Stringy Dynamics from an Amplitudes Bootstrap

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Amplitudes 2023
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The amplitudes bootstrap

- Bootstrap method for discovering physics:

  ✗ geometry/thought experiments?

  ✓ math question about the S-matrix

  → get gauge theory, gravity
  ("elevator-free" derivation of GR)

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  ✓ math question about the S-matrix

  → get gauge theory, gravity
  (“elevator-free” derivation of GR)

- What is the analogue for string theory?
  What is the math question for which string amplitudes are the answer?

- Ultimate goal: Is string theory in some sense unique?

String amplitudes

- What do string amplitudes do?
    - Ultraviolet-complete low-energy physics by taming Planck-scale pathologies in amplitudes.
    - Accomplish this by adding a tower of massive higher-spin degrees of freedom. (Cannot add just one higher-spin state without making the problem worse. e.g., CEMZ [1407.5597])

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- So string theory answers the question of how to build an amplitude exchanging higher-spin modes consistently at high energies:

Veneziano amplitude: (1968)

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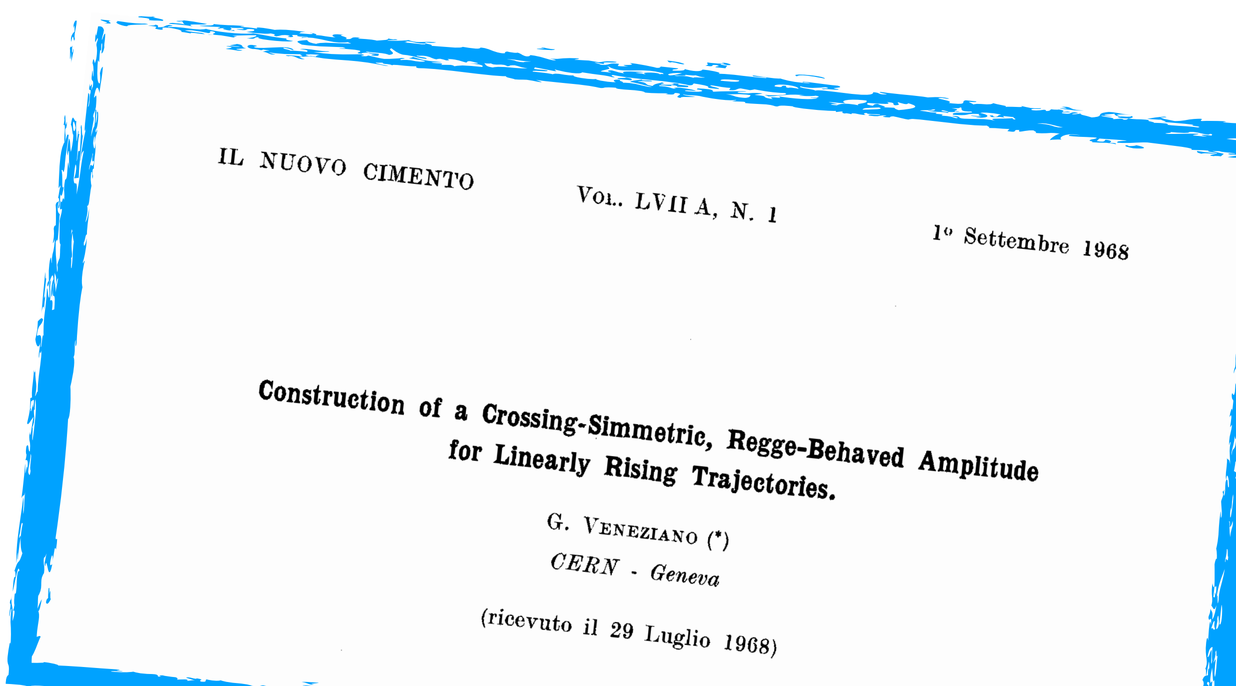
IL NUOVO CIMENTO VOL. LVII A, N. 1 1° Settembre 1968

Construction of a Crossing-Simmetric, Regge-Behaved Amplitude for Linearly Rising Trajectories.

G. VENEZIANO (*)

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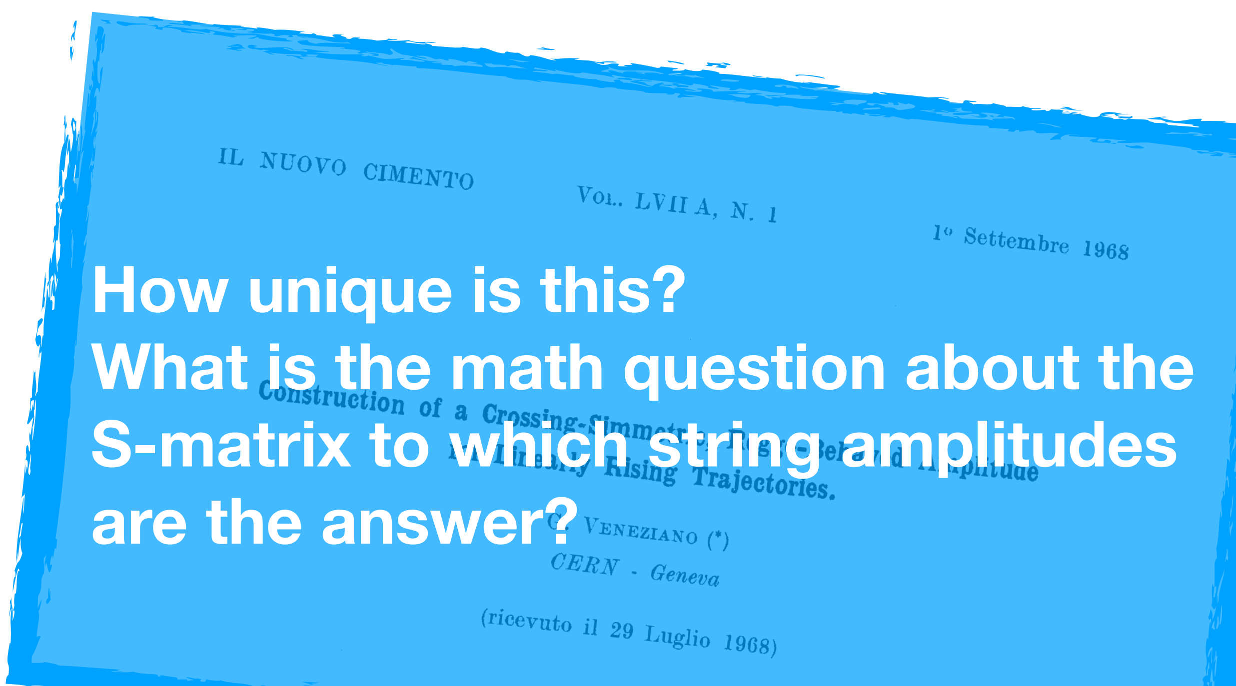
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Construction of a Crossing-Simmetric,

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How unique is this?
What is the math question about the S-matrix to which string amplitudes are the answer?

Bootstrap and QCD prehistory

- Veneziano's amplitude was constructed to solve several phenomenological puzzles of the late 1960s:
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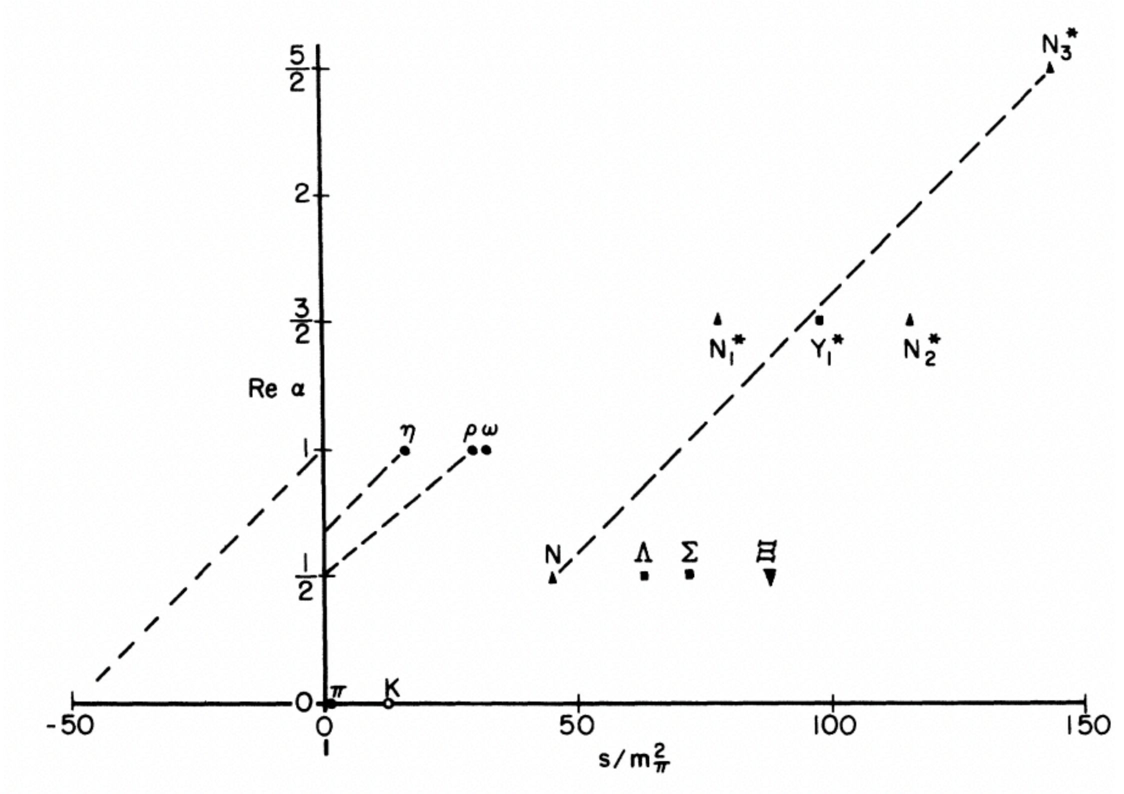
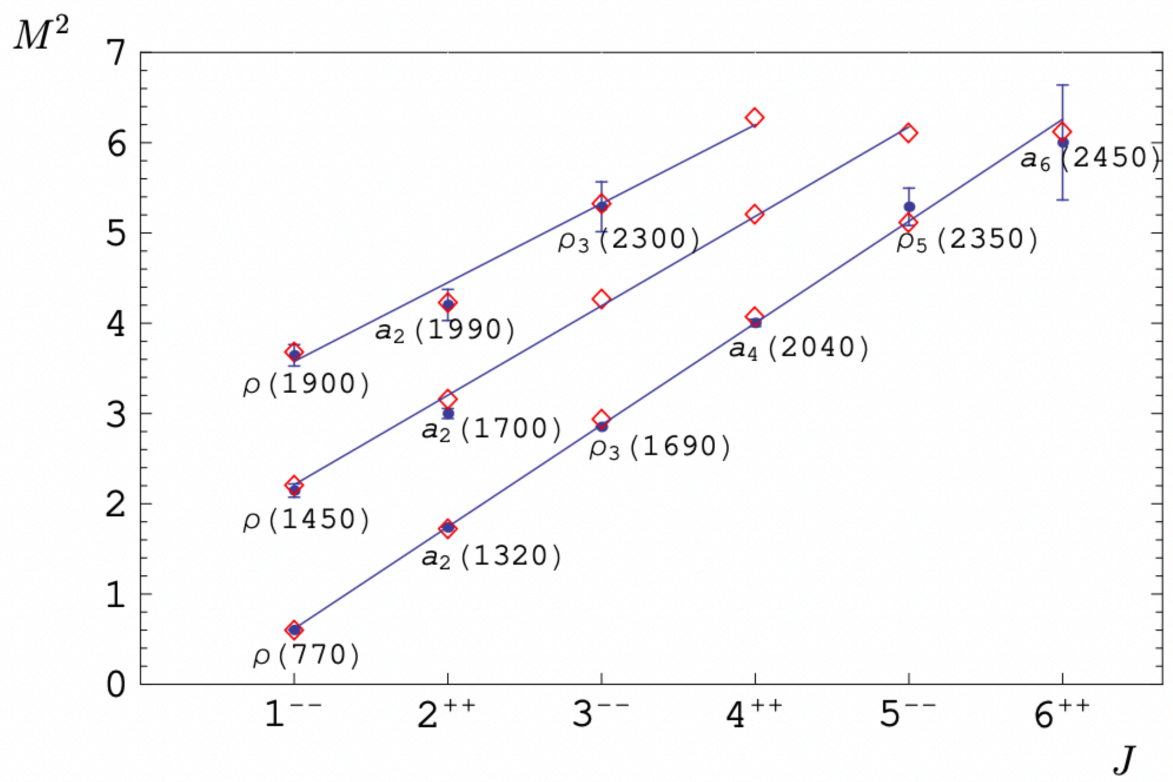


