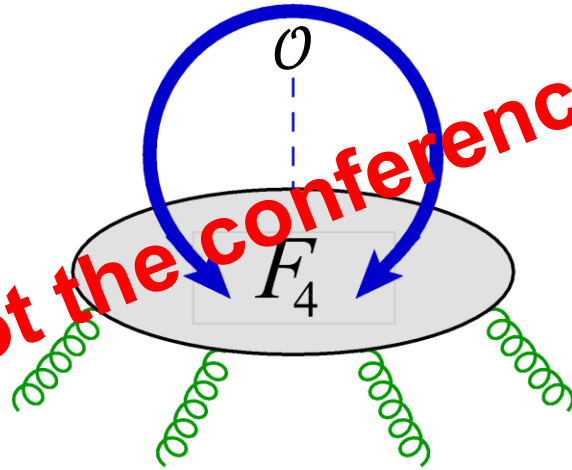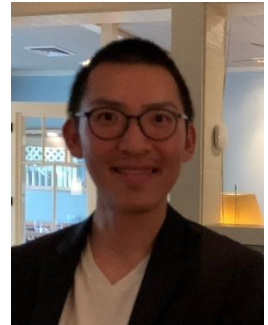# Antipodal Self-Duality

This is not the conference summary!

Lance Dixon (SLAC)

Andy Liu

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm
2112.06243, 2204.11901, 2212.02410, and in progress
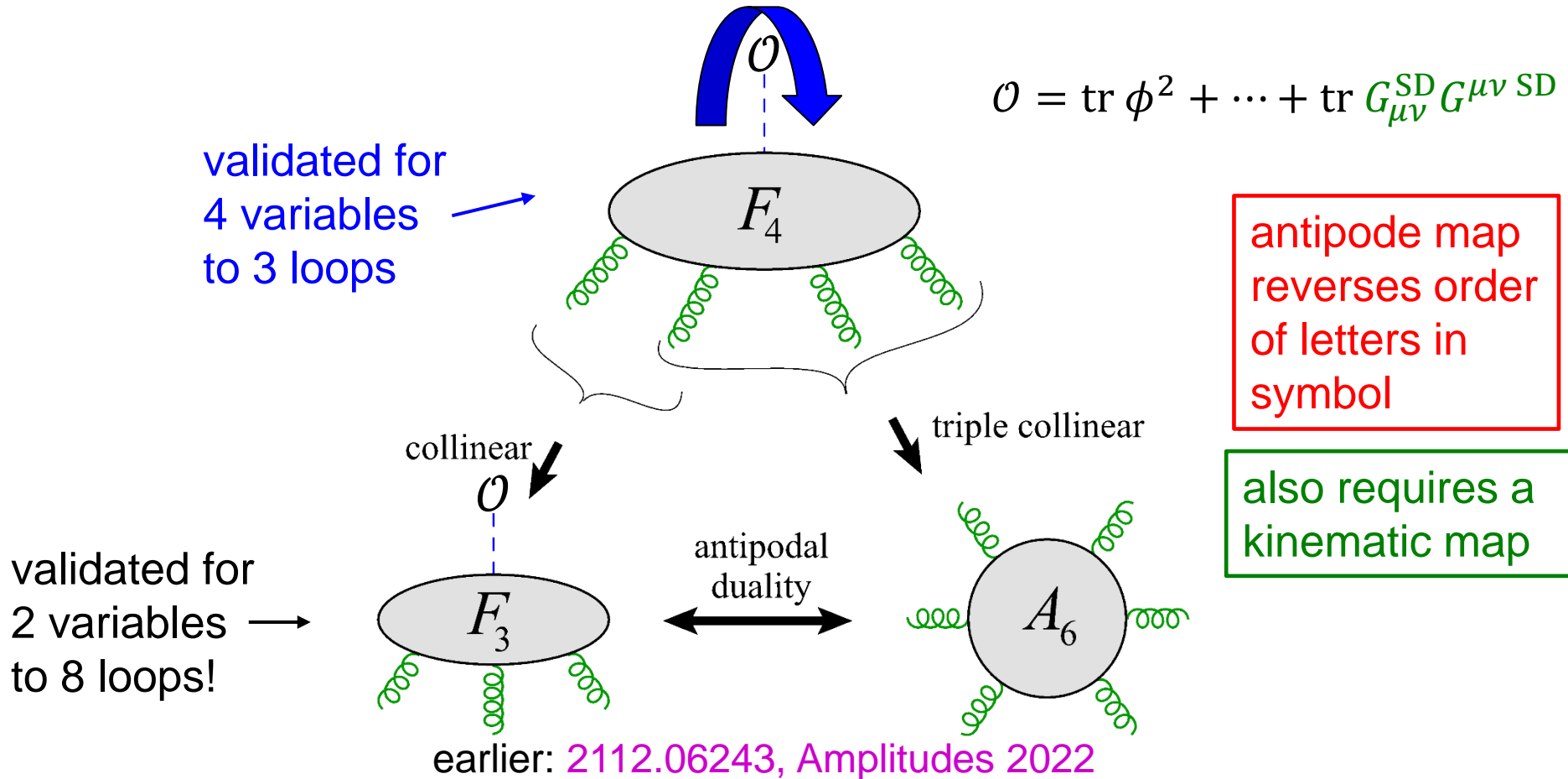
Amplitudes @ CERN
11 August 2023

This talk in one picture:
Antipodal Self Duality
for a 4-point form factor in planar N=4 SYM

$$\mathcal{O} = \mathrm{tr}\,\phi^2 + \cdots + \mathrm{tr}\,G_{\mu\nu}^{\mathrm{SD}} G^{\mu\nu\,\mathrm{SD}}$$

validated for 4 variables to 3 loops

antipode map reverses order of letters in symbol

also requires a kinematic map

collinear

validated for 2 variables to 8 loops!

triple collinear

antipodal duality

earlier: 2112.06243, Amplitudes 2022

Multiple polylogarithms (MPLs)

- Characterize all the form factors & amplitudes playing a role here.
- At L loops, the results are weight $n = 2L$ MPLs, defined as iterated integrals by

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

and $\quad G(\vec{0}_n, x) = \frac{(\ln x)^n}{n!}$

Hopf algebra

Goncharov; Brown; Goncharov, Spradlin, Vergu, Volovich; Duhr, Gangl, Rhodes

- Differential definition: $\boxed{dF = \sum_{s_k \in \mathcal{L}} F^{s_k} \, d \ln s_k}$

- Hopf algebra "co-acts" on space of MPLs,
$$\Delta : F \;\to\; F \otimes F$$

- Derivative dF is one piece of Δ: $\boxed{\Delta_{n-1,1} F = \sum_{s_k \in \mathcal{L}} F^{s_k} \otimes \ln s_k}$

- So we refer to F^{s_k} as a $\{n\text{-}1,1\}$ coproduct of F

- **s_k are letters in the symbol alphabet** $\mathcal{L}$

Iterate to get **symbol**

- Apply {n-1,1} coaction iteratively:
- Define {n-2,1,1} double coproducts, F^{s_k,s_j}, via derivatives of {n-1,1} single coproducts F^{s_j}:

$$dF^{s_j} \equiv \sum_{s_k \in \mathcal{L}} F^{s_k,s_j} \, d \ln s_k$$

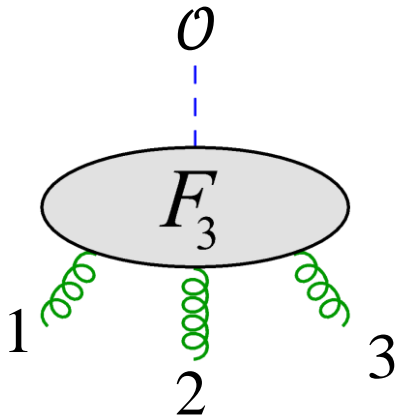- And so on for {n-m,1,…,1} m^{th} coproducts of F.
- Maximal iteration, n times for weight n function, is the symbol,   ["ln" is implicit in s_{i_k}]

$$\mathcal{S}[F] \equiv \sum_{s_{i_1},…,s_{i_n} \in \mathcal{L}} F^{s_{i_1},…,s_{i_n}} \, s_{i_1} \otimes … \otimes s_{i_n}$$

where now $F^{s_{i_1},…,s_{i_n}}$ are just rational numbers
Goncharov, Spradlin, Vergu, Volovich, 1006.5703

