Tropicalised Braid Automorphisms of Grassmannian Cluster Algebras

Based on work with

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Grassmannian $Gr(k,n)$

$k \times n$ matrix (Z_i^A) mod sl_k and column rescalings.

The case $k=4$ describes kinematics for n-pt amplitudes in planar $\mathcal{N}=4$ SYM.

Then $\dim = 3n - 15$

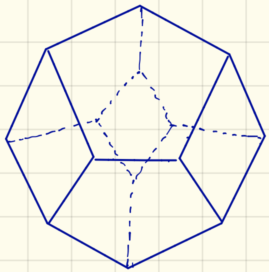
$\uparrow$

sl_4 – dual conformal symmetry

Columns Z_i are momentum twistors (Hodges 2008)

Later we will also consider $Gr(k, 2k)$ (middle)

Cluster Algebras (Fomin, Zelevinsky 2002)



$Gr^+(k,n)$ described by cluster algebra. (Scott 2006)

$\langle ijkl \rangle \geq 0$

Relevance for amplitudes in planar $\mathcal{N}=4$ (Golden, Goncharov, Spradlin, Vergu, Volovich 2013)

for $n = 6, 7$, L loops, 'polylogs' of weight $2L$

Symbol alphabet = {cluster variables} & Adjacency Relations (c.f. Steinmann)
(JMD, Foster, Gürdoğan 2017)

for $n = 8, 9$ also square roots $\dfrac{A + B\sqrt{\Delta}}{A - B\sqrt{\Delta}}$ (four mass box)

for $n = 10, \ldots$ ellipitcs appear (Caron-Huot, Larsen 2012)
not understood from cluster algebra.

Relevant for bootstrap.

Cluster Algebras

$$\langle 14\rangle\langle 14\rangle' = \langle 13\rangle\langle 45\rangle + \langle 15\rangle\langle 34\rangle = \langle 14\rangle\langle 35\rangle$$

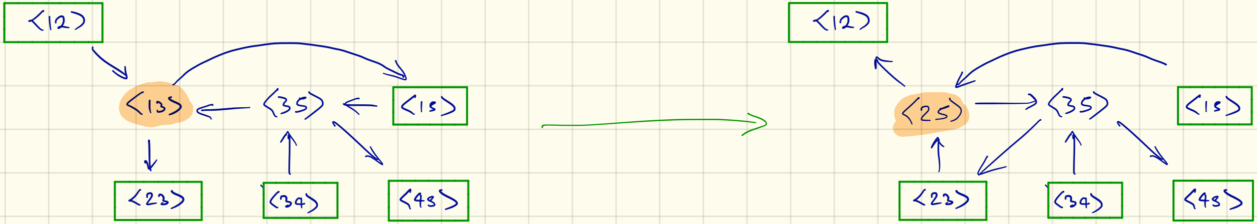
$$\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
& \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul]
\end{tikzcd}
\xrightarrow{\text{mutation}}
\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[d] \arrow[rr, bend left] & \langle 35\rangle \arrow[l] \arrow[dr] & \boxed{\langle 15\rangle} \arrow[l] \\
& \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[u] & \boxed{\langle 45\rangle}
\end{tikzcd}$$

g-vectors

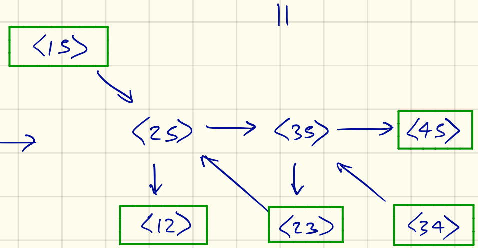
$$(1,0) \longrightarrow (0,1) \qquad \longrightarrow \qquad (1,0) \longleftarrow (0,-1)$$

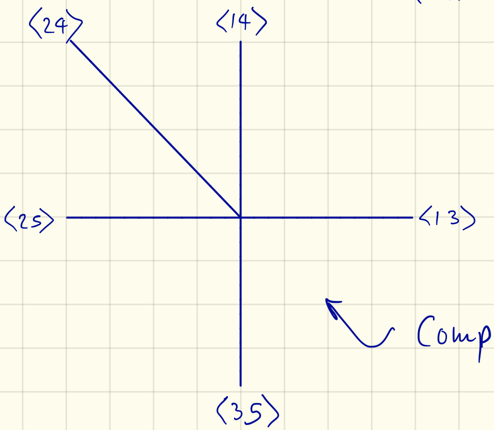
Cluster Automorphisms

$$\langle 13\rangle\langle 13\rangle' = \langle 12\rangle\langle 35\rangle + \langle 15\rangle\langle 23\rangle = \langle 13\rangle\langle 25\rangle$$