Chew, Frautschi (1962)



Ebert, Faustov, Galkin [0903.5183]

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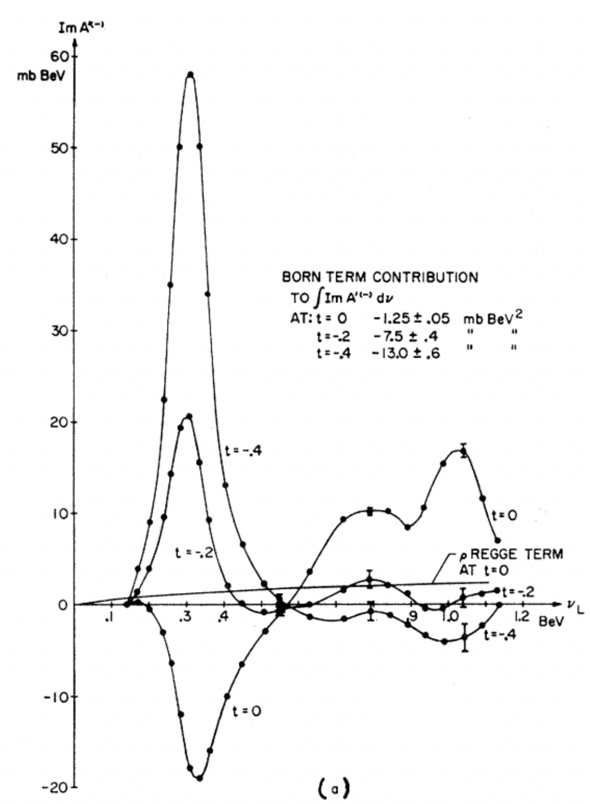
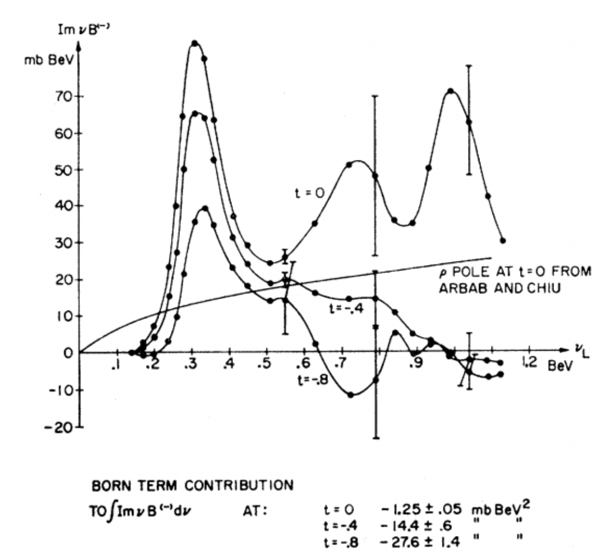
    - Observations had shown the existence of Regge trajectories in meson spectra, $m^2 \propto J$ (Chew-Frautschi plot)
    - Dolen-Horn-Schmid duality (1967): "dual resonance"





Physical constraints

- In this talk, we will derive a class of four-point scalar amplitudes (i.e., *dynamics*) by inputting a *spectrum* of tree-level exchanged states m_n^2, along with a set of *physical constraints*:

i) *Crossing Symmetry*

ii) *Polynomial Residues*

iii) *High-Energy Boundedness*

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- We will find that the dynamics of string theory—the distinctive form of string amplitudes—arise naturally.

- However, we will find that four-point string amplitudes exist in an infinite space of new objects that accomplish the same mathematical miracles: generalizations of Veneziano amplitudes.

Crossing symmetry

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i) Crossing Symmetry

Colored scalars $\Longrightarrow$ cyclic invariant amplitude:
$A(s,t) = A(t,s)$

Polynomial residues

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Each exchanged state with mass m_n^2 exhibits a finite tower of spins, which we take to run from 0 to n, so that

$$R_n(t) = \operatorname*{Res}_{s=m_n^2} A(s,t) = \sum_{m=0}^{n} \lambda_{n,m} t^m$$

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We considered other polynomials for generalizations of the Coon amplitude in Cheung, **GR** [2210.12163], but will leave the bootstrap approach to arbitrary polynomials to future work.

Non-polynomial residues would describe an infinitely-extended object; can be of interest, e.g., for the EFThedron.

Huang, **GR** [2203.00696]; Caron-Huot, Van Duong [2011.02957]; Arkani-Hamed, Huang, Huang [2012.15849]

High-energy boundedness

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We assume by fiat that $A \to 0$ in the Regge limit:
$s \to \infty$, $t < m_0^2$ fixed

Equivalently, no pole at infinity, so we can write a dispersion relation:

$$A(s,t) = \frac{1}{2\pi i} \oint_{s=s'} \mathrm{d}s' \, \frac{A(s',t)}{s'-s} = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}s' \, \frac{\mathrm{Im}\, A(s',t)}{s'-s}$$

Dual resonance

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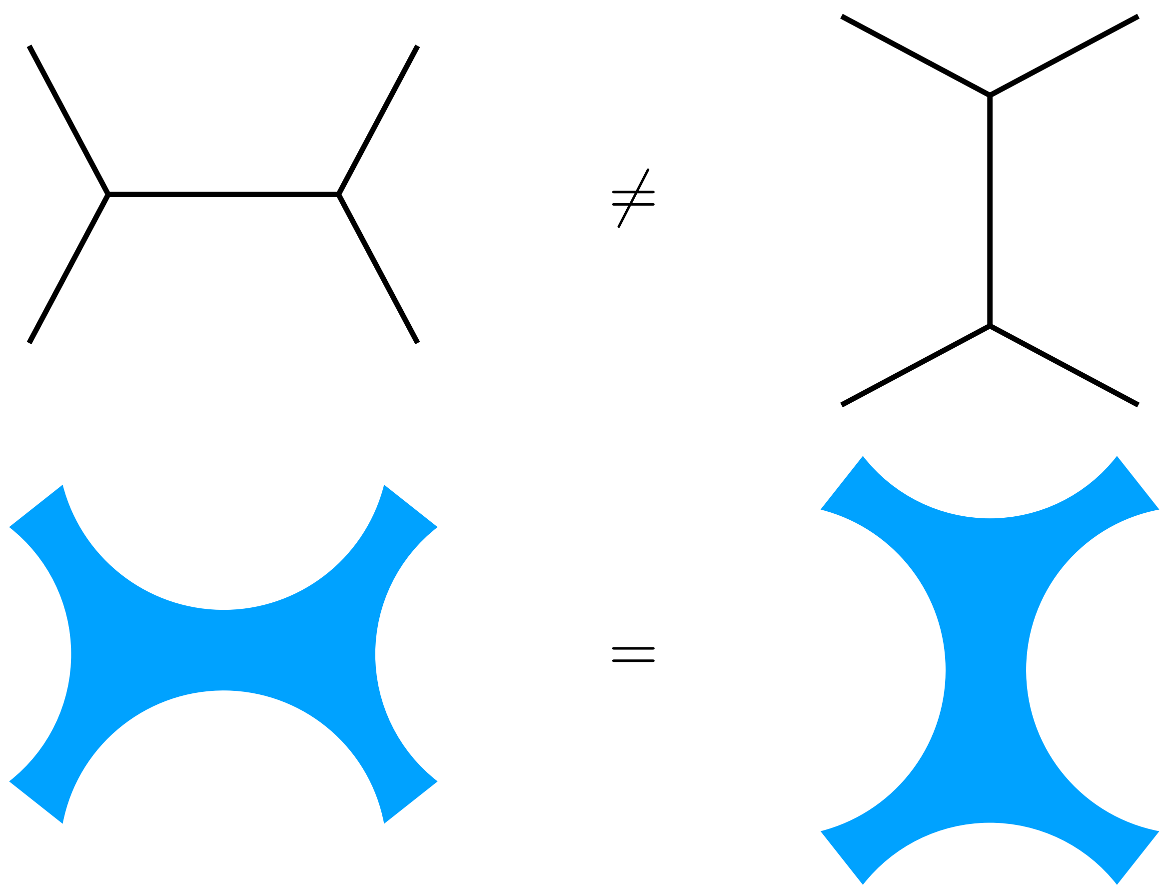
Tree-level exchanges: $\mathrm{Im}\, A(s',t) = \pi \sum_{n=0}^{\infty} \delta(s' - m_n^2) R_n(t)$

$\Longrightarrow$ Dual resonance:

$$A(s,t) = \sum_{n=0}^{\infty} \frac{R_n(t)}{m_n^2 - s} = \sum_{n=0}^{\infty} \frac{R_n(s)}{m_n^2 - t}$$

Dual resonance

- Dual resonance is a hallmark of string amplitudes that differentiates them from field theory:



- We're asking: What's the full space of dual resonant functions?

Integer Spectrum Bootstrap

Integer spectrum bootstrap

- Let's first consider the case of an integer spectrum:

$$m_n^2 = n$$

- Implementing crossing symmetry in the dual resonant ansatz is a two-variable problem:

$$A(s,t) = \sum_{n=0}^{\infty} \frac{R_n(t)}{n-s} = \sum_{n=0}^{\infty} \frac{R_n(s)}{n-t} = A(t,s)$$

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- Turn into a single-variable problem by choosing special kinematics,

$$t = s-k, \qquad k \in \mathbb{N}$$

Crossing becomes:

$$A(s,s-k) = A(s-k,s) \implies \sum_{n=0}^{\infty} \frac{R_n(s-k)}{n-s} = \sum_{n=0}^{\infty} \frac{R_n(s)}{n+k-s}$$

$$\implies \sum_{n=k}^{\infty} \frac{R_n(s-k) - R_{n-k}(s)}{n-s} = -\sum_{n=0}^{k-1} \frac{R_n(s-k)}{n-s}$$

Integer spectrum bootstrap

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finite number of terms,
no poles at $s = n \leq k$

Integer spectrum bootstrap

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demanding
no poles at $s = n \leq k$
yields

$$R_n(n-k) = R_{n-k}(n), \qquad 1 \leq k \leq n$$

Caveats:

- Strictly speaking, neither necessary nor sufficient for crossing:
- Not sufficient: We chose special kinematics $t = s - k$, so crossing could conceivably hold on that choice but not away from it.
- Not necessary: s-channel representation of the amplitude converges for $t < m_0^2$ or as a residue on generic t. For $t \in \mathbb{N}$, could break down.
- We will take the residue constraint above as motivation and see what we find. All subsequent examples will indeed satisfy this constraint and converge.

Integer spectrum bootstrap

We have n conditions

$$R_n(n-k) = R_{n-k}(n), \qquad 1 \le k \le n$$

on the $n+1$ free parameters in the residue ansatz:

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Defining $\lambda_m \equiv \lambda_{m,m}$ and for brevity writing $x! \equiv \Gamma(x+1)$ for $x \in \mathbb{C}$, we find the general solution:

$$R_n(t) = \sum_{m=0}^{n} \frac{\lambda_m}{m!} \frac{t!}{(t-m)!} \frac{n!}{(n-m)!}$$

Remarkably, we numerically find that choosing λ_m such that the s-channel representation of $A(s,t)$ converges always yields a crossing-symmetric amplitude: **an infinite-parameter family of dual resonant amplitudes.**

Veneziano amplitude

- Let us choose $\lambda_m = \dfrac{1}{m!}$
- The Vandermonde identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

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$$= -\frac{1}{s} \sum_{n=0}^{\infty} \frac{(-s)_n (1+t)_n}{(1-s)_n} \frac{1}{n!}$$

definition of Pochhammer symbol:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

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$$= -\frac{1}{s}\,{}_2F_1\left[\begin{matrix} -s,\ 1+t \\ 1-s \end{matrix};1\right]$$

definition of hypergeometric function:

$${}_2F_1\left[\begin{smallmatrix} a,b \\ c \end{smallmatrix};z\right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

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- From the definition of the generalized hypergeometric function,

$${}_mF_n\left[\begin{array}{c} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{array} ; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_m)_k}{(b_1)_k \cdots (b_n)_k} \frac{z^k}{k!}$$

the amplitude becomes

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- Using a Thomae transformation,

$$A(s,t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)}\, {}_3F_2\left[\begin{array}{c} -s,\ -t,\ r \\ -s-t,\ 1+r \end{array} ; 1\right]$$

New hypergeometric amplitude

Partial wave unitarity

- In order for our amplitude to correspond to a physical theory, its residues must obey partial wave unitarity.

