Bespoke Dual Resonance

Clifford Cheung
Caltech

2210.12163 (CC, Remmen)
2302.12263 (CC, Remmen)
2308.03833 (CC, Remmen)

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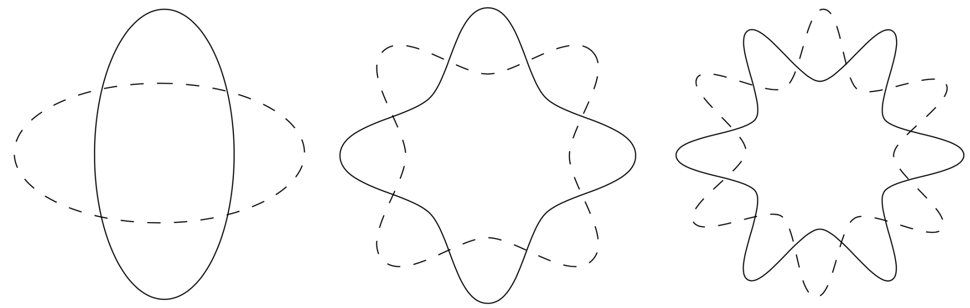
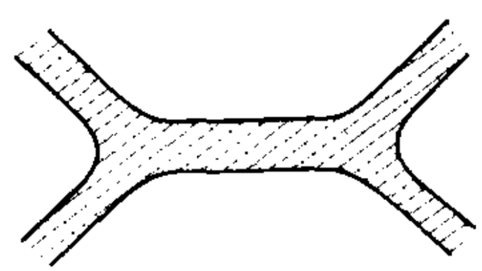
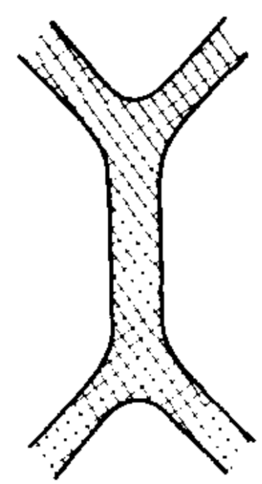
2308.03833 (CC, Remmen) ← posted this week

String amplitudes are miraculous objects!

$A(s,t) \quad \sim$  infinite spin tower

$A(s \to \infty, t) \quad \sim \quad s^{t+\mathrm{const}}$ high energy softness

$A(s,t) \quad \sim$  $\sim$  dual resonance

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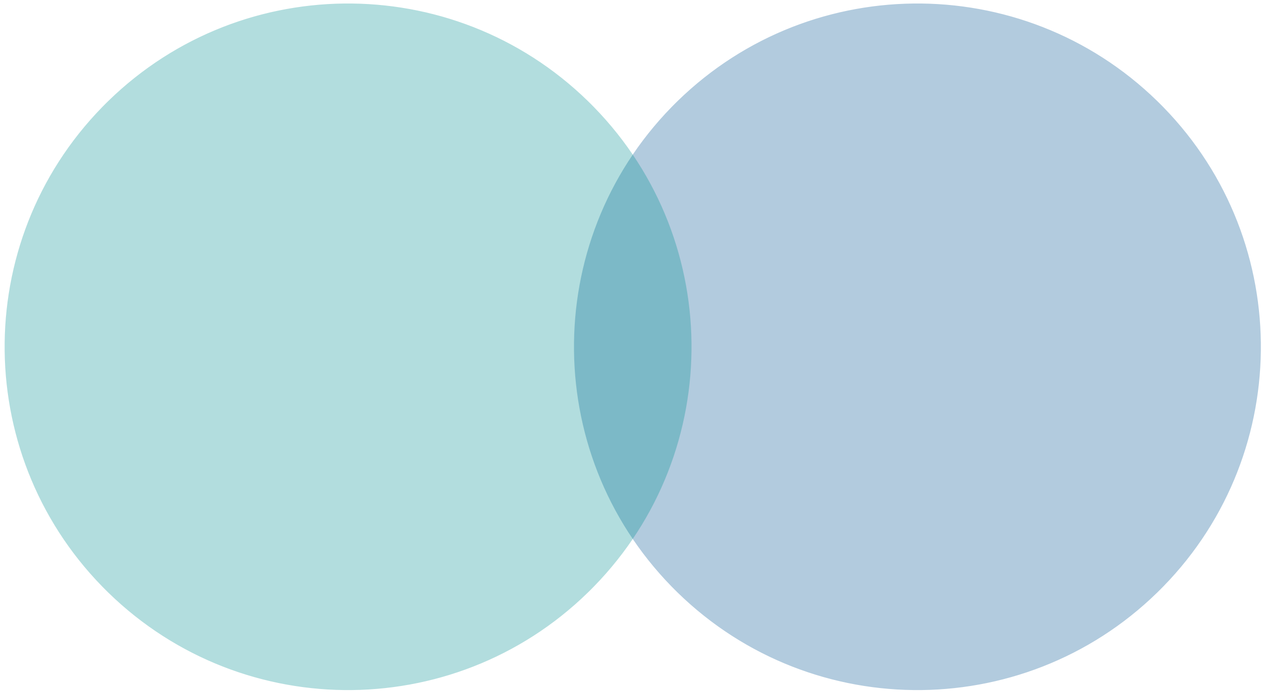

