Large Charge 't Hooft Limit

Shota Komatsu

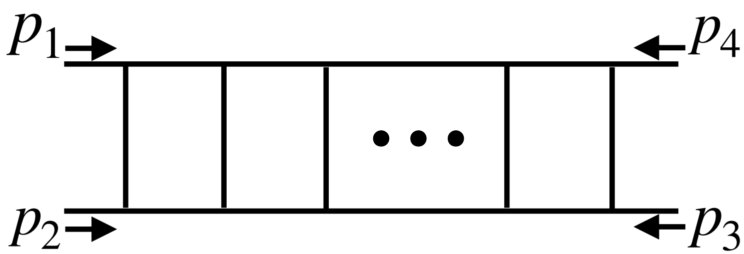
Based on **arXiv: 2306.00929** with **Joao Caetano** (CERN), **Yifan Wang** (NYU)
& works / discussion in progress + Jingxiang Wu (Oxford) (1/16-BPS)
+ Nicola Dondi (Bern) ("dual" description)

Let’s start with some beautiful formula

Resummation of off-shell massless ladders

[Broadhurst, Davydychev '10]

$$\sum_n \lambda^n$$



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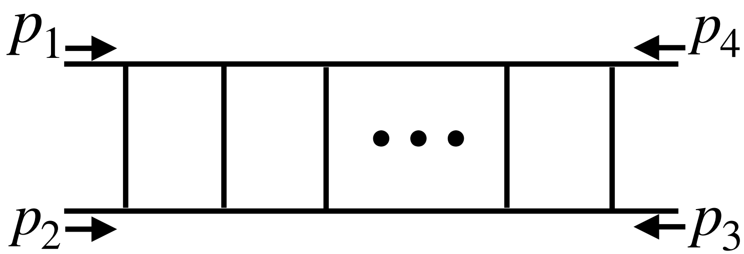
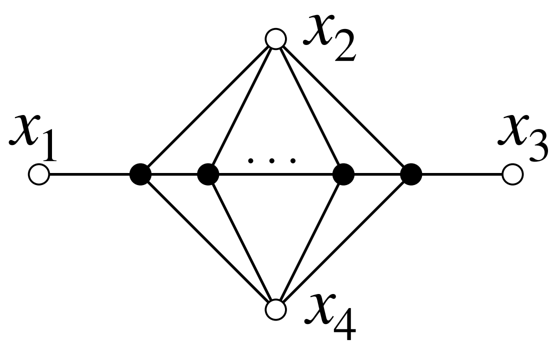


$$= \sum_n \lambda^n$$



$$x_{i,i+1} = p_i$$

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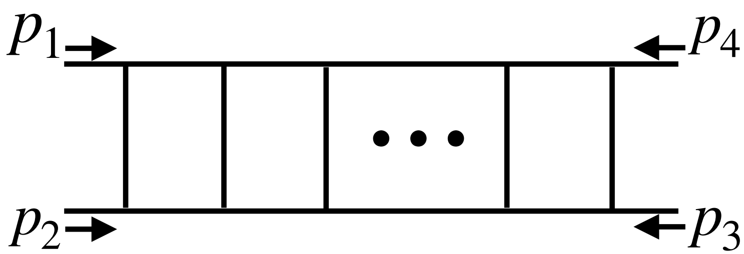

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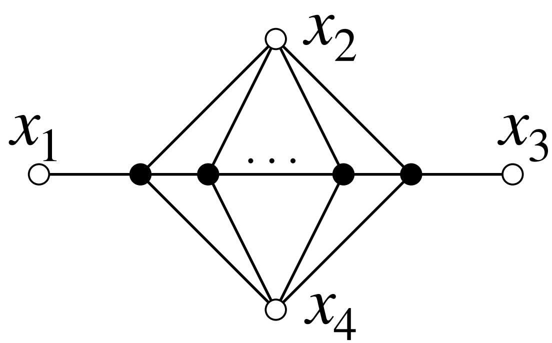

$$x_{i,i+1} = p_i$$

$$= \sum_n \lambda^n \; F^{(n)}(z, \bar{z})$$

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

$$F^{(n)}(z, \bar{z}) = \sum_{r=0}^{k} \frac{(-1)^r (2k-r)!}{k!(k-r)!r!} (\log(z\bar{z}))^r \left(\mathrm{Li}_{2k-r}(z) - \mathrm{Li}_{2k-r}(\bar{z})\right)$$