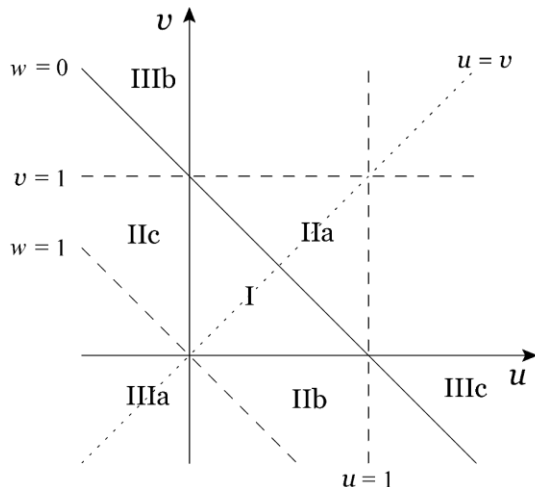# 3-gluon form factor depends on 2 dimensionless variables u, v

$$k_i^2 = 0 \qquad s_{ij} = (k_i + k_j)^2$$

$$k_1 + k_2 + k_3 = -k_{\mathcal{O}}$$

$$s_{123} = s_{12} + s_{23} + s_{31} = q^2$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}} = 1 - u - v$$

I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

$D_3 \equiv S_3$ **dihedral symmetry** generated by:

a. cycle: $i \to i + 1 \pmod 3$, or
$$u \to v \to w \to u$$

b. flip: $u \leftrightarrow v$

N=4 amplitude is S_3 invariant

F_3 symbol alphabet has 6 letters

$$\mathcal{L} = \{\, a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w}\,\}$$

- Symbols of form factor $F_3^{(L)}$ at $L = 1, 2$ loops are just 1 and 2 terms, plus D_3 dihedral images(!!!):

$$\mathcal{S}\left[F_3^{(1)}\right] = (-1)\, b \otimes d + \text{dihedral}$$

$$\mathcal{S}\left[F_3^{(2)}\right] = 4\, b \otimes d \otimes d \otimes d + 2\, b \otimes b \otimes b \otimes d + \text{dihedral}$$

Brandhuber, Travaglini, Yang, 1201.4170
LD, Gürdoğan A. McLeod, M. Wilhelm, 2204.11901

known to 8 loops

dihedral cycle: $a \to b \to c \to a,$ $d \to e \to f \to d$
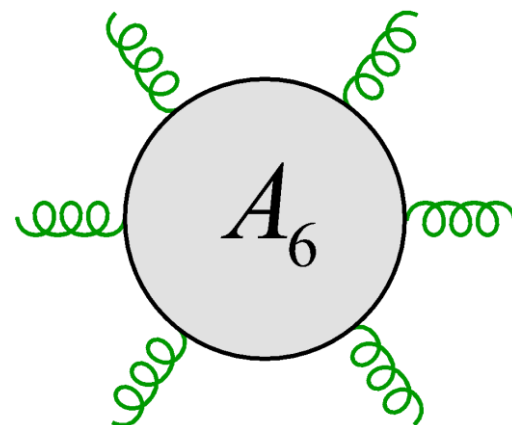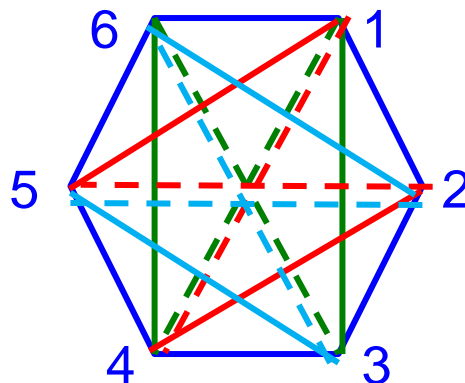dihedral flip: $a \leftrightarrow b,$ $d \leftrightarrow e$

6-gluon amplitude

- Dual to Wilson hexagon, invariant under dual conformal transformations, so it can only depend on 3 dual conformal cross ratios, $\hat{u}, \hat{v}, \hat{w}$:

$$\hat{u} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

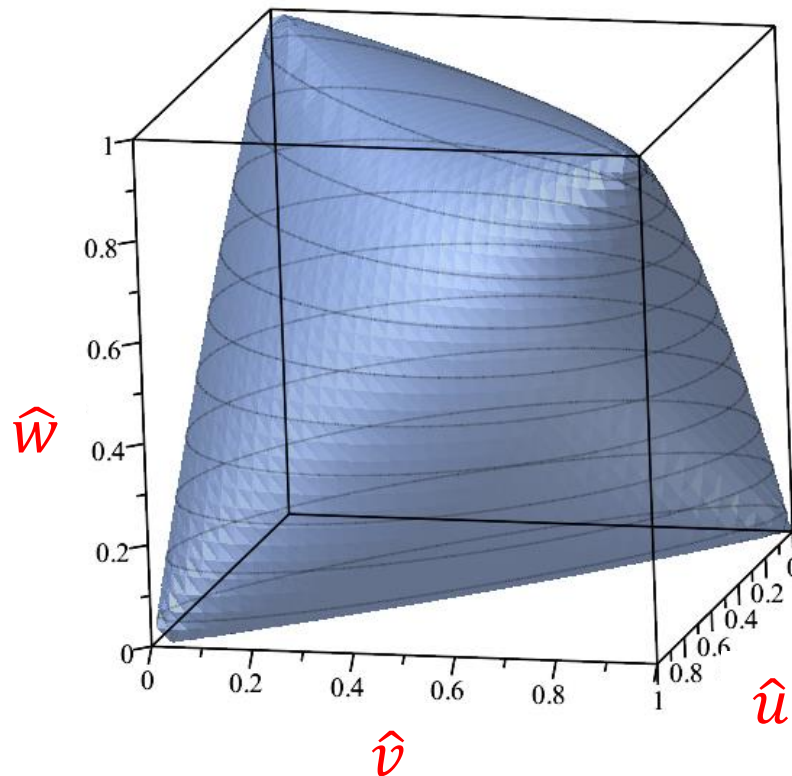$$\hat{v} = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

$$\hat{w} = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$

D_6 dihedral symmetry: cycle $(\mathrm{mod}\ 6)$ and flip, but it acts on $\hat{u}, \hat{v}, \hat{w}$ as $D_3 = S_3$

Parity-preserving surface

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

kinematics lies in a 3d subspace of 4d spacetime
→ parity invariant

6-gluon symbol alphabet

- $\mathcal{L}_6 = \{\, \hat{u}, \hat{v}, \hat{w}, 1 - \hat{u}, 1 - \hat{v}, 1 - \hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w \}$ $\qquad$ 1 for $\Delta = 0$

$\rightarrow \mathcal{L}_6' = \{\, \hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}}, \; \hat{b} = \frac{\hat{v}}{\hat{w}u}, \; \hat{c} = \frac{\hat{w}}{\hat{u}\hat{v}}, \; \hat{d} = \frac{1-\hat{u}}{\hat{u}}, \; \hat{e} = \frac{1-\hat{v}}{\hat{v}}, \; \hat{f} = \frac{1-\hat{w}}{\hat{w}} \}$

- Symbols of amplitude $A_6^{(L)}$ on $\Delta = 0$ at $L = 1, 2$ loops are just 1 and 2 terms, plus D_3 dihedral images(!!!):

$$\mathcal{S}\left[A_6^{(1)}\right] = (-\frac{1}{2})\hat{b} \otimes \hat{d} + \text{dihedral}$$

$$\mathcal{S}\left[A_6^{(2)}\right] = \hat{b} \otimes \hat{d} \otimes \hat{d} \otimes \hat{d} + \frac{1}{2}\hat{b} \otimes \hat{b} \otimes \hat{b} \otimes \hat{d} + \text{dihedral}$$

…

Goncharov, Spradlin, Vergu, Volovich, 1006.5703, …,
Caron-Huot, LD, Dulat, McLeod, von Hippel, 1903.10890

was known to 7 loops

Antipodal duality

$$F_3^{(L)}(u, v, w) = S\left(A_6^{(L)}(\hat{u}, \hat{v}, \hat{w})\right)$$

Antipode map S, at symbol level, reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

Kinematic map in terms of underlying variables is:

$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

Maps $u + v + w = 1$ to parity-preserving surface

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

Kinematic map on letters

$$\sqrt{\hat{a}} = d, \qquad \hat{d} = a, \qquad \textit{plus cyclic relations}$$
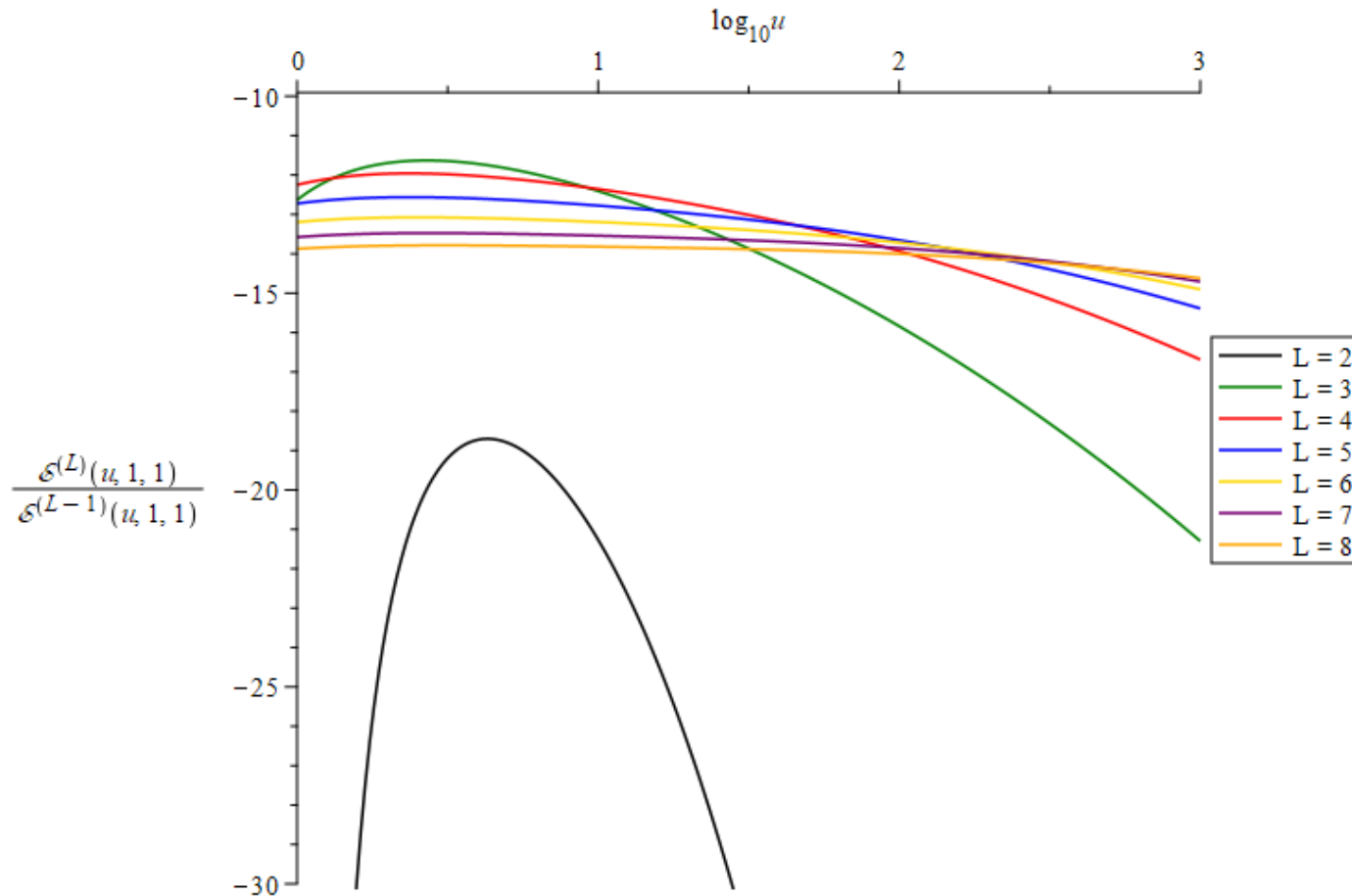
Works through 8 loops (even beyond symbol)!

L	number of terms in symbol
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

But why?!

Exploit to compute A_6 at 8 loops

$$\frac{\mathcal{E}^{(L)}(u, 1, 1)}{\mathcal{E}^{(L-1)}(u, 1, 1)}$$

Legend: L = 2, L = 3, L = 4, L = 5, L = 6, L = 7, L = 8
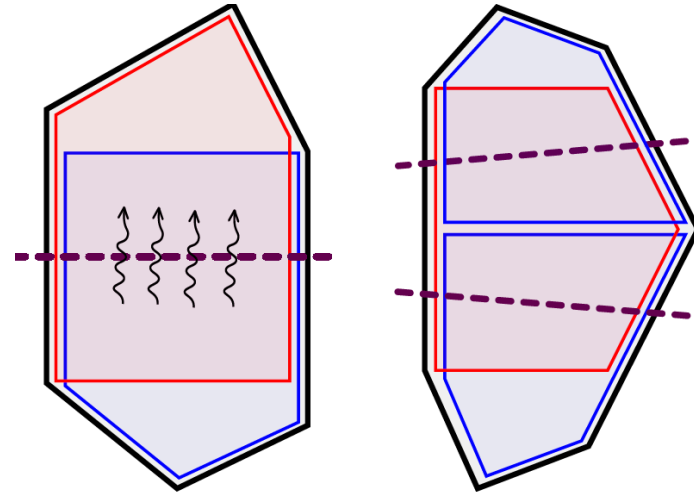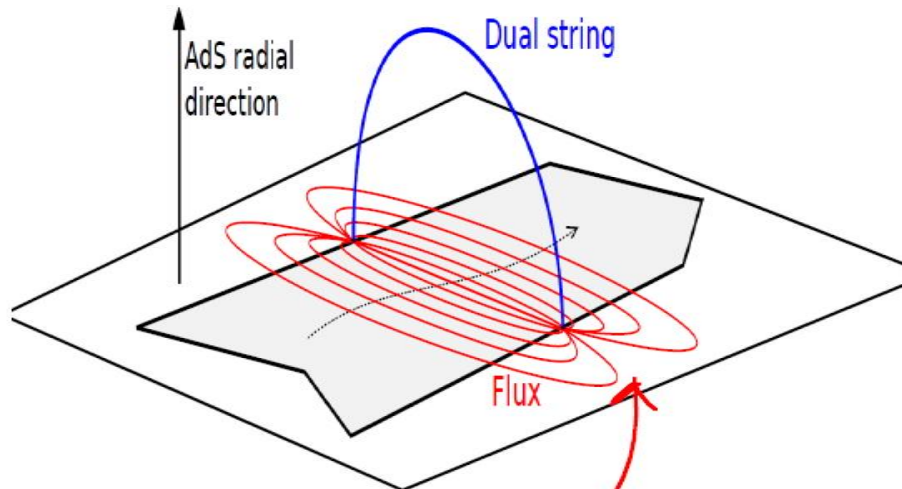
$\log_{10} u$

The Flux Tube (Pentagon) OPE

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987

- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

The Form Factor OPE
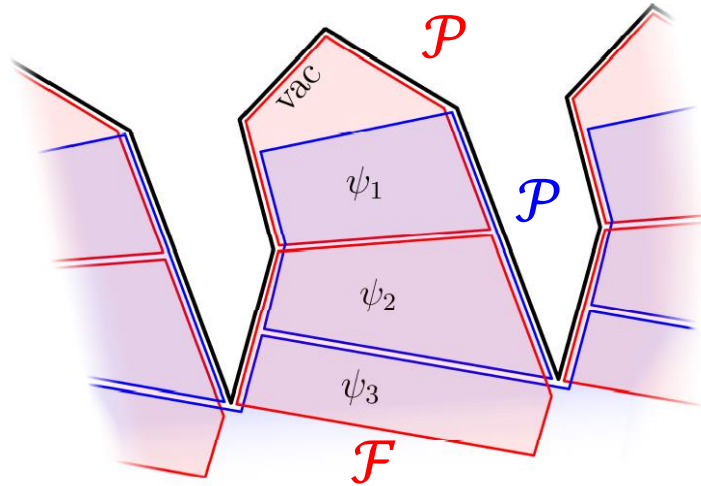
- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

 Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$.

