LCSRs in Rare *B* Decays

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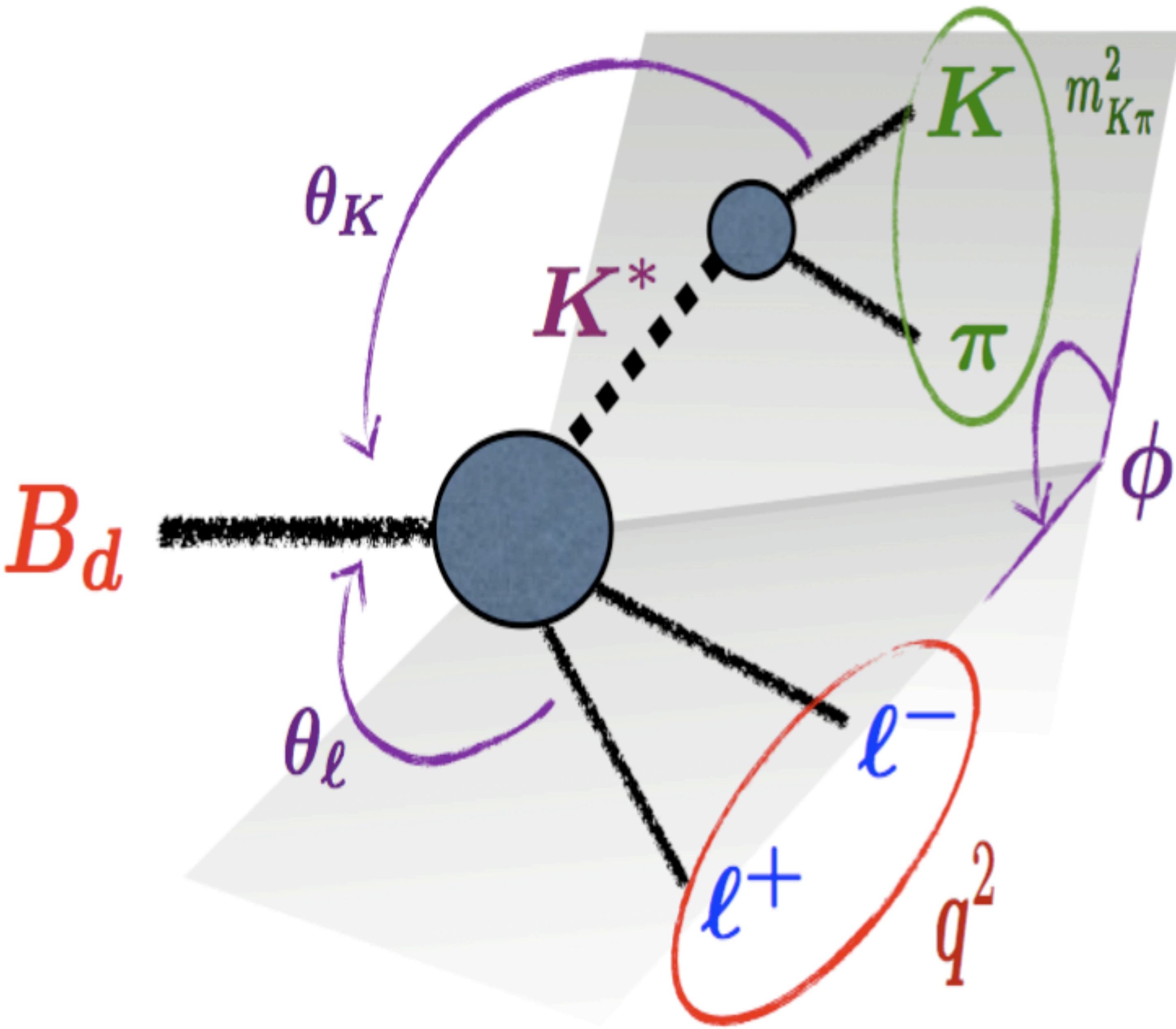
θ_K
K
$m^2_{K\pi}$
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literature collaboration LHCb and not t detector and not t design

Observation of $J/\psi p$ Resonances Consistent with Pentaquark States in $\Lambda_b^0 \to J/\psi K^- p$ Decays #1

LHCb Collaboration • Roel Aaij (CERN) et al. (Jul 13, 2015)

Published in: *Phys.Rev.Lett.* 115 (2015) 072001 • e-Print: 1507.03414 [hep-ex]

pdf links DOI cite claim reference search 1,545 citations

Test of lepton universality using $B^+ \to K^+\ell^+\ell^-$ decays #2

LHCb Collaboration • Roel Aaij (NIKHEF, Amsterdam) et al. (Jun 25, 2014)

Published in: *Phys.Rev.Lett.* 113 (2014) 151601 • e-Print: 1406.6482 [hep-ex]

pdf DOI cite claim reference search 1,297 citations

Test of lepton universality with $B^0 \to K^{*0}\ell^+\ell^-$ decays #3

LHCb Collaboration • R. Aaij (CERN) et al. (May 16, 2017)

