Classical Dynamics of Vortex Solitons from Perturbative Scattering Amplitudes

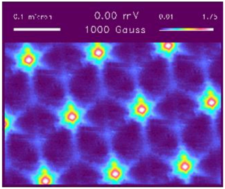
Callum R. T. Jones

Based on [2305.08902]

UCLA

Vortices and Vortex Strings

Superconductivity
[Ginzburg, Landau 1950]
[Abrikosov 1957]



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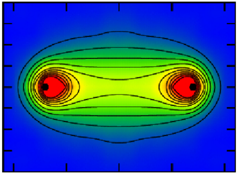

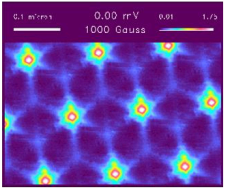

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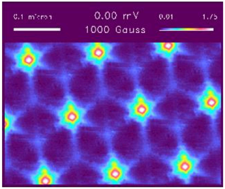

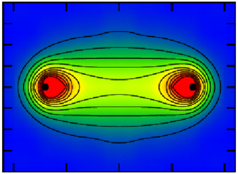

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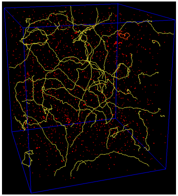

Cosmic strings [Kibble 1976]

Scattering Amplitudes for *Superconductors*

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Abelian Higgs (Landau-Ginzburg) in $d = 2 + 1$

$$S = \int \mathrm{d}^3 x \left[-\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 - \frac{\mu^2}{8} \left(|\phi|^2 - v^2 \right)^2 \right], \qquad D_\mu = \partial_\mu + ieA_\mu .$$

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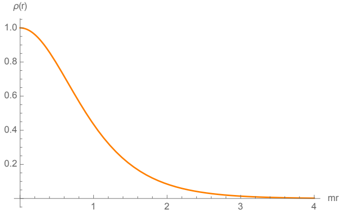
Elementary particle spectrum: *massive photon* $A_\mu(x)$ and *Higgs boson* $\sigma(x)$

$$m_\gamma = \sqrt{2} e v, \qquad m_\sigma = \frac{v\mu}{\sqrt{2}}.$$

Abrikosov-Nielsen-Olesen (ANO) Vortex

$$\phi(x) = \left(v + \frac{\rho(r)}{\sqrt{2}} \right) e^{iN\theta},$$

$$A_i(x) = \frac{\epsilon_{ij} x_j}{r^2} A(r)$$



Spectrum of “particle-like” topological solitons called **ANO vortices**.

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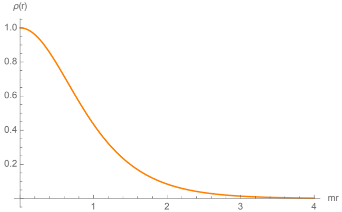

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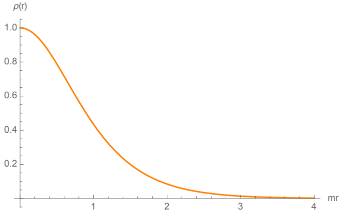

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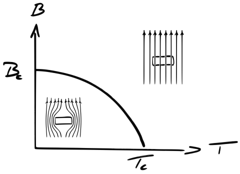
No known analytic solution for profile functions [Nielsen, Olesen 1973]

$$\sigma(\mathrm{x}) \sim K_0(mr), \qquad A(\mathrm{x}) \sim K_1(mr), \qquad r \to \infty.$$

Critical Superconductivity



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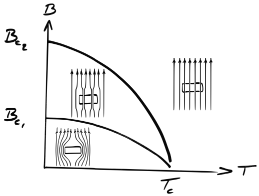

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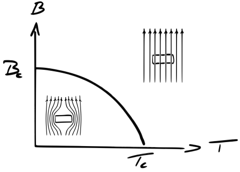

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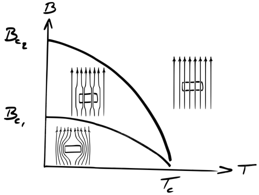

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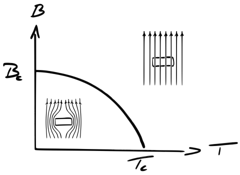