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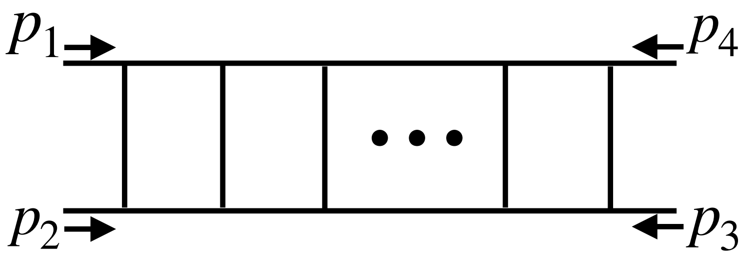


$$\overset{\lambda\to\infty}{\sim} e^{-\sqrt{\lambda}}$$

cf. [Arkani-Hamed, Henn, Trnka '21]

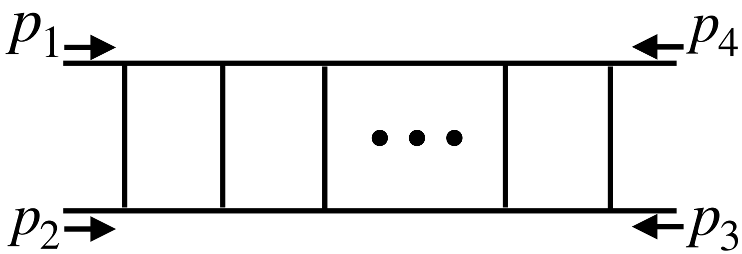
- Exponential suppression at strong coupling.
- Reminiscent of "stringy" behavior often seen in AdS/CFT

$$\langle W \rangle \overset{\lambda\to\infty}{\sim} e^{-\sqrt{\lambda}\text{Area}}$$

- A physical set up which leads to this resummation was not known.

Refinement [Caetano, SK, Wang]

$$\sum_n \lambda^n$$



$$= \sum_m W(\varphi + 2\pi m) + W(2\pi - \varphi + 2\pi m)$$

$$W(x) = \frac{4xK_1\left(\sqrt{\lambda(x^2+\sigma^2)}\right)}{\sin x\sqrt{x^2+\sigma^2}}$$

$$z = e^{\sigma + i\varphi}$$

Refinement [Caetano, SK, Wang]

$$\sum_n \text{Massless loops}$$

$$= \sum_m \text{Massive propagators}$$

Why?

Punchline

- I will present a physical setup which naturally explains all these features.

- Large charge 't Hooft limit of $\mathcal{N}=4$ super Yang-Mills

Far from the planar limit: SU(2) gauge group

Large number of particles

cf. Talk by Korchemsky

- The limit is surprisingly rich and exhibits surprising similarity with the planar limit.