Published in: *JHEP* 08 (2017) 055 • e-Print: 1705.05802 [hep-ex]

pdf links DOI cite datasets claim reference search 1,220 citations

Measurement of the ratio of branching fractions $\mathcal{B}(\bar{B}^0 \to D^{*+}\tau^-\bar{\nu}_\tau)/\mathcal{B}(\bar{B}^0 \to D^{*+}\mu^-\bar{\nu}_\mu)$ #4

LHCb Collaboration • Roel Aaij (CERN) et al. (Jun 29, 2015)

Published in: *Phys.Rev.Lett.* 115 (2015) 11, 111803, *Phys.Rev.Lett.* 115 (2015) 15, 159901 (erratum) • e-Print: 1506.08614 [hep-ex]

pdf links DOI cite claim reference search 1,147 citations

Angular analysis of the $B^0 \to K^{*0}\mu^+\mu^-$ decay using 3 fb^{-1} of integrated luminosity #5

LHCb Collaboration • Roel Aaij (CERN) et al. (Dec 14, 2015)

Published in: *JHEP* 02 (2016) 104 • e-Print: 1512.04442 [hep-ex]

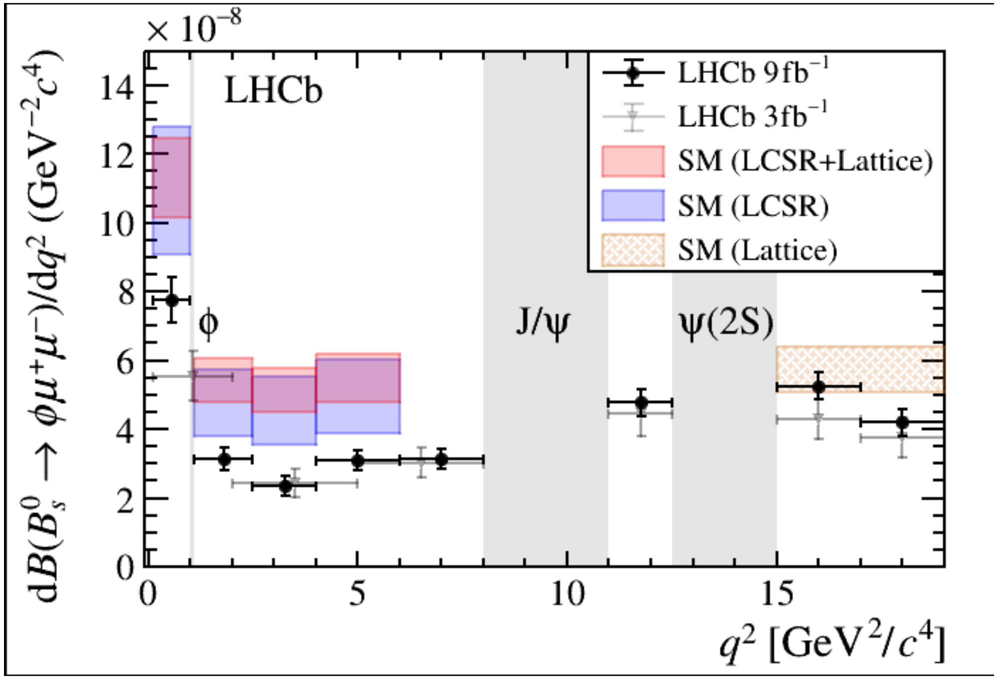
pdf links DOI cite datasets claim reference search 901 citations

Measurement of Form-Factor-Independent Observables in the Decay $B^0 \to K^{*0}\mu^+\mu^-$ #6

LHCb Collaboration • R Aaij (NIKHEF, Amsterdam) et al. (Aug 7, 2013)

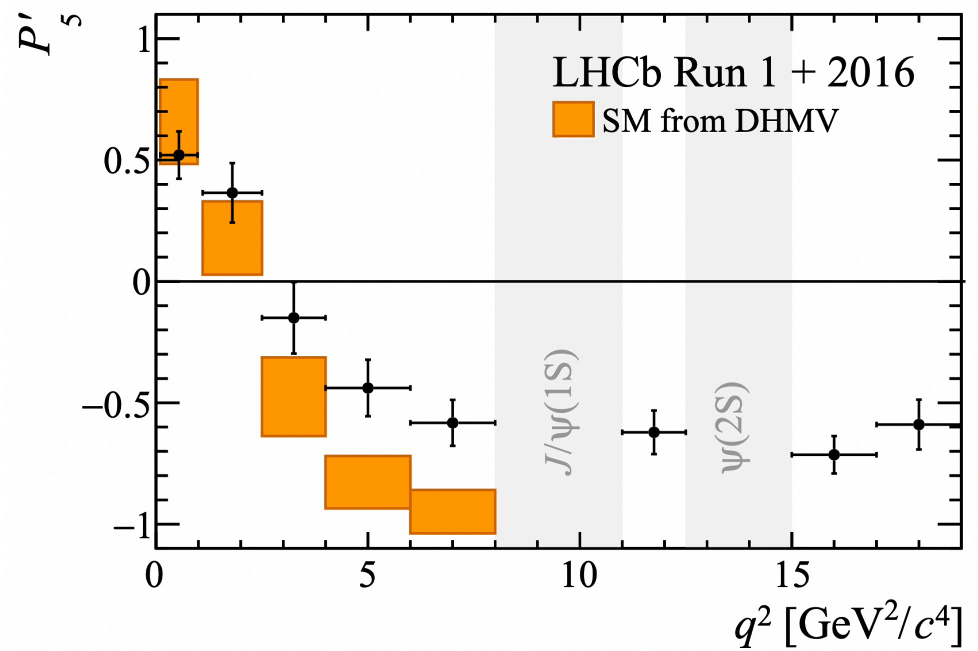
Published in: *Phys.Rev.Lett.* 111 (2013) 191801 • e-Print: 1308.1707 [hep-ex]