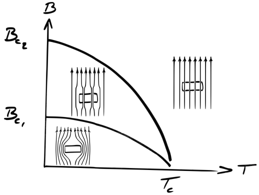

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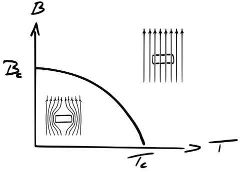

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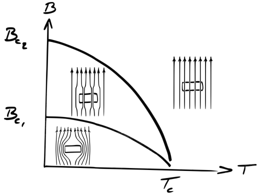

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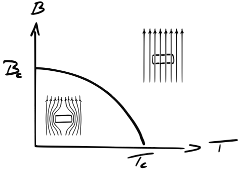

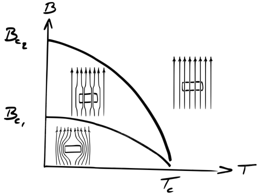

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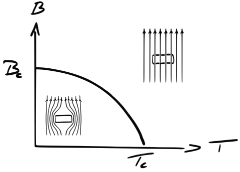

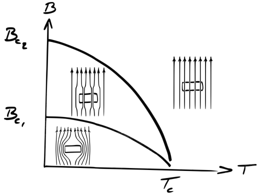

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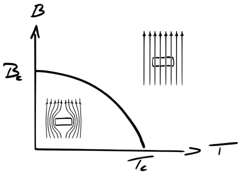

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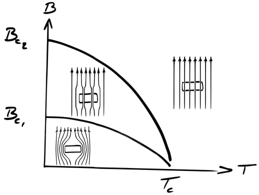

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	Nb	InBi	PbTl	TaN
m_σ/m_γ	1.10-1.45	1.07-2.06	0.61-1.47	0.49-2.16

Small Winding Expansion

Look for spherically symmetric solution to BPS equations:

$$\rho(r) = \left(v^2 - \frac{1}{e}\frac{A'(r)}{r}\right)^{1/2}, \qquad A(r) \equiv \frac{1}{e}\Big(a(\xi) + N\Big), \qquad \xi \equiv \sqrt{2}evr$$

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then **resum** the perturbative expansion in N:

$$Z_1^{\text{perturbative}} = 1.7078629\ldots, \qquad Z_1^{\text{numerical}} = 1.707864175\ldots$$

Separation of Scales

Relevant scales in the problem

$$R_{\text{Compton}} \sim \frac{1}{M}, \qquad R_{\text{interaction}} \sim \frac{1}{m}, \qquad R_{\text{core}} \sim \frac{\sqrt{N}}{m}$$

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Physical picture: classical point-particle vortices interacting by exchanging mediators over distance $r \sim m^{-1}$.

An Effective Field Theory of Vortex Solitons

UV Theory:

Critical AHM

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Relativistic EFT:

Critical AHM + *point-particle* vortex

$A_\mu, \sigma, \Phi, \Phi^*$

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↓

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↓

Non-Relativistic EFT:

Point-particle vortex

Φ, Φ^*

$R_{\text{core}} \ll R_{\text{interaction}}$

$R_{\text{Compton}} \ll R_{\text{interaction}}$

$v \ll c$

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$$S_{\mathrm{REFT}}[\Phi, \Phi^*, \sigma, A_\mu] = S_{\mathrm{AHM}}[\sigma, A_\mu] + S_{\mathrm{Vortex}}[\Phi, \Phi^*, \sigma, A_\mu].$$

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$$\begin{aligned} S_{\text{AHM}}[\sigma, A_\mu] = \int \mathrm{d}^3x \Bigg[& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}m^2\sigma^2 \\ & + \sqrt{\frac{\pi}{2}} M^{3/2}\left(\frac{m}{M}\right)^2 N^{1/2}\sigma A_\mu A^\mu - \sqrt{\frac{\pi}{8}} M^{3/2}\left(\frac{m}{M}\right)^2 N^{1/2}\sigma^3 \\ & + \frac{\pi}{4}M\left(\frac{m}{M}\right)^2 N\sigma^2 A_\mu A^\mu - \frac{\pi}{16}M\left(\frac{m}{M}\right)^2 N\sigma^4 \Bigg], \end{aligned}$$