- Later, we'll be much more general, but as a preliminary check, let's consider massless external particles in $D = 4$ dimensions.

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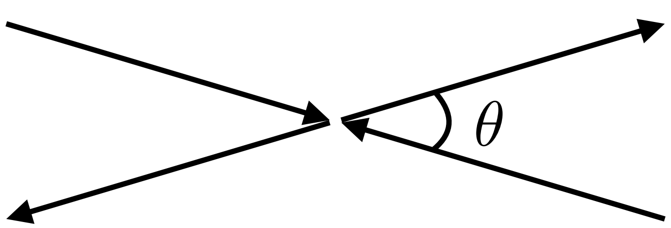
- Partial wave decomposition:

$$R(n,t) = \sum_{\ell=0}^{n} a_{n,\ell} P_\ell(\cos\theta)$$

Legendre polynomials

Scattering angle $\cos\theta = 1 + \frac{2t}{s}$



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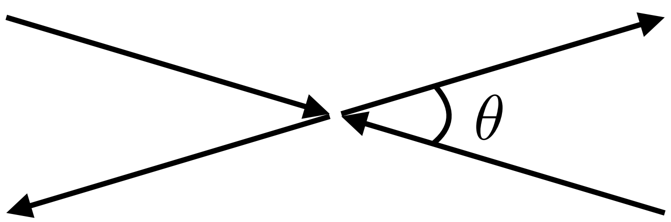

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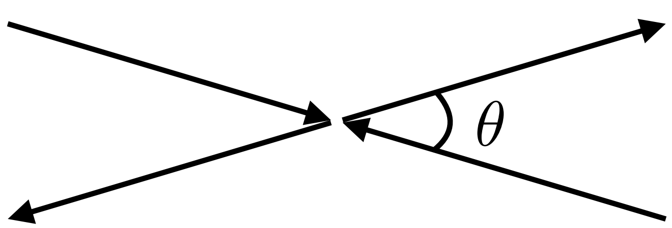
$$a_{n,\ell} = (2\ell+1)\frac{r!}{(r+n)!}\sum_{j=\ell}^{n}\sum_{s=0}^{\lfloor (j-\ell)/2\rfloor} S_1(n,j)\frac{2^{\ell-j}(\ell+s)!j!n^{\ell+2s}(2-n+2r)^{j-\ell-2s}}{(j-\ell-2s)!s!(2\ell+2s+1)!}$$



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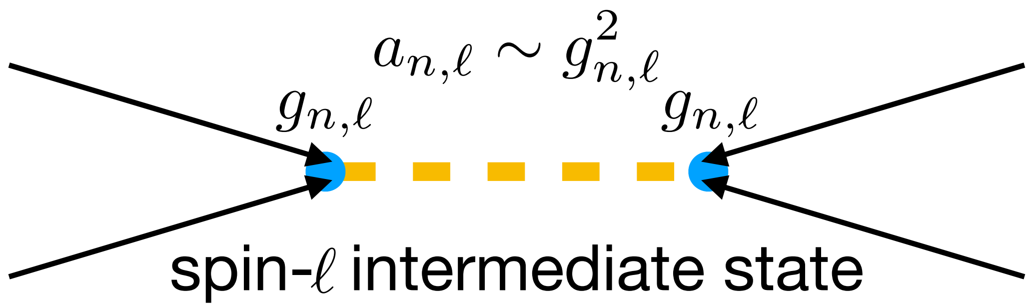
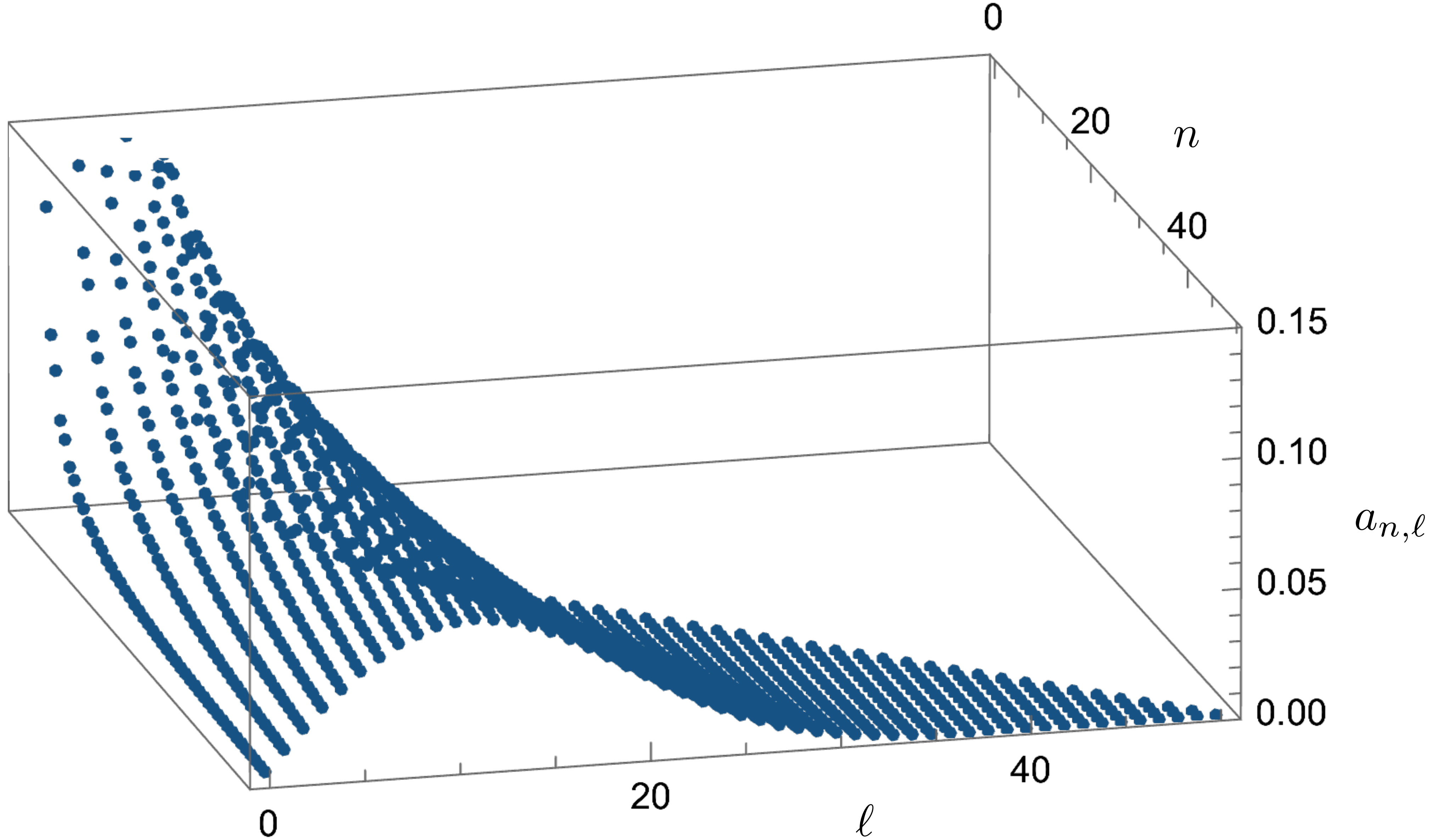
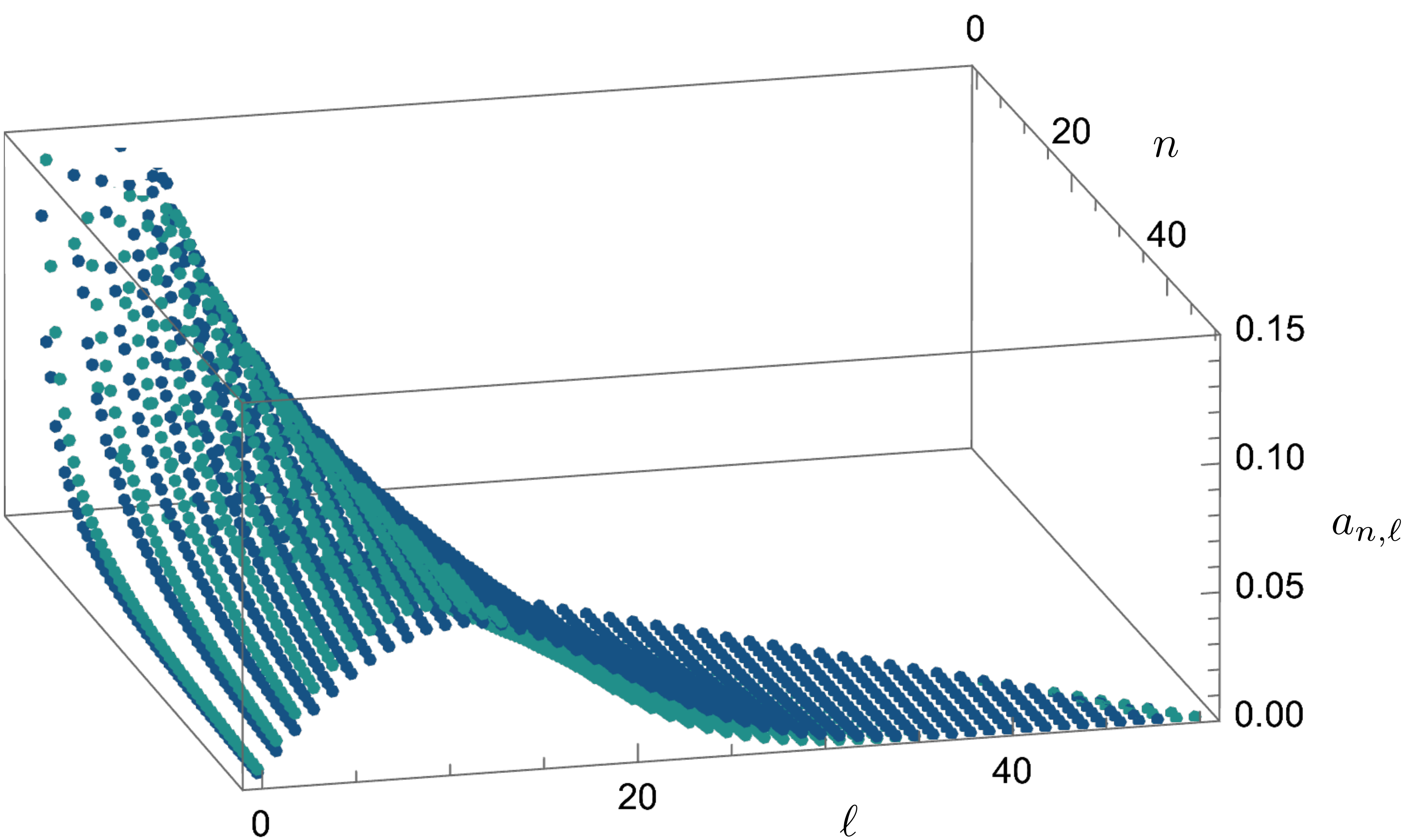


