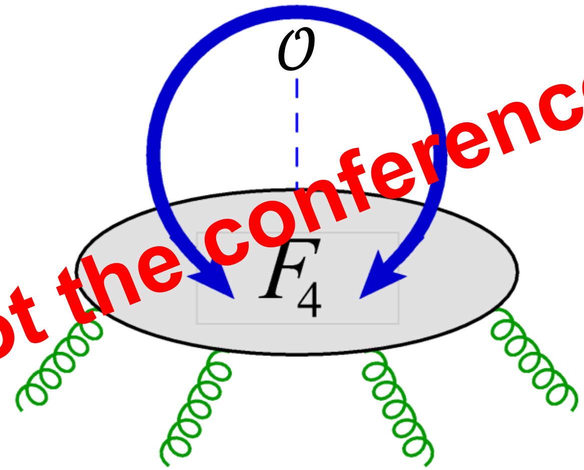
Antipodal Self-Duality

This is not the conference summary!



Lance Dixon (SLAC)

Andy Liu

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm

2112.06243, 2204.11901, 2212.02410, and in progress

Amplitudes @ CERN

11 August 2023

NATIONAL ACCELERATOR LABORATORY

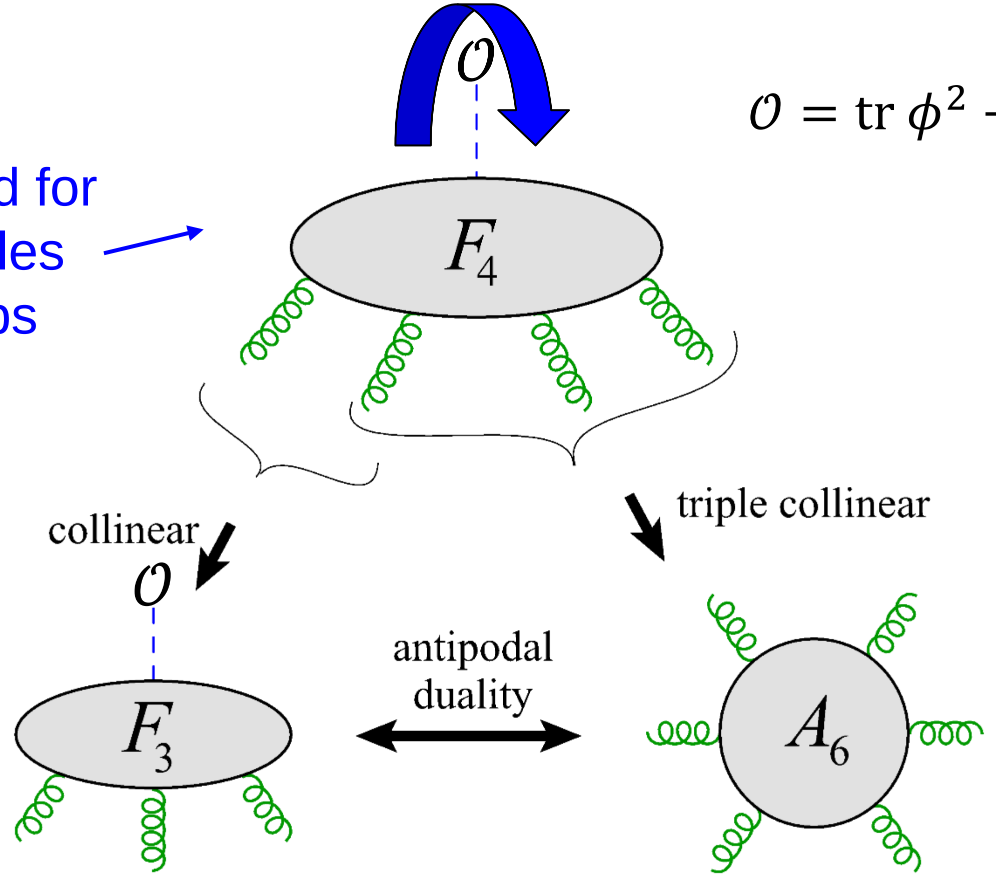
This talk in one picture:
Antipodal Self Duality
for a 4-point form factor in planar N=4 SYM

$\mathcal{O} = \mathrm{tr}\,\phi^2 + \cdots + \mathrm{tr}\, G_{\mu\nu}^{\mathrm{SD}} G^{\mu\nu\,\mathrm{SD}}$

validated for 4 variables to 3 loops

antipode map reverses order of letters in symbol

also requires a kinematic map



validated for 2 variables to 8 loops!

earlier: 2112.06243, Amplitudes 2022

Multiple polylogarithms (MPLs)

- Characterize all the form factors & amplitudes playing a role here.
- At L loops, the results are weight $n = 2L$ MPLs, defined as iterated integrals by

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

and $G(\vec{0}_n, x) = \frac{(\ln x)^n}{n!}$

Hopf algebra

Goncharov; Brown; Goncharov, Spradlin, Vergu, Volovich; Duhr, Gangl, Rhodes

- Differential definition: $dF = \sum_{s_k \in \mathcal{L}} F^{s_k} \, d \ln s_k$

- Hopf algebra "co-acts" on space of MPLs,

$$\Delta: F \to F \otimes F$$

- Derivative dF is one piece of Δ: $\Delta_{n-1,1} F = \sum_{s_k \in \mathcal{L}} F^{s_k} \otimes \ln s_k$

- So we refer to F^{s_k} as a $\{n\text{-}1,1\}$ coproduct of F

- **s_k are letters in the symbol alphabet $\mathcal{L}$**

Iterate to get **symbol**

e.g. Spiering talk

- Apply $\{n\text{-}1,1\}$ coaction iteratively:
- Define $\{n\text{-}2,1,1\}$ double coproducts, F^{s_k,s_j},

via derivatives of $\{n\text{-}1,1\}$ single coproducts F^{s_j}:

$$dF^{s_j} \equiv \sum_{s_k \in \mathcal{L}} F^{s_k,s_j} \, d \ln s_k$$

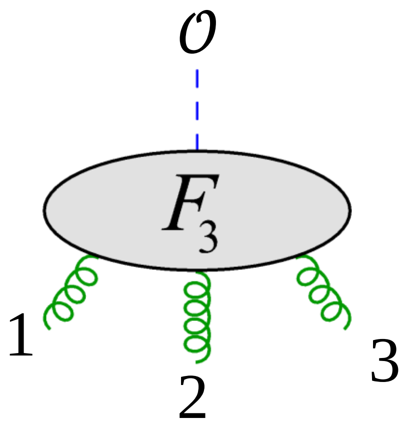
- And so on for $\{n\text{-}m,1,\ldots,1\}$ m^{th} coproducts of F.
- Maximal iteration, n times for weight n function, is

the symbol, $[\text{"ln" is implicit in } s_{i_k}]$

$$\mathcal{S}[F] \equiv \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}} \, s_{i_1} \otimes \ldots \otimes s_{i_n}$$

where now $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers

Goncharov, Spradlin, Vergu, Volovich, 1006.5703



3-gluon form factor depends on 2 dimensionless variables u, v

$k_i^2 = 0 \quad s_{ij} = (k_i + k_j)^2$

$k_1 + k_2 + k_3 = -k_{\mathcal{O}}$

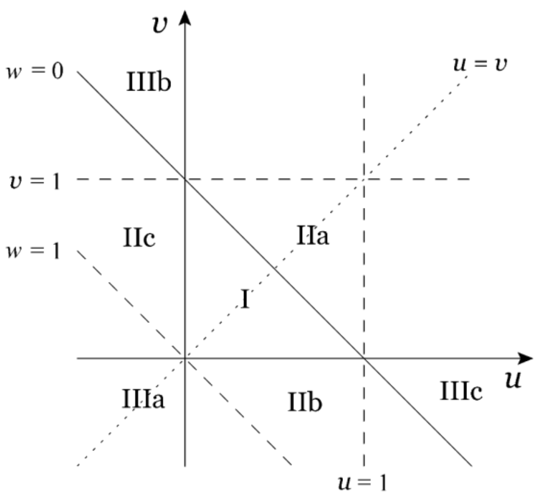
$s_{123} = s_{12} + s_{23} + s_{31} = q^2$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}} = 1 - u - v$$