$$\begin{aligned} S_{\text{Vortex}}[\mathbf{\Phi}, \mathbf{\Phi}^*, \sigma, A_\mu] = \int \mathrm{d}^3x \Big[& |\partial_\mu \mathbf{\Phi}|^2 - M^2|\mathbf{\Phi}|^2 \\ & + i g_e A^\mu \mathbf{\Phi}^* \partial_\mu \mathbf{\Phi} + i g_m \epsilon^{\mu\nu\rho} F_{\mu\nu} \mathbf{\Phi}^* \partial_\rho \mathbf{\Phi} + g_s \sigma |\mathbf{\Phi}|^2 + \text{c.c.} \Big], \\ & + \text{“finite size” terms.} \end{aligned}$$

$$\mathcal{M}_3\left(\gamma^\pm, \mathbf{\Phi}, \overline{\mathbf{\Phi}}\right) = g_e\left(p \cdot \varepsilon_\pm(q)\right) \pm g_m\left(p \cdot \varepsilon_\pm(q)\right)$$

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No Chern-Simons coupling in AHM $\Rightarrow$ no "electric" charge $g_e = 0$.

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Manifest hidden $\mathcal{N} = 2$ supersymmetry in *on-shell* superspace.

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$$\Rightarrow \quad \boxed{g_s = 4g_m}$$

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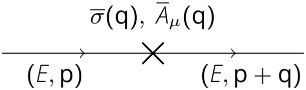
UV: Expand around classical back-grounds $\overline{\sigma}$ and $\overline{A}_\mu$



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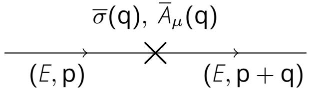
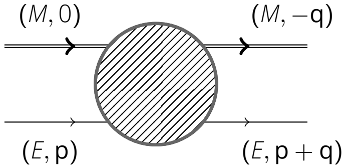


REFT: Calculate (classical) vortex-probe amplitude at $\mathcal{O}(\mathfrak{g}_{s,m})$



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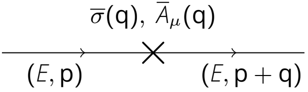

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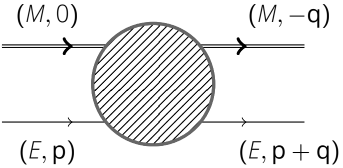

$$\bar{\sigma}(\mathrm{x}) = \frac{1}{2M}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} e^{i\mathrm{q}\cdot\mathrm{x}} \mathcal{M}^{(\text{probe})}_{\text{REFT}}(\mathrm{q})\Big|_{\mathfrak{g}_s}, \quad \bar{A}_i(\mathrm{x}) = -\frac{i}{8\mathfrak{m}M}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} e^{i\mathrm{q}\cdot\mathrm{x}} \frac{\epsilon_{ij} q_j}{\mathrm{q}^2} \mathcal{M}^{(\text{probe})}_{\text{REFT}}(\mathrm{q})\Big|_{\mathfrak{g}_m}.$$

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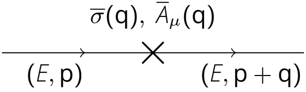

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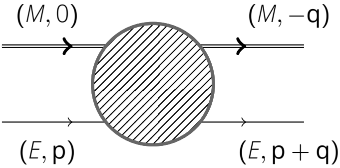

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Tree-level matching: $g_s = -4\sqrt{2\pi} M^{3/2} N^{1/2}, \quad g_m = -\sqrt{2\pi} M^{-1/2} \left(\frac{m}{M}\right)^{-1} N^{1/2}.$

Classical Solitons from the S-Matrix

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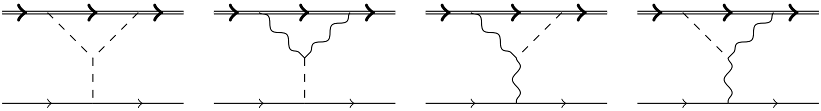

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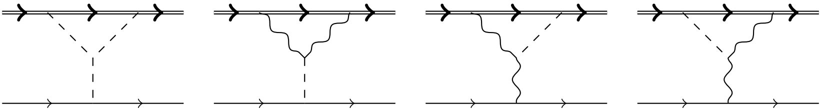