Partial wave unitarity



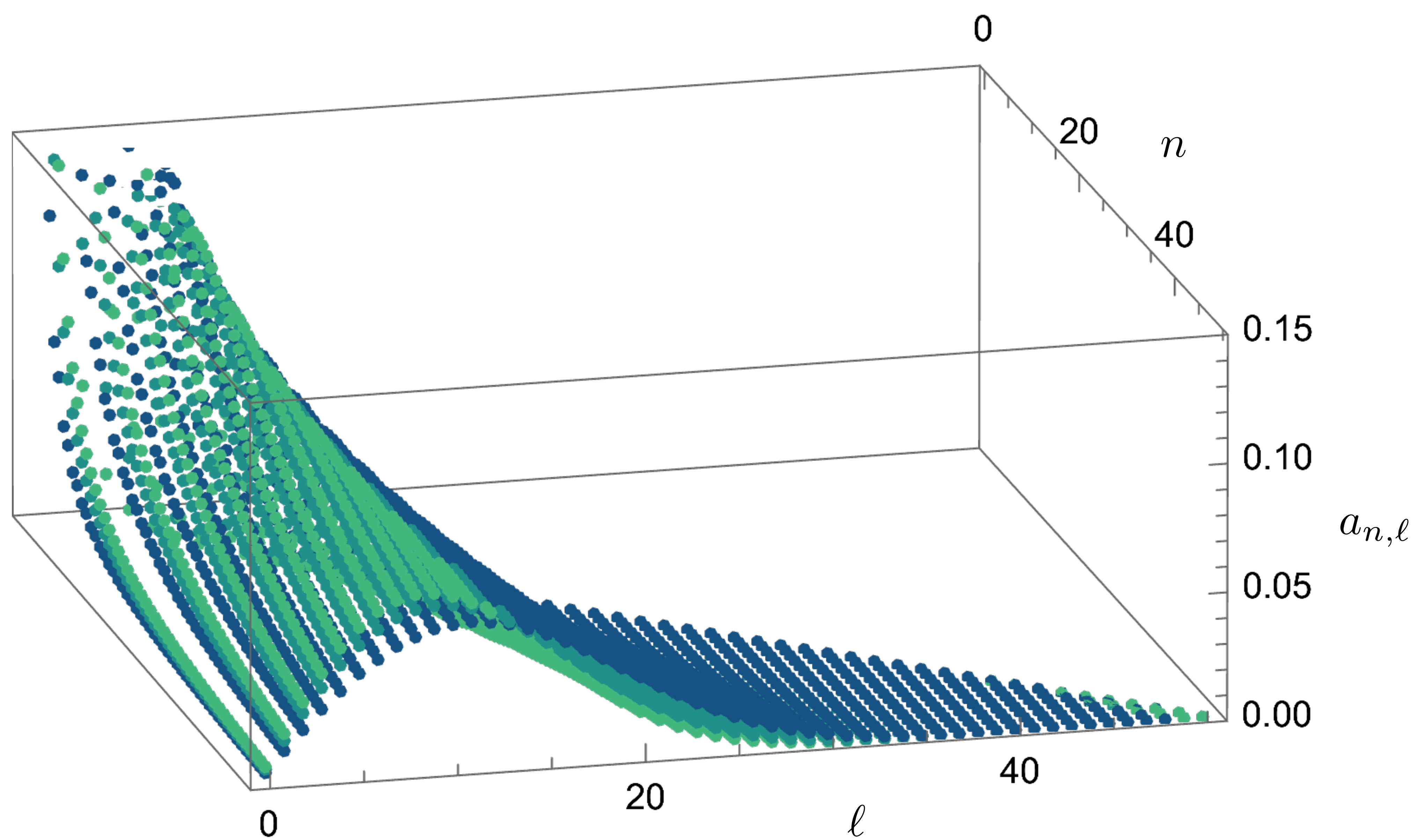
$r = 0$

Partial wave unitarity



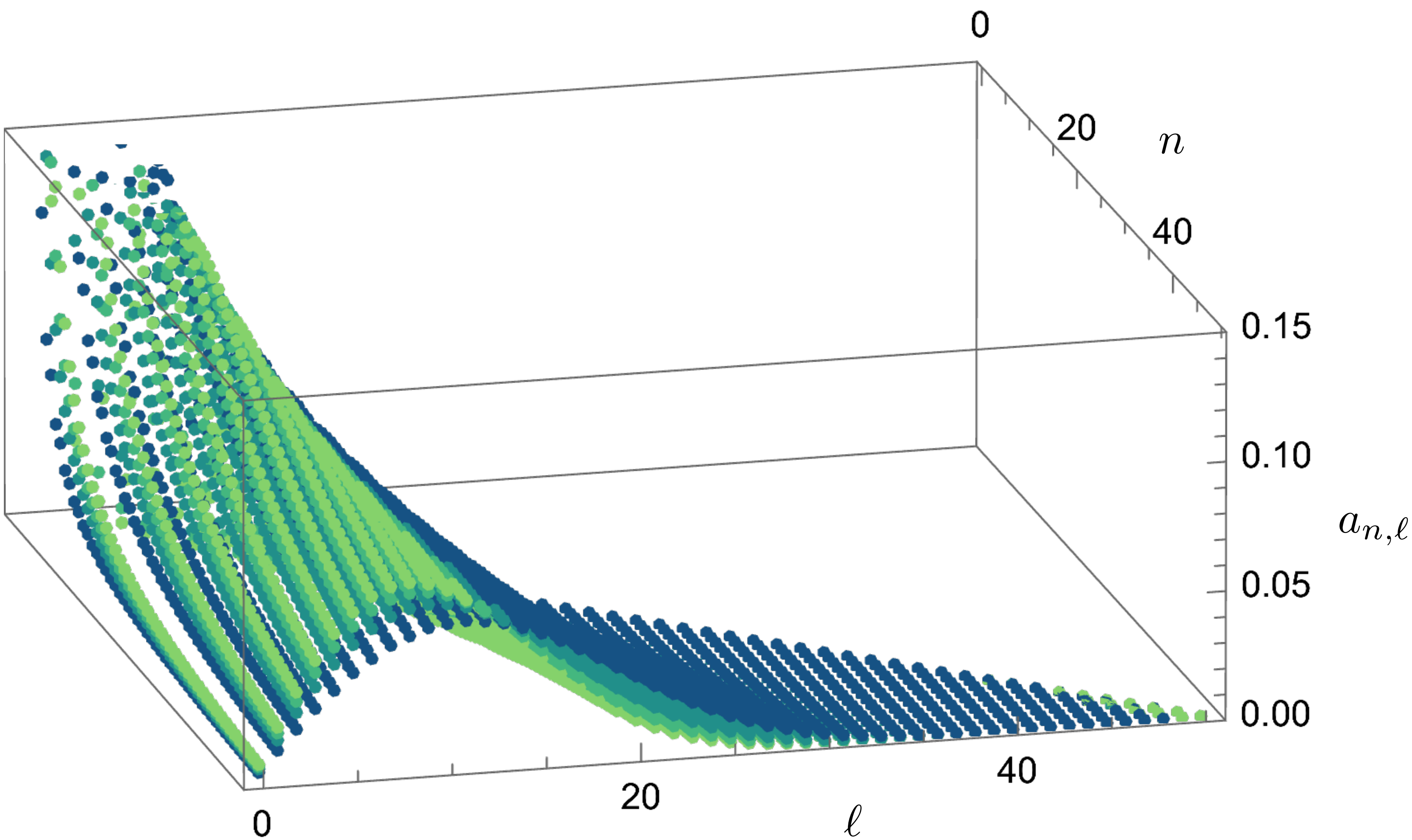
$r = 1$

Partial wave unitarity



$r = 2$

Partial wave unitarity



$r = 3$

Hard scattering

- In the high-energy, fixed-angle limit,

$$|s|,\ |t| \to \infty, \qquad t/s \text{ fixed}$$

the hypergeometric amplitude exhibits the scaling:

$$A(s,t) \sim e^{B(s,t)} + \frac{r}{st} + \cdots\ , \qquad B(s,t) = (s+t)\log(s+t) - s\log s - t\log t + \cdots$$

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- In the physical region, $\cos\theta = 1 + \frac{2t}{s} \in [-1,1]$, one has $B<0$, so the amplitude falls off as a power law $\sim r/st$, unless $r=0$, where the exponential decay of the string amplitude obtains.

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- In the unphysical $t>0$ region, $B>0$ and we find the scaling:

Caron-Huot, Komargodski, Sever, Zhiboedov [1607.04253]

$$\log A \sim (s+t)\log(s+t) - s\log s - t\log t$$

Regge limit

- In the Regge limit,

$$s \to \infty, \qquad t \text{ fixed}$$

the hypergeometric amplitude exhibits the scaling:

$$A(s,t) \sim s^{J(s,t)} + \frac{r}{(1+t)s} + \cdots, \qquad J(s,t) = t + \cdots$$

Scaling for $t > 0$ or $t < 0$.

- Expected $j + t$ Regge scaling when dressed with external polarizations

A worldsheet interpretation?

Remarkably, the hypergeometric amplitude has an integral representation,

$$A(s,t) = r \int_0^1 \int_0^1 \mathrm{d}x \, \mathrm{d}y \, \frac{x^{-s-1} y^{r-1} (1-xy)^t}{(1-x)^{t+1}}$$

reminiscent the Koba-Nielsen form for the Veneziano amplitude,

4-point: $$A^{(4)}_{\mathrm{Ven}} = \int_0^1 \mathrm{d}x \, \frac{x^{-s-1}}{(1-x)^{t+1}}$$

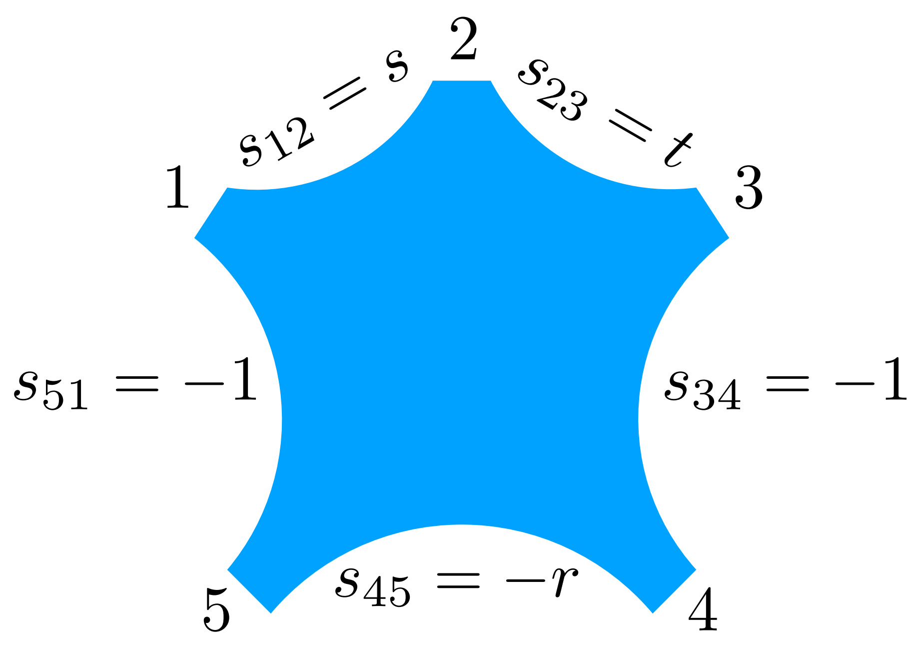
A worldsheet interpretation?

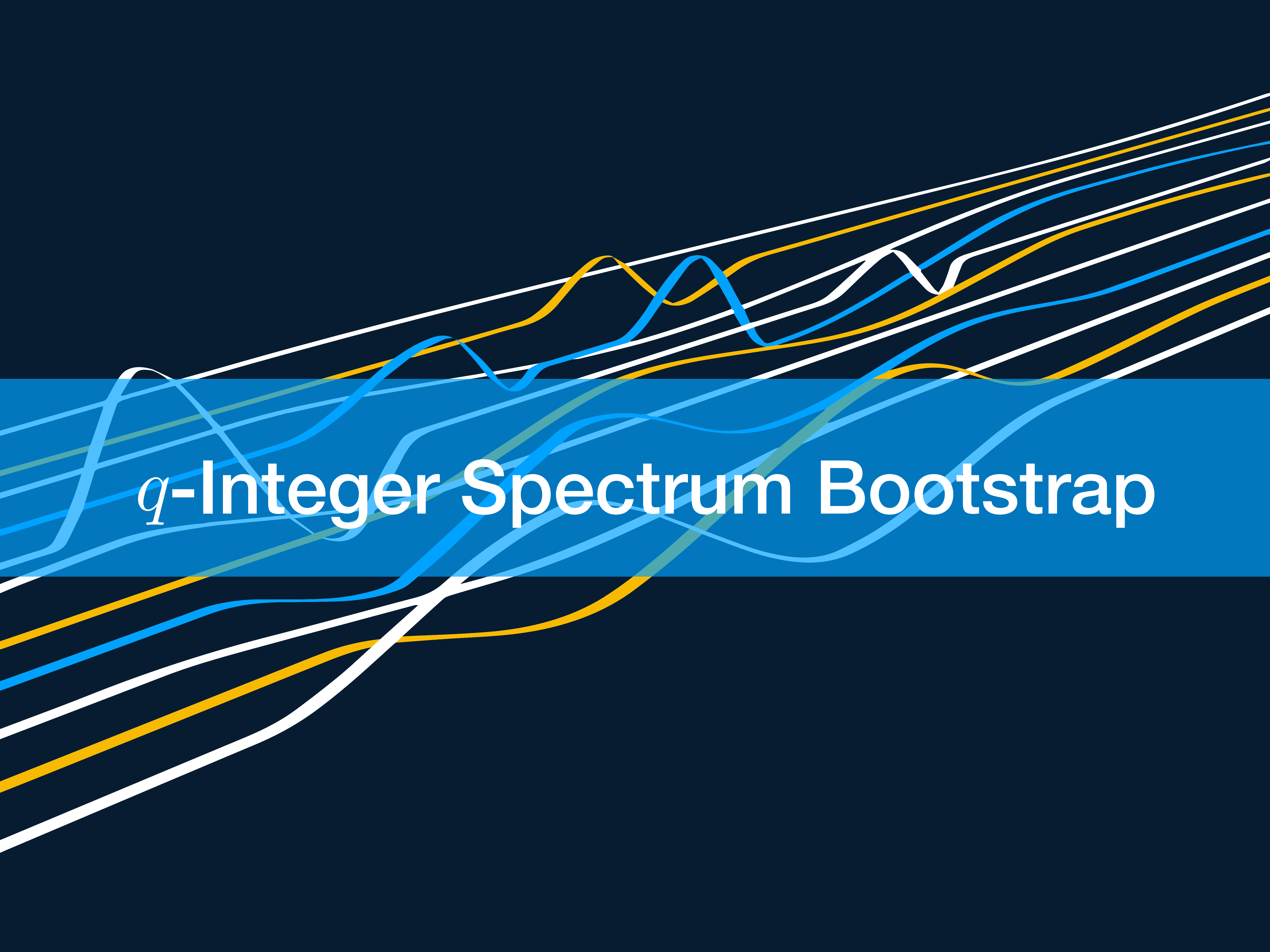
Remarkably, the hypergeometric amplitude has an integral representation,