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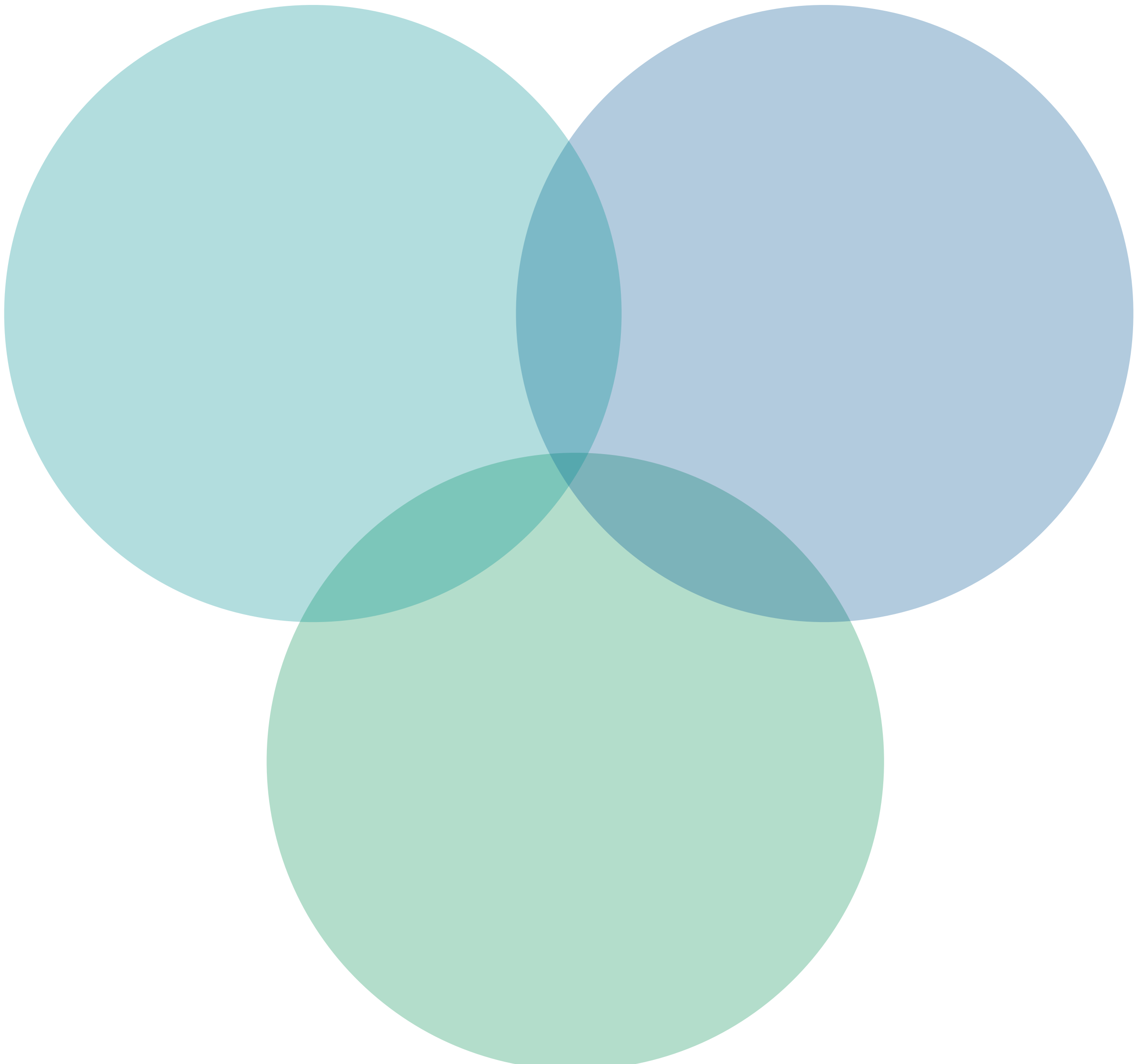

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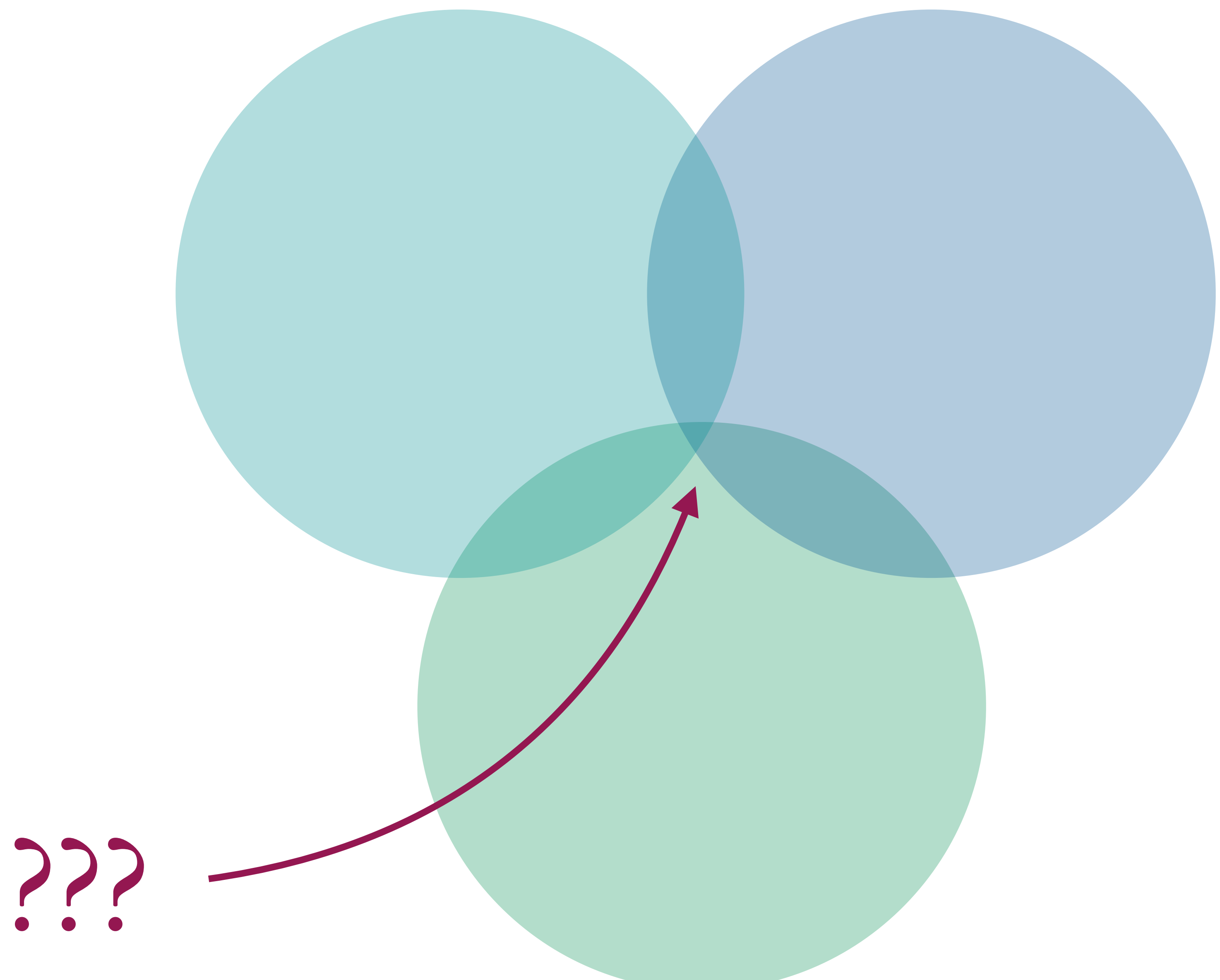