Cyclic rotation of initial cluster
(automorphism)





g-vectors: $(-1,0) \longrightarrow (0,-1)$

Complete g-vector fan

Tropicalised Automorphisms

$$\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
& \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul]
\end{tikzcd}$$

$\downarrow$ ($\mathcal{X}$-coords)

$$\frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle} \longrightarrow \frac{\langle 15\rangle\langle 34\rangle}{\langle 13\rangle\langle 45\rangle} \xrightarrow{\text{cyclic}} \frac{\langle 25\rangle\langle 34\rangle}{\langle 23\rangle\langle 45\rangle} \longrightarrow \frac{\langle 12\rangle\langle 45\rangle}{\langle 24\rangle\langle 15\rangle}$$

$$\frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle} = x_1, \qquad \frac{\langle 15\rangle\langle 34\rangle}{\langle 13\rangle\langle 45\rangle} = x_2, \qquad \frac{\langle 25\rangle\langle 34\rangle}{\langle 23\rangle\langle 45\rangle} = \frac{1+x_2+x_1x_2}{x_1}, \qquad \frac{\langle 12\rangle\langle 45\rangle}{\langle 24\rangle\langle 15\rangle} = \frac{1}{x_2(1+x_1)}$$

Tropicalise:

$$x_1 \longmapsto \min(0, x_2, x_1+x_2) - x_1$$

$$x_2 \longmapsto -x_2 - \min(0, x_1)$$

Get action of C^{-1} on g-vectors:

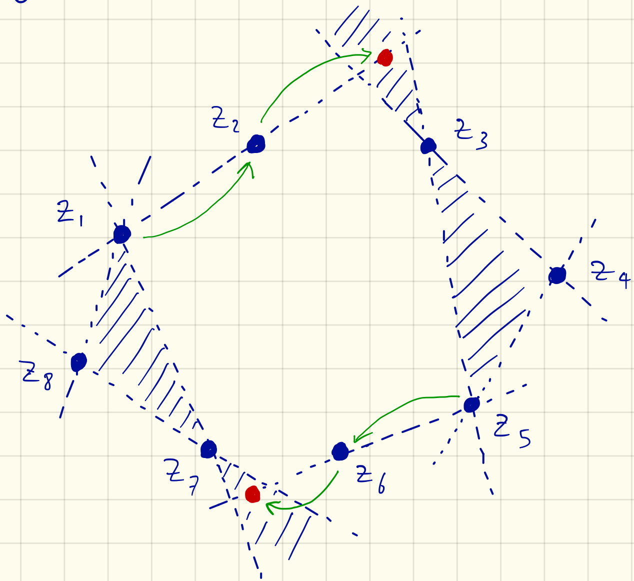
$$(1,0) \longmapsto (-1,0)$$

$$(0,1) \longmapsto (0,-1) \longmapsto (-1,1)$$

Braid Automorphisms (Fraser, 2017) (Fock, Goncharov)

When k not coprime to n have more exotic (quasi) automorphisms.

e.g. in $Gr(4,8)$

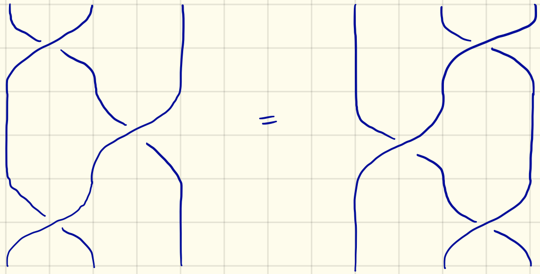


$$\sigma_1 : \quad z_1 \longmapsto z_2$$
$$z_2 \longmapsto z_1\langle 2345\rangle - z_2\langle 1345\rangle$$
$$z_5 \longmapsto z_6$$
$$z_6 \longmapsto z_5\langle 6781\rangle - z_6\langle 5781\rangle$$

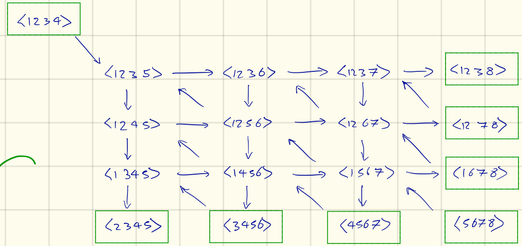
& $\sigma_2, \sigma_3, \sigma_4$ sim

They obey braid relations:

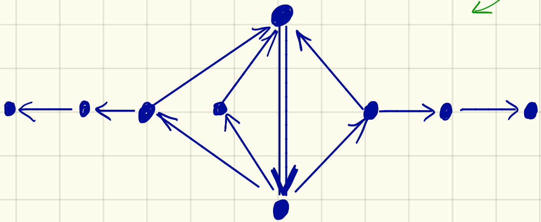
$$\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1}$$



Application : $Gr(4,8)$



mutations



Infinite sequences of mutations

g-vectors converge to limit rays

$\wr$

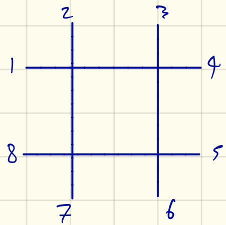
Associated to square roots $\sqrt{\Delta} \longleftrightarrow g_\infty$

2019: Arkani-Hamed, Lam, Spradlin

JMD, Foster, Gürdoğan, Kalousios

Henke, Papathanasiou

Simplest are:



Braid fixed points

Iterate σ_1 on random vector : $\longrightarrow$ goes to $g_\infty^{(1)}$

(Four-mass box)

$\therefore$ $\sigma_1 g_\infty^{(1)} = g_\infty^{(1)}$ also $\sigma_3 g_\infty^{(1)} = g_\infty^{(1)}$

sim $\sigma_2 g_\infty^{(1)} = g_\infty^{(2)}$ $\sigma_4 g_\infty^{(2)} = g_\infty^{(2)}$

New interpretation of square roots ! $\sqrt{\Delta}$ $\Delta = p^2 - 4q$

Better :

$$\sigma_2 g_\infty^{(1)} = g_\infty^{(3)}$$

$$\sigma_1 \underbrace{g_\infty^{(2)}}_{P \text{ deg } 2} = \underbrace{g_\infty^{(4)}}_{P \text{ deg } 4}$$

$(\sigma_3 \simeq \sigma_1$

$\sigma_4 \simeq \sigma_2)$

can generate more ...

Application (of application)

Early and Li (2023) counted limit rays by degree.

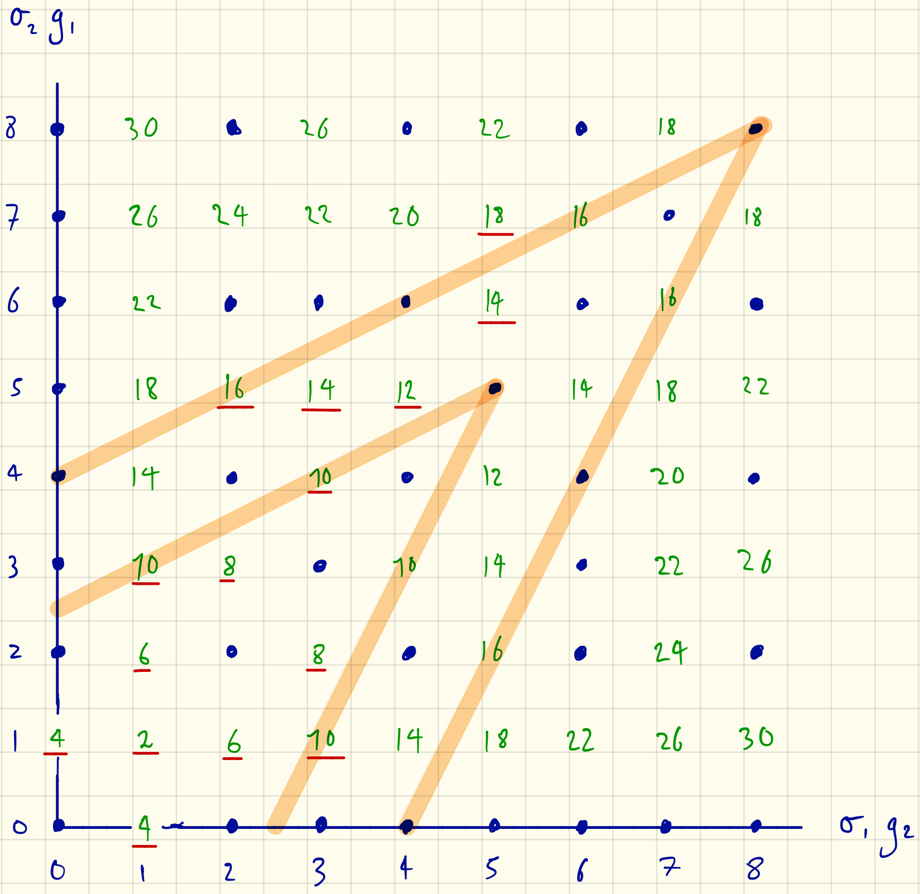
Deg	1	2	3	4	5	6	7	8	9	10	11	12
#	0	2	0	2	0	4	0	4	0	8	0	4