$$A(s,t) = r\int_0^1\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\frac{x^{-s-1}y^{r-1}(1-xy)^t}{(1-x)^{t+1}}$$

reminiscent the Koba-Nielsen form for the Veneziano amplitude,

5-point: $$A^{(5)}_{\text{Ven}} = \int_0^1\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\frac{x^{-s_{12}-1}y^{-s_{45}-1}(1-xy)^{s_{23}+s_{34}-s_{51}}}{(1-x)^{s_{23}+1}(1-y)^{s_{34}+1}}$$

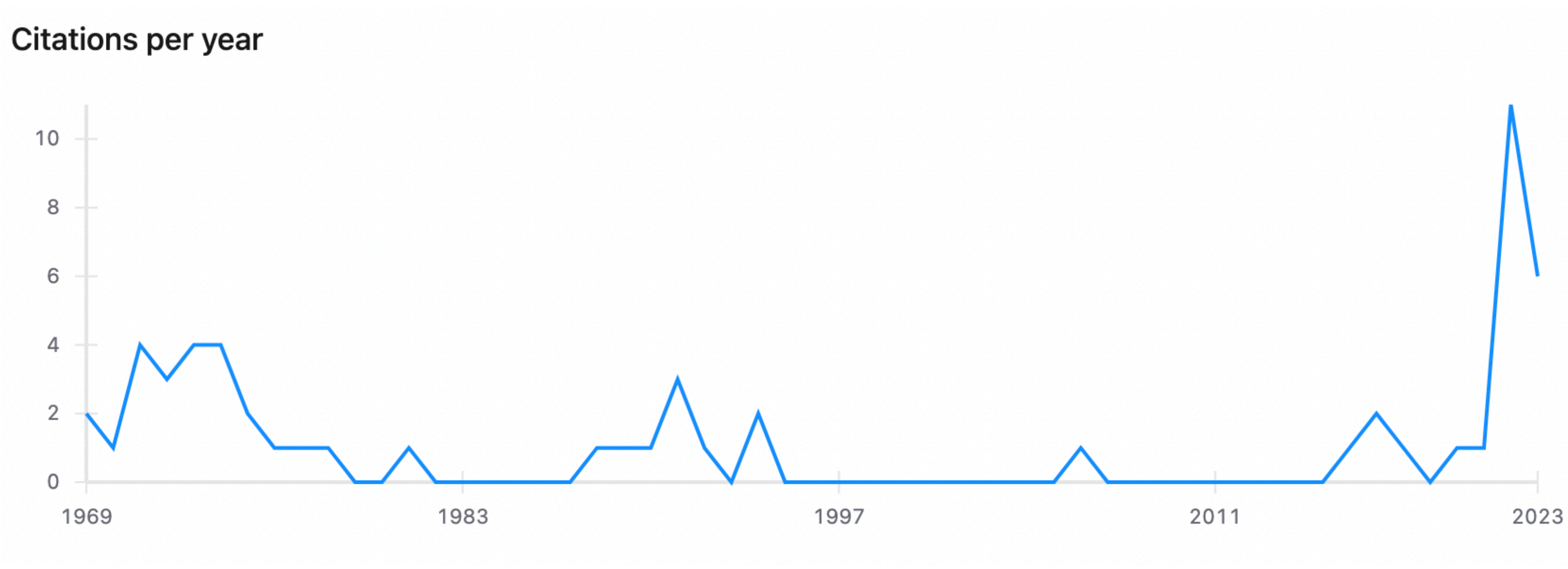



q-Integer Spectrum Bootstrap

Coon amplitudes

- Historically, string amplitudes *predate* the realization that the theory was about strings at all. Exploring amplitudes can lead to new physics, as we've seen from the hypergeometric amplitude.

- Also satisfying our physical constraints is the q-deformed generalization of Veneziano discovered by Coon (1969), unfortunately forgotten for decades:



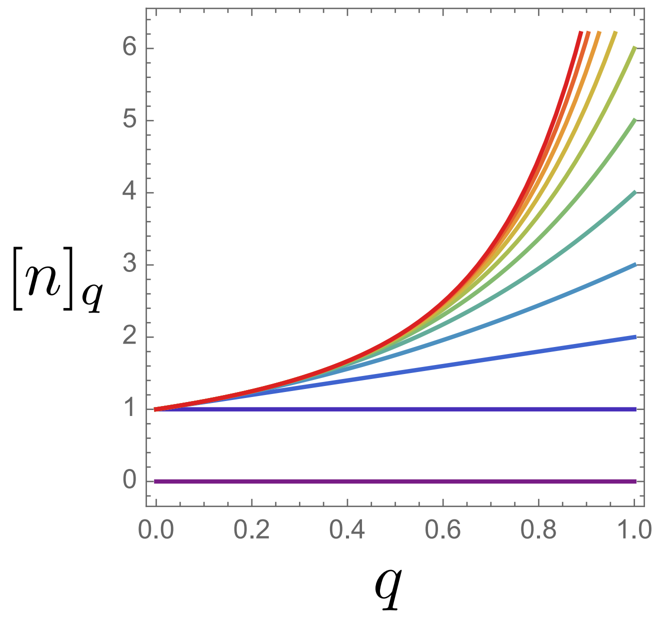
Recent surge of interest:

- unitarity
  Figueroa, Tourkine [2201.12331]; Bhardwaj et al. [2212.00764]; Jepsen [2303.02149]
- string amplitudes with similar properties
  Maldacena, **GR** [2207.06426]
- construction and generalization
  Cheung, **GR** [2210.12163, 2302.12263]; Geiser, Lindwasser [2207.08855, 2210.14920]

q-bootstrapping

- Let's bootstrap an amplitude with spectrum given by the q-deformed integers:

$$m_n^2 = [n]_q = \frac{1-q^n}{1-q}$$



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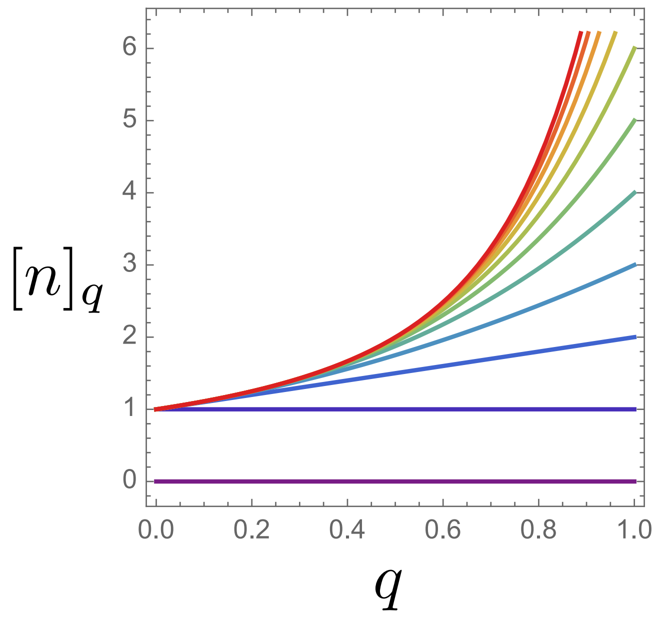


- Define new auxiliary kinematic variables:

$$\begin{aligned} s &= [\sigma]_q \\ t &= [\tau]_q \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \sigma &= \frac{\log[1-(1-q)s]}{\log q} \\ \tau &= \frac{\log[1-(1-q)t]}{\log q} \end{aligned}$$

- Assume dual resonant ansatz:

$$A(\sigma, \tau) = \sum_{n=0}^{\infty} \frac{R_n([\tau]_q)}{[n-\sigma]_q}$$

$$\frac{1}{[n-\sigma]_q} = \frac{1-(1-q)s}{[n]_q - s} \quad \Longrightarrow \quad \text{poles at } \; m_n^2 = [n]_q$$

q-bootstrapping

- As before, choose special kinematics:

$$\tau = \sigma - k$$

- Imposing crossing $A(\sigma, \sigma - k) = A(\sigma - k, \sigma)$

$$\Longrightarrow \sum_{n=k}^{\infty} \frac{R_n([\sigma - k]_q) - R_{n-k}([\sigma]_q)}{[n-\sigma]_q} = -\sum_{n=0}^{k-1} \frac{R_n([\sigma - k]_q)}{[n-\sigma]_q}$$

- q-generalized residue constraint:

$$R_n([n-k]_q = R_{n-k}([n]_q), \qquad 1 \leq k \leq n$$

- General solution:

$$R_n([\tau]_q) = \sum_{m=0}^{n} \frac{\lambda_m q^{\frac{m(m-1)}{2}}}{[m]_q!} \frac{[\tau]_q!}{[\tau - m]_q!} \frac{[n]_q!}{[n-m]_q!}$$

Prefactor dressing

- We still have the freedom to dress each term in the sum with a prefactor that does not change the poles or residues:

$$q^{\tau(\sigma-n)} \xrightarrow{\sigma\to n} 1$$

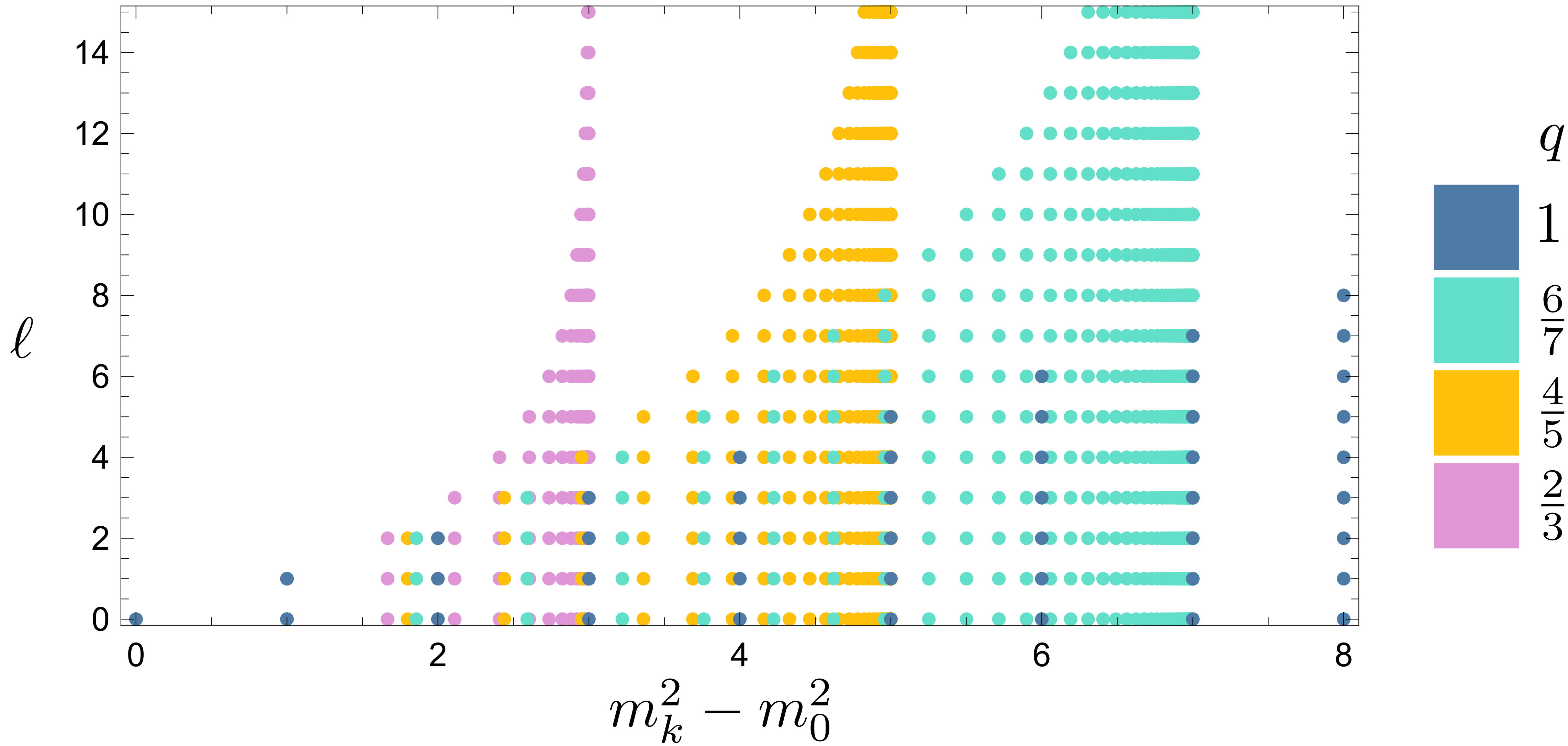
- Doing so, and choosing $\lambda_m = \dfrac{q^{\frac{m(m+1)}{2}}}{[m]_q!}$, yields a dual resonant, crossing symmetric amplitude:

$$A(\sigma,\tau) = \sum_{n=0}^{\infty} \frac{q^{\tau(\sigma-n)}}{[n-\sigma]_q} \binom{\tau+n}{n}_q$$

$$= q^{\sigma\tau} \frac{\Gamma_q(-\sigma)\Gamma_q(-\tau)}{\Gamma_q(-\sigma-\tau)}$$

Coon amplitude

Chew-Frautschi plot



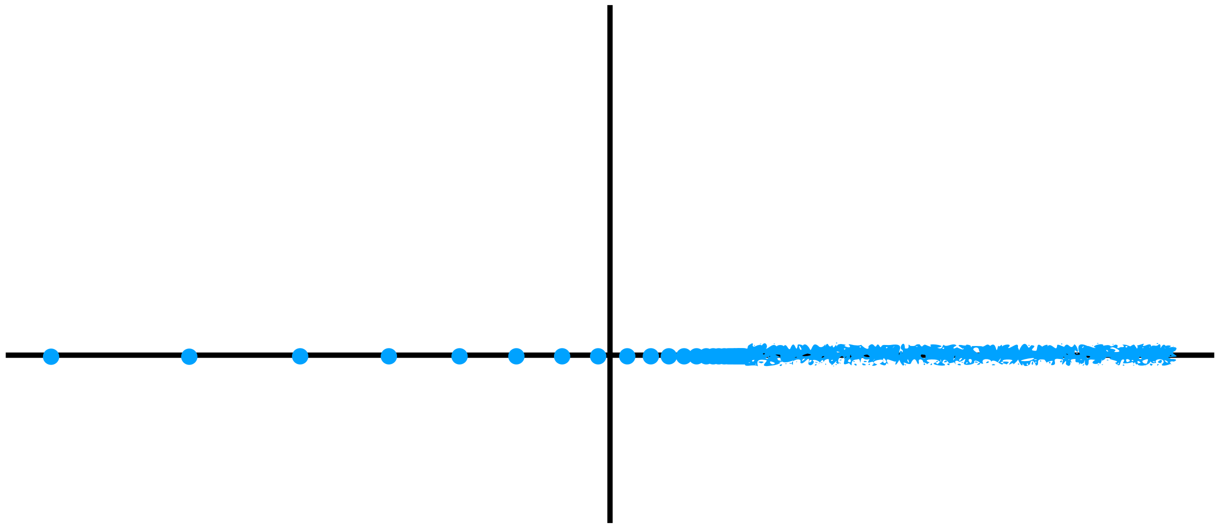
Branch cuts

- At high energies, the Coon amplitude goes like

$$\log A \sim -\log(-s)\log(-t)$$

indicating a branch cut with a double log.

- In Maldacena, **GR** [2207.06426], we built string constructions with exactly the $\log A \sim -\log(-s)\log(-t)$ form at high energies, so this is physically healthy.



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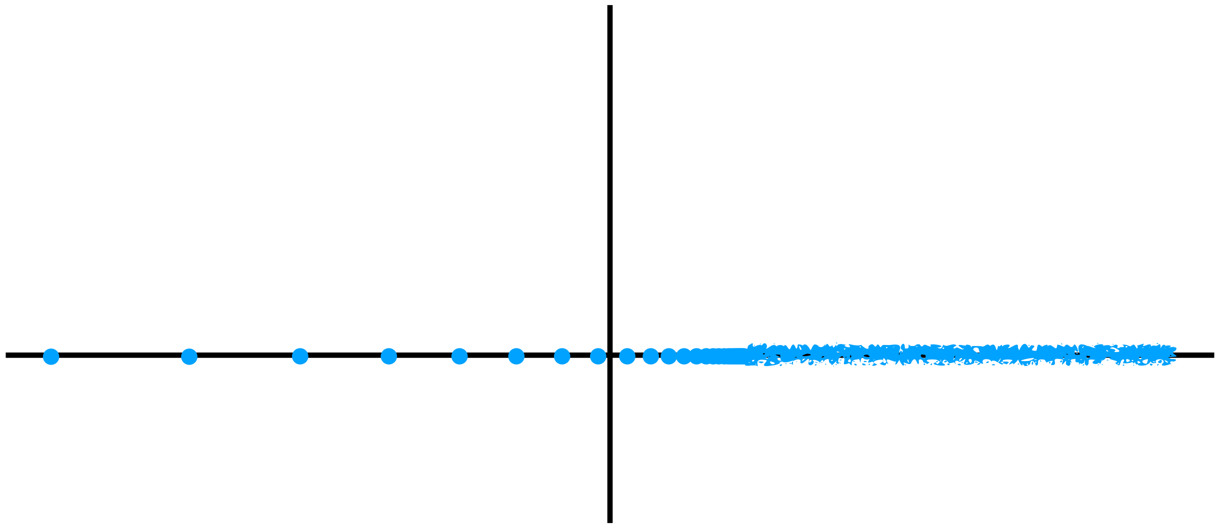
- Branch cuts above an accumulation point suggest an ionization process.
  Figueroa, Tourkine [2201.12331]

- Recent work has shown an apparent breakdown of unitarity exponentially near the branch point, but this may possibly be cured by choosing a different dressing prefactor. Jepsen [2303.02149]



q-hypergeometric amplitude

- Let's consider a more general choice for the λ_m:

$$\lambda_m = q^{\frac{m(m+1)}{2}+rm}\frac{[r]_q!}{[m+r]_q!}$$

$$\implies R_n([\tau]_q) = \frac{[\tau+n+r]_q![r]_q!}{[\tau+r]_q![n+r]_q!}$$

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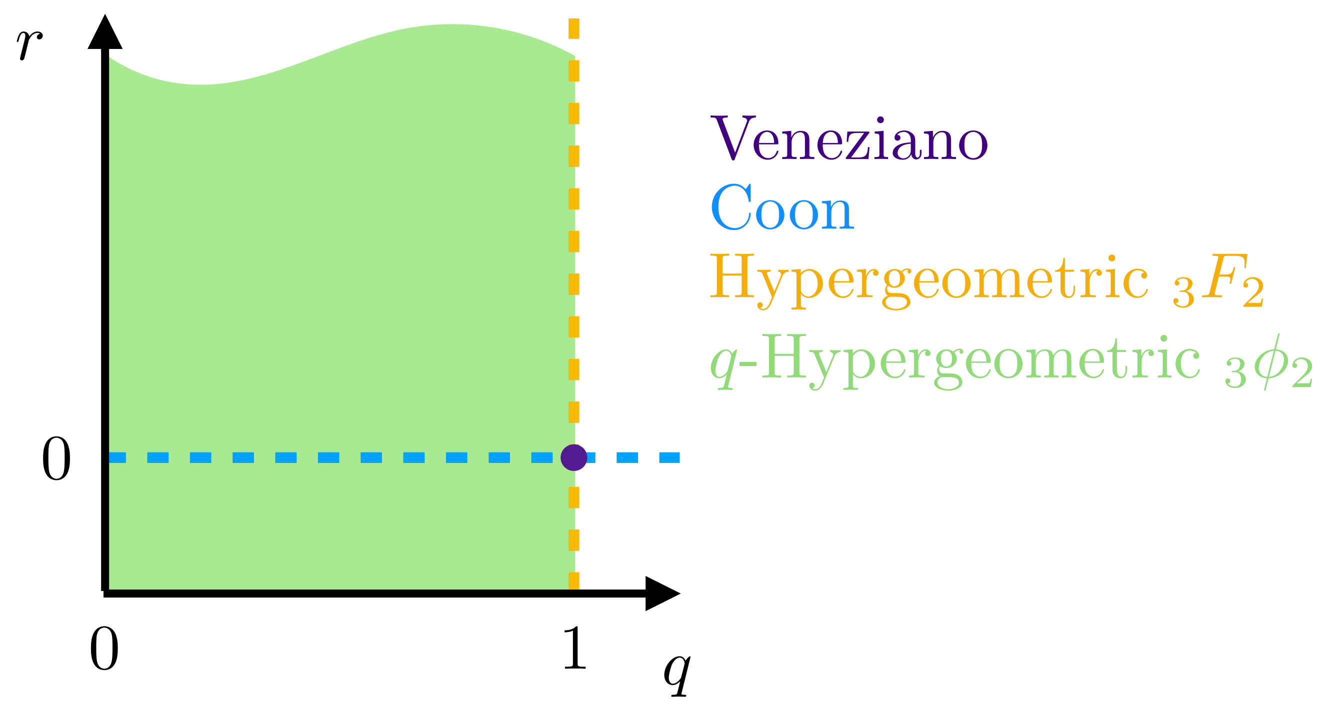
- Then using a q-Thomae (-Kummer-Whipple) transformation, we have a new class of crossing symmetric, dual resonant amplitudes:

$$A(\sigma,\tau) = \sum_{n=0}^{\infty}\frac{q^{\tau(\sigma-n)}R_n([\tau]_q)}{[n-\sigma]_q} = q^{\sigma\tau}\frac{\Gamma_q(-\sigma)\Gamma_q(-\tau)}{\Gamma_q(-\sigma-\tau)}\,{}_3\phi_2\left[\begin{matrix}q^{-\sigma}, q^{-\tau}, q^{r} \\ q^{-\sigma-\tau}, q^{1+r}\end{matrix}; q; q\right]$$