pdf DOI cite claim reference search 738 citations

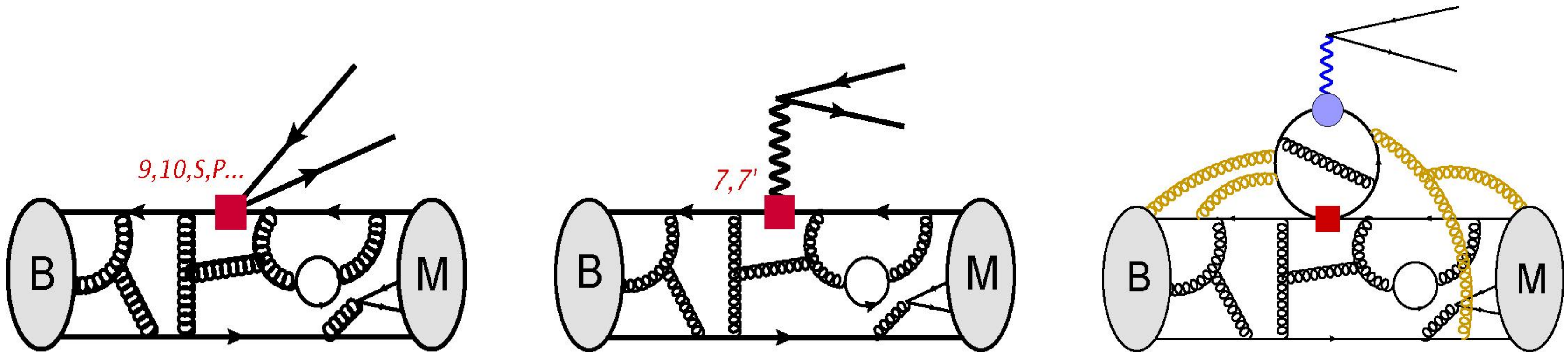


Branching fractions

Angular observables



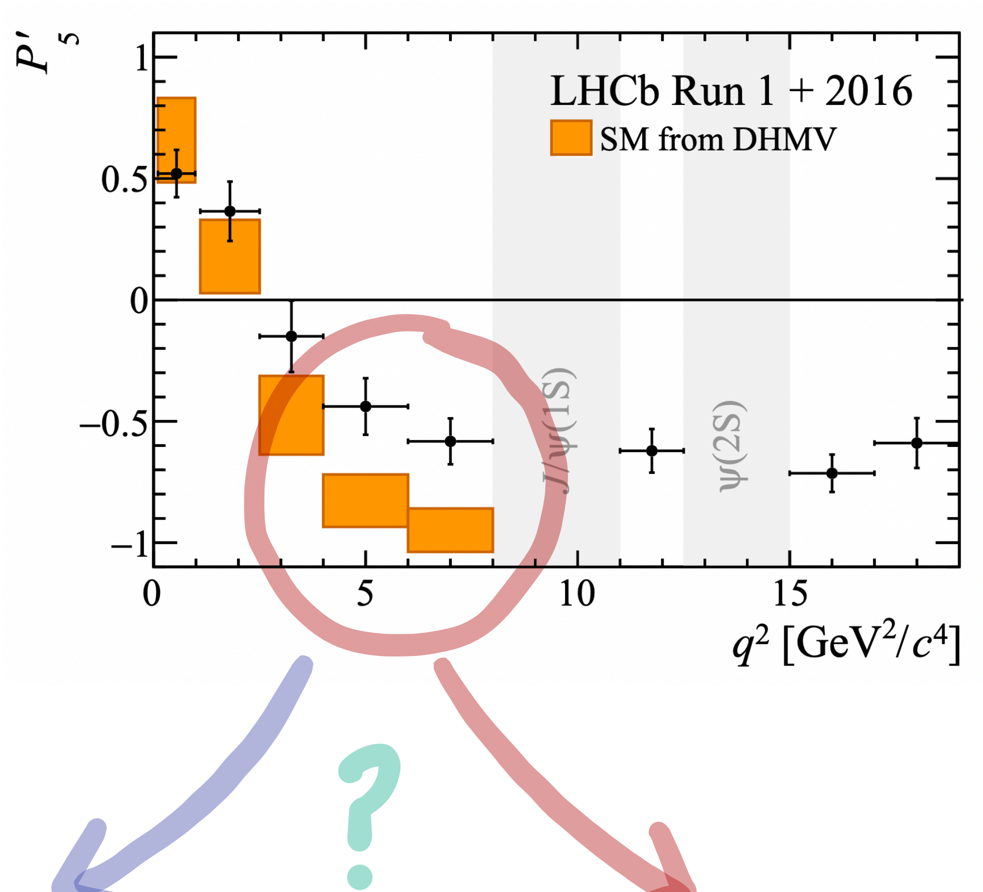
Anatomy of $B \to M_\lambda \ell^+ \ell^-$ EFT Amplitudes