$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i + 1 \pmod 3$, or
$u \to v \to w \to u$

b. flip: $u \leftrightarrow v$



I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

F_3 symbol alphabet has 6 letters

$$\mathcal{L} = \{\, a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w} \}$$

- Symbols of form factor $F_3^{(L)}$ at $L = 1, 2$ loops are just 1 and 2 terms, plus D_3 dihedral images(!!!):

$$\mathcal{S}\left[F_3^{(1)}\right] = (-1)\, b \otimes d + \text{dihedral}$$

$$\mathcal{S}\left[F_3^{(2)}\right] = 4\, b \otimes d \otimes d \otimes d + 2\, b \otimes b \otimes b \otimes d + \text{dihedral}$$

Brandhuber, Travaglini, Yang, 1201.4170

LD, Gürdoğan A. McLeod, M. Wilhelm, 2204.11901

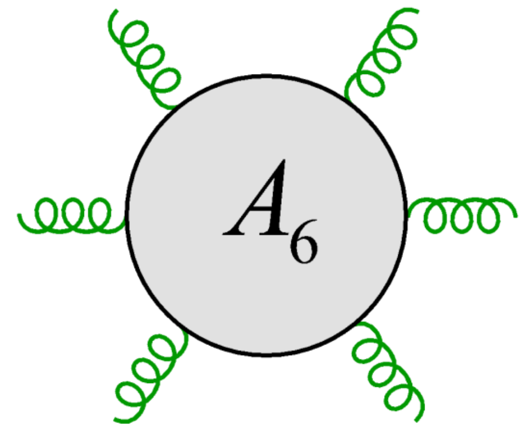
known to 8 loops

dihedral cycle: $a \to b \to c \to a,\quad d \to e \to f \to d$

dihedral flip: $a \leftrightarrow b,\quad d \leftrightarrow$ e

6-gluon amplitude

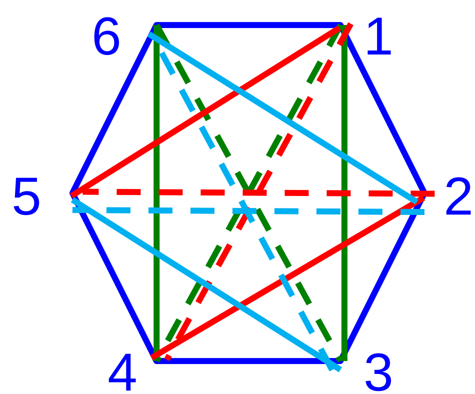
- Dual to Wilson hexagon, invariant under dual conformal transformations, so it can only depend on 3 dual conformal cross ratios, $\hat{u}, \hat{v}, \widehat{w}$:



$$\widehat{u} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

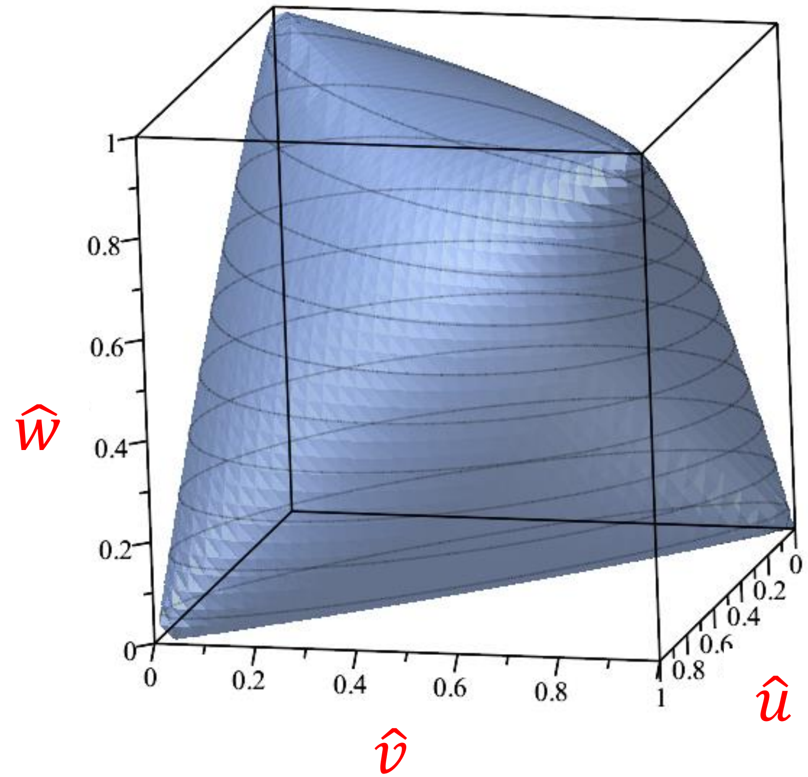
$$\widehat{v} = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

$$\widehat{w} = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$



D_6 dihedral symmetry: cycle $(\mathrm{mod}\ 6)$ and flip, but it acts on $\hat{u}, \hat{v}, \widehat{w}$ as $D_3 = S_3$

Parity-preserving surface



$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

kinematics lies in a 3d subspace of 4d spacetime
→ parity invariant

6-gluon symbol alphabet

- $\mathcal{L}_6 = \{ \hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w \}$ → 1 for $\Delta = 0$

→ $\mathcal{L}_6' = \{ \hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}},\ \hat{b} = \frac{\hat{v}}{\hat{w}u},\ \hat{c} = \frac{\hat{w}}{\hat{u}\hat{v}},\ \hat{d} = \frac{1-\hat{u}}{\hat{u}},\ \hat{e} = \frac{1-\hat{v}}{\hat{v}},\ \hat{f} = \frac{1-\hat{w}}{\hat{w}} \}$

- Symbols of amplitude $A_6^{(L)}$ on $\Delta = 0$ at $L = 1, 2$ loops are just 1 and 2 terms, plus D_3 dihedral images(!!!):

$$\mathcal{S}\left[A_6^{(1)}\right] = (-\frac{1}{2})\hat{b}\otimes\hat{d} + \text{dihedral}$$

$$\mathcal{S}\left[A_6^{(2)}\right] = \hat{b}\otimes\hat{d}\otimes\hat{d}\otimes\hat{d} + \frac{1}{2}\hat{b}\otimes\hat{b}\otimes\hat{b}\otimes\hat{d} + \text{dihedral}$$

…

was known to 7 loops

Goncharov, Spradlin, Vergu, Volovich, 1006.5703, …,
Caron-Huot, LD, Dulat, McLeod, von Hippel, 1903.10890

Antipodal duality

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

$$F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)$$

Antipode map S, at symbol level, reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

Kinematic map in terms of underlying variables is:

$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

Maps $u + v + w = 1$ to parity-preserving surface

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

Kinematic map on letters

$\sqrt{\hat{a}} = d, \quad \hat{d} = a, \qquad$ *plus cyclic relations*

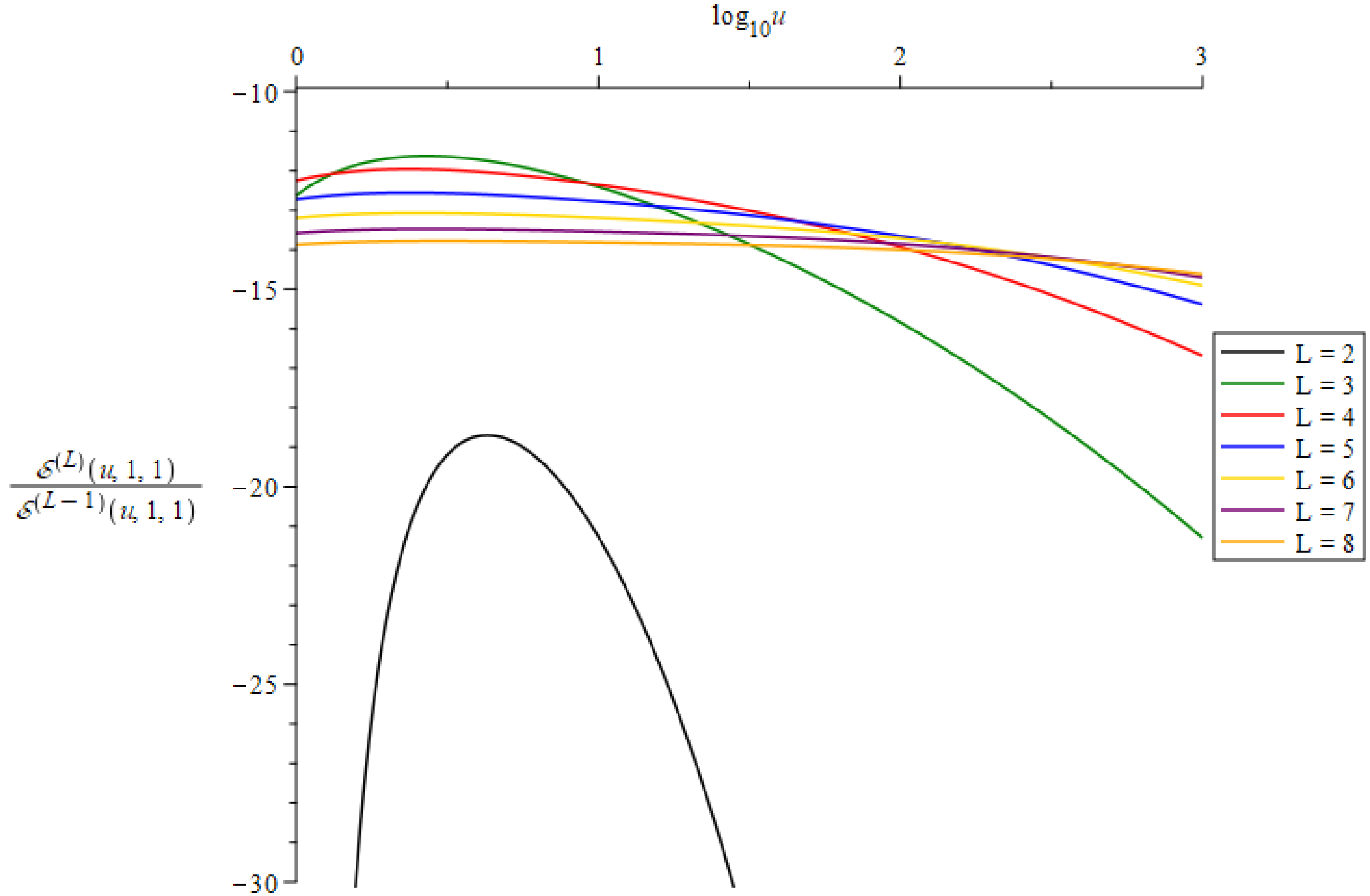
Works through 8 loops (even beyond symbol)!

L	number of terms in symbol
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

But why?!

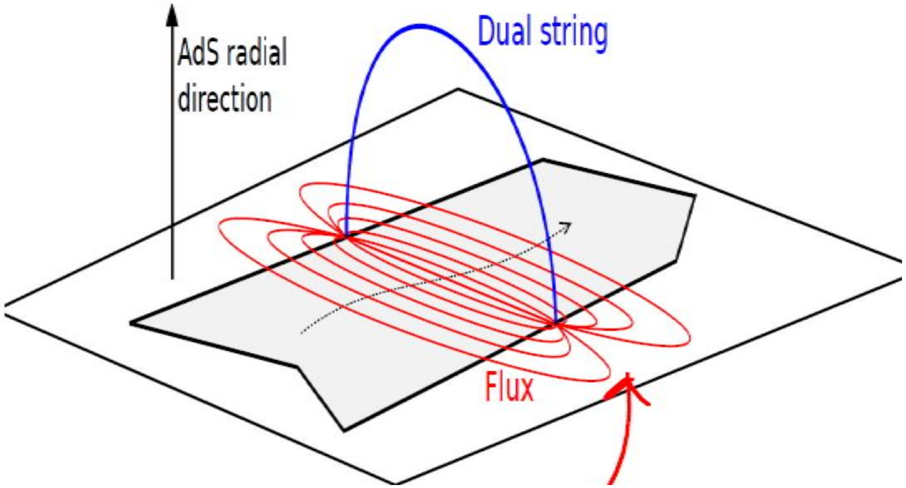
Exploit to compute A_6 at 8 loops

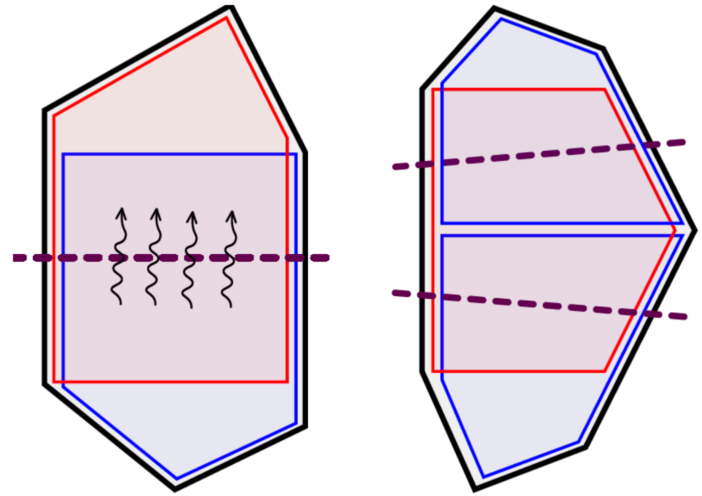
Dixon, Liu, 2308.nnnnn



The Flux Tube (Pentagon) OPE

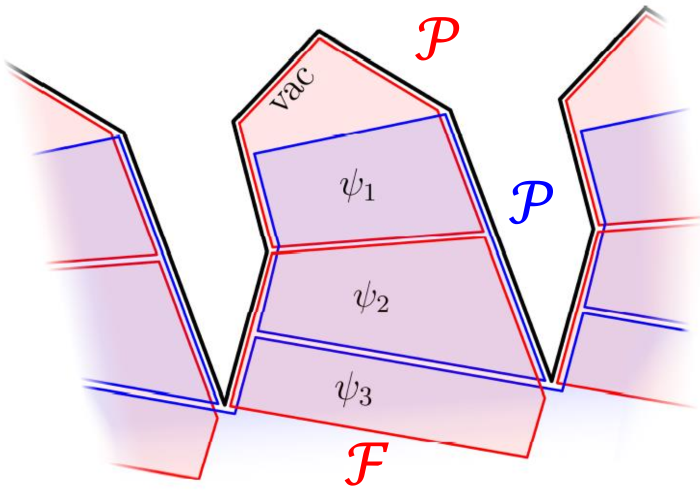
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in ’t Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

The Form Factor OPE



- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$.
- For $\mathrm{tr}\phi^2$: **Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569**

OPE representation

- 6-gluon amplitude:

$$\mathcal{W}_{\text{hex}} = \sum_{\mathbf{a}} \int d\mathbf{u}\, P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{a}}(\bar{\mathbf{u}}|0)\, e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi}$$

$T = e^{-\tau}, S = e^{\sigma}, F = e^{i\phi}$. $\quad v = \frac{T^2}{1+T^2} \to 0$,

weak-coupling, $E = k + \mathcal{O}(g^2)$ → expansion in T^k

- 3-gluon form factor: $\psi = helicity\ 0\ pairs\ of\ states$

$$\mathcal{W}_3 = \sum\!\!\!\!\!\!\int_{\psi} e^{-E_\psi \tau + ip_\psi \sigma} \mathcal{P}(0|\psi) \mathcal{F}(\psi)$$

weak-coupling → expansion in T^{2k} (no azimuthal angle ϕ)