1. Crossing Symmetry

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$$R(n,t) = \operatorname*{Res}_{s=\mu(n)} A(s,t) = \sum_{\ell=0}^{\infty} a_{n,\ell}\, G_{\ell}^{(D)}(\cos\theta)$$

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$$\underset{\text{level label}}{R(n,t)} = \underset{\substack{s=\mu(n) \\ \text{mass spectrum}}}{\operatorname{Res}} A(s,t) = \sum_{\ell=0}^{\infty} a_{n,\ell}\, G_{\ell}^{(D)}(\cos\theta)$$

level
label

mass
spectrum

$a_{n,l} \geq 0$
positivity

However, conditions I. + II. still allow for a huge class of very peculiar amplitudes.

$$A(s,t) = \frac{1}{(s-m^2)(t-m^2)(u-m^2)}$$

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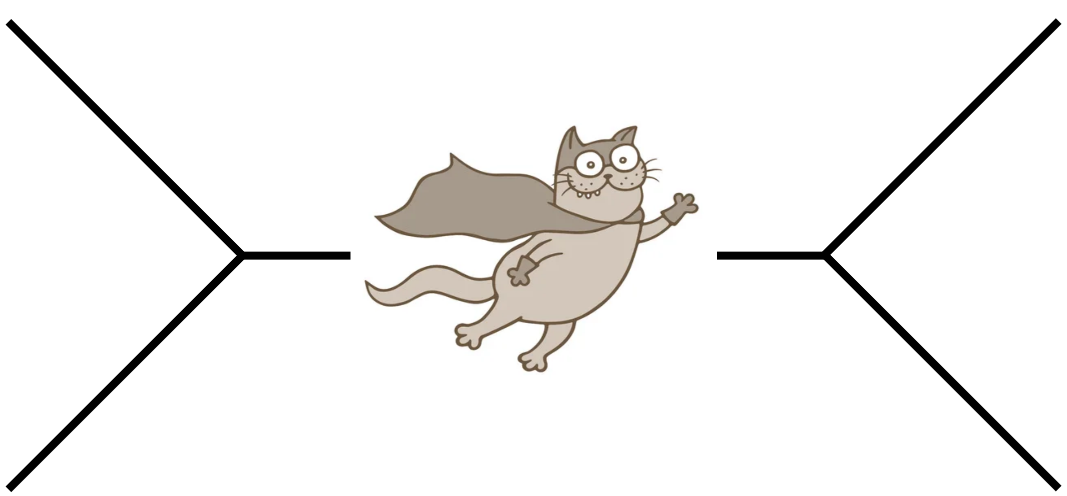

" exchange of a narrow cat in the s-channel "

III. Polynomial Residues

$$R(n,t) = \sum_{m=0}^{n} \lambda(n,k) t^k$$

Explicitly, the Veneziano amplitude has

$$\lambda_V(n,k) \quad = \quad \frac{1}{n\,!} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}$$

We assume here that level n carries up to spin n.

IV. Tame Ultraviolet Behavior

$$A(s,t) \quad = \quad \oint_{s'=s} \frac{ds'}{2\pi i}\frac{A(s',t)}{s'-s}$$

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for some value of t

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strict dual resonance

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strict dual resonance

Even if the boundary term is nonzero, we can sometimes still reabsorb it into a single channel.

$$A(s,t) = \sum_{n=0}^{\infty} \frac{R(n,s,t)}{\mu(n) - s}$$

effective dual resonance

Even with I. + II. + III. + IV. , there is a vast space of viable amplitudes (see Grant Remmen's talk).

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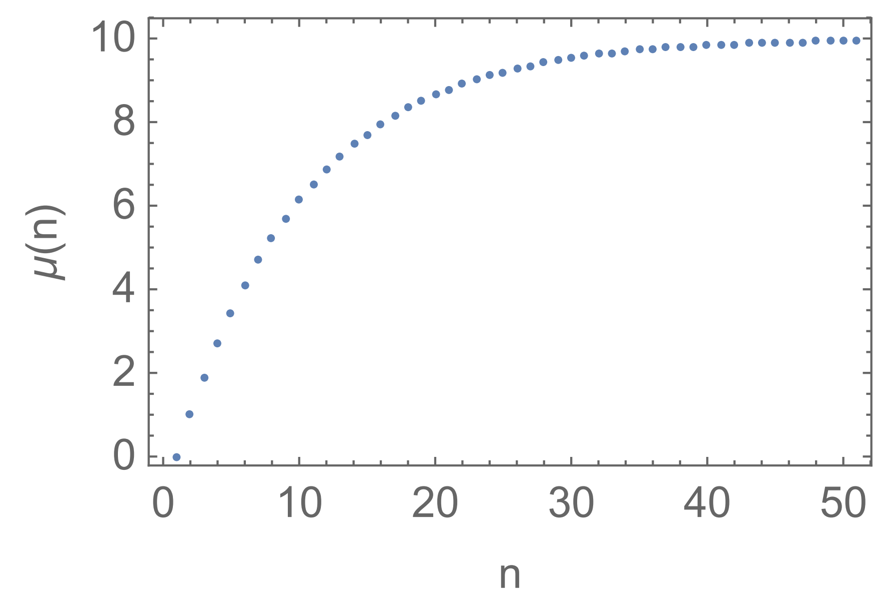
- Coon Amplitudes

accumulation point!