$$\mathcal{A}_\lambda^{L,R} = \mathcal{N}_\lambda \left\{ (C_9 \mp C_{10}) \mathcal{F}_\lambda(q^2) + \frac{2 m_b M_B}{q^2} \left[C_7 \mathcal{F}_\lambda^T(q^2) - 16\pi^2 \frac{M_B}{m_b} \mathcal{H}_\lambda(q^2) \right] \right\} + \mathcal{O}(\alpha^2)$$

- Local (Form Factors) : $\mathcal{F}_\lambda^{(T)}(q^2) = \langle \bar{M}_\lambda(k) | \, \bar{s}\, \Gamma_\lambda^{(T)} \, b \, | \bar{B}(k+q) \rangle$
- Non-Local : $\mathcal{H}_\lambda(q^2) = i\, \mathcal{P}_\mu^\lambda \int d^4x \, e^{iq\cdot x} \, \langle \bar{M}_\lambda(k) | T\{ j^\mu_{\rm em}(x), \mathcal{C}_i \, \mathcal{O}_i(0) \} | \bar{B}(q+k) \rangle$

The BSM/QCD dichotomy



$C_i \neq C_i^{SM}$ (BSM)

$\langle \mathcal{O}_i \rangle \neq \langle \mathcal{O}_i \rangle^{\text{calculated}}$ (QCD)

Here we are discussing the problem of calculating

- Local Form Factors :

$$\mathcal{F}_\lambda^{(T)}(q^2) = \langle \bar{M}_\lambda(k)|\, \bar{s}\, \Gamma_\lambda^{(T)}\, b\, |\bar{B}(k+q)\rangle$$

- Non-Local Form Factors:

$$\mathcal{H}^\mu(q^2) = i \int d^4x\, e^{iq\cdot x}\, \langle \bar{M}_\lambda(k)| T\{ j^\mu_{\rm em}(x), [\bar{b}_L \gamma^\nu c_L][\bar{c}_L \gamma_\nu s](0)\}|\bar{B}(q+k)\rangle$$

In **both cases** the *“modern” strategy* is

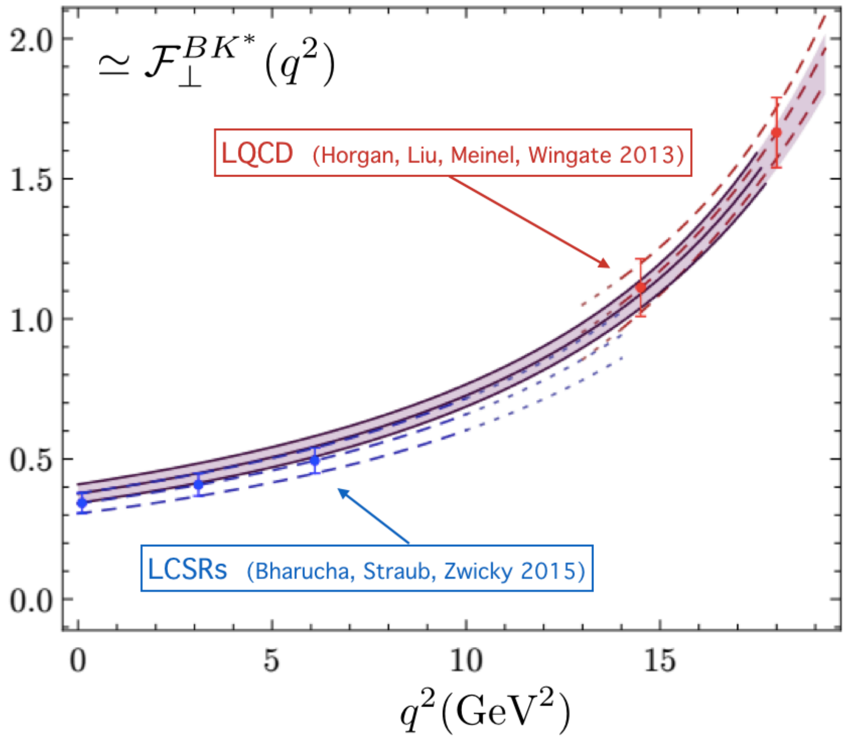
- **Calculate/extract** the form factors in optimal/feasible kinematic regions (**not** necessarily physical or the regions we are interested in)
  $\longrightarrow$ **“data”**