- For $\mathrm{tr}\phi^2$: **Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569**

OPE representation

- ## 6-gluon amplitude:

$$\mathcal{W}_{\text{hex}} = \sum_{\mathbf{a}} \int d\mathbf{u} \, P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{a}}(\bar{\mathbf{u}}|0) \, e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi}$$

$$T = e^{-\tau}, S = e^{\sigma}, F = e^{i\phi}. \quad v = \frac{T^2}{1+T^2} \to 0,$$

weak-coupling, $E = k + \mathcal{O}(g^2)$ → expansion in T^k

- ## 3-gluon form factor:

$\psi = helicity \ 0 \ pairs \ of \ states$

$$\mathcal{W}_3 = \sum_{\psi} \oint e^{-E_\psi \tau + ip_\psi \sigma} \mathcal{P}(0|\psi) \mathcal{F}(\psi)$$

weak-coupling → expansion in T^{2k} (no azimuthal angle ϕ)

"OPE" coordinates simplify kinematic map

Basso, Sever, Vieira, 1303.1396, 1306.2058,…;
Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

- Amplitude:

$$\hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})},$$

$$\hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2, \qquad \hat{w} = \frac{\hat{T}^2}{1 + \hat{T}^2}$$

($\hat{F} = 1$ for $\Delta = 0$)

- Form factor:

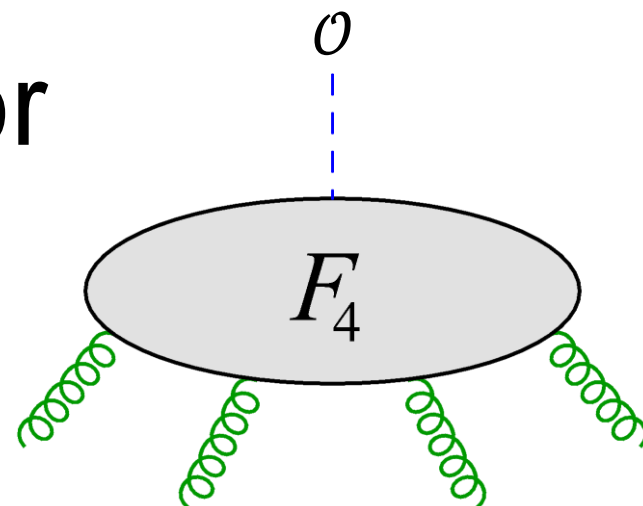$$u = \frac{1}{1 + S^2 + T^2}, \qquad v = \frac{T^2}{1 + T^2},$$

$$w = \frac{1}{(1 + T^2)(1 + S^{-2}(1 + T^2))},$$

- Kinematic map
  for antipodal duality →

$$\hat{T} = \frac{T}{S}, \qquad \hat{S} = \frac{1}{TS}$$

$$T = \sqrt{\frac{\hat{T}}{\hat{S}}}, \qquad S = \sqrt{\frac{1}{\hat{T}\hat{S}}}$$

Four-gluon form factor

Depends on 5 kinematical variables instead of 2.

Even just at two loops, contains state-of-the art loop integrals → 113 possible symbol letters!
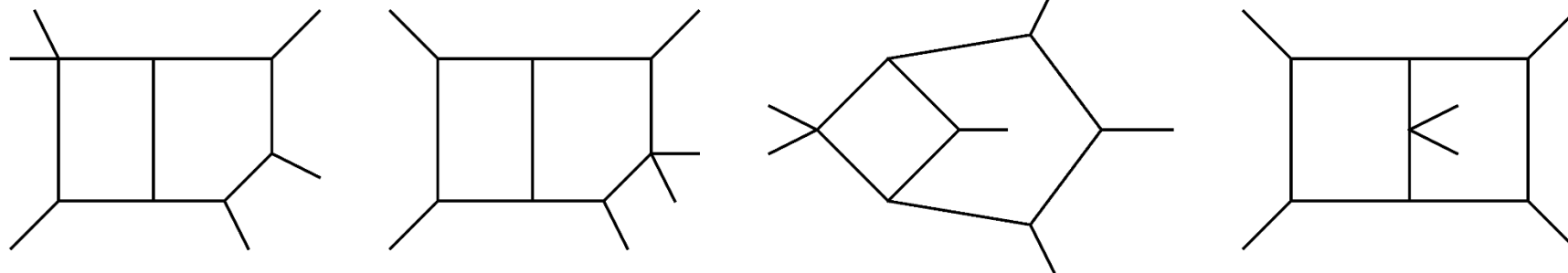
Abreu, Ita, Moriello, Page, Tschernow, 2005.04195;
Abreu, Ita, Page, Tschernow, 2107.14180;
Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, 2306.15431

4-gluon form factor depends on 5 dimensionless variables

$\mathcal{O}$

F_4

1   2   3   4

$$u_i = \frac{s_{i,i+1}}{s_{1234}} \qquad v_i = \frac{s_{i,i+1,i+2}}{s_{1234}}$$
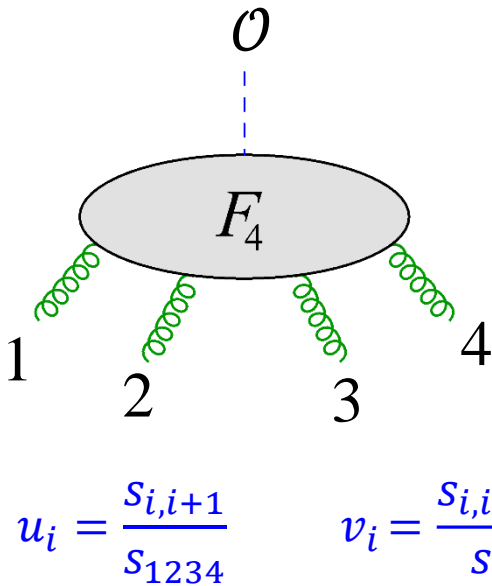
8 variables, obeying 3 linear relations from momentum conservation:
$-u_1 + u_3 + v_4 + v_1 = 1,$ etc.

- Only **8** of **113** letters (u_i, v_i) are allowed in first entry by a branch-cut condition
- One loop symbol is

$$\mathcal{S}\left[F_4^{(1)}\right] = 2\, v_1 \otimes (1 - v_1) + \frac{u_1}{u_2 u_4} \otimes u_1 + \frac{u_1}{v_4 v_1} \otimes \frac{u_1 - v_4 v_1}{u_1}$$
$$+ \text{cyclic}$$

Bootstrapping symbol of $F_4^{(2)}$

- Impose the following conditions:

1. first-entry is u_i, v_i

2. weight 4 integrable symbol

3. D_4 dihedral invariance

4. invariance under 5 different $\sqrt{} \to -\sqrt{}$

5. when 2 particles become collinear, $R_4^{(2)} \to R_3^{(2)}$

→ **A unique symbol!**

- Reminiscent of rigidity of 7-loop amplitude

  Drummond, Papathanasiou, Spradlin, 1412.3763

Checks of $F_4^{(2)}$

1. As the operator momentum becomes light-like, $k_{\mathcal{O}}^2 \to 0$, $R_4^{(2)}$ agrees with Guo, Wang, Yang, 2209.06816

2. In triple collinear limit, 4-gluon form factor becomes the triple-collinear splitting amplitude, which is also the 6-gluon amplitude(!):