New q-hypergeometric amplitude

q-hypergeometric amplitude

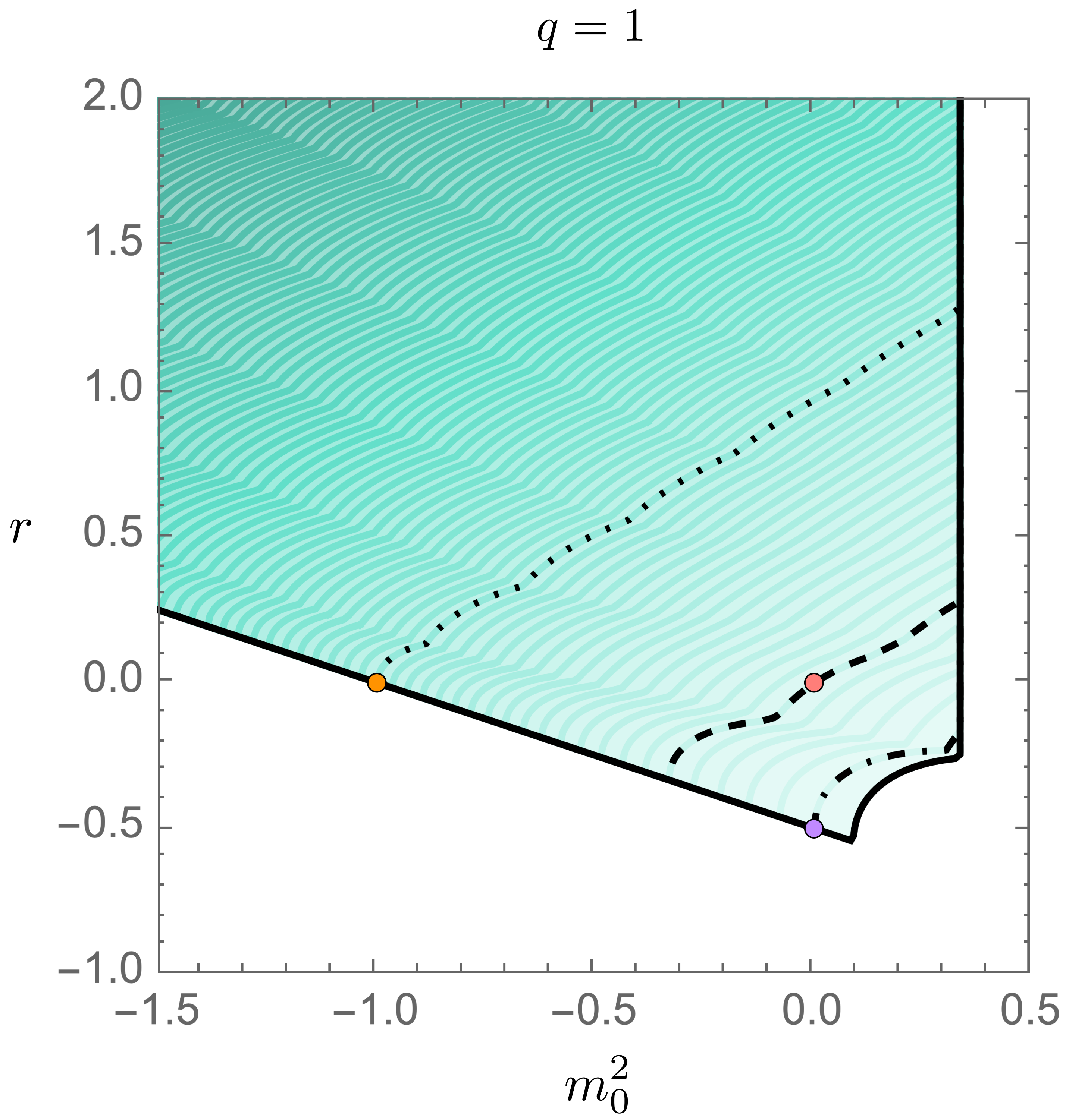
- Generalizes all of our previous amplitudes:



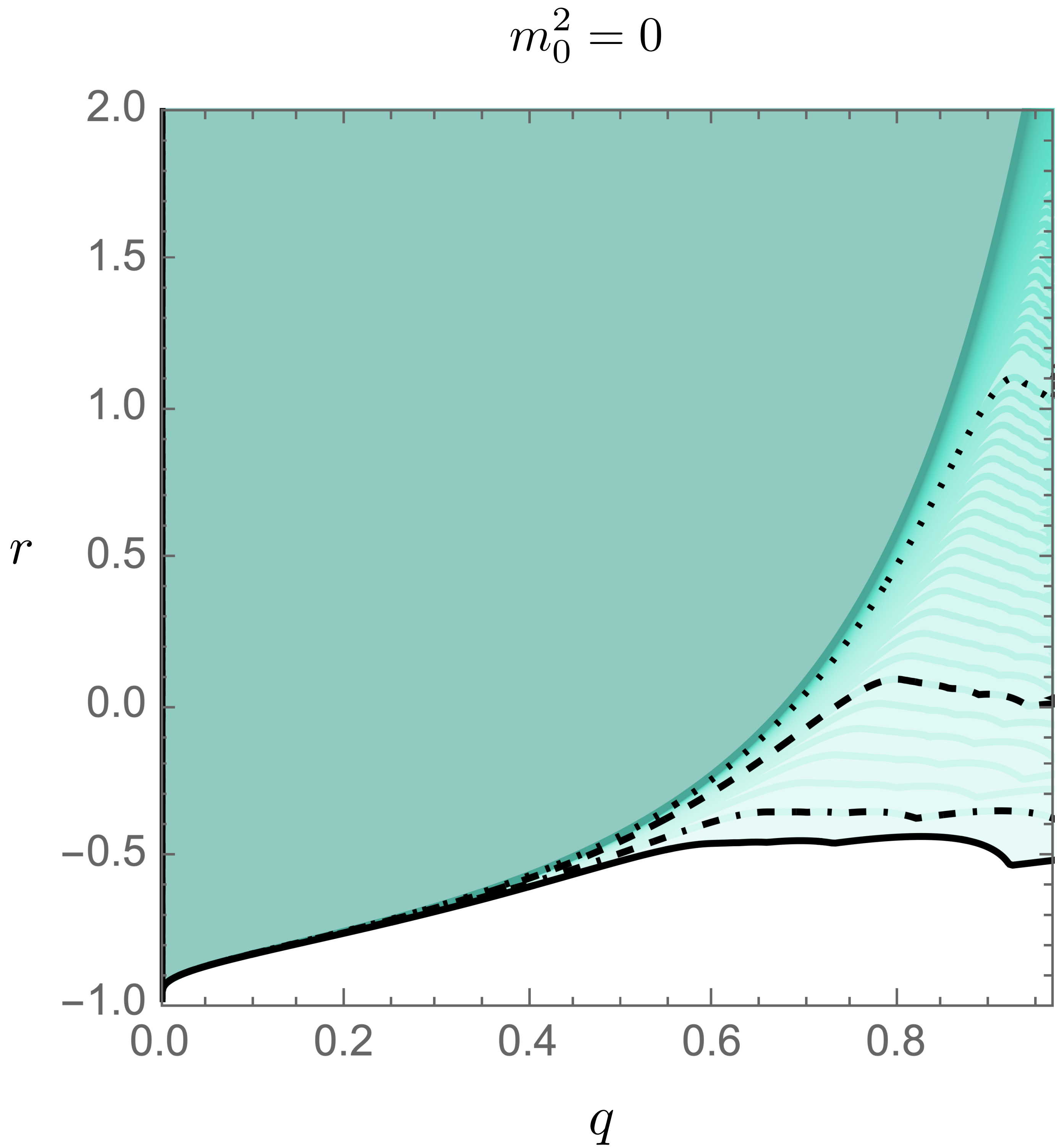
$$A(\sigma,\tau) = \sum_{n=0}^{\infty} \frac{q^{\tau(\sigma-n)} R_n([\tau]_q)}{[n-\sigma]_q} = q^{\sigma\tau} \frac{\Gamma_q(-\sigma)\Gamma_q(-\tau)}{\Gamma_q(-\sigma-\tau)} \, {}_3\phi_2 \left[\begin{matrix} q^{-\sigma}, q^{-\tau}, q^{r} \\ q^{-\sigma-\tau}, q^{1+r} \end{matrix} ; q; q \right]$$