Extract classical solution from matching:

$$\bar{\sigma}(\mathrm{x}) = \sqrt{\frac{2M}{\pi N}}\left[-\left(N+\frac{\pi}{3\sqrt{3}}N^2\right)K_0(mr) - \frac{N^2}{2}\left[K_0(mr)\right]^2 \right.$$
$$\left. +2N^2\left(K_0(mr)\int_{mr}^{\infty} \mathrm{d}\xi\ \xi I_1(\xi)K_0(\xi)K_1(\xi) + I_0(mr)\int_{mr}^{\infty}\mathrm{d}\xi\ \xi K_0(\xi)\left[K_1(\xi)\right]^2\right)\right] + \mathcal{O}\left(N^{5/2}\right),$$

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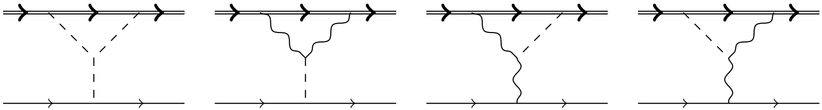

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Agrees with perturbative solution of BPS equations [de Vega, Schaposnik 1976] ✓

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Loop integrals evaluated using *method of regions* [Beneke, Smirnov 1998]

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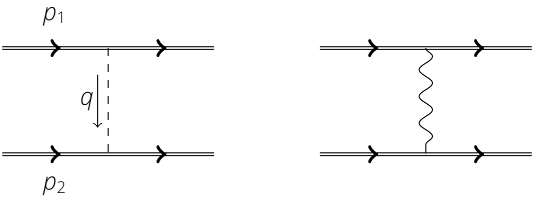
Tree-Level Potential



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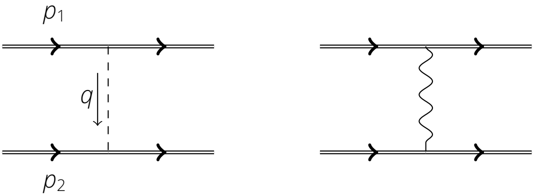

$$\mathcal{M}^{\text{tree}} = \underbrace{-\frac{32\pi MN\left(M^2 - (p_1 \cdot p_2)\right)}{q^2 - m^2}}_{\text{classical potential}} + \underbrace{\frac{8\pi MNm^2}{q^2 - m^2}}_{\text{quantum}} + \underbrace{\frac{32\pi MN}{m^2}\left(p_1 \cdot p_2 + \frac{1}{4}\left(q^2 + m^2\right)\right)}_{\text{short-range "\textit{Darwin}" terms } \sim \ \delta^{(2)}(\mathrm{x})}.$$

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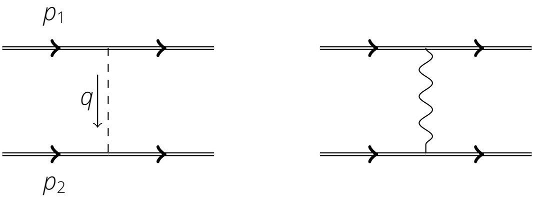

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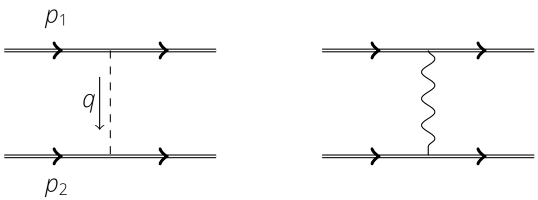

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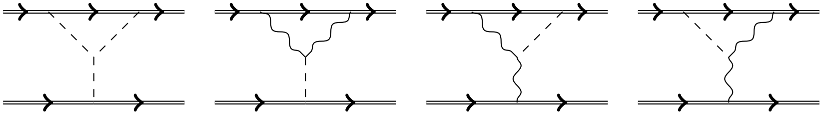
Vanishes as $\mathbf{p}\to 0$ as expected for BPS vortices.