"OPE" coordinates simplify kinematic map

Basso, Sever, Vieira, 1303.1396, 1306.2058,…;
Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

- Amplitude:

$$\hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})},$$

$$\hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2, \qquad \hat{w} = \frac{\hat{T}^2}{1 + \hat{T}^2}$$

$(\hat{F} = 1 \text{ for } \Delta = 0)$

- Form factor:

$$u = \frac{1}{1 + S^2 + T^2}, \qquad v = \frac{T^2}{1 + T^2},$$

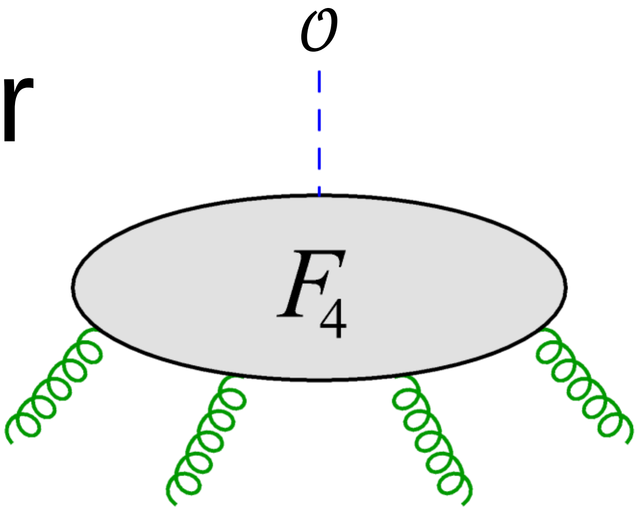
$$w = \frac{1}{(1 + T^2)(1 + S^{-2}(1 + T^2))},$$

- Kinematic map for antipodal duality →

$$\hat{T} = \frac{T}{S}, \qquad \hat{S} = \frac{1}{TS}$$

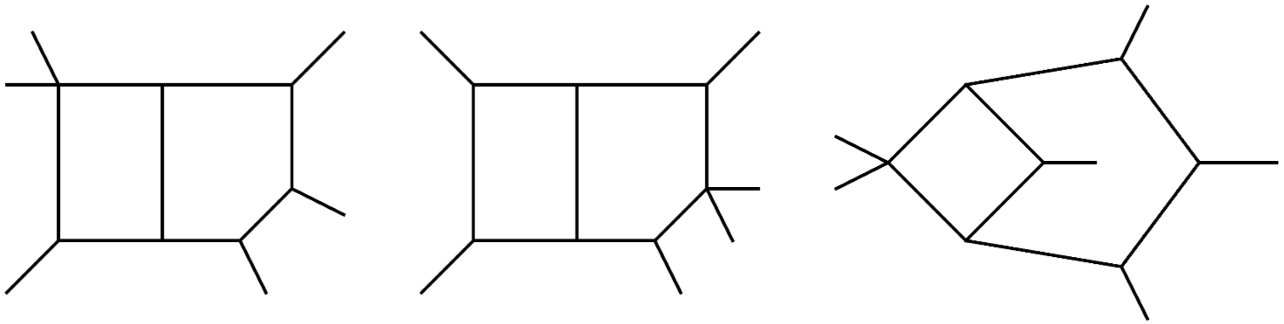
$$T = \sqrt{\frac{\hat{T}}{\hat{S}}}, \qquad S = \sqrt{\frac{1}{\hat{T}\hat{S}}}$$

Four-gluon form factor

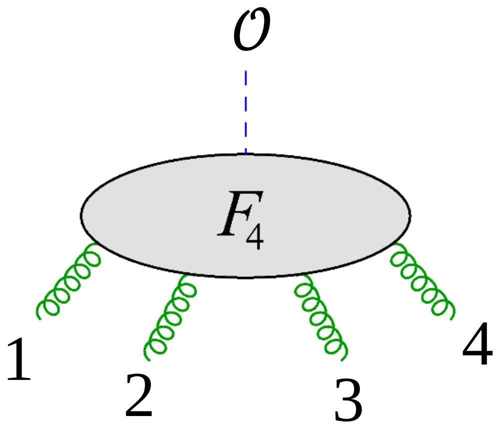


Depends on 5 kinematical variables instead of 2.

Even just at two loops, contains state-of-the art loop integrals → 113 possible symbol letters!



Abreu, Ita, Moriello, Page, Tschernow, 2005.04195;
Abreu, Ita, Page, Tschernow, 2107.14180;
Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, 2306.15431



4-gluon form factor depends on 5 dimensionless variables

$u_i = \frac{s_{i,i+1}}{s_{1234}}$ $\qquad$ $v_i = \frac{s_{i,i+1,i+2}}{s_{1234}}$

8 variables, obeying 3 linear relations from momentum conservation:
$-u_1 + u_3 + v_4 + v_1 = 1$, etc.

- Only 8 of 113 letters (u_i, v_i) are allowed in first entry by a branch-cut condition
- One loop symbol is

$$\mathcal{S}\left[F_4^{(1)}\right] = 2\, v_1 \otimes (1 - v_1) + \frac{u_1}{u_2 u_4} \otimes u_1 + \frac{u_1}{v_4 v_1} \otimes \frac{u_1 - v_4 v_1}{u_1} + \text{cyclic}$$

Bootstrapping symbol of $F_4^{(2)}$

- Impose the following conditions:

1. first-entry is u_i, v_i
2. weight 4 integrable symbol
3. D_4 dihedral invariance
4. invariance under 5 different $\sqrt{} \to -\sqrt{}$
5. when 2 particles become collinear, $R_4^{(2)} \to R_3^{(2)}$

→ **A unique symbol!**

- Reminiscent of rigidity of 7-loop amplitude

  Drummond, Papathanasiou, Spradlin, 1412.3763

Checks of $F_4^{(2)}$

1. As the operator momentum becomes light-like, $k_{\mathcal{O}}^2 \to 0$, $R_4^{(2)}$ agrees with Guo, Wang, Yang, 2209.06816
2. In triple collinear limit, 4-gluon form factor becomes the triple-collinear splitting amplitude, which is also the 6-gluon amplitude(!):