Outline

1. Intro / motivation
2. Large charge limit vs large charge ’t Hooft limit
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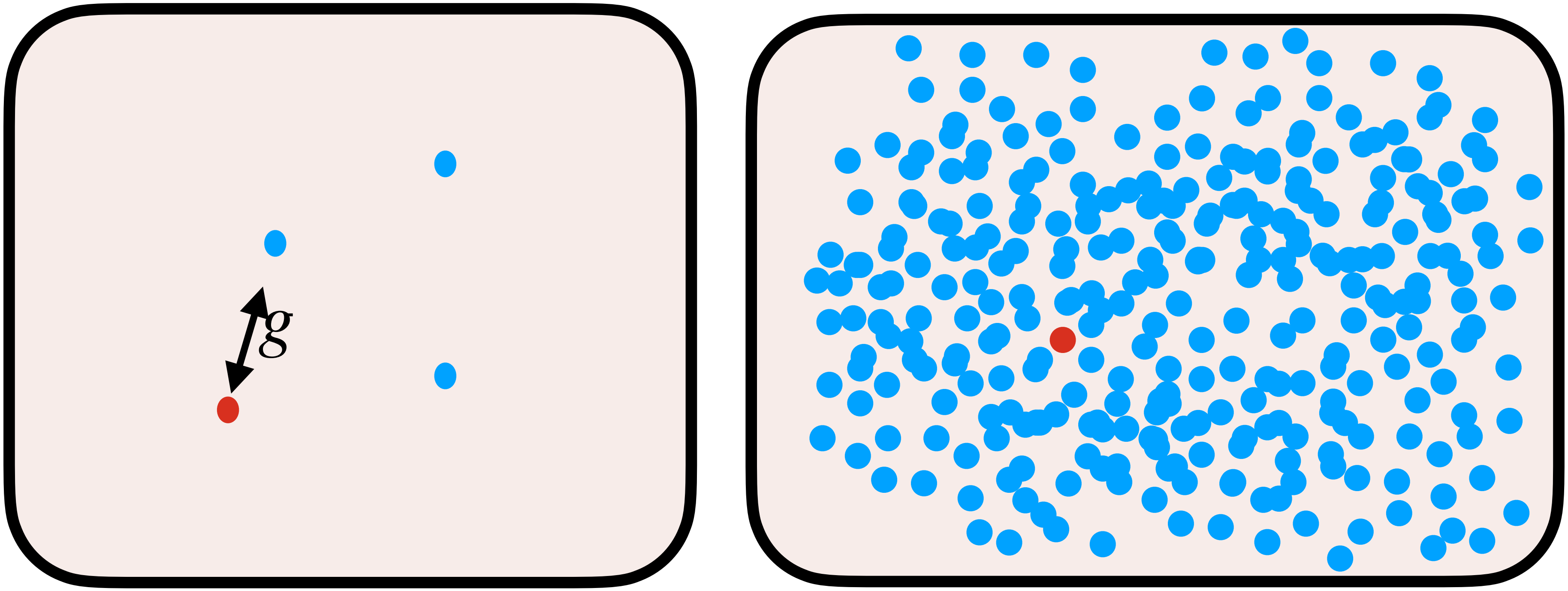
Systems with large # of d.o.f

- Systems with large # of d.o.f often exhibit emergent phenomena.

- Two ways to introduce large # of d.o.f.
  1. Consider a family of theories (parametrized by "N_c") and take $N_c \to \infty$.
  2. Consider a state in a given theory in which a large number of particles are excited.

- The former is the large N_c limit. The latter is common in cond-mat.

Large # of particles



For g and N small:
perturbation around free system

Effective interaction strength
$\lambda_{\text{eff}} \sim g\, N$

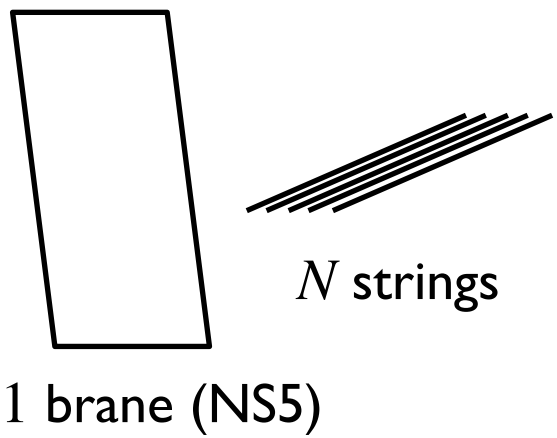
Large # of particles

$$\lambda_{\text{eff}} \sim g\, N$$

- Even if the fundamental interaction is weak ($g \ll 1$), the sector with a large # of particles can be strongly coupled.
- Suggests a double scaling limit, λ_{eff} : fixed, and $N \to \infty$.
- Formally this looks like a 't Hooft limit…..
- Is there any similarity with the standard 't Hooft limit….?

“Duality”

[Polchinski, Silverstein ’12]



$\longleftrightarrow \quad AdS_3 \times S^3 \times T^4$

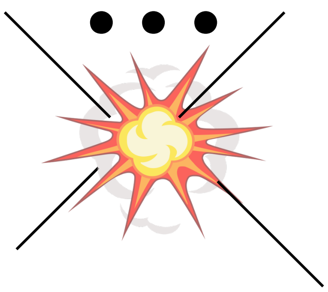
- When viewed from fundamental strings, this is a standard large N limit of 2d theory.
- When viewed from NS5, this is a large density state in 6d.
- It suggests that the large charge and the large N_c can be sometimes dual descriptions of the same system.
- There is also a diagrammatic understanding based on open-closed-open triality.

[Gopakumar, unpublished], [Gopakumar, Mazenc ’22]
[Goel, Verlinde ’21], [WIP]

Large # of particles

- Relevant in various physically interesting set ups.