$\# \sim 2\,\phi\left(\frac{\deg}{2}\right)$ deg even (0 if deg odd)

Euler totient function !

$0 \quad \frac{1}{20} \quad \frac{1}{10} \quad \frac{3}{20} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{3}{10} \quad \frac{7}{20} \quad \frac{2}{5} \quad \frac{9}{20} \quad \frac{1}{2} \quad \ldots\ldots$

$\phi(20) = 2 \times 4 = 8$



Green number $2k$
= order of polynomial
for each ray

Blue dot • means
a multiple of another ray

Orange highlighted
line
means all rays of a
given order

Note we have
Euler totient $\phi(k)$
of them.
(c.f. rationals between
0 and 1 with
minimal denominators)

Application : General massless kinematics ? $\quad p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}$

Spinor helicity ideal

$$\langle ij\rangle\langle kl\rangle - \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle jk\rangle = 0$$

$$[ij][kl] - [ik][jl] + [il][jk] = 0$$

$$\sum_{k=1}^{n} \langle ik\rangle [kj] = 0$$

We can embed into $Gr(n-2, 2n-4)$ (middle Grassmannian)

e.g. $n=5$

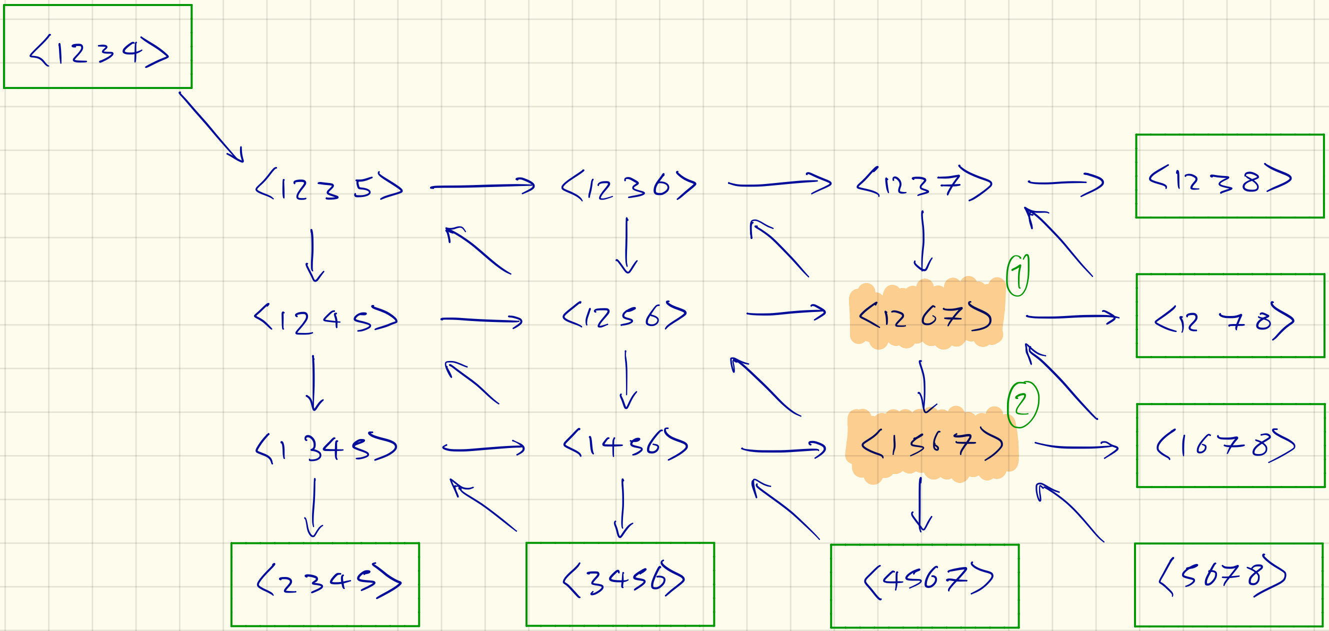
$$\left.\begin{aligned} \langle ij\rangle &\longrightarrow (ij6) \\ [ij] &\longrightarrow (-1)^{j-i-1} (klm) \end{aligned}\right\} \quad Gr(3,6)$$

