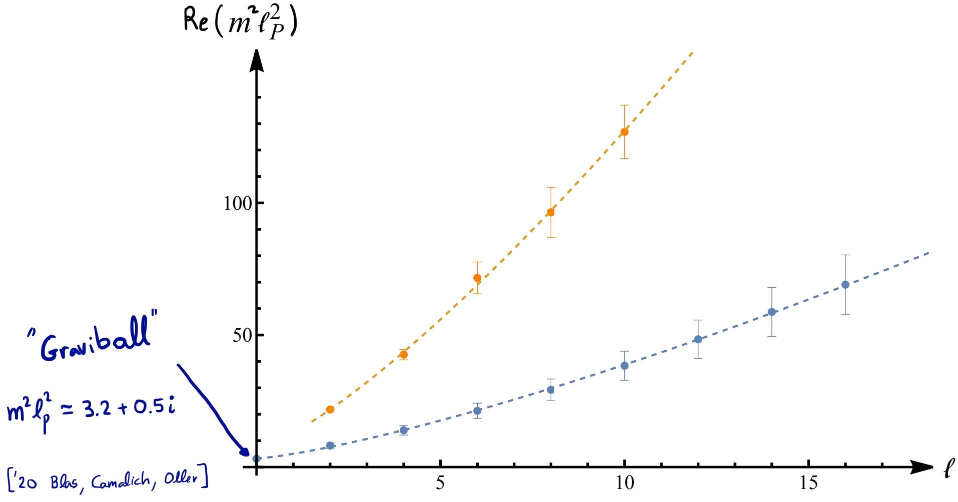
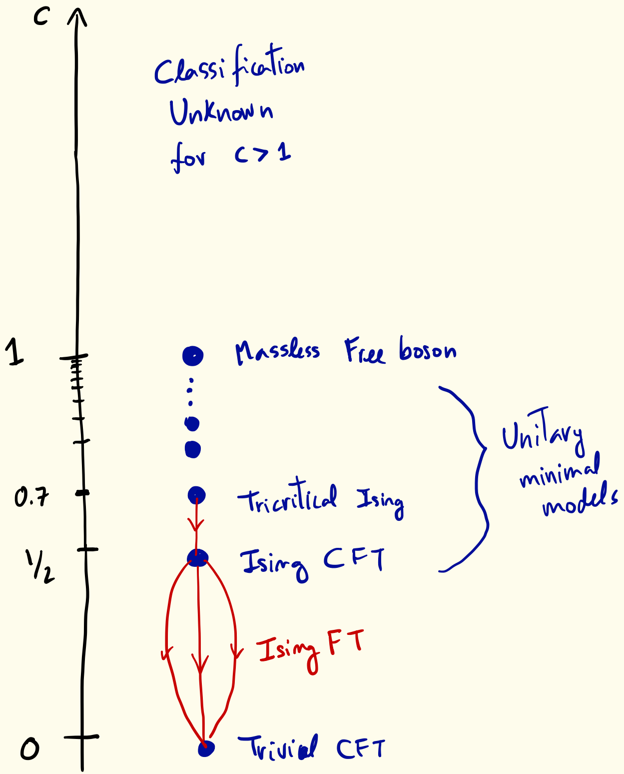


Figure 6. The first two Regge trajectories in 10d for N=30 and L=200. The error bars represent the widths and the resonances lie on curved trajectories that scale approximately like $\ell^{1.3}$. More details in appendix F.