Cond-mat	• Large chemical potential, high density
High energy (ph)	• Amplitudes with many legs • multi-particle production
High energy (th)	• Large charge state in CFT



Set up: $\mathcal{N} = 4$ SYM at large charge

- 4d $\mathcal{N} = 4$ SYM with $SU(2)$ gauge group.

cf. Talk by Korchemsky

- 1/2 BPS operator $\mathbf{tr}\left(\phi^J\right)$ with large R-charge J and take

$$J \to \infty \text{ with fixed } \lambda_J = g_{\rm YM}^2 J.$$

[Bourget, Rodriguez-Gomez, Russo'18]

- Study near BPS spectrum and correlation functions (heavy-heavy-light-light etc) in this limit.

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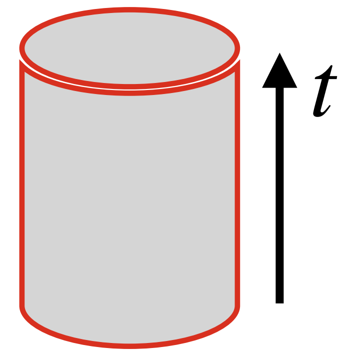
Physics of large charge limit in CFT$_d$

[Alvarez-Gaume, Cuomo, Dondi, Giombi, Hellerman, Kalogerakis, Loukas, Monin, Orlando, Pirtskhalava, Rattazzi, Reffert, Sannino, Watanabe. . . .]

- $\langle \mathcal{O}_J \bar{\mathcal{O}}_J \rangle$ at large charge $\leftrightarrow$ large charge state on $R_t \times S_L^{d-1}$

  $\mathcal{O}_J$: minimal dim op. for a given charge J



- $E_{\rm state} = \dfrac{\Delta_{\rm min}}{L} \quad \to \quad \epsilon_{\rm state} = \dfrac{\Delta_{\rm min}}{L^d}, \quad j_{\rm min} = \dfrac{J}{L^{d-1}}$

- Large charge limit: $J \to \infty, L \to \infty$ with $j_{\rm min}$ finite.

- Lowest energy state in flat space with finite charge density.

Physics of large charge limit in CFT_d

$$\epsilon_{\text{state}} = \frac{\Delta_{\text{min}}}{L^d}, \quad j_{\text{min}} = \frac{J}{L^{d-1}}$$

- Generic (nonsupersymmetric) CFT_d : $j_{\text{state}} \sim \epsilon_{\text{state}} \sim O(1)$

$$\Delta \overset{J\to\infty}{\sim} J^{\frac{d}{d-1}}$$

[Alvarez-Gaume, Cuomo, Dondi, Giombi, Hellerman, Kalogerakis, Loukas, Monin, Orlando, Pirtskhalava, Rattazzi, Reffert, Sannino, Watanabe. . . .]

- CFT with a moduli space of vacua: $j_{\text{state}} \sim O(1),\ \epsilon_{\text{state}} = 0.$

e.g. BPS operator in SUSY CFT: $\Delta \propto J$

[Arias-Tamargo, Beccaria, Bourget, Hellerman, Maeda, Orlando, Rodriguez-Gomez, Reffert, Russo, Watanabe. . . .]

Large charge limit from path integral

- Consider two point function of charged scalars $O_J = \phi^J$

$$\int \mathscr{D}\phi \exp\left(-S + \underline{J \log(\phi)\delta^d(x - x_1) + J \log(\bar{\phi})\delta^d(x - x_2)}\right)$$

- $J \to \infty$: path integral dominated by a saddle point

- In theories with coupling constant g_{YM} : $\langle\phi\rangle \sim g_{\text{YM}}\sqrt{J}$

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- In theories with coupling constant g_{YM} : $\langle \phi \rangle \sim g_{\mathrm{YM}}\sqrt{J}$

- In supersymmetric gauge theories, the theory is effectively on the Coulomb branch.