$$R_4^{(2)} \to \hat{R}_6^{(2)}$$
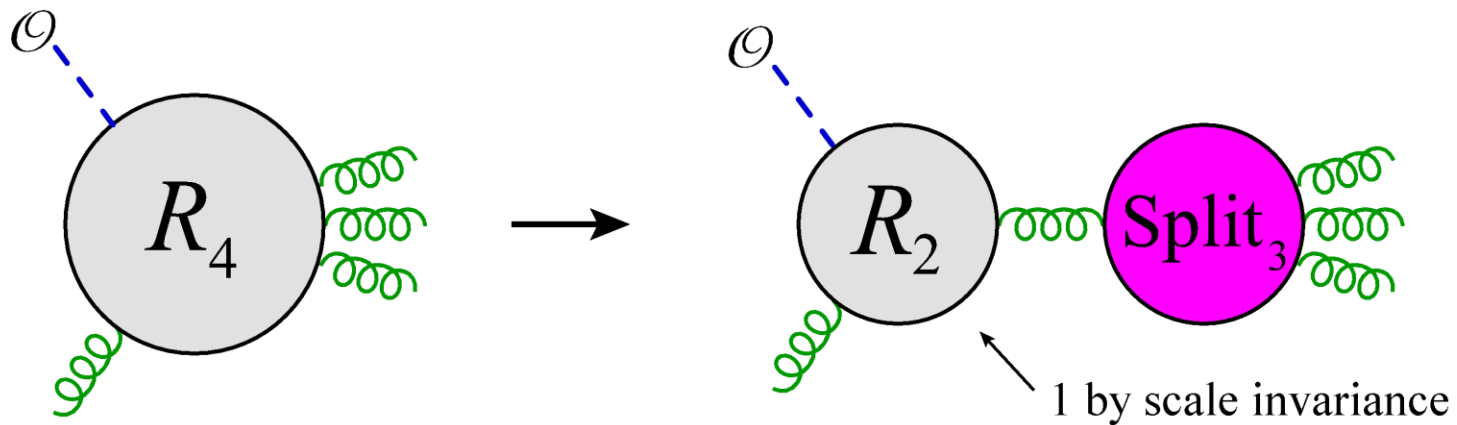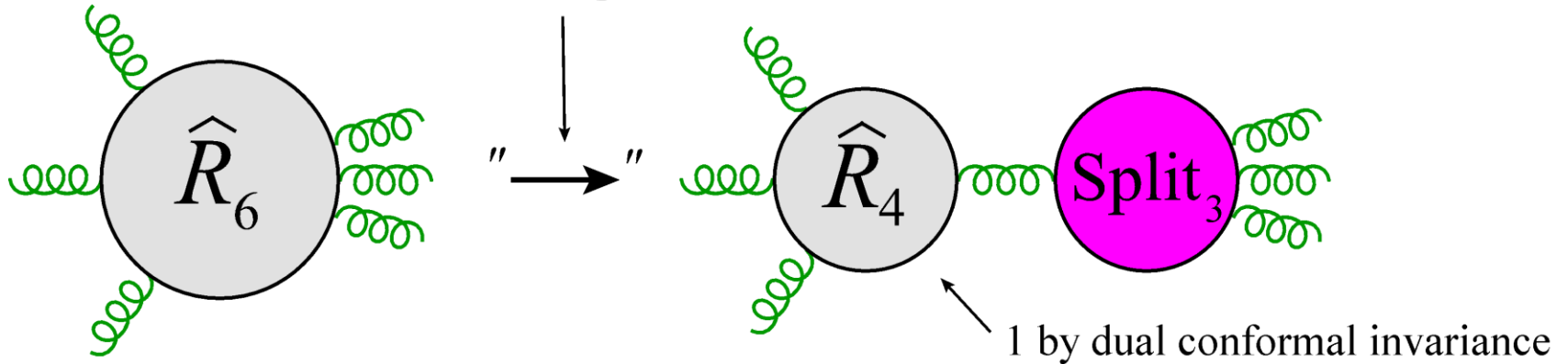   Bern et al., 0803.1465, or FFOPE

3. More generally, we check nontrivial FFOPE predictions Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

Why does form factor → amplitude in triple-collinear limit?

dual conformal transformations map
all kinematics to triple collinear limit!

Bern et al., 0803.1465

1 by dual conformal invariance

1 by scale invariance

Antipodal Self Duality

Given an antipodal duality relating 2-collinear and 3-collinear limits of F_4, it's natural to search for a self-duality of F_4 that holds for all parity-preserving bulk kinematics

ASD
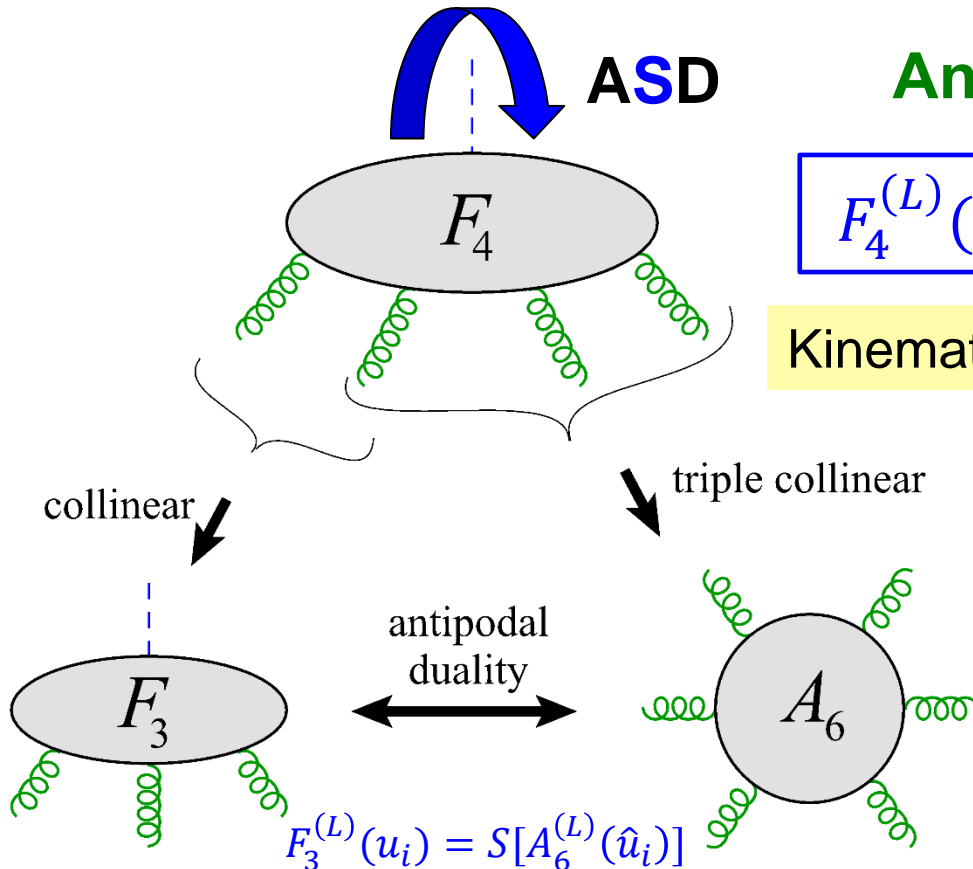
And it's there!

$$F_4^{(L)}(u_i, v_i) = S[F_4^{(L)}(g(u_i), g(v_i))]$$

Kinematic map g simple in FFOPE variables:

$$g: \quad T_2 \to \frac{T}{S}, \quad S_2 \to \frac{1}{TS}$$

$$T \to \sqrt{\frac{T_2}{S_2}}, \qquad S \to \sqrt{\frac{1}{T_2 S_2}}$$

$$F_2 = 1$$

collinear

triple collinear

antipodal duality

$$F_3^{(L)}(u_i) = S[A_6^{(L)}(\hat{u}_i)]$$

ASD beyond 2 loops

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 23mm.nnnnn

- Bootstrapped symbol of F_4 at 3 loops, using same 113 letter alphabet.

- We again find a **unique result**, which obeys all the FFOPE predictions we could check.

- 2 loop symbol uses only 34 of the letters   [3,784 terms]

- 3 loop symbol uses only 88 of the letters [3,621,202 terms]

- A**S**D **holds at 3 loops!**

- 4 loops in progress

Summary & Open Questions

- 6-gluon amplitude and 3-gluon form factor in planar N=4 SYM are related by a strange new antipodal duality, swapping role of branch cuts and derivatives
- Embedded in a 4-gluon form factor self-duality!

- Who ordered that?
- Underlying physical reason for this duality?
- 3-dimensions seems to play a crucial role (parity preserving surfaces). Why?
- (How) does it hold at strong coupling?
- Where else might it hold? E.g. $\text{tr}\phi^3$ operator form factors?
  Tumanov poster; Basso, Tumanov; Basso, LD, Tumanov to appear
- How much more can we exploit it to learn more about both amplitudes and form factors?

The End of This Conference

Many, many thanks!!!

To CERN-TH Secretariat

To CERN-TH

To CERN-IT

And especially to the Fab Four:
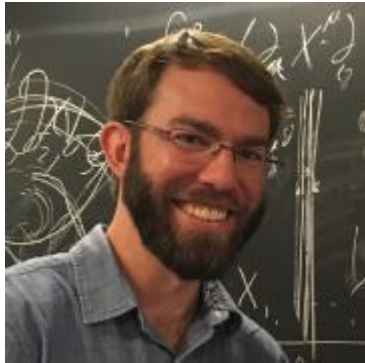
Andrew   Ben   Lorenzo   Samuel
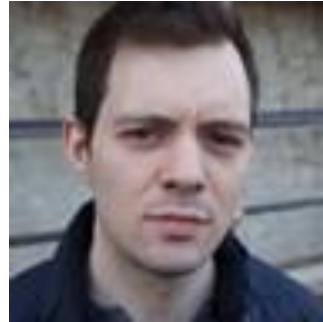
See you all in Princeton next year!

Safe Travels!