1-loop Potential



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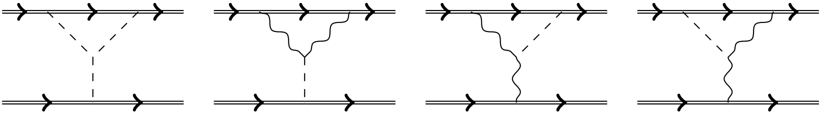
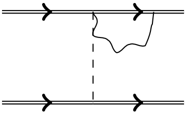


Renormalize UV divergent *pinch* contributions:



$\longrightarrow \quad 0.$

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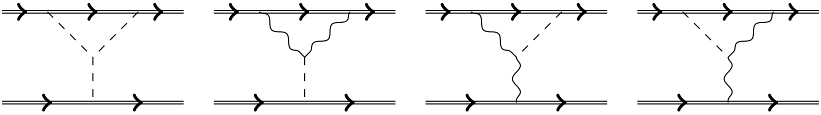

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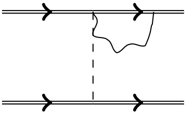

$\longrightarrow \quad 0.$

Expand in soft region: $\quad q^\mu \sim l^\mu \sim m$

$$i\mathcal{M}^{\text{1-loop}}_{\Upsilon} =$$

$$128\pi^2 M^2 N^2 \left[\left(M^2 - 2(p_1 \cdot p_2) \right) + \frac{2m^2 \left(M^2 - (p_1 \cdot p_2) \right)}{q^2 - m^2} \right] \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{[l^2 - m^2][(l+q)^2 - m^2] \underbrace{[p_1 \cdot l + i0]}_{Eikonal}}.$$

Seagull Contributions

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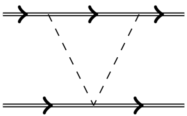
Problem: Does *not* vanish in static limit.

Solution: Need to add *seagull* vertex

$$S_{\text{Vortex}} \to S_{\text{Vortex}} + \int \mathrm{d}^3 x \left[-2\pi M N \sigma^2 |\Phi|^2 \right] ,$$



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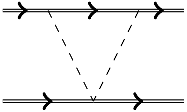

$$i\mathcal{M}^{\text{1-loop}}_{\underline{\Upsilon}+\underline{\curlyvee}} = 256\pi^2 M^2 N^2 \left(M^2 - (p_1 \cdot p_2)\right) \left[1 + \frac{m^2}{q^2 - m^2} \right] \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{[l^2 - m^2][(l+q)^2 - m^2][p_1 \cdot l + i0]} .$$

Velocity Expansion and Resummation

Expand master soft integrals in potential region: $\mathbf{p} \sim v, \quad \omega \sim v$

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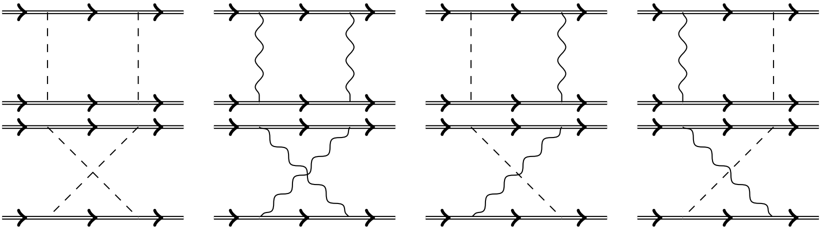
and *resum* the complete velocity dependence

$$= \frac{i}{2E} \int \frac{d^2 \mathbf{l}}{(2\pi)^2} \frac{1}{[\mathbf{l}^2 + m^2][(\mathbf{l}+\mathbf{q})^2 + m^2][\mathbf{p} \cdot \mathbf{l} - i0]^2} - \frac{2i}{ME(M+E)} \int \frac{d^2 \mathbf{l}}{(2\pi)^2} \frac{1}{[\mathbf{l}^2 + m^2]^2[(\mathbf{l}+\mathbf{q})^2 + m^2]}.$$

Box Diagrams and Iteration

Add boxes and crossed-boxes:



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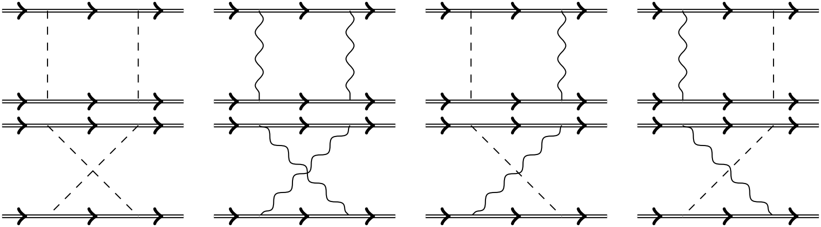
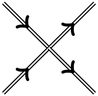
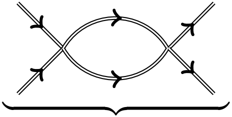


Match to amplitude in NREFT:

$M_{\text{NREFT}} =$  +  + …

"iteration"

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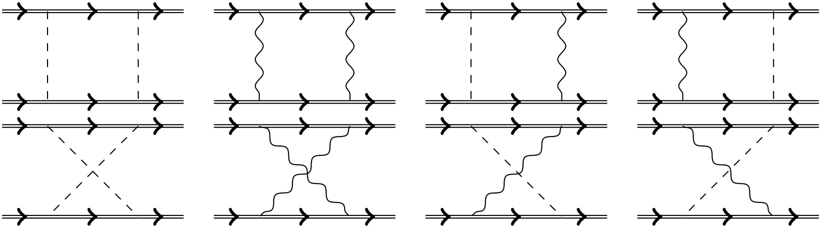

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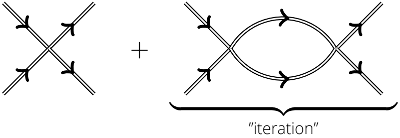

Spurious branch cut must match between REFT boxes and NREFT iteration. ✓

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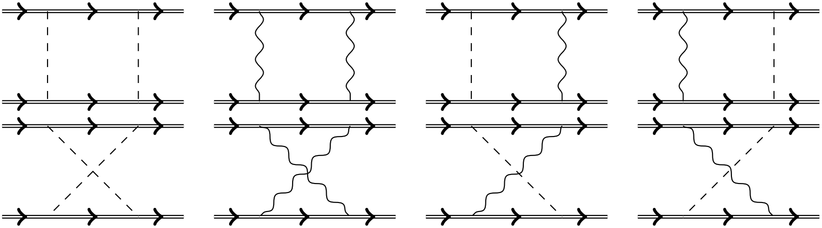

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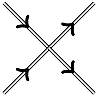
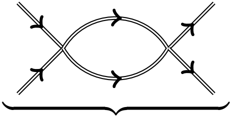
$M_{\mathrm{NREFT}} =$  $+$  $+ \quad \ldots$

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Box, crossed-box and iteration contributions vanish in static limit. ✓

Main Result: 1-loop Vortex-Vortex Potential

$$V(\mathrm{p},\mathrm{x}) = \frac{8MN\mathrm{p}^2}{\mathrm{p}^2+M^2}K_0(mr) + \frac{16\pi MN^2\mathrm{p}^2}{3\sqrt{3}\left(\mathrm{p}^2+M^2\right)}K_0(mr) + \frac{16M^2N^2\mathrm{p}^2(\mathrm{p}^2+4M^2)}{(\mathrm{p}^2+M^2)^{5/2}}K_0(mr)^2$$

$$+\frac{32MN^2\mathrm{p}^2}{\mathrm{p}^2+M^2}\left(1-\frac{M}{(\mathrm{p}^2+M^2)^{1/2}}\right)mrK_0(mr)K_1(mr)$$

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Consistency with limiting cases:

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- Metric on moduli space should be *Kähler* [Samols 1991] ✓

Moduli Space Metric

Truncate at $\mathcal{O}\left(\mathbf{p}^2\right)$; calculate effective Lagrangian

$$L(\dot{\mathrm{x}}_1, \mathrm{x}_1; \dot{\mathrm{x}}_2, \mathrm{x}_2) = \frac{1}{2}M\dot{\mathrm{x}}_1^2 + \frac{1}{2}M\dot{\mathrm{x}}_2^2 - \tilde{U}\left(r_{12}\right)\left|\dot{\mathrm{x}}_1 - \dot{\mathrm{x}}_2\right|^2 + \mathcal{O}\left(v^4\right),$$

where

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First ever *analytic* information beyond leading asymptotic approximation!

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Thank you!

UCLA