- Mass of BPS W-bosons: $m_W \sim g_{\mathrm{YM}}\sqrt{J} \ \to \infty \ (J \to \infty)$

- Derivative expansion of (Coulomb branch) EFT = $1/J$ expansion

$$\frac{1}{p^2+m_W^2} = \frac{1}{m_W^2} - \frac{p^2}{m_W^4} + \cdots \sim \frac{\#}{J} + \frac{\#p^2}{J^2} + \cdots$$

Large charge 't Hooft limit

- Large charge limit + EFT : powerful, universal predictions
- But it is insensitive to physics of massive (BPS) excitations
- Alternative limit: [Bourget,Rodriguez-Gomez, Russo'18]

$$J \to \infty \text{ with } \lambda_J = g_{\mathrm{YM}}^2 J \text{ fixed.}$$

- $m_W \sim \sqrt{\lambda_J}$ is finite. They contribute to obs even at $J \to \infty$.
- Results in the literature so far: supersymmetric (BPS) observables.
- Our work: non-supersymmetric (near BPS) observables.

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Heavy-heavy-(light)^n correlators

- Consider higher-point functions (HHLLLL….)

$$\langle O_J(0) O_{i_1}(x_1) \ldots O_{i_n}(x_n) O_J(\infty) \rangle = \langle J | O_{i_1}(x_1) \ldots O_{i_n}(x_n) | J \rangle$$

- **Examples**: light BPS $\mathbf{tr}\left((Y^I \phi_I)^\ell\right)$, Konishi $K \sim \mathbf{tr}\left(\phi^I \phi_I\right)$

- Large charge 't Hooft limit:

$$\langle J | O_{i_1}(x_1) \ldots O_{i_n}(x_n) | J \rangle \to \langle O_{i_1}(x_1) \ldots O_{i_n}(x_n) \rangle_{\textbf{large charge bkd}}$$

$$\langle \phi \rangle \neq 0$$

- Basic building block: propagator in the background

$$\langle \phi(x) \phi(y) \rangle_{\mathrm{bkd}}$$

Propagator in the background

$$\langle\phi(x)\phi(y)\rangle_{\text{bkd}}$$

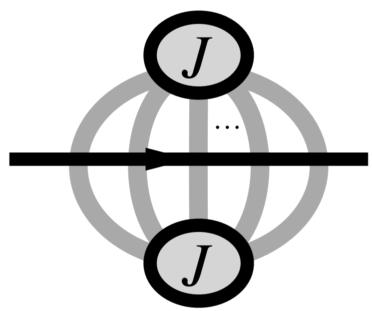
- Satisfies a (complicated-looking) differential equation.

$$\mathscr{D}\langle\phi(x)\phi(y)\rangle_{\text{bkd}} = \delta^4(x-y)$$

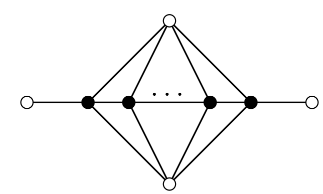
- Surprisingly, the solution is given by a resummation of ladders!

[Giombi, Hyman '20]

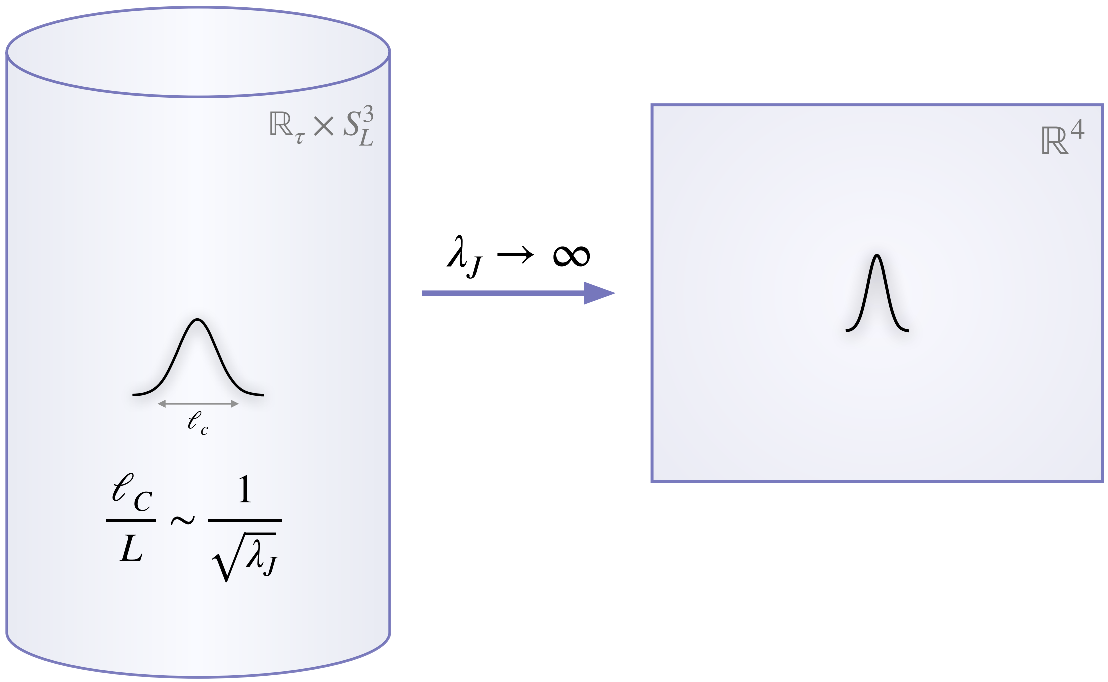
$$\langle\phi\phi\rangle_{\text{bkd}} \sim \sum$$