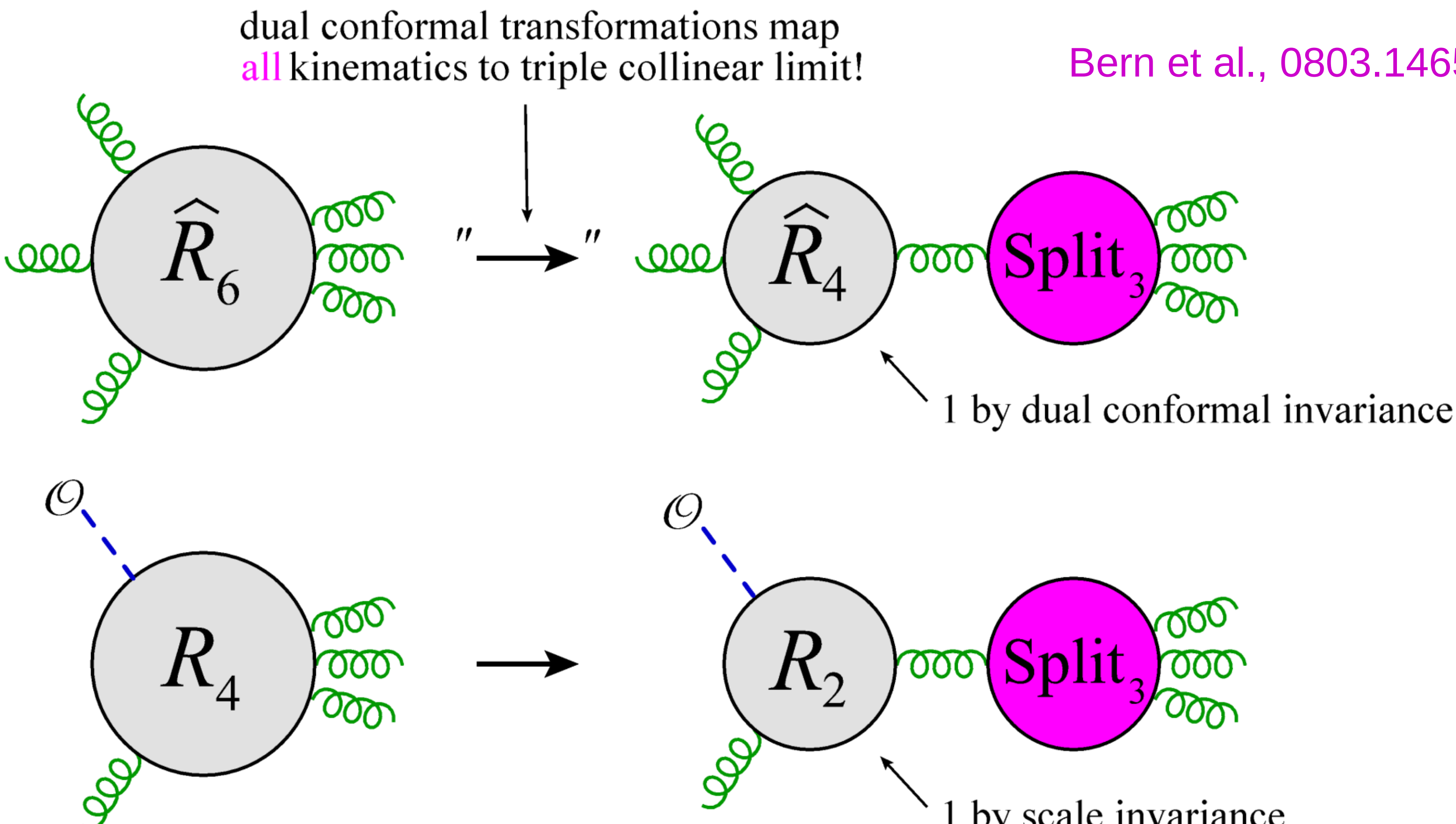
$$R_4^{(2)} \to \hat{R}_6^{(2)}$$

Bern et al., 0803.1465, or FFOPE

3. More generally, we check nontrivial FFOPE predictions Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

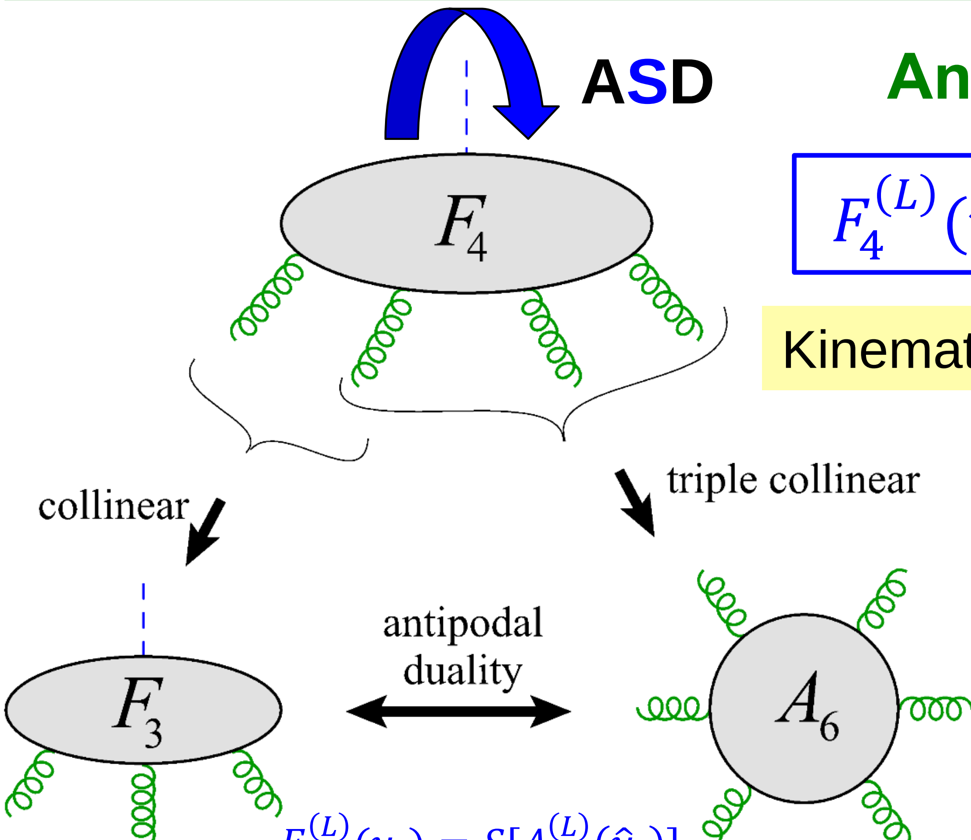
Why does form factor → amplitude in triple-collinear limit?

Bern et al., 0803.1465



Antipodal Self Duality

Given an antipodal duality relating 2-collinear and 3-collinear limits of F_4, it's natural to search for a self-duality of F_4 that holds for all parity-preserving bulk kinematics



$F_3^{(L)}(u_i) = S[A_6^{(L)}(\hat{u}_i)]$

And it's there!

$$F_4^{(L)}(u_i, v_i) = S[F_4^{(L)}(g(u_i), g(v_i))]$$

Kinematic map g simple in FFOPE variables:

$$g: \quad T_2 \to \frac{T}{S}, \quad S_2 \to \frac{1}{TS}$$

$$T \to \sqrt{\frac{T_2}{S_2}}, \qquad S \to \sqrt{\frac{1}{T_2 S_2}}$$

$$F_2 = 1$$

ASD beyond 2 loops

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 23mm.nnnnn

- Bootstrapped symbol of F_4 at 3 loops, using same 113 letter alphabet.
- We again find a **unique result**, which obeys all the FFOPE predictions we could check.
- 2 loop symbol uses only 34 of the letters [3,784 terms]
- 3 loop symbol uses only 88 of the letters [3,621,202 terms]
- **ASD holds at 3 loops!**
- 4 loops in progress

Summary & Open Questions

- 6-gluon amplitude and 3-gluon form factor in planar N=4 SYM are related by a strange new antipodal duality, swapping role of branch cuts and derivatives
- Embedded in a 4-gluon form factor self-duality!

- Who ordered that?
- Underlying physical reason for this duality?
- 3-dimensions seems to play a crucial role (parity preserving surfaces). Why?
- (How) does it hold at strong coupling?
- Where else might it hold? E.g. $\mathrm{tr}\phi^3$ operator form factors?
  Tumanov poster; Basso, Tumanov; Basso, LD, Tumanov to appear
- How much more can we exploit it to learn more about both amplitudes and form factors?

The End of This Conference

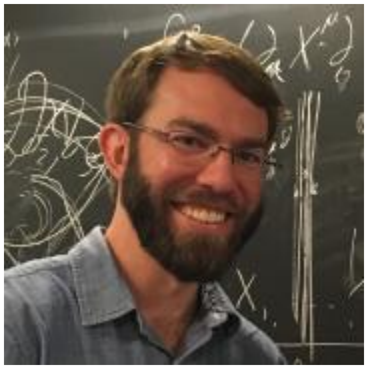
Many, many thanks!!!

To CERN-TH Secretariat

To CERN-TH

To CERN-IT

And especially to the Fab Four:



Andrew

Ben

Lorenzo

Samuel

See you all in Princeton next year!

Safe Travels!