$$\mu(n) = [n]_q = \frac{1 - q^n}{1 - q}$$

$$= 1 + q + \cdots + q^n$$



It is an open question if these are fully consistent.

- Hypergeometric Amplitudes

$$A(s,t) \quad \sim \quad A_V(s,t)\ {}_3F_2(\ \cdots\ )$$

$$\sim \sum_{a,b} k_{ab} \frac{\Gamma(-s+a)\Gamma(-s+a)}{\Gamma(-s-t+b)}$$

Many amplitudes can be expanded in a basis of satellite Venezianos. Of course, just since x^n is a basis for $f(x)$ doesn't mean they behave the same.

All known dual resonant amplitudes exhibit an integer or q-integer spectrum. Is this required?

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"Fibonacci spectrum"

$$\mu(n) = [n]_{-\varphi^{-2}} = F_n/\varphi^{n-1}$$

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$$\mu(n) = (135\ \mathrm{MeV})^2, (498\ \mathrm{MeV})^2, \ldots$$

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No. We have closed-form tree amplitudes with all the "string miracles" for an arbitrary spectrum.

Toy Model

Let's try a leading order guess. Just remap s and t.

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$$A_V(s,t)$$

$$\mu(n) = n$$

linear Regge spectrum

Let's try a leading order guess. Just remap s and t.

$$A_V(s,t) \longrightarrow A_V\left(-\delta+\sqrt{s},-\delta+\sqrt{t}\right)$$

$$\mu(n)=n \longrightarrow \mu(n)=(n+\delta)^2$$

linear Regge spectrum

Kaluza-Klein spectrum

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linear Regge spectrum

Kaluza-Klein spectrum

However, this object has annoying branch cuts.

What if we sum over square root branches?

$$A(s,t) = \sum_{\substack{\sigma = -\delta \pm \sqrt{s} \\ \tau = -\delta \pm \sqrt{t}}} A_V(\sigma, \tau)$$

Sum over branch cuts to cancel them, e.g. like in

$$f(\sqrt{x}) + f(-\sqrt{x}) \quad = \quad \text{even function of } \sqrt{\mathrm{x}}$$

“Galois sum” the Veneziano dual representation:

$$A_V(s,t) = \sum_n \left[\frac{1}{n-s} \right] \left[R_V(n,t) \right]$$

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simple pole in s

$$\frac{2(n+\delta)}{(n+\delta)^2 - s}$$

polynomial in t

$$n = 0: \quad 2$$

$$n = 1: \quad 2(1-\delta)$$

$$n = 2: \quad (1-\delta)(2-\delta) + t$$

"Galois sum" the leading asymptotics to obtain:

$$A(s \to \infty, t) \quad \sim \sum_{\substack{\sigma = -\delta \pm \sqrt{s} \\ \tau = -\delta \pm \sqrt{t}}} \sigma^{\tau} + \cdots$$

$$\sim \left(\sqrt{s}\right)^{-\delta+\sqrt{t}} + \left(\sqrt{s}\right)^{-\delta-\sqrt{t}}$$

decay phase

Require $\delta < 0$ so this term vanishes for some t.

We can recast the "Galois sum" as a "dlog form"!

$$A(s,t) = \frac{1}{2\pi i}\oint d\sigma \left[\frac{1}{\sigma+\delta-\sqrt{s}} + \frac{1}{\sigma+\delta+\sqrt{s}}\right]$$
$$\times \frac{1}{2\pi i}\oint d\tau \left[\frac{1}{\tau+\delta-\sqrt{t}} + \frac{1}{\tau+\delta+\sqrt{t}}\right] \times A_V(\sigma,\tau)$$

$$= \left(\frac{1}{2\pi i}\right)^2 \oint d\log\left((\sigma+\delta)^2 - s\right) d\log\left((\tau+\delta)^2 - t\right) A_V(\sigma,\tau)$$

Our construction generalizes to any spectrum.

General Model

The crux of our framework is a "spectral curve".

$$f(\mu, \nu) \quad = \quad P(\nu) - \mu Q(\nu) \quad = \quad 0$$

kinematic level polynomials

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kinematic level polynomials

The zero locus defines the important quantities:

A) spectrum

$$\mu(\nu) = P(\nu)\,/\,Q(\nu)$$

B) inverse spectrum

$$\nu_\alpha(\mu) \quad \text{for} \quad \alpha = 0,1,2,\ldots$$

Consider the asymptotic behavior of spectrum.

$$\lim_{\nu\to\infty} \mu(\nu) = \lim_{\nu\to\infty} \frac{P(\nu)}{Q(\nu)} \sim \nu^{|P|-|Q|}$$

$|P| \leq |Q|$ accumulation point spectrum

$|P| > |Q|$ divergent spectrum

Let us assume "asymptotic uniqueness of strings".

$$|P| = |Q| + 1$$

Define an amplitude with a "bespoke" spectrum.