Bootstrapping the a-anomaly in 4d

$\boxed{d=2}$

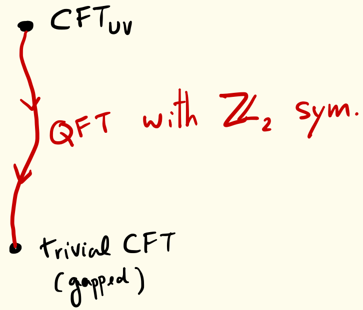


$\boxed{c_{UV} > c_{IR}}$ ['86 Zamolodchikov]

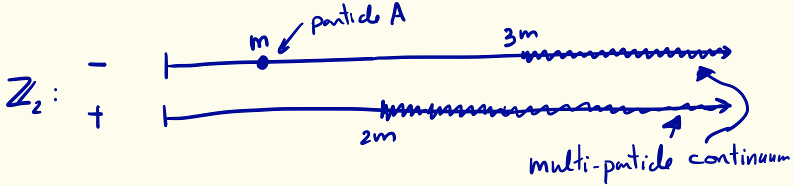
$\boxed{d=4}$

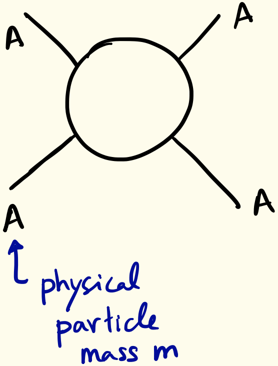
$\boxed{a_{UV} > a_{IR}}$ ['88 Cardy]
['11 Komargodski, Schwimmer]

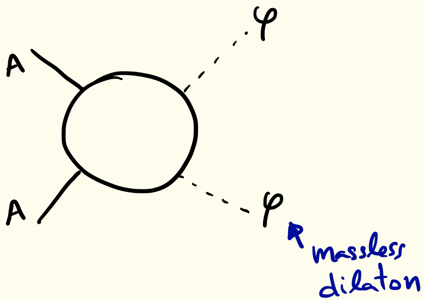
S-matrix Bootstrap setup

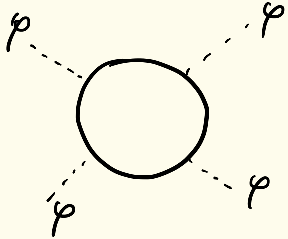


Mass Spectrum









$$T_{\varphi\varphi\to\varphi\varphi} = \frac{1}{f^4}\, a^{UV} \left(s^2+t^2+u^2\right) + O(s^3)$$

['11 Komargodski, Schwimmer]

Results

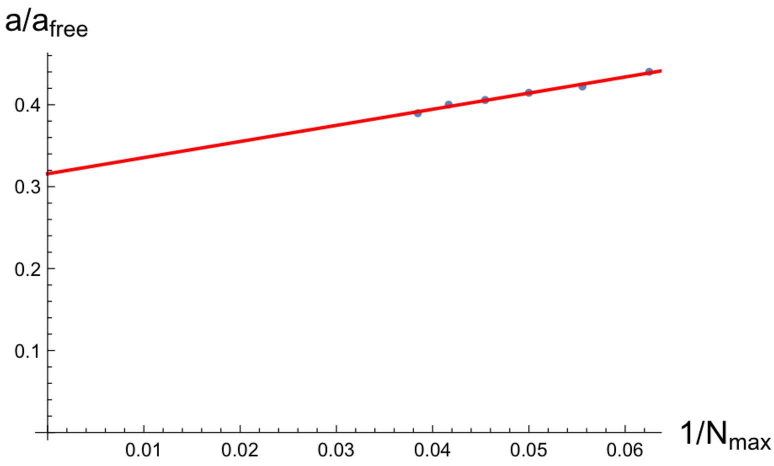
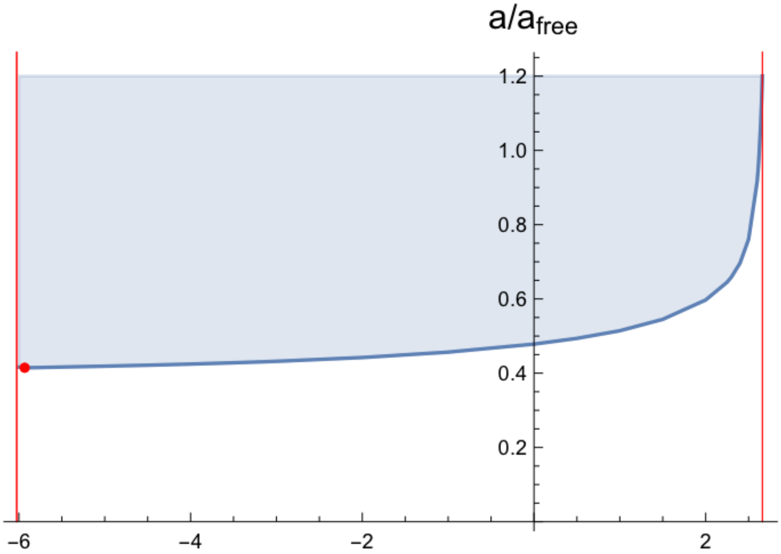