Extra Slides

FFOPE kinematical variables for F_4

$$u_1 = \frac{T^2 T_2^2}{(T^2+1)(S^2+T^2+T_2^2+1)}$$

$$u_2 = \{1 + T^2 + \frac{S^2[(1+F_2^2)S_2T_2 + F_2(1+S_2^2+T^2+T_2^2)]}{F_2 S_2^2}\}^{-1}$$

$$u_3 = \frac{S^2}{(T^2+1)(S^2+T^2+T_2^2+1)}$$

$$u_4 = \frac{S^2T^2}{S_2^2} u_2$$

$$v_1 = \frac{T_2^2+1}{S^2+T^2+T_2^2+1}$$

- OPE limit takes $T, T_2 \to 0$, interpolates between 2-collinear limit $T_2 \to 0$ and 3-collinear limit $T \to 0$,

AD explains many patterns in F_3

- Every term in the symbol starts with a, b, c; never d, e, f
- Physical reason related to causality, which dictates where branch cuts can appear: only for $(p_i + p_j)^2 \sim 0$
- Empirically, 12 pairs of adjacent letters are forbidden:

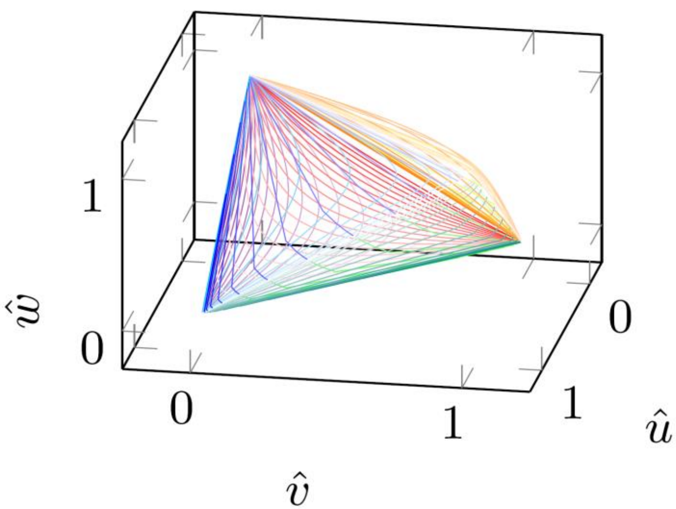
$$\ldots a \otimes d \ldots, \quad \ldots b \otimes e \ldots, \quad \ldots c \otimes f \ldots$$
$$\ldots d \otimes a \ldots, \quad \ldots e \otimes b \ldots, \quad \ldots f \otimes c \ldots$$
$$\ldots d \otimes e \ldots, \quad \ldots e \otimes f \ldots, \quad \ldots f \otimes d \ldots$$
$$\ldots e \otimes d \ldots, \quad \ldots f \otimes e \ldots, \quad \ldots d \otimes f \ldots$$

- Resemble constraints from causality: Steinmann relations Steinmann, Helv. Phys. Acta (1960)
- But not really, which mystified us for a while…
- However, the relations are antipodally dual to the (extended) Steinmann relations for A_6 !!

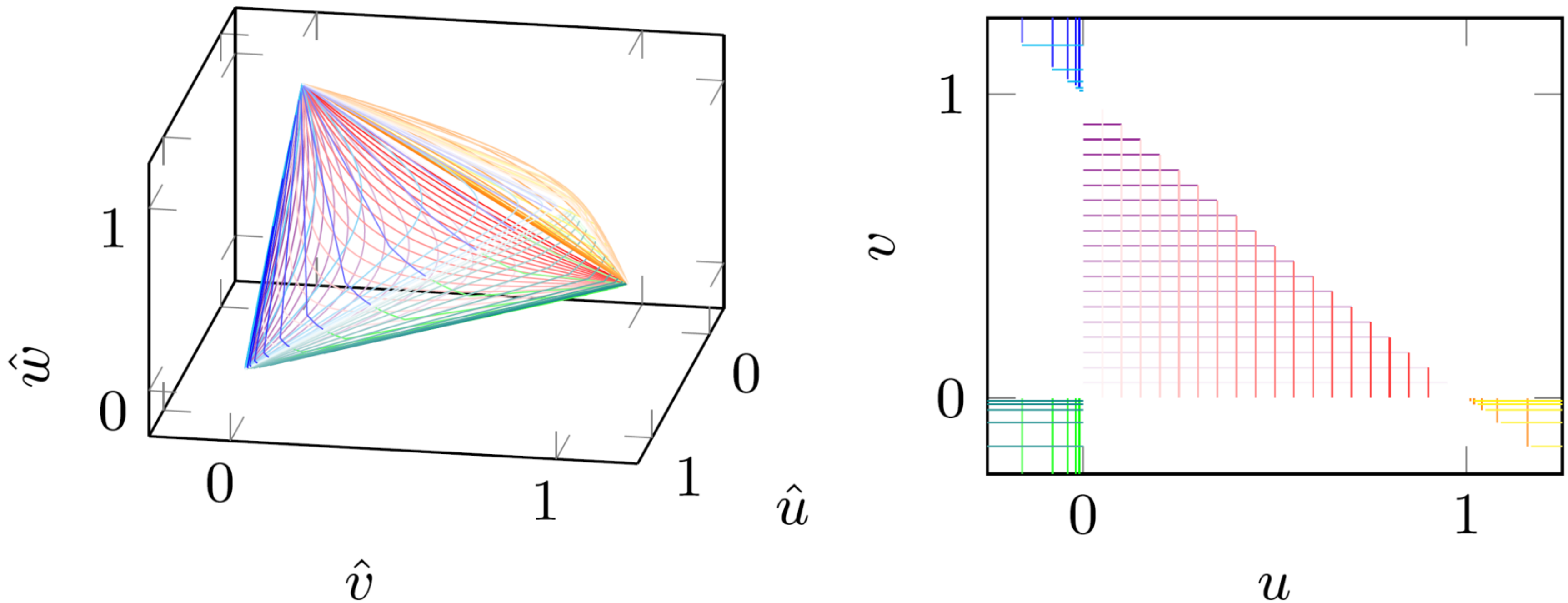
Exploit/test antipodal duality at 8 loops

LD, Y.-T. Liu, 2308.nnnnn

- Given form factor, antipodal duality determines symbol of MHV 6 gluon amplitude at 8 loops on $\Delta = 0$ surface.
- Lift symbol into bulk. Only 3 free parameters!
- 2 killed at origin, $(\hat{u}, \hat{v}, \hat{w}) \to (0,0,0)$
- last killed in process of lifting to full function level
- Need one OPE data point to kill one beyond-symbol ambiguity $\propto \zeta_8$



Antipodal kinematic map covers entire phase space for 3-gluon form factor



- Soft is dual to collinear; collinear is dual to soft
- White regions in (u, v) map to some of $\hat{u}, \hat{v}, \hat{w} > 1$

Values of HPLs {0,1} at $u = 1$

- Classical polylogs

evaluate to Riemann zeta values

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

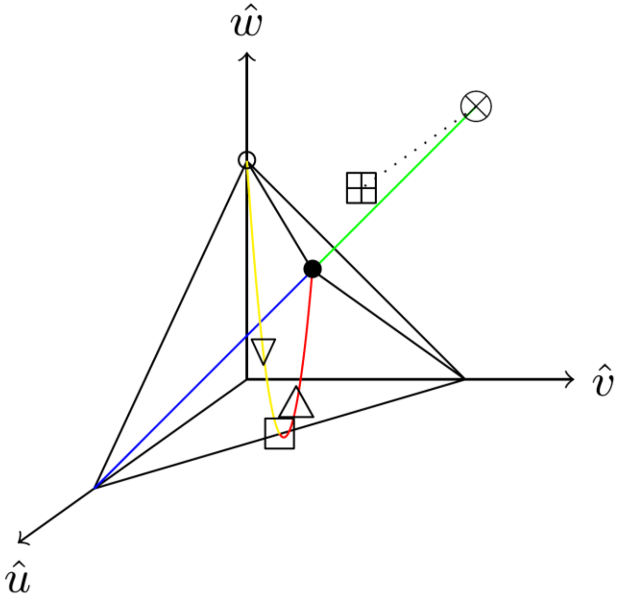
$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

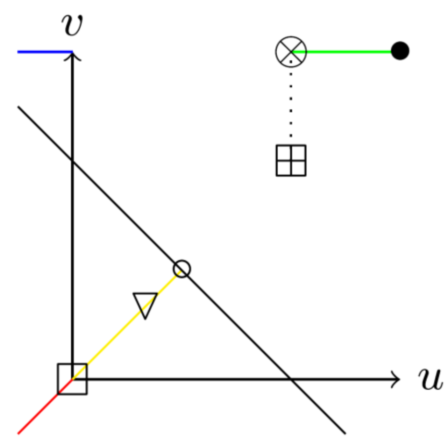
- All reducible to Riemann zeta values until weight 8.
  Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$

Many special dual points

There is an "f" alphabet at all these points: a way of writing multiple zeta values (MZV's) so that coaction is manifest.