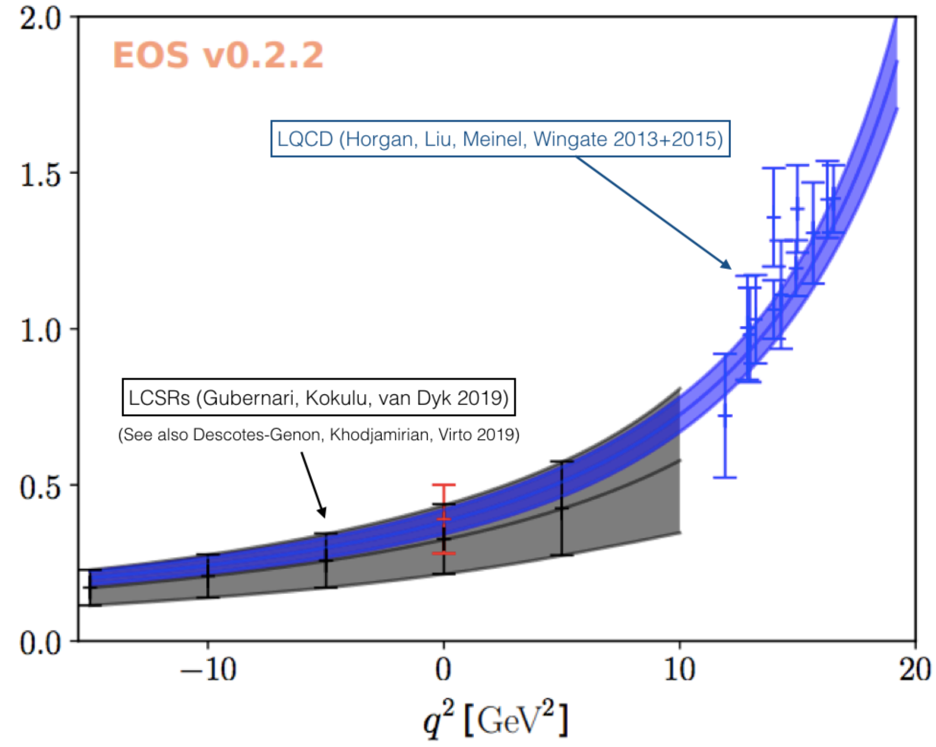
- **Parametrize** q^2 dependence by means of a rigorous **analytic expansion**

- **Fit** the (truncated) parametrization to the **“data”**

- Control the truncation error by means of a **dispersive bound**.