$$= \sum_k \lambda_J^k F^{(k)}(z,\bar{z})$$

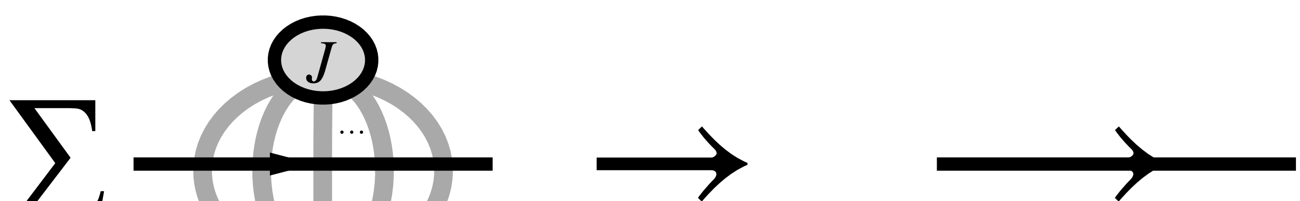


Physical explanation of $e^{-\sqrt{\lambda_J}}$



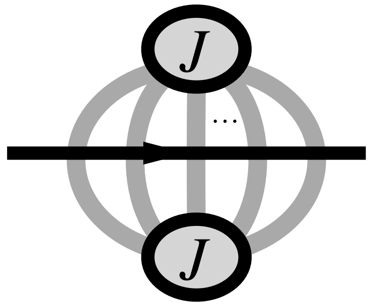
Large charge state on a cylinder

Coulomb branch in flat space



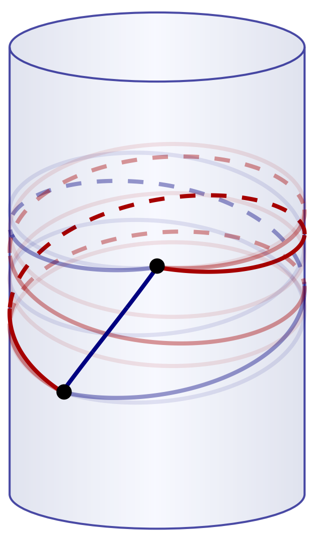
Propagation of massive W-boson

Physical explanation of $e^{-\sqrt{\lambda_J}}$

$\sum$ 

$$W(x) = \frac{x K_1(\sqrt{\lambda_J}\sqrt{x^2+\sigma^2})}{\sin x \sqrt{x^2+\sigma^2}}$$

$$= \frac{(1-z)(1-\bar{z})}{\sqrt{z\bar{z}}} \sum_{n=0}^{\infty} W(\varphi + 2\pi n) + W(2\pi - \varphi + 2\pi n)$$



Worldline instantons wrapping cylinder

Some other nice results

Result for Konishi 3pt

- Using the propagator, we computed the 3pt function $\langle \mathcal{O}_J \, K \, \mathcal{O}_J \rangle$

$$\left(K \sim \mathrm{tr}\left(\phi^I \phi_I\right) \right)$$

$$C_{KJJ} = -8\lambda + 4\sqrt{\lambda} \int_0^\infty dw \frac{4\sqrt{\lambda}w - J_1(8\sqrt{\lambda}w)}{\sinh^2(w)}$$

- Weak coupling expansion: finite radius of convergence $|\lambda| < 1/16$.