F. Brown, 1102.1310; O. Schnetz, HyperlogProcedures





	$(\hat{u},\hat{v},\hat{w})$	(u,v,w)	functions
$\bigtriangledown$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\sqrt[6]{1}$
$\square$	$(\frac{1}{2},\frac{1}{2},0)$	$(0,0,1)$	$\mathrm{Li}_2(\frac{1}{2})$ + logs
$\bullet$	$(1,1,1)$	$\lim_{u\to\infty}(u,u,1-2u)$	MZVs
$\circ$	$(0,0,1)$	$(\frac{1}{2},\frac{1}{2},0)$	MZVs + logs
$\bigtriangleup$	$(\frac{3}{4},\frac{3}{4},\frac{1}{4})$	$(-1,-1,3)$	$\sqrt[6]{1}$
$\boxplus$	(∞,∞,∞)	$(1,1,-1)$	alternating sums
$\otimes$	$\lim_{\hat{v}\to\infty}(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(1,v,-v)$	MZVs
green–blue line	$(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(u,v,1-u-v)$	HPL$\{0,1\}$
red–yellow line	$(\hat{u},\hat{u},(1-2\hat{u})^2)$	$(u,u,1-2u)$	HPL$\{-1,0,1\}$

Simplest point

- $(\hat{u}, \hat{v}, \hat{w}) = (1,1,1) \Leftrightarrow u, v \to \infty$
- At this point,

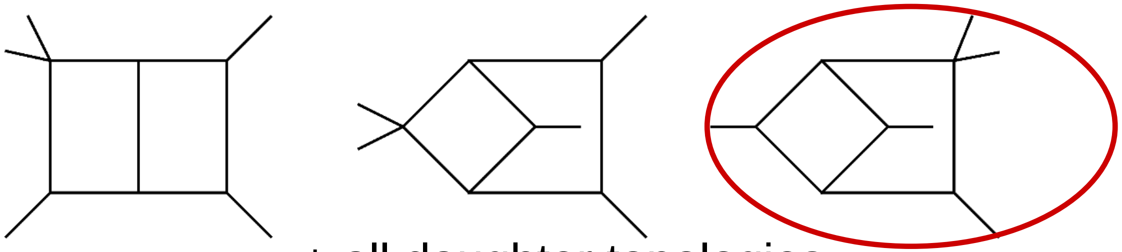
$$A_6^{(1)}(\cdot) = 0 \qquad F_3^{(1)}(\cdot) = 8\zeta_2$$
$$A_6^{(2)}(\cdot) = -9\zeta_4 \qquad F_3^{(2)}(\cdot) = 31\zeta_4$$
$$A_6^{(3)}(\cdot) = 121\zeta_6 \qquad F_3^{(3)}(\cdot) = -145\zeta_6$$
$$A_6^{(4)}(\cdot) = 120 f_{3,5} - 48\zeta_2 f_{3,3} - \frac{6381}{4}\zeta_8 \qquad F_3^{(4)}(\cdot) = 120 f_{5,3} + \frac{11363}{4}\zeta_8$$
$$A_6^{(5)}(\cdot) = -2688 f_{3,7} - 1560 f_{5,5} + \mathcal{O}(\pi^2) \qquad F_3^{(5)}(\cdot) = -2688 f_{7,3} - 1560 f_{5,5} + \mathcal{O}(\pi^2)$$
$$A_6^{(6)}(\cdot) = 48528 f_{3,9} + 37296 f_{5,7} + 21120 f_{7,5} + \mathcal{O}(\pi^2) \qquad F_3^{(6)}(\cdot) = 48528 f_{9,3} + 37296 f_{7,5} + 21120 f_{5,7} + \mathcal{O}(\pi^2)$$

- Reversing ordering of letters in *f*-alphabet, blue values show that antipodal duality holds beyond symbol level, modulo $i\pi$
- modulo $i\pi$ is best we can get from mathematical antipode map

Meaning for integrals?

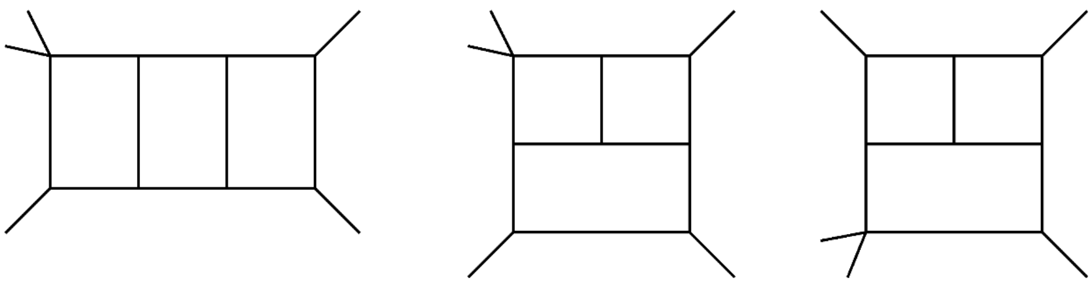
LD, McLeod, Wilhelm, 2012.12286; Chicherin, Henn, Papathanasiou, 2012.12285

Gehrmann, Remiddi, hep-ph/0008287, hep-ph/0101124



doesn't contribute to planar N=4 SYM form factor

+ all daughter topologies



DiVita, Mastrolia, Schubert, Yundin, 1408.3107

Canko, Syrrakos, 2112.14275 [some]

all have

~~... $d \otimes e$ ...~~

+ dihedral

half of the adjacency constraints seen in planar N=4 SYM

Why?

(No Steinmann relations for massless 2-particle cuts…)

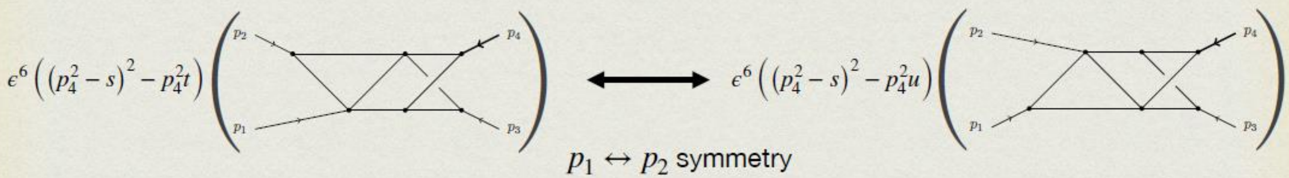
Other 3 loop integrals have new letters

New letters in the alphabet

Henn, Lim, Torres Bobadilla, 2302.12776

[Henn, Lim, WJT (2023)]

$$\vec{\alpha} = \{\alpha_0, \ldots, \alpha_8\} = \left\{p_4^2, s, t, -p_4^2+s+t, -p_4^2+s, -p_4^2+t, s+t, -\left(p_4^2-s\right)^2+p_4^2 t, s^2-p_4^2(s-t)\right\}.$$