Local Form Factors



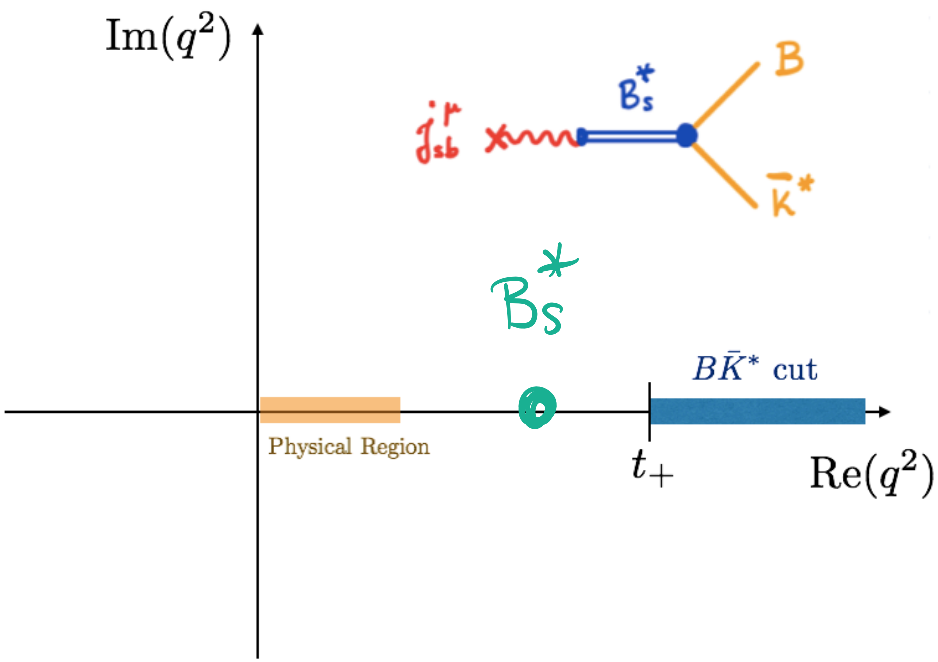


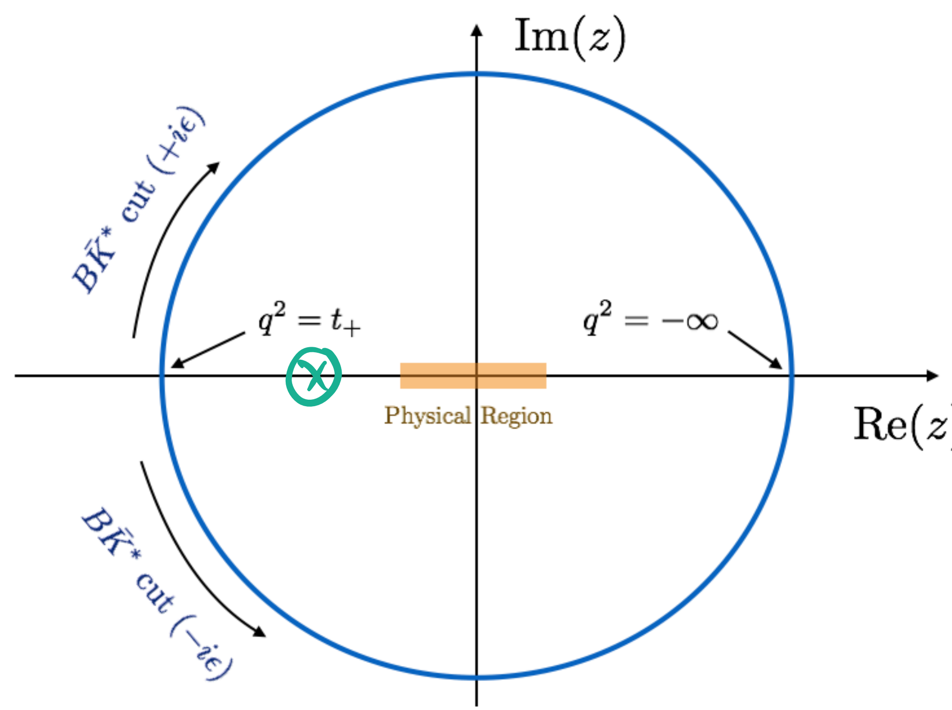
- Two main approaches: (1) Lattice QCD (large q^2 ***) (2) LCSRs (low q^2)
- Two approaches to LCSRs, in terms of (1) K^* LCDAs (2) B LCDAs
- q^2 dependence parametrized via a (dispersively-bounded) z-expansion

Form Factors : q^2-dependence from analyticity

Bourrely, Caprini, Lellouch; Boyd, Grinstein, Lebed; Caprini, Lellouch, Neubert; ...

- Conformal mapping : $$z(q^2) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$$





- "z-parametrization" : $\hat{\mathcal{F}}_\lambda^{(T)}(q^2(z))$ is **analytic in** $|z| < 1$ $\quad (|z_{\rm phys}| < 0.15)$

$$\mathcal{F}_\lambda^{(T)}(q^2) = \frac{1}{(q^2 - m^2_{B_s^*})} \sum_k \alpha_k\, z(q^2)^k$$

Form Factors : Dispersive Bounds (BGL)

Boyd, Grinstein, Lebed 1997; Bharucha, Feldmann, Wick 2014

1. One starts with the two-point function

$$\Pi_\Gamma^{\mu\nu}(q) \equiv i\int d^4x\, e^{iq\cdot x}\langle 0|T\{J_\Gamma^\mu(x)J_\Gamma^{\dagger,\nu}(0)\}|0\rangle = \Pi_\Gamma^{(J=0)}(q^2)\left[\frac{q^\mu q^\nu}{q^2}\right] + \Pi_\Gamma^{(J=1)}(q^2)\left[g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right]$$

2. The invariant functions fulfil a once-subtracted dispersion relation:

$$\chi_\Gamma^{(\lambda)}(Q^2) = \left[\frac{\partial}{\partial q^2}\right]\Pi_\Gamma^{(\lambda)}(q^2)\Big|_{q^2=Q^2} = \frac{1}{\pi}\int_0^\infty ds\,\frac{\mathrm{Im}\Pi_\Gamma^{(\lambda)}(s)}{(s-Q^2)^2}\,.$$

3. The function $\chi_\Gamma^{(\lambda)}(Q^2)$ can be calculated in an OPE at a suitable substraction point Q^2
Bharucha, Feldmann, Wick 2014

4. The discontinuity of $\Pi_\Gamma^{(\lambda)}(q^2)$ is the spectral function:

$$\mathrm{Im}\Pi_\Gamma^{(\lambda)}(s) \sim \sum_H \langle 0|J^\mu|H\rangle\langle H|J^{\nu\dagger}|0\rangle \sim f_{B_s^*}^2 + |F^{BK}|^2 + |F^{BK^*}|^2 + |F^{B_s\phi}|^2 + \cdots$$

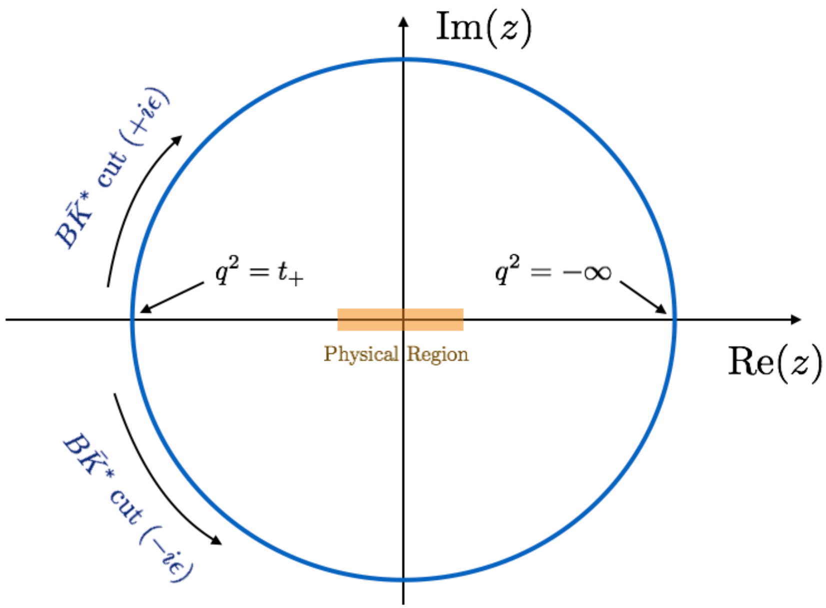
(up to phase-space functions...)