Extra Slides

FFOPE kinematical variables for F_4

$$u_1 = \frac{T^2 T_2^2}{(T^2 + 1)(S^2 + T^2 + T_2^2 + 1)}$$

$$u_2 = \left\{ 1 + T^2 + \frac{S^2[(1 + F_2^2)S_2 T_2 + F_2(1 + S_2^2 + T^2 + T_2^2)]}{F_2 S_2^2} \right\}^{-1}$$

$$u_3 = \frac{S^2}{(T^2 + 1)(S^2 + T^2 + T_2^2 + 1)}$$

$$u_4 = \frac{S^2 T^2}{S_2^2} u_2$$

$$v_1 = \frac{T_2^2 + 1}{S^2 + T^2 + T_2^2 + 1}$$

- OPE limit takes $T, T_2 \to 0$, interpolates between 2-collinear limit $T_2 \to 0$ and 3-collinear limit $T \to 0$,

AD explains many patterns in F_3

- Every term in the symbol starts with a, b, c; never d, e, f
- Physical reason related to causality, which dictates where branch cuts can appear: only for $(p_i + p_j)^2 \sim 0$
- Empirically, 12 pairs of adjacent letters are forbidden:

$$\ldots a \otimes d \ldots, \qquad \ldots b \otimes e \ldots, \qquad \ldots c \otimes f$$
$$\ldots d \otimes a \ldots, \qquad e \otimes b \qquad, \qquad \ldots f \otimes c \ldots$$
$$\ldots d \otimes e \ldots \qquad \ldots e \otimes f \ldots, \qquad f \otimes d \ldots$$
$$\ldots e \otimes d \ldots, \qquad \ldots f \otimes e \ldots, \qquad \ldots d \otimes f \ldots$$

- Resemble constraints from causality:
  Steinmann relations   Steinmann, Helv. Phys. Acta (1960)
- But not really, which mystified us for a while…
- However, the relations are antipodally dual to the (extended) Steinmann relations for A_6 !!

Exploit/test antipodal duality at 8 loops

- Given form factor, antipodal duality determines symbol of MHV 6 gluon amplitude at 8 loops on $\Delta = 0$ surface.

- Lift symbol into bulk.   Only 3 free parameters!

- 2 killed at origin, $(\hat{u}, \hat{v}, \hat{w}) \to (0,0,0)$

- last killed in process of lifting to full function level

- Need one OPE data point to kill one beyond-symbol ambiguity $\propto \zeta_8$
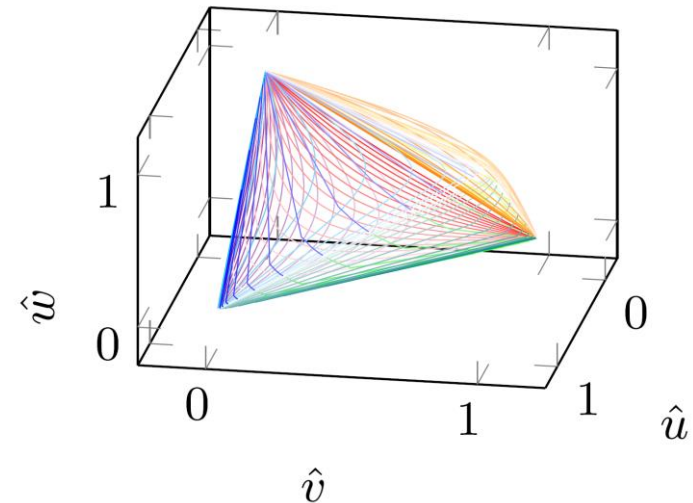
Antipodal kinematic map covers entire phase space for 3-gluon form factor
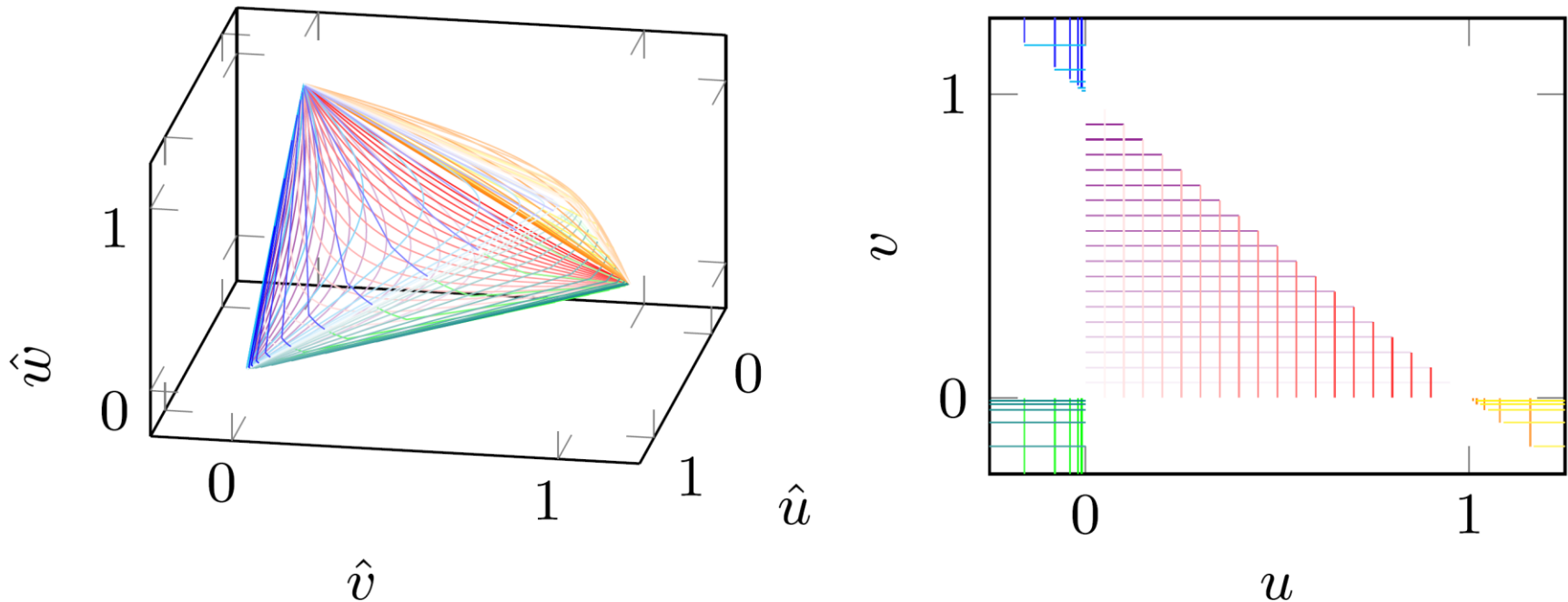
- Soft is dual to collinear; collinear is dual to soft
- White regions in (u, v) map to some of $\hat{u}, \hat{v}, \hat{w} > 1$

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

evaluate to Riemann zeta values

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1, n_2, \ldots, n_m} = \sum_{k_1 > k_2 > \cdots > k_m > 0} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $\quad n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

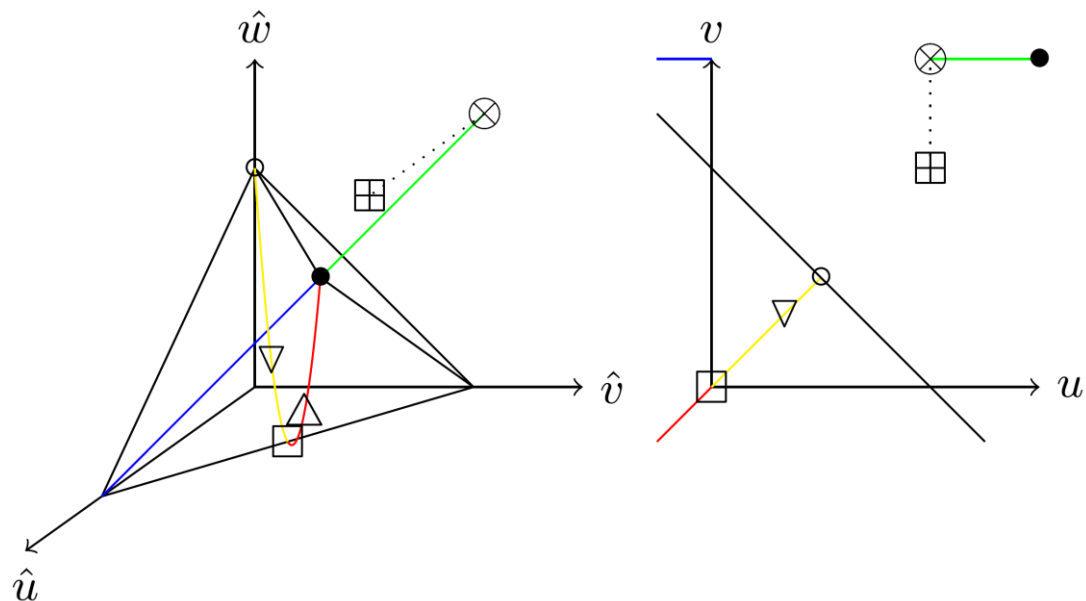$$\zeta_{n_1} \zeta_{n_2} = \zeta_{n_1, n_2} + \zeta_{n_2, n_1} + \zeta_{n_1 + n_2}$$

- All reducible to Riemann zeta values until weight 8.
  Irreducible MZVs: $\zeta_{5,3}, \; \zeta_{7,3}, \; \zeta_{5,3,3}, \; \zeta_{9,3}, \; \zeta_{6,4,1,1}, \; \cdots$

Many special dual points

There is an "f" alphabet at all these points: a way of writing multiple zeta values (MZV's) so that coaction is manifest.