Start appearing at weight 4

$$\mathcal{S}\left(f_{B1}^{41}\right)\Big|_{\epsilon^4} = 6\left[\alpha_1 \otimes \alpha_1 \otimes \frac{\alpha_2}{\alpha_4} \otimes \alpha_7 - \alpha_1 \otimes \alpha_1 \otimes \alpha_4 \otimes \alpha_7 + \alpha_1 \otimes \frac{\alpha_4}{\alpha_2} \otimes \frac{\alpha_3}{\alpha_1\alpha_4} \otimes \alpha_7 + \alpha_2 \otimes \alpha_1 \otimes \frac{\alpha_1\alpha_4}{\alpha_3} \otimes \alpha_7 + \alpha_2 \otimes \alpha_5 \otimes \frac{\alpha_3}{\alpha_1} - \frac{1}{2}\alpha_2 \otimes \alpha_5 \otimes \alpha_2 \otimes \alpha_7 + \ldots\right]$$

William J. Torres Bobadilla talk at LoopFest 2023 13

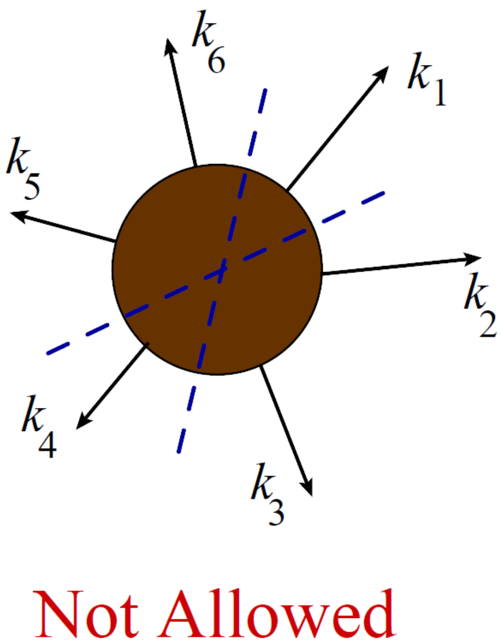
Other integrals lose the adjacency conditions

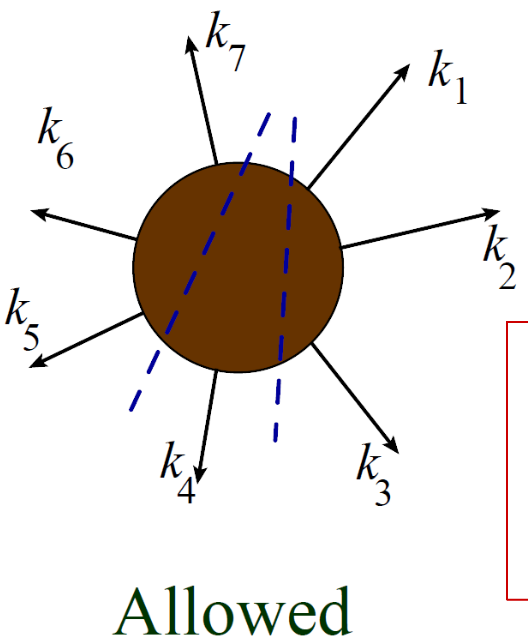
~~$a \otimes d$ …, … $d \otimes e$ …~~

???

Steinmann relations

- Amplitudes should not have overlapping branch cuts:





can't apply to 2 particle cuts in massless case because they are not independent

$$\mathrm{Disc}_{s_{234}} \mathrm{Disc}_{s_{123}} A_6(\hat{u}, \hat{v}, \widehat{w}) = 0$$

if you remove IR divergences properly

Steinmann + DCI consequences

$$\hat{u} = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad \hat{v} = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad \hat{w} = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$ are not ideal,

so switch to $\hat{a} \equiv \frac{\hat{u}}{\hat{v}\hat{w}} = (s_{234}^2)^2 \times \left[s_{i,i+1}\, stuff\right]$

$$\hat{b} \equiv \frac{\hat{v}}{\hat{w}\hat{u}}, \qquad \hat{c} \equiv \frac{\hat{w}}{\hat{u}\hat{v}}$$

$$\boxed{\mathrm{Disc}_{\hat{b}}\, \mathrm{Disc}_{\hat{a}}\, A_6(\hat{u},\hat{v},\hat{w}) = 0}$$

Should hold on any Riemann sheet (?)

Discontinuities via symbol

- Discontinuities commute with derivatives; discontinuities act on left entry of symbol, while derivatives act on right

$$\mathcal{S}[\mathrm{Disc}_{\hat{a}} F] = 2\pi i \, \hat{a} \otimes \ldots$$

- $\mathrm{Disc}_{\hat{b}} \mathrm{Disc}_{\hat{a}} A_6(\hat{u}, \hat{v}, \hat{w}) = 0$ (+ dihedral images)

means $\mathcal{S}[A_6]$ cannot contain any terms of the form $\hat{a} \otimes \hat{b} \otimes \ldots$

- But we actually find more generally, for any adjacent slots,

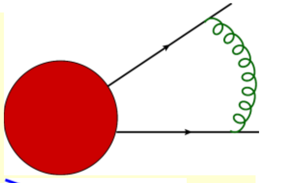
$$\ldots \otimes \hat{a} \otimes \hat{b} \otimes \ldots$$

Caron-Huot, LD, McLeod, von Hippel, Papathanasiou, 1806.01361, 1906.07116

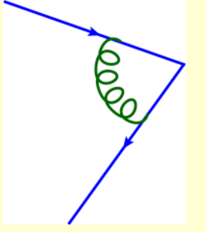
- "Extended Steinmann relations".
- With first entry condition, also find $\ldots \otimes \hat{a} \otimes \hat{d} \otimes \ldots$
- equivalent to "cluster adjacency" for $A_3 = \mathrm{Gr}(4,6)$ cluster algebra
Drummond, Foster, Gürdoğan, 1710.10953

Removing Amplitude (or Form Factor) Infrared Divergences

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\Gamma_{\rm cusp}$



– known to all orders in planar N=4 SYM:
Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

$$\mathcal{E}_6(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\rm BDS-like}(s_{i,i+1},\epsilon)} = \exp\left[\frac{\Gamma_{\rm cusp}}{4}\mathcal{E}_6^{(1)} + R_6\right]$$

remainder function

BDS & BDS-like normalization for $\mathcal{F}_3$

$$\frac{\mathcal{F}_3}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\mathrm{cusp}}^{(L)}}{4} + \mathcal{O}(\epsilon)\right)M^{1-\mathrm{loop}}(L\epsilon) + C^{(L)} + R^{(L)}(u,v,w)\right]\right\}$$

BDS ansatz

remainder function only a function of u, v, w; vanishes in all collinear limits, but no adjac... y constraints

split 1-loop amplitude judiciously:

$$\frac{\mathcal{F}_3^{1-\mathrm{loop}}}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} \equiv M^{1-\mathrm{loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u,v,w)$$

$$M(\epsilon) = -\frac{1}{\epsilon^2}\sum_{i=1}^{3}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} - \frac{7}{2}\zeta_2 + \sum^{3} \ldots$$

$$\mathcal{E}^{(1)}(u,v \ldots \ldots \left(1-\frac{1}{v}\right) + \mathrm{Li}_2\left(1-\frac{1}{w}\right)\Big]$$

$$\mathcal{E}^{(1),u} + \mathcal{E}^{(1),1-u} = 0$$

$\mathcal{E}$ obeys "adjacency constraints"

Now div... ...y:

$$\frac{\mathcal{F}_3^{\mathrm{BDS-like}}}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\mathrm{cusp}}}{4} + \mathcal{O}(\epsilon)\right)M(L\epsilon) + C^{(L)}\right]\right\} \Rightarrow$$

$$\mathcal{E} = \exp\left[\frac{\Gamma_{\mathrm{cusp}}}{4}\mathcal{E}^{(1)} + R\right]$$

Number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts ($2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L=1$	1	3	1														
$L=2$	1	3	6	3	1												
$L=3$	1	3	9	12	6	3	1										
$L=4$	1	3	9	21	24	12	6	3	1								
$L=5$	1	3	9	21	46	45	24	12	6	3	1						
$L=6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L=7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L=8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- Properly normalized L loop N=4 form factors $\mathcal{E}^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $\mathcal{E}^{(L)}$ also obeys multiple-final-entry relations, saturation on right