The two-body contributions are, e.g.

$$\chi_A^{(J=1)}\Big|_{BK^*} = \frac{\eta^{B\to K^*}}{24\pi^2}\int_{(M_B+M_{K^*})^2}^{\infty} ds \frac{\lambda_{\rm kin}^{1/2}}{s^2(s-Q^2)^3}\Big[s\,(M_B+M_{K^*})^2 \boxed{|A_1^{B\to K^*}|^2} + 32\, M_B^2 M_{K^*}^2 \boxed{|A_{12}^{B\to K^*}|^2}\Big]$$

In order to simplify the bound, it is thus convenient to reparametrize:

$$\hat{\mathcal{F}}_\lambda^{B\to M}(q^2) = \mathcal{B}_{\mathcal{F}}(z)\phi_{\mathcal{F}}(z)\mathcal{F}_\lambda^{B\to M}(q^2) = \sum_k \alpha_k^{\mathcal{F}}\, z(q^2)^k$$



$$\sum_{B\to M}\int_{-\pi}^{+\pi} d\theta \left|\hat{\mathcal{F}}_\lambda^{B\to M}(e^{i\theta})\right|^2 < 1$$

$$\boxed{\sum_{\mathcal{F},k} |\alpha_k^{\mathcal{F}}|^2 < 1}$$

Local Form Factors: 2 variations

Gubernari, Reboud, van Dyk, Virto 2023

1. "Polarized" 2-point function decomposition

$$\Pi_\Gamma^{\mu\nu}(q) = \sum_{\lambda=t,\perp,\|,0} \epsilon_\lambda^{\mu}\epsilon_\lambda^{\nu*}\, \Pi_\Gamma^{(\lambda)}(q^2)$$

- This is the bound used in the literature:

$$\chi_A^{(J=1)}\big|_{BK^*} = \frac{\eta^{B\to K^*}}{24\pi^2}\int_{(M_B+M_{K^*})^2}^{\infty} ds \frac{\lambda_{\rm kin}^{1/2}}{s^2(s-Q^2)^3}\left[s\,(M_B+M_{K^*})^2 \boxed{|A_1^{B\to K^*}|^2} + 32\, M_B^2 M_{K^*}^2 \boxed{|A_{12}^{B\to K^*}|^2}\right]$$

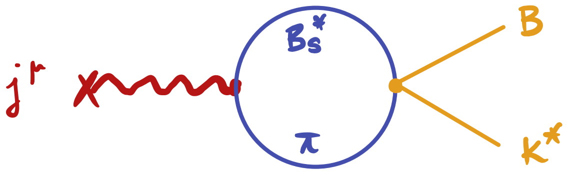
- And this is what we propose:

$$\chi_A^{(0)}\big|_{\bar B K^*} = \frac{\eta^{B\to K^*}}{\pi^2}\int_{(M_B+M_{K^*})^2}^{\infty} ds \frac{\lambda_{\rm kin}^{1/2}}{s^2(s-Q^2)^3} 4\, M_B^2 M_{K^*}^2 \boxed{|A_{12}^{B\to K^*}|^2},$$

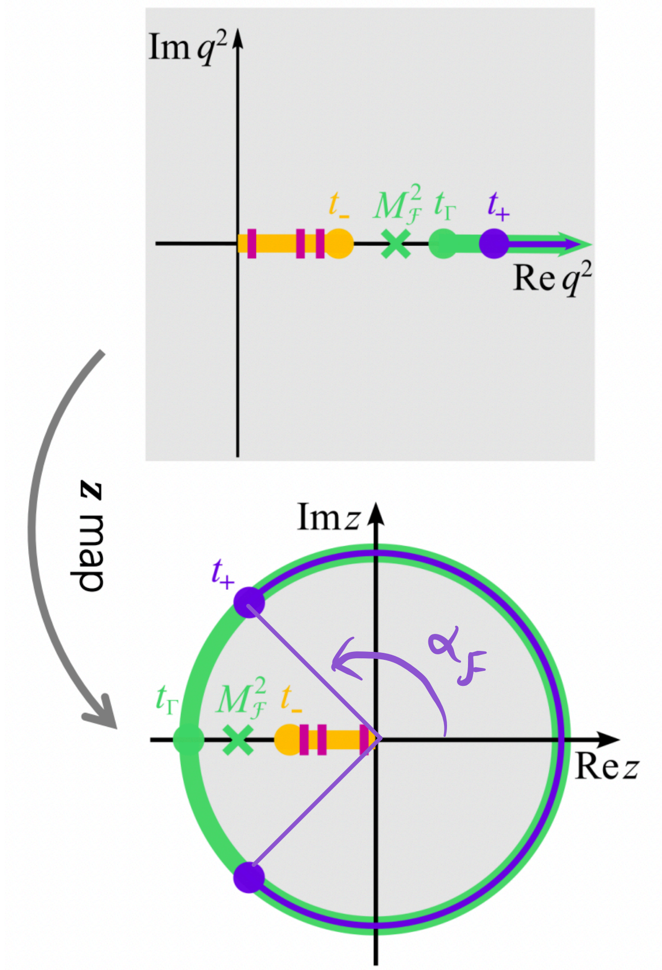
$$\chi_A^{(\|)}\big|_{\bar B K^*} = \frac{\eta^{B\to K^*}}{8\pi^2}\int_{(M_B+M_{K^*})^2}^{\infty} ds \frac{\lambda_{\rm kin}^{1/2}}{s^2(s-Q^2)^3} s\,(M_B+M_{K^*})^2 \boxed{|A_1^{B\to K^*}|^2},$$