  "magnon" becoming massless, tachyonic instability $\sqrt{1+16\lambda}$

- Strong coupling expansion:

$$C_{KJJ} = \frac{\lambda}{2\pi^2}\left(\gamma_E + \log\frac{\lambda}{4\pi^2}\right) - \frac{2\lambda}{\pi^2}\sum_{n=1}^{\infty}\left(K_1(2n\sqrt{\lambda}) + K_0(2n\sqrt{\lambda})\right)$$

- Similar expressions seen in the planar limit (cusp anomalous dim etc)

Spectrum around large charge operator

- The spectrum of excitations around the large charge operator.

$$\mathcal{O} = \mathrm{tr}\left(\phi^J\right) + \text{corrections}$$

- Result:

  (Gauge invariant) combinations of "magnons" with energy

$$E_{\mathrm{magnon}} = \sqrt{1 + 16\lambda_J}$$

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- Result:

  (Gauge invariant) combinations of "magnons" with energy

$$E_{\text{magnon}} = \sqrt{1 + 16\lambda_J}$$

- Planar limit:

  (Gauge invariant) combinations of magnons with momentum p

$$E_{\text{magnon}} = \sqrt{1 + 16\lambda_J \sin^2 p/2}$$

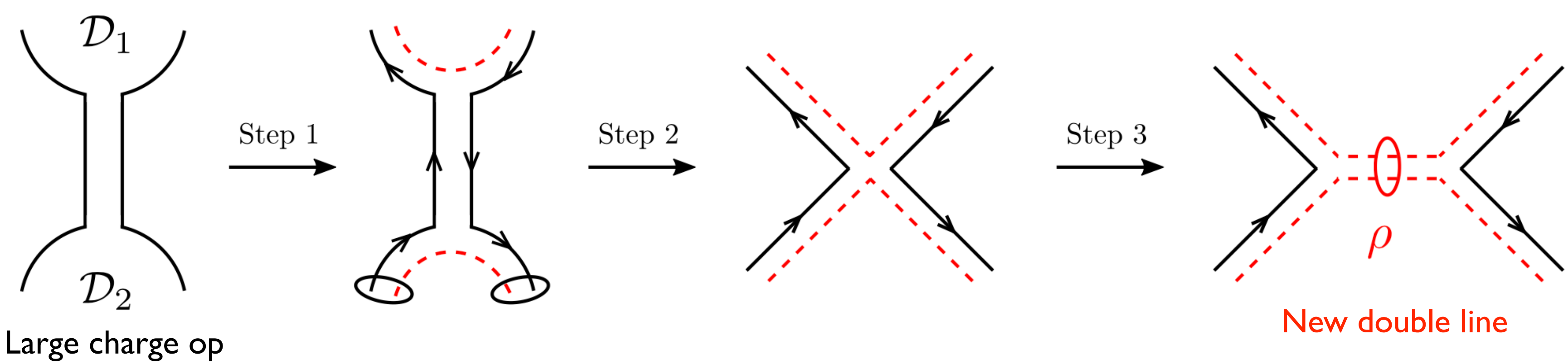
- In both limits, the spectrum is fully fixed by centrally-extended $\mathfrak{psu}(2|2)^2$ [Beisert '06]

Diagrammatic interpretation of "duality"

- Large charge limit and large N limit are sometimes dual.
  [Polchinski, Silverstein '12]

- **Q:** Is there a "planar" expansion at large charge?

- **A:** planar "dual graphs" (at least in some cases) [WIP + Dondi]



- Studied before in the conventional large N limit: "open-closed-open triality"

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Conclusion

- Large charge 't Hooft limit provides an interesting solvable corner of $\mathcal{N} = 4$ SYM.

  Many more to explore!

- Underlying centrally extended symmetry.

- Various observables can be computed exactly as a function of λ_J.

  Integrated correlators in terms of $J \times J$ matrix integral.

- Other limits? Large spin 't Hooft limit $g_{\text{YM}}^2 \log S$

  Effective description in terms of 2d YM. [Alday, Maldacena 08]

  Similarity with BFKL limit.

  It's likely that the limit can be studied also in QCD(?)

- Holography for large density, large # of particles, large # of legs?

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- Holography for large density, large # of particles, large # of legs?
- We might not need large N for AdS/CFT. AdS/CFT for collider?