$$A(s,t) = \sum_{\alpha,\beta} A_V(\nu_\alpha(s), \nu_\beta(t))$$

$$= \left(\frac{1}{2\pi i}\right)^2 \oint \sum_{\alpha,\beta} \frac{d\sigma}{\sigma - \nu_\alpha(s)} \frac{d\tau}{\tau - \nu_\beta(t)} A_V(\sigma,\tau)$$

$$= \oint \frac{d\log f(s,\sigma)}{2\pi i} \oint \frac{d\log f(t,\tau)}{2\pi i} A_V(\sigma,\tau),$$

As before, we "Galois sum" the Regge behavior.

$$A_\infty(t) = \lim_{s\to\infty} A(s,t) \sim \lim_{s\to\infty} \sum_{\alpha,\beta} \nu_\alpha(s)^{\nu_\beta(t)}$$

$A_\infty(t)$ converges if the condition below is satisfied.
It is typically constant, but can be made zero.

$$\mathrm{Re}\left(\nu_\beta(t)\right) < 0 \quad \text{for} \quad \text{all} \quad \beta$$

"Hurwitz stability"

Next “Galois sum” the dual resonant form.

$$A(s,t) = \sum_{\alpha,\beta} A_V(\nu_\alpha(s), \nu_\beta(t))$$

polynomial in t

$$= \sum_{n=0}^{\infty} \left(\sum_{\alpha} \frac{1}{n - \nu_\alpha(s)} \right) \left(\sum_{\beta} R_V(n, \nu_\beta(t)) \right)$$

rational polynomial in s

We have derived expressions for dual resonant amplitudes with customizable spectra!

No explicit roots (cf. the quintic)? No problem.

i) power sums of roots

$$\sum_{\alpha} \nu_\alpha(t)^k$$

ii) elementary symmetric polynomials

$$1, \quad \sum_{\alpha} \nu_\alpha(t), \quad \sum_{\alpha<\beta} \nu_\alpha(t)\nu_\beta(t), \quad \sum_{\alpha<\beta<\gamma} \nu_\alpha(t)\nu_\beta(t)\nu_\gamma(t),$$

iii) coefficients of spectral curve

$$f(t,\nu) \;=\; P(\nu) - tQ(\nu) \;=\; 0$$

Example A:
Simplest Nonlinear Model

The simplest model with a nonlinear spectrum:

$$\mu(\nu) = \frac{P(\nu)}{Q(\nu)} = \frac{\nu^2 + p_1\nu + p_2}{\nu + q_2}$$

$$\nu_\pm(\mu) = \frac{1}{2}\left[\mu - p_1 \pm \sqrt{p_1^2 - 4p_2 - 2(p_1 - 2q_2)\mu + \mu^2}\right]$$

$$\longrightarrow \qquad A(s,t) = \sum_{\alpha,\beta=\pm} A_V(\nu_\alpha(s), \nu_\beta(t))$$

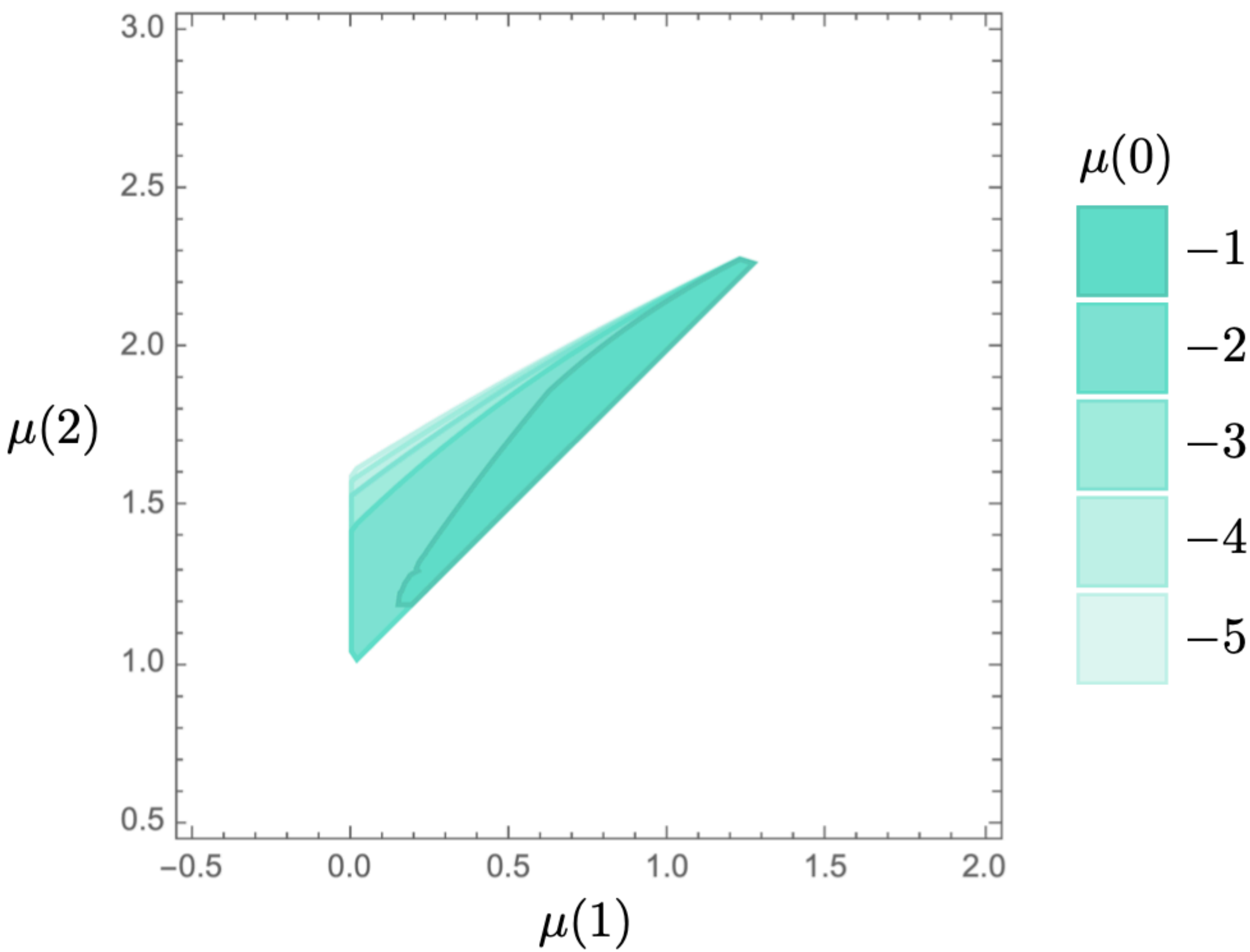
In general, the space of parameters is big. But we can map three parameters to three masses.