F. Brown, 1102.1310; O. Schnetz, HyperlogProcedures

	$(\hat{u}, \hat{v}, \hat{w})$	(u, v, w)	functions
▽	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\sqrt[6]{1}$
□	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(0, 0, 1)$	$\mathrm{Li}_2(\frac{1}{2}) + \text{logs}$
●	$(1, 1, 1)$	$\lim_{u \to \infty}(u, u, 1-2u)$	MZVs
○	$(0, 0, 1)$	$(\frac{1}{2}, \frac{1}{2}, 0)$	MZVs + logs
△	$(\frac{3}{4}, \frac{3}{4}, \frac{1}{4})$	$(-1, -1, 3)$	$\sqrt[6]{1}$
⊞	(∞, ∞, ∞)	$(1, 1, -1)$	alternating sums
⊗	$\lim_{\hat{v} \to \infty}(1, \hat{v}, \hat{v})$	$\lim_{v \to \infty}(1, v, -v)$	MZVs
—(blue)	$(1, \hat{v}, \hat{v})$	$\lim_{v \to \infty}(u, v, 1-u-v)$	HPL$\{0, 1\}$
—(red)	$(\hat{u}, \hat{u}, (1-2\hat{u})^2)$	$(u, u, 1-2u)$	HPL$\{-1, 0, 1\}$

Simplest point

- $(\hat{u}, \hat{v}, \hat{w}) = (1,1,1) \iff u, v \to \infty$

- At this point,

$$A_6^{(1)}(\cdot) = 0 \qquad F_3^{(1)}(\cdot) = 8\zeta_2$$

$$A_6^{(2)}(\cdot) = -9\zeta_4 \qquad F_3^{(2)}(\cdot) = 31\zeta_4$$

$$A_6^{(3)}(\cdot) = 121\zeta_6 \qquad F_3^{(3)}(\cdot) = -145\zeta_6$$

$$A_6^{(4)}(\cdot) = 120 f_{3,5} - 48\zeta_2 f_{3,3} - \frac{6381}{4}\zeta_8 \qquad F_3^{(4)}(\cdot) = 120 f_{5,3} + \frac{11363}{4}\zeta_8$$

$$A_6^{(5)}(\cdot) = -2688 f_{3,7} - 1560 f_{5,5} + \mathcal{O}(\pi^2) \qquad F_3^{(5)}(\cdot) = -2688 f_{7,3} - 1560 f_{5,5} + \mathcal{O}(\pi^2)$$

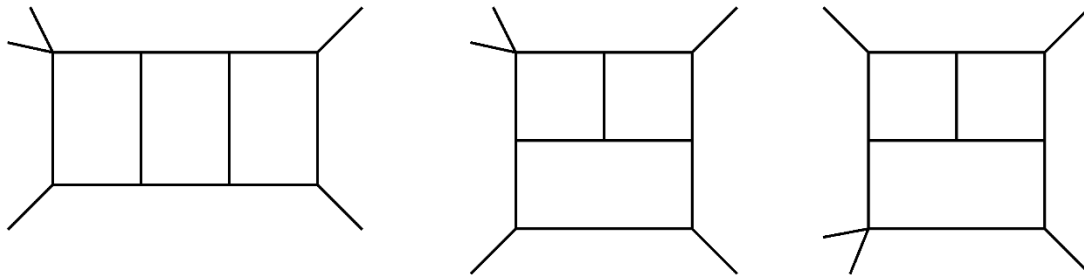$$A_6^{(6)}(\cdot) = 48528 f_{3,9} + 37296 f_{5,7} + 21120 f_{7,5} + \mathcal{O}(\pi^2) \qquad F_3^{(6)}(\cdot) = 48528 f_{9,3} + 37296 f_{7,5} + 21120 f_{5,7} + \mathcal{O}(\pi^2)$$

- Reversing ordering of letters in f-alphabet, blue values show that antipodal duality holds beyond symbol level, modulo $i\pi$

- modulo $i\pi$ is best we can get from mathematical antipode map

Meaning for integrals?

LD, McLeod, Wilhelm, 2012.12286; Chicherin, Henn, Papathanasiou, 2012.12285

Gehrmann, Remiddi, hep-ph/0008287, hep-ph/0101124

doesn't contribute to planar N=4 SYM form factor

+ all daughter topologies

all have

$...d\!\!\!/\otimes e...$

+ dihedral

half of the adjacency constraints seen in planar N=4 SYM
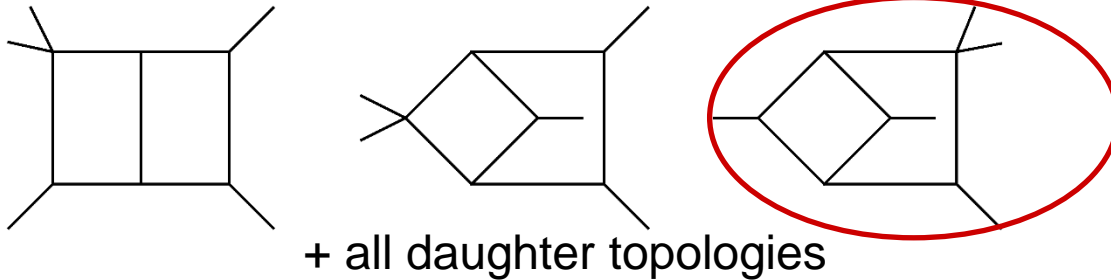
DiVita, Mastrolia, Schubert, Yundin, 1408.3107

Canko, Syrrakos, 2112.14275 [some]

Why?

(No Steinmann relations for massless 2-particle cuts…)

Other 3 loop integrals have new letters

Other integrals lose the adjacency conditions

New letters in the alphabet

Henn, Lim, Torres Bobadilla, 2302.12776

[Henn, Lim, WJT (2023)]

$$\vec{\alpha} = \{\alpha_0, \dots, \alpha_8\} = \left\{ p_4^2, s, t, -p_4^2 + s + t, -p_4^2 + s, -p_4^2 + t, s + t, -(p_4^2 - s)^2 + p_4^2 t, s^2 - p_4^2(s - t) \right\}.$$

$$\epsilon^6 \left((p_4^2 - s)^2 - p_4^2 t \right) \left(\raisebox{-1em}{\rule{0pt}{2em}} \right) \longleftrightarrow \epsilon^6 \left((p_4^2 - s)^2 - p_4^2 u \right) \left(\raisebox{-1em}{\rule{0pt}{2em}} \right)$$

$p_1 \leftrightarrow p_2$ symmetry
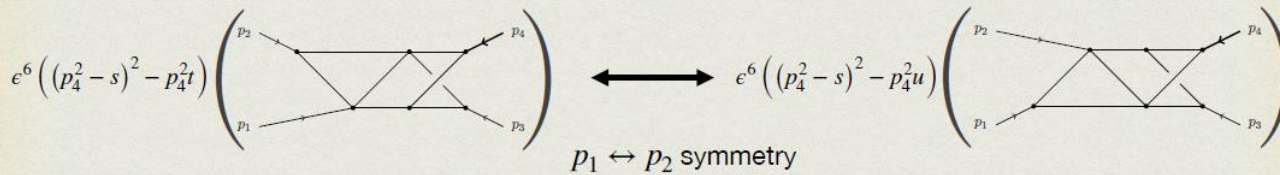
Start appearing at weight 4

$$\mathcal{S}\left(f_{B1}^{41}\right)\Big|_{\epsilon^4} = 6\left[\alpha_1 \otimes \alpha_1 \otimes \frac{\alpha_2}{\alpha_4} \otimes \alpha_7 - \alpha_1 \otimes \alpha_1 \otimes \alpha_4 \otimes \alpha_7 + \alpha_1 \otimes \frac{\alpha_4}{\alpha_2} \otimes \frac{\alpha_3}{\alpha_1 \alpha_4} \otimes \alpha_7 \right.$$
$$\left. + \alpha_2 \otimes \alpha_1 \otimes \frac{\alpha_1 \alpha_4}{\alpha_3} \otimes \alpha_7 + \alpha_2 \otimes \alpha_5 \otimes \frac{\alpha_3}{\alpha_1} - \frac{1}{2} \alpha_2 \otimes \alpha_5 \otimes \alpha_2 \otimes \alpha_7 + \dots \right]$$

$a \otimes d \quad , \dots d \otimes e$

???