Figure 5. Minimum possible value of the a-anomaly without any further assumptions as a function of $1/N_{max}$ with $L_{max} = N_{max} + 10$. The numerical results are depicted by blue points. Linear extrapolation to $N_{max} \to \infty$ depicted by the red line gives 0.316 ± 0.015 for the minimum of a/a_{free}.

$$\boxed{\frac{a}{a_{FB}} \gtrsim 0.32}$$



$N_{max} = 20$

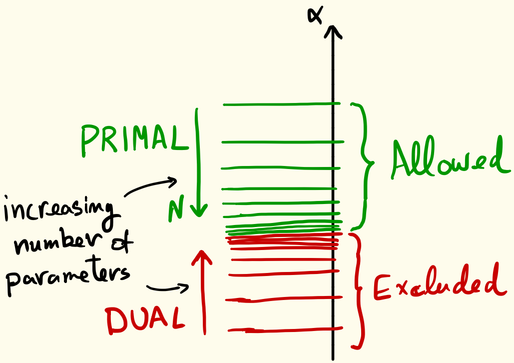
$L_{max} = 30$

quartic coupling

$$\lambda_0 \equiv \frac{1}{32\pi} T_{AA\to AA}\left(s=t=u=\frac{4}{3}m^2\right)$$

Discussion

– Dual formulation would be very useful.



['77 Lopez, Mennessier]

[...]

['19 Cordova, He, Kruczenski, Vieira]

['20 Guerrieri, Homrich, Vieira]

['21 Guerrieri, Elias Miró]

D = 2

['21 He, Kruczenski]

['21 Guerrieri, Sever]

D > 2

Unitarity: $\boxed{2\,\mathrm{Im}\, f_\ell(s) \geq |f_\ell(s)|^2}$ $\iff$ $|\underbrace{S_\ell(s)}_{=1+if_\ell(s)}|^2 \leq 1$

vs.

Positivity: $\boxed{\mathrm{Im}\, f_\ell(s) \geq 0}$

$\rightarrow$ Unitarity is stronger than positivity.

Example: Supergraviton scattering ($d=10$)

$$\begin{cases} \text{Unitarity} \Rightarrow \alpha \gtrsim 0.14 \\ \text{positivity} \Rightarrow \alpha \geq 0 \end{cases}$$

Future work in graviton scattering

- Other spacetime dimensions

  $5 \leq d \leq 8$ Need to take into account log's from loops

  $d = 4$ IR divergences

- Inelastic effects from $2 \to 3$ and/or Black Hole production

- Other Wilson coefficients

- No SUSY.

  ['21 Bern, Kosmopoulos, Zhiboedov]
  ['22 Caron-Huot, Li, Parra-Martinez, Simmons-Duffin]

Big open questions

- Prove (or drop) Maximal Analyticity
  - in perturbation theory? ['21 Correia, Sever, Zhiboedov]
  - from the flat space limit of QFT in AdS? ['23 van Rees, Zhao]

- Anomalous thresholds

- Multi-particle amplitudes $(n \to m)$

Thank You!