$$p_1, \;\; p_2, \;\; q_2$$

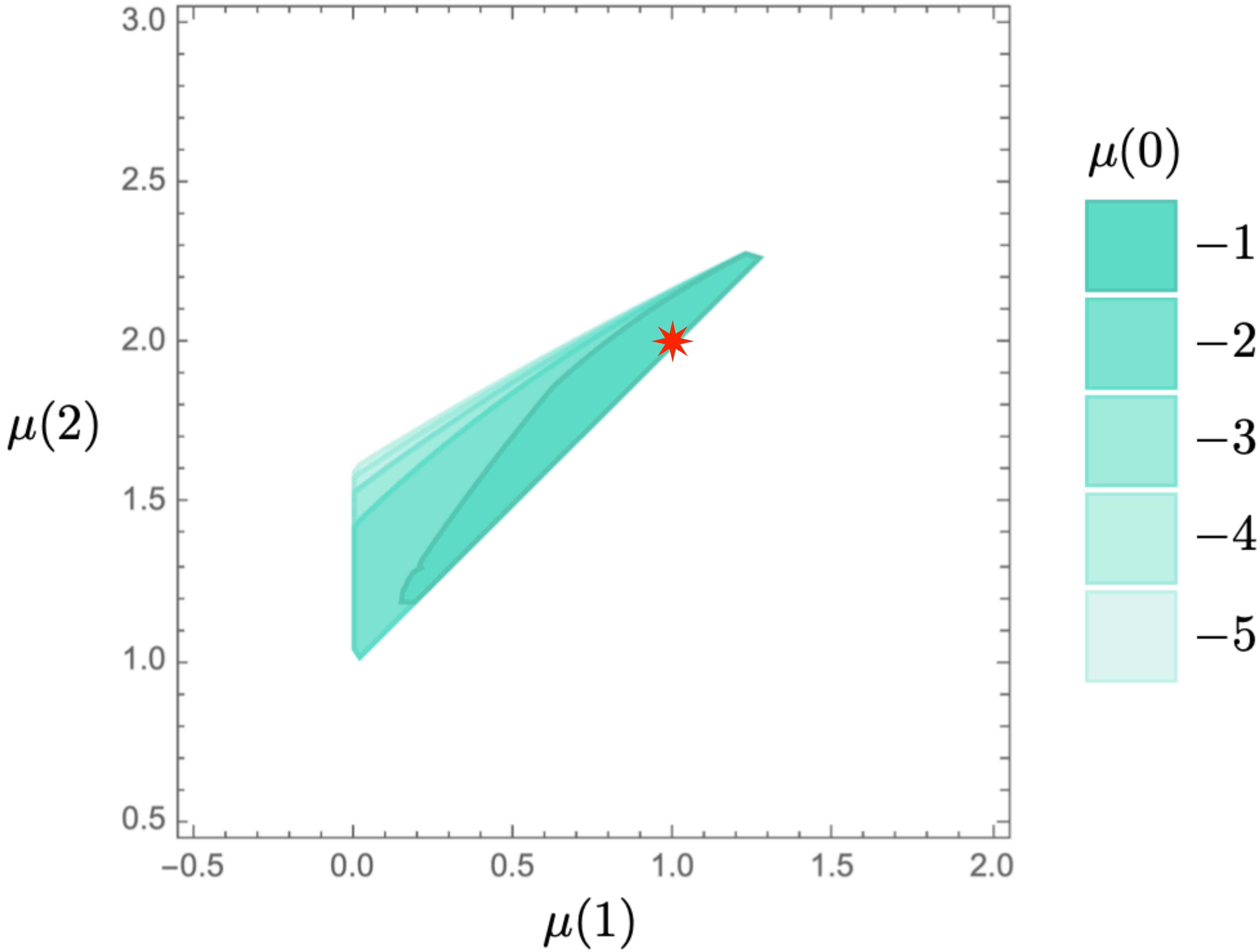
$$\big\downarrow$$

$$\mu(0), \;\; \mu(1), \;\; \mu(2)$$

For partial wave unitarity bounds, let us assume dimension $D = 4$ and external mass $m_{\text{ext}}^2 = 0$.



3.0
2.5
2.0
1.5
1.0
0.5
−0.5
0.0
0.5
1.0
1.5
2.0
$\mu(2)$
$\mu(1)$
$\mu(0)$
−1
−2
−3
−4
−5



3.0
2.5
2.0
1.5
1.0
0.5
−0.5
0.0
0.5
1.0
1.5
2.0
μ(1)
μ(2)
μ(0)
−1
−2
−3
−4
−5

Example B:
Post-Regge Model

A model with a post-Regge expanded spectrum:

$$\mu(\nu) = \frac{P(\nu)}{Q(\nu)} = \frac{\sum_{i=0}^{h} \kappa_i (\nu - \nu_*)^{h-i}}{(\nu - \nu_*)^{h-1}}$$