William J. Torres Bobadilla   talk at LoopFest 2023   13
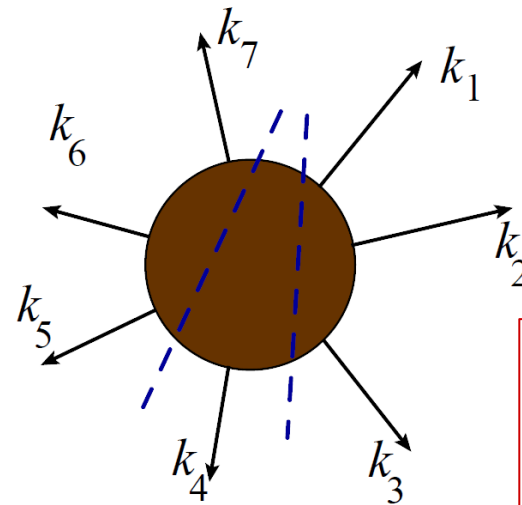
Steinmann relations

- Amplitudes should not have overlapping branch cuts:
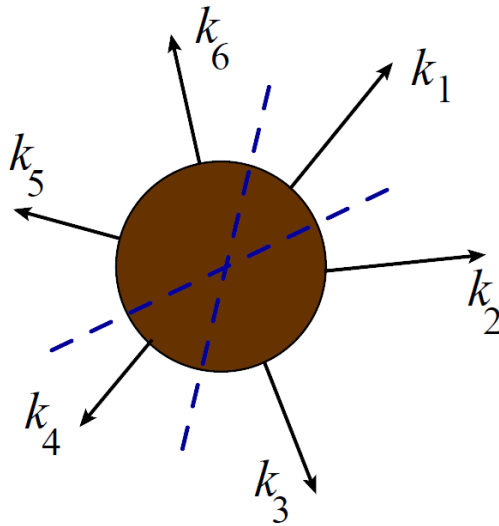
Not Allowed

Allowed

can't apply to 2 particle cuts in massless case because they are not independent

$$\mathrm{Disc}_{s_{234}}\,\mathrm{Disc}_{s_{123}}\,A_6(\hat{u},\hat{v},\hat{w}) = 0$$

if you remove IR divergences properly

Steinmann + DCI consequences

$$\hat{u} = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad \hat{v} = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad \widehat{w} = \frac{s_{34}s_{61}}{s_{345}s_{234}} \quad \text{are not ideal,}$$

so switch to $\hat{a} \equiv \frac{\hat{u}}{\hat{v}\widehat{w}} = (s_{234}^2)^2 \times \left[s_{i,i+1} \; stuff \right]$

$$\hat{b} \equiv \frac{\hat{v}}{\widehat{w}\hat{u}}, \qquad \hat{c} \equiv \frac{\widehat{w}}{\hat{u}\hat{v}}$$

$$\boxed{\mathrm{Disc}_{\hat{b}} \, \mathrm{Disc}_{\hat{a}} \, A_6(\hat{u}, \hat{v}, \widehat{w}) = 0}$$

Should hold on any Riemann sheet (?)

Discontinuities via symbol

- Discontinuities commute with derivatives; discontinuities act on left entry of symbol, while derivatives act on right

$$\mathcal{S}\left[\mathrm{Disc}_{\hat{a}} F\right] = 2\pi i\,\hat{a} \otimes \ldots$$

- $\mathrm{Disc}_{\hat{b}}\,\mathrm{Disc}_{\hat{a}} A_6(\hat{u}, \hat{v}, \widehat{w}) = 0$ (+ dihedral images)

means $\mathcal{S}[A_6]$ cannot contain any terms of the form $\hat{a} \otimes \hat{b} \otimes \ldots$

- But we actually find more generally, for any adjacent slots,

$$\ldots \otimes \hat{a} \otimes \hat{b} \otimes \ldots$$

Caron-Huot, LD, McLeod, von Hippel, Papathanasiou, 1806.01361, 1906.07116

- "Extended Steinmann relations".

- With first entry condition, also find $\ldots \otimes \hat{a} \otimes \hat{d} \otimes \ldots$

- equivalent to "cluster adjacency" for $A_3 = \mathrm{Gr}(4,6)$ cluster algebra
Drummond, Foster, Gürdoğan, 1710.10953
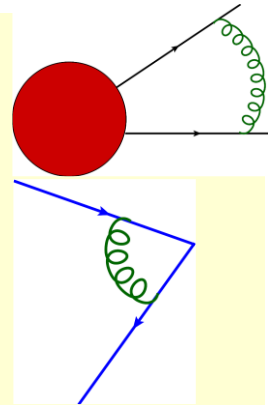
Removing Amplitude (or Form Factor) Infrared Divergences

- On-shell amplitudes IR divergent due to long-range gluons

- Polygonal Wilson loops UV divergent at cusps, anomalous dimension Γ_{cusp}
  - **known to all orders in planar N=4 SYM:**
  Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

$$\mathcal{E}_6(u_i) = \lim_{\epsilon \to 0} \frac{\mathcal{A}_6(s_{i,i+1}, \epsilon)}{\mathcal{A}_6^{\text{BDS-like}}(s_{i,i+1}, \epsilon)} = \exp\left[\frac{\Gamma_{\text{cusp}}}{4} \mathcal{E}_6^{(1)} + R_6\right]$$

remainder function

BDS & BDS-like normalization for $\mathcal{F}_3$

$$\frac{\mathcal{F}_3}{\mathcal{F}_3^{\text{MHV, tree}}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\text{cusp}}^{(L)}}{4} + \mathcal{O}(\epsilon)\right)M^{1-\text{loop}}(L\epsilon) + C^{(L)} + R^{(L)}(u,v,w)\right]\right\}$$

BDS ansatz

remainder function only a function of u, v, w; vanishes in all collinear limits, but no adjacency constraints

split 1-loop amplitude judiciously:

$$\frac{\mathcal{F}_3^{1-\text{loop}}}{\mathcal{F}_3^{\text{MHV, tree}}} \equiv M^{1-\text{loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u,v,w)$$

$$M(\epsilon) = -\frac{1}{\epsilon^2}\sum_{i=1}^{3}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} - \frac{7}{2}\zeta_2 + \sum_{}^{3}$$

$\mathcal{E}^{(1)}(u,v \ldots$ $\ldots \left(1 - \frac{\ }{v}\right) + \text{Li}_2\left(1 - \frac{1}{w}\right)\right]$ $\mathcal{E}^{(1),u} + \mathcal{E}^{(1),1-u} = 0$

$\mathcal{E}$ obeys "adjacency constraints"

Now div ...

$$\frac{\mathcal{F}_3^{\text{BDS-like}}}{\mathcal{F}_3^{\text{MHV, tree}}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\text{cusp}}}{4} + \mathcal{O}(\epsilon)\right)M(L\epsilon) + C^{(L)}\right]\right\} \Rightarrow \boxed{\mathcal{E} = \exp\left[\frac{\Gamma_{\text{cusp}}}{4}\mathcal{E}^{(1)} + R\right]}$$

Number of (symbol-level) linearly independent $\{n, 1, ..., 1\}$ coproducts $(2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L = 1$	1	3	1														
$L = 2$	1	3	6	3	1												
$L = 3$	1	3	9	12	6	3	1										
$L = 4$	1	3	9	21	24	12	6	3	1								
$L = 5$	1	3	9	21	46	45	24	12	6	3	1						
$L = 6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L = 7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L = 8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- **Properly normalized** L loop N=4 form factors $\mathcal{E}^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left

- $\mathcal{E}^{(L)}$ also obeys multiple-final-entry relations, saturation on right