New q-hypergeometric amplitude

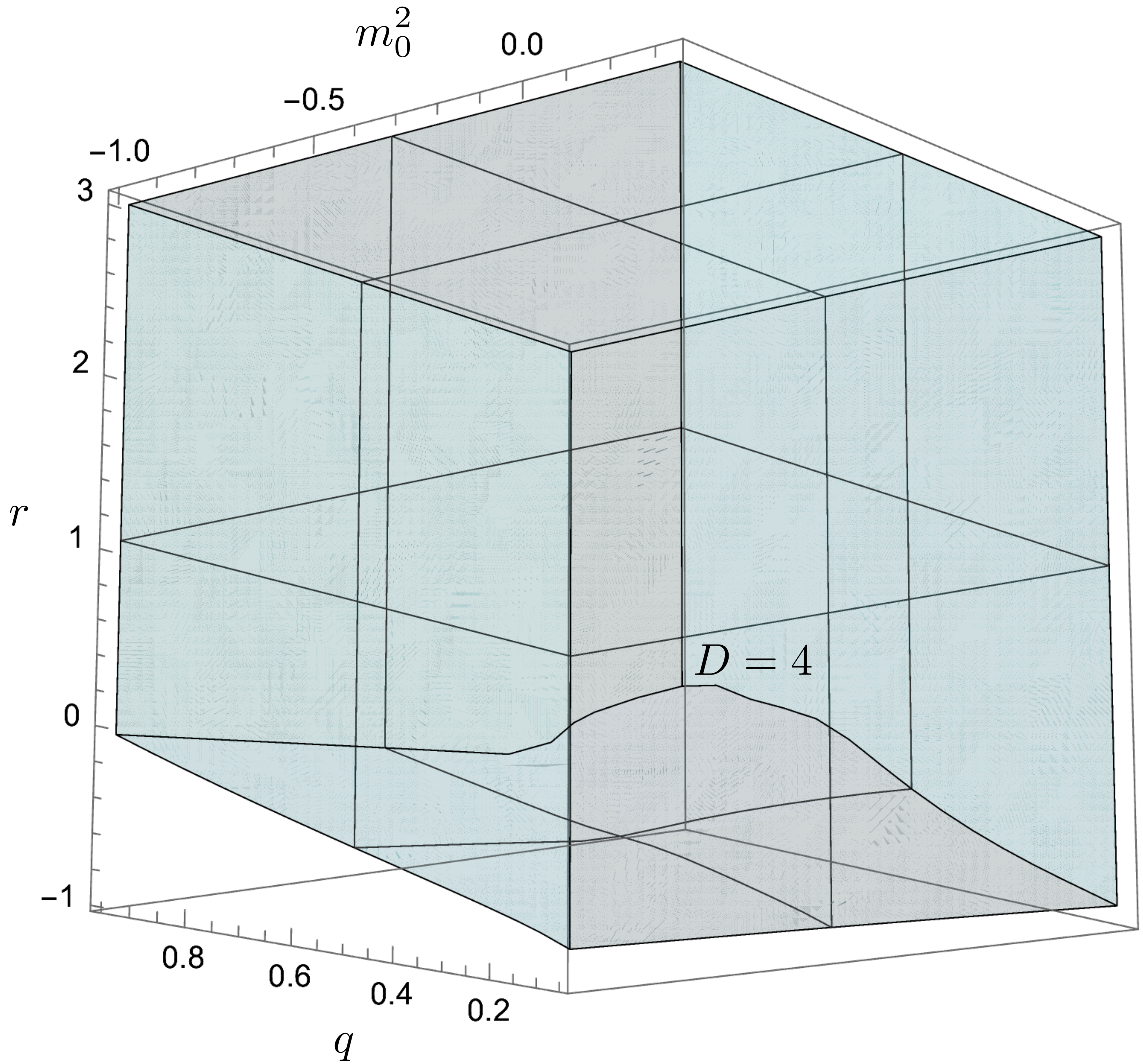
Partial wave unitarity



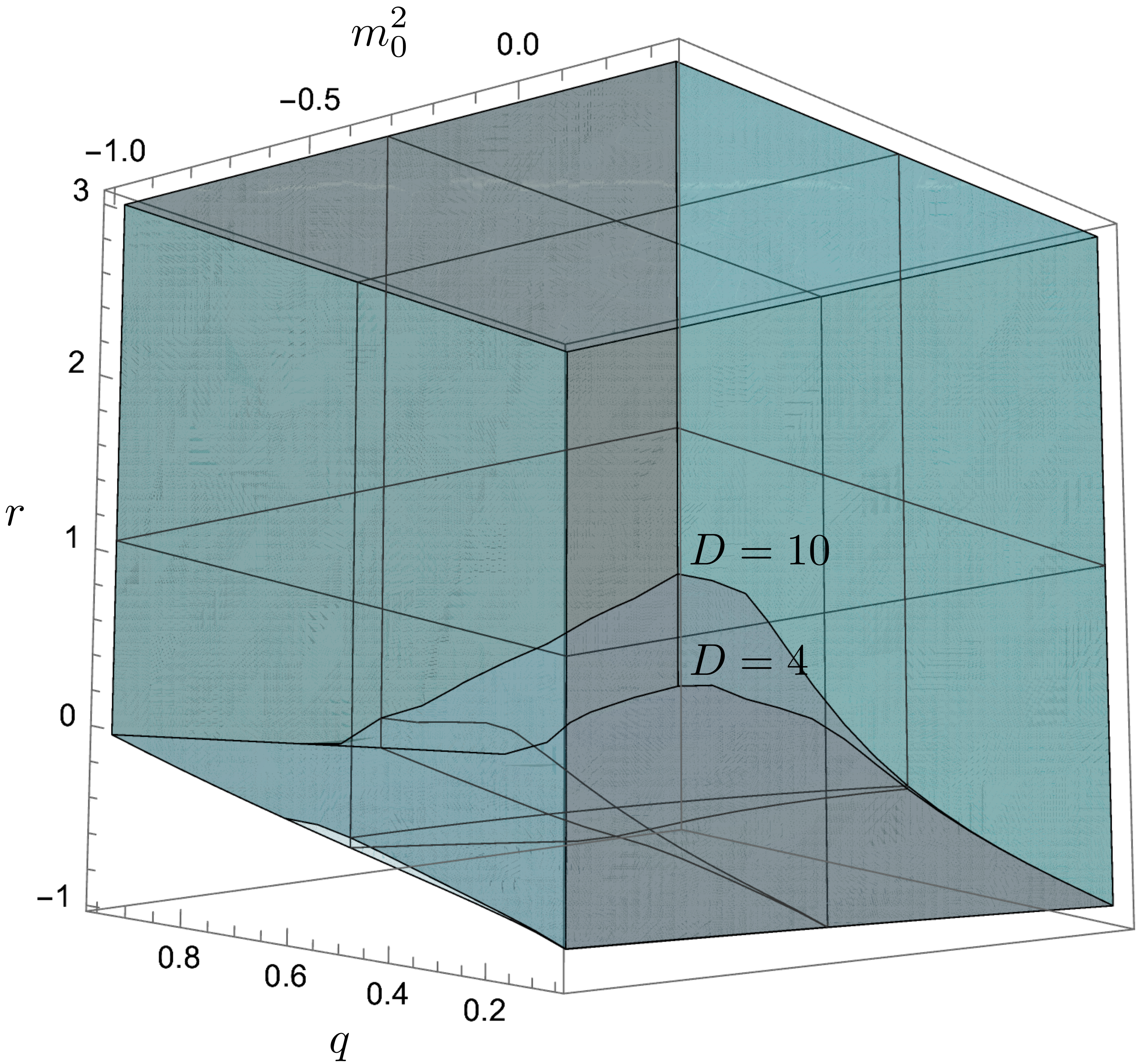
Partial wave unitarity



Partial wave unitarity



Partial wave unitarity



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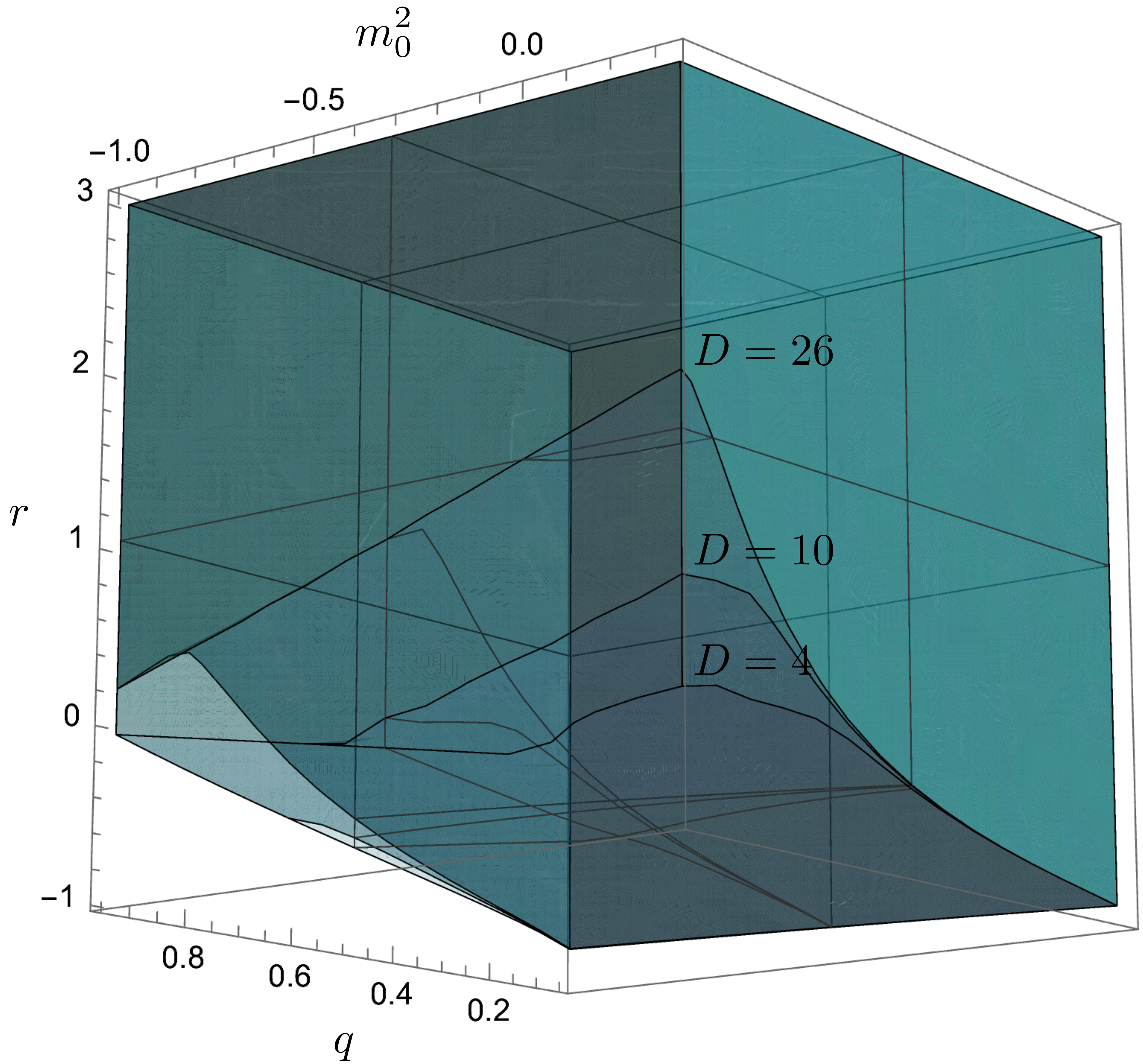
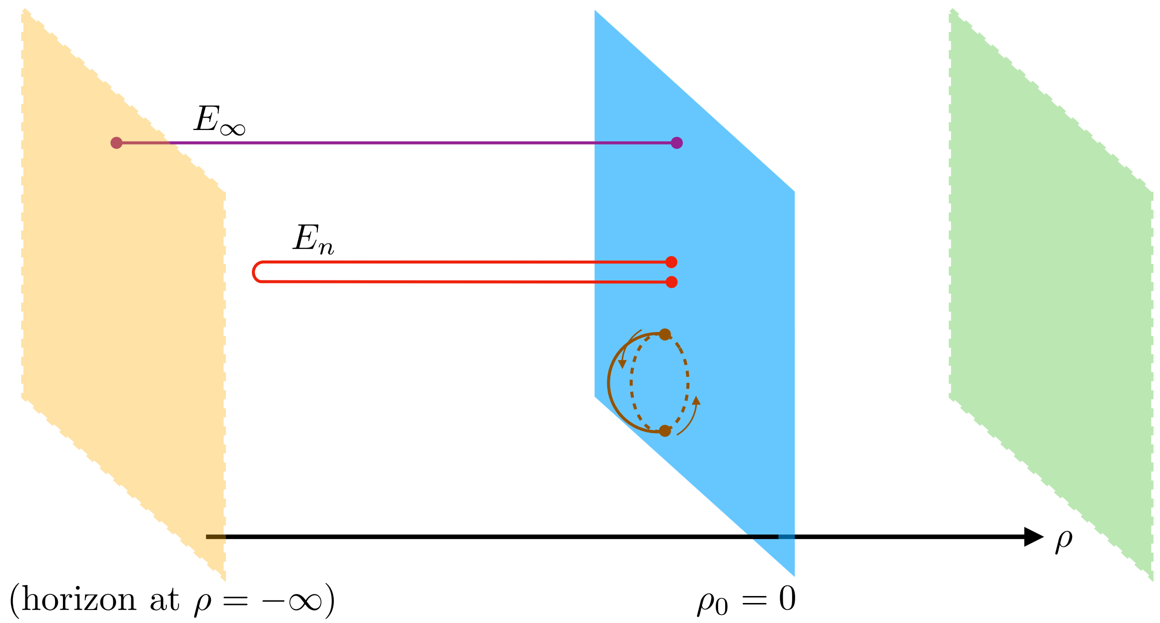
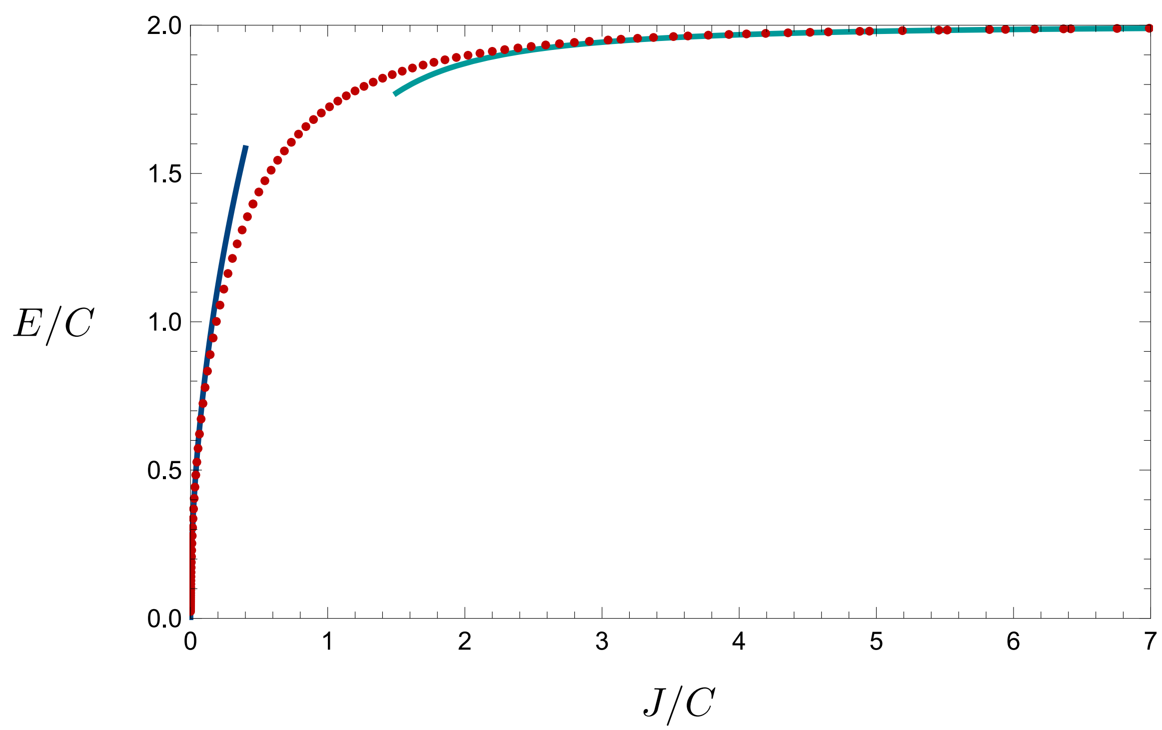


Discussion

Accumulation points in string theory

- What is the underlying physical interpretation of these amplitudes (e.g., strings)?





- For example, in Maldacena, **GR** [2207.06426] we constructed string backgrounds (scattering open strings attached to a D-brane in AdS) that exhibited various features seen in the Coon amplitude (accumulation points, high-energy behavior).

- Bootstrap amplitude constructions may therefore point the way toward new structures within string theory itself.

Conclusions

- We have derived an infinite-parameter family of amplitudes obeying:
    - Lorentz invariance
    - Crossing symmetry
    - Polynomial residues
    - UV boundedness: dual resonance
    - Regge spectrum

Conclusions

- We have derived an infinite-parameter family of amplitudes obeying:
    - Lorentz invariance
    - Crossing symmetry
    - Polynomial residues
    - UV boundedness: dual resonance
    - Regge spectrum
- For particular choices of the parameters, we get the Veneziano amplitude and a compelling one-parameter generalization: hypergeometric amplitudes
- q-deformation: derived Coon amplitude and q-hypergeometric generalization
- Partial wave unitarity satisfied in large regions of parameter space

Future directions

- Even broader spaces of dual resonant amplitudes?
- Higher-point generalization of hypergeometric amplitudes
- Systematic analysis of unitary regions of parameter space and critical dimensions
- Underlying physical system giving rise to our new amplitudes

To prove that string theory is the unique theory of quantum gravity, we should explore whether it can be bootstrapped from first principles. Alternative structures that we find along the way can help us understand the mechanism by which strings become inevitable and give us insights into new structures within string theory itself.

Questions