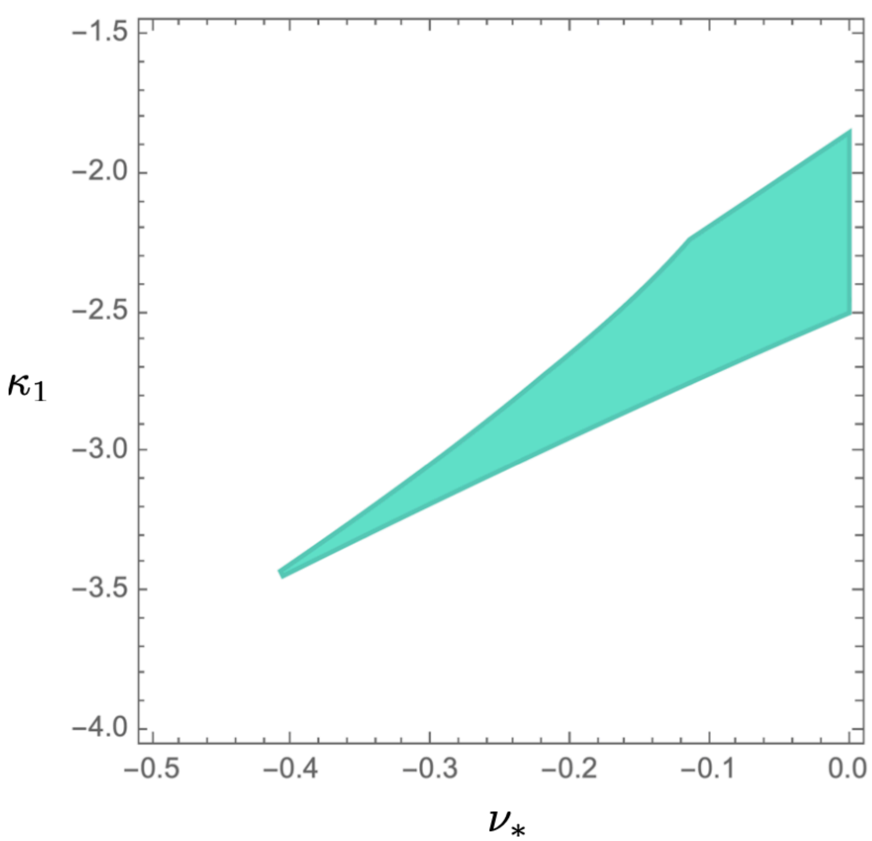
$$= (\nu - \nu_*) + \kappa_1 + \frac{\kappa_2}{\nu - \nu_*} + \frac{\kappa_3}{(\nu - \nu_*)^2} + \cdots$$

$$\longrightarrow \quad A(s,t) = \sum_{\alpha,\beta} A_V(\nu_\alpha(s), \nu_\beta(t))$$

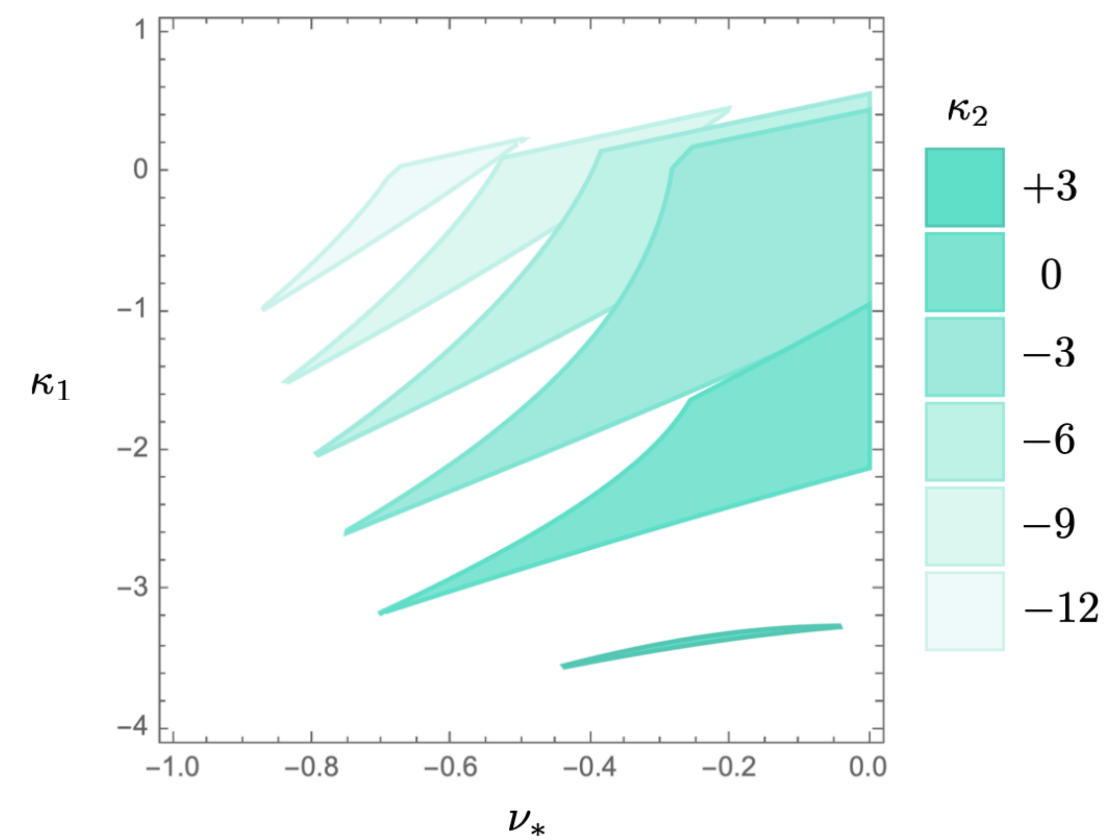
$h = 2$

$$-1.2293 \lesssim \nu_* < 0$$

$h = 3$



$h = 4$



Higher-Point

The generalization to higher-point is obvious:

$$A_4 = \sum_{\alpha_{12},\alpha_{23}} A_V(\nu_{\alpha_{12}}(s_{12}), \nu_{\alpha_{23}}(s_{23})),$$

$$A_5 = \sum_{\substack{\alpha_{12},\alpha_{23} \\ \alpha_{34},\alpha_{45},\alpha_{51}}} A_V(\nu_{\alpha_{12}}(s_{12}), \nu_{\alpha_{23}}(s_{23}), \nu_{\alpha_{34}}(s_{34}), \nu_{\alpha_{45}}(s_{45}), \nu_{\alpha_{51}}(s_{51}))$$

Folding each planar Mandelstam by ν maps the stringy linear spectrum to an arbitrary spectrum.