Back up slides

Centrally-extended $\mathfrak{psu}(2|2)^2$

- Consider a large charge BPS op and perturb it:

$$|Z\ldots Z\chi Z\ldots Z\chi Z\ldots Z\rangle$$

- Subgroup of superconformal symmetry preserved by large charge 1/2 BPS state → $\mathfrak{psu}(2|2)^2$

- Excitations are classified by irreps of $\mathfrak{psu}(2|2)^2$.

- But (as expected) it is not powerful enough to constrain the dynamics. No g_{YM} dependence.

- The actual symmetry is larger: centrally-extended $\mathfrak{psu}(2|2)^2$

Centrally-extended $\mathfrak{psu}(2|2)^2$

- In general superconformal algebra, $\{Q, Q\} = 0$.

- This is true when acting on gauge inv states. But not true for individual fields.

$$\{Q, Q\}\chi \sim [Z, \chi]$$

- Maximally centrally extended $\mathfrak{psu}(2|2)^2$

$$\{Q, S\} \sim D - \hat{J}$$

$$\{Q^a{}_\alpha, Q^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\, P\,, \quad \{S^\alpha{}_a, S^\beta{}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta}\, K$$

$$P \cdot \chi \equiv [Z, \chi]\,, \quad K \cdot \chi \equiv [Z^{-1}, \chi]$$

Centrally-extended $\mathfrak{psu}(2|2)^2$

$$\{Q^a{}_\alpha, Q^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\, P\,, \quad \{S^\alpha{}_a, S^\beta{}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta}\, K$$

$$P\cdot\chi \equiv [Z,\chi]\,, \quad K\cdot\chi \equiv [Z^{-1},\chi]$$

- In the planar limit, P and K can be identified with translations on spin chain.

- In the large charge 't Hooft limit, the action of P and K are determined by VEV of Z induced by the charge.

$$\langle Z\rangle = \begin{pmatrix} \frac{g_{\rm YM}\sqrt{J}}{2\pi}e^{i\varphi} & 0 \\ 0 & -\frac{g_{\rm YM}\sqrt{J}}{2\pi}e^{i\varphi} \end{pmatrix}$$

- $\{Q,Q\}\,m^{\pm} \sim \pm 2\lambda m^{\pm}$, $\{Q,Q\}\,m^0 = 0$.

$$\chi = \begin{pmatrix} m^0 & m^+ \\ m^- & -m^0 \end{pmatrix}$$

Centrally-extended $\mathfrak{psu}(2|2)^2$

$$\{Q^a{}_\alpha, Q^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\, P\,, \quad \{S^\alpha{}_a, S^\beta{}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta}\, K$$

$$\{Q, S\} \sim D - \hat{J}$$

- By requiring that the centrally-extended algebra closes on $|Z\ldots m^{\pm}\ldots Z\rangle$ (BPS rep)

$$(D-\hat{J})|Z\ldots m^{\pm}\ldots Z\rangle = \sqrt{1+16\,\lambda}\,|Z\ldots m^{\pm}\ldots Z\rangle$$

$$(D-\hat{J})|Z\ldots m^{0}\ldots Z\rangle = |Z\ldots m^{0}\ldots Z\rangle$$

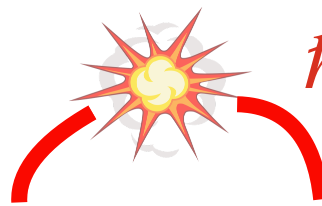
- To construct a gauge inv state, we require $\#m^+ = \#m^-$

e.g. $|Z\ldots Z\, m^- m^- m^0\, Z\ldots Z\, m^+ m^+\, Z\ldots Z\rangle$

Energy = $\sum$ individual energies $\quad \Delta - J = 1 + 4\sqrt{1+16\,\lambda}$

(interactions $\sim 1/J$)

Spectrum at $1/J$



$$\hbar \sim \frac{1}{J}$$

$$|Z\ldots Z\, m^- m^- m^0 Z\ldots Z\, m^+ Z\ldots Z\, m^+ Z\ldots Z\rangle$$

- At $1/J$, there is 2-body interaction among excitations ("magnons").
- Since we are not in the planar limit, the interaction is "all-to-all".
- Centrally-extended $\mathfrak{psu}(2|2)$ is still powerful: It determines the interaction Hamiltonian up to a few overall coefficient.

Result for Konishi 3pt