(Bossinger, JMD, Glew, 2022)

$n=6$

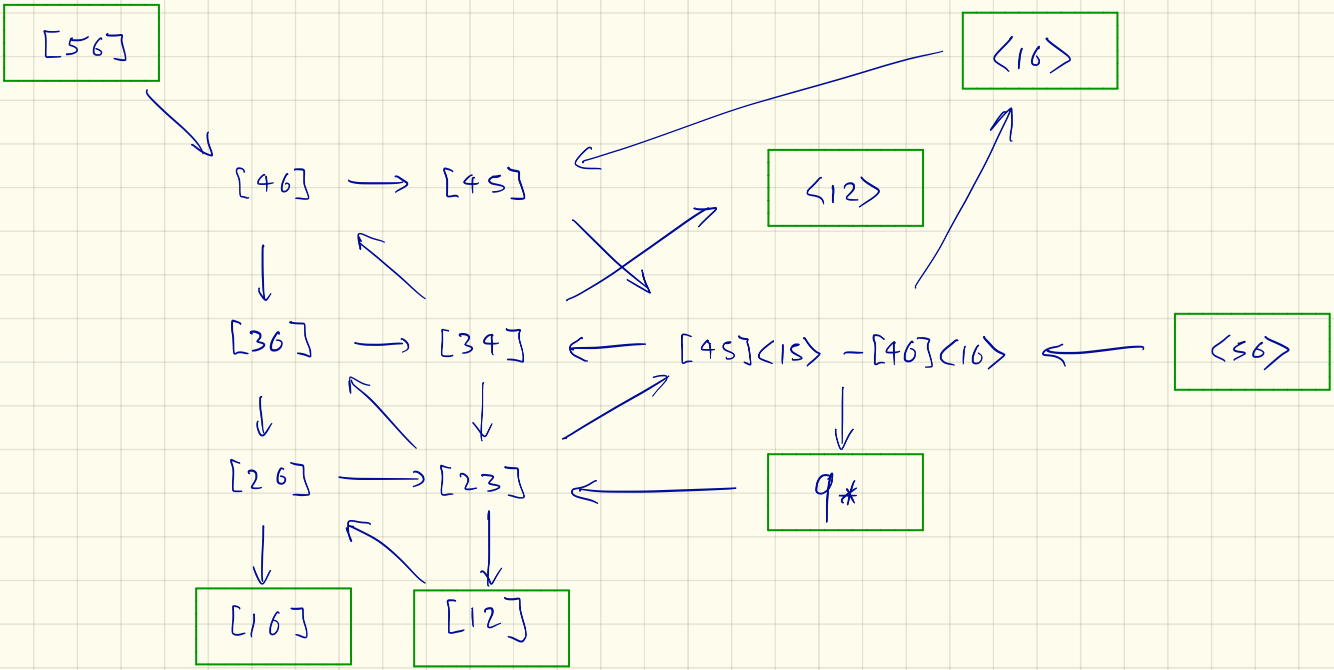
$$\left.\begin{aligned} \langle ij\rangle &\longrightarrow (ij78) \\ [ij] &\longrightarrow (-1)^{j-i-1} (klmr) \end{aligned}\right\} \quad F_{2,4,6} \subset Gr(4,8)$$

Gr (4,8) initial cluster



Mutate twice and freeze a node

$F_{2,4,6}$ initial cluster:



Do these characterise singularities for general massless amps?

$n=4$: $Gr(2,4) \xrightarrow{\text{perms}}$ HPLs $\{0,1,-1\}$

$n=5$: $Gr(3,6) \xrightarrow{\text{perms}}$ Pentagon fn alphabet (except W_{31} !)

$n=6$: $F_{2,4,6} \subset Gr(4,8) \xrightarrow{\text{perms}}$? .... at least some of partially known 6 pt. alphabet... (incl. square roots)

⋮

<u>Note</u>: σ_1 is an automorphism of all of these spaces

Need to explore further!