Now "Galois sum" the worldsheet representation.

$$A_V(s_{12}, s_{23}) \;=\; \int_0^1 dx\, x^{-s_{12}-1}(1-x)^{-s_{23}-1}$$

$$A(s,t) \;=\; \int_0^1 dx \left(\sum_{\alpha_{12}} x^{-\nu_{\alpha_{12}}(s_{12})-1} \right) \left(\sum_{\alpha_{23}} (1-x)^{-\nu_{\alpha_{23}}(s_{23})-1} \right)$$

Since the vertex operators in the original string amplitude factorize, the branch cuts cancel!

Conclusions

Summary of Amplitudes Properties

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- arbitrary customizable spectrum

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- tame ultraviolet behavior
- dual resonance
- worldsheet + dlog representation
- higher-point generalization

The Takeaway:

Option i) String theory isn't unique,

or,

Option ii) These miraculous amplitudes properties just are not that constraining!

Future Prospects

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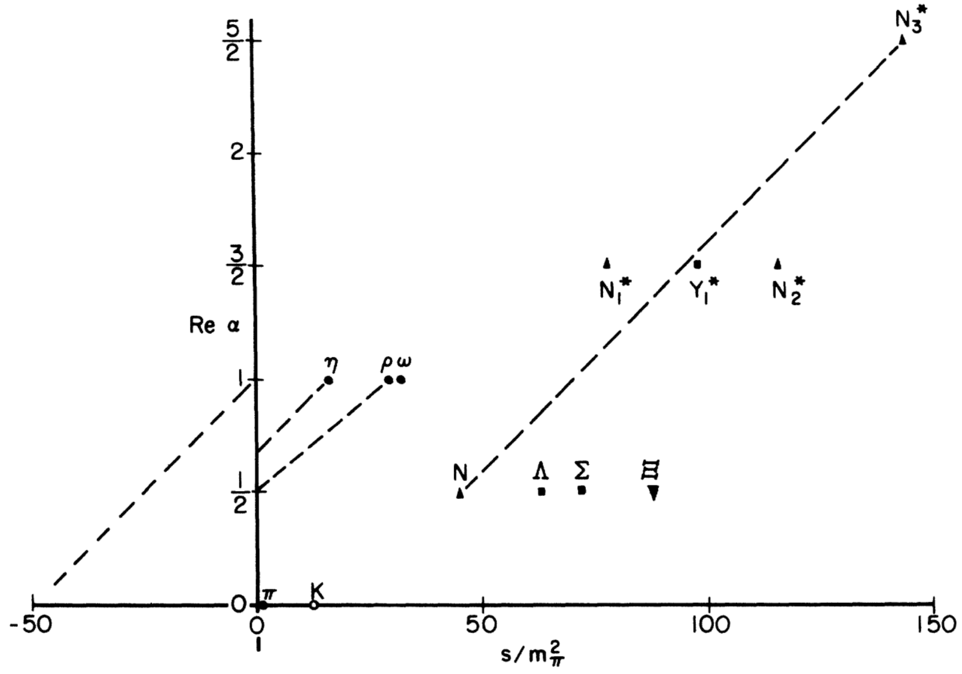
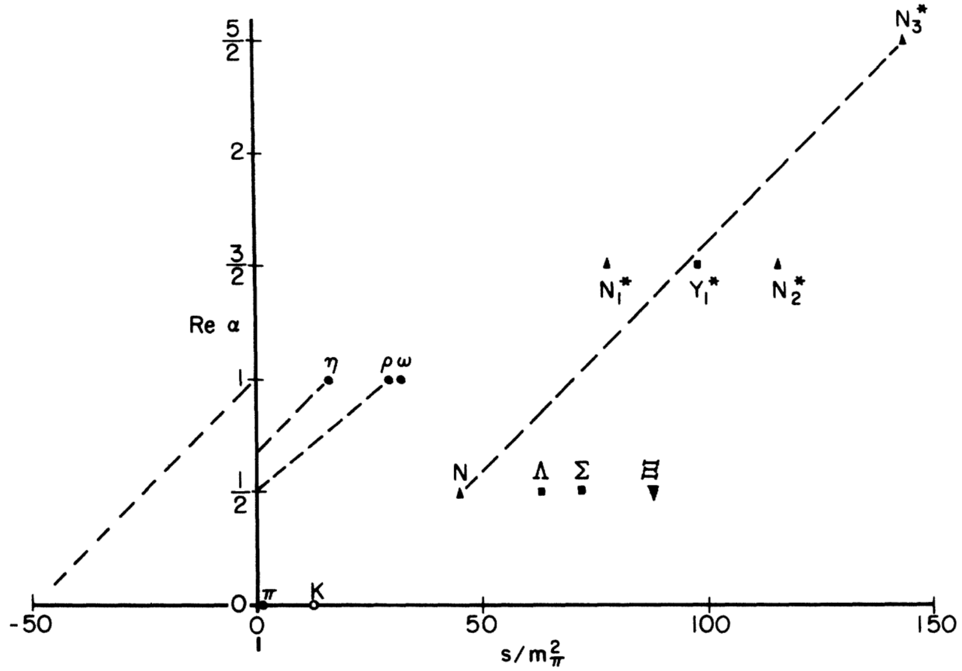


Future Prospects

- unitarity for higher-point + massive poles
- monodromy relations
- extension to gravity
- application to QCD?
- is there an actual theory?



Thank You!