Flynn, Jüttner, Tsang 2023; Talk by J.Flynn @ this workshop

Gubernari, Reboud, van Dyk, Virto 2023



2. Correct threshold, different from trivial one:



$$\hat{\mathcal{F}}_\lambda^{B\to M}(q^2) = \mathcal{B}_\mathcal{F}(z)\phi_\mathcal{F}(z)\mathcal{F}_\lambda^{B\to M}(q^2) = \sum_k \alpha_k^\mathcal{F}\, p_k^\mathcal{F}(z)$$

$$\int_{-\alpha_\mathcal{F}}^{+\alpha_\mathcal{F}} d\theta\, p_m^\mathcal{F}(e^{i\theta}) p_n^\mathcal{F}(e^{-i\theta}) = \delta_{mn}$$

$$\sum_{B\to M} \int_{-\alpha_\mathcal{F}}^{+\alpha_\mathcal{F}} d\theta \left| \hat{\mathcal{F}}_\lambda^{B\to M}(e^{i\theta}) \right|^2 < 1$$

$$\boxed{\sum_{\mathcal{F},k} |\alpha_k^\mathcal{F}|^2 < 1}$$

Flynn, Jüttner, Tsang 2023; Talk by J.Flynn @ this workshop

Gubernari, Reboud, van Dyk, Virto 2023

FOR FIT SEE TALK BY MÉRIL REBOUD

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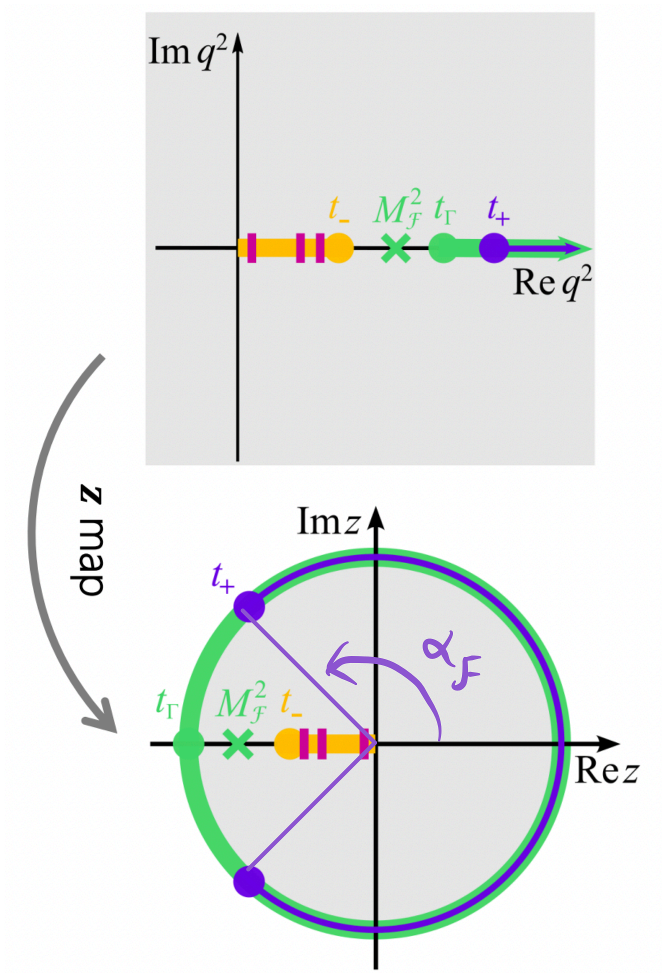


$$\hat{\mathcal{F}}_\lambda^{B\to M}(q^2) = \mathcal{B}_{\mathcal{F}}(z)\phi_{\mathcal{F}}(z)\mathcal{F}_\lambda^{B\to M}(q^2) = \sum_k \alpha_k^{\mathcal{F}}\, p_k^{\mathcal{F}}(z)$$

$$\int_{-\alpha_{\mathcal{F}}}^{+\alpha_{\mathcal{F}}} d\theta\, p_m^{\mathcal{F}}(e^{i\theta}) p_n^{\mathcal{F}}(e^{-i\theta}) = \delta_{mn}$$

$$\sum_{B\to M} \int_{-\alpha_{\mathcal{F}}}^{+\alpha_{\mathcal{F}}} d\theta \left|\hat{\mathcal{F}}_\lambda^{B\to M}(e^{i\theta})\right|^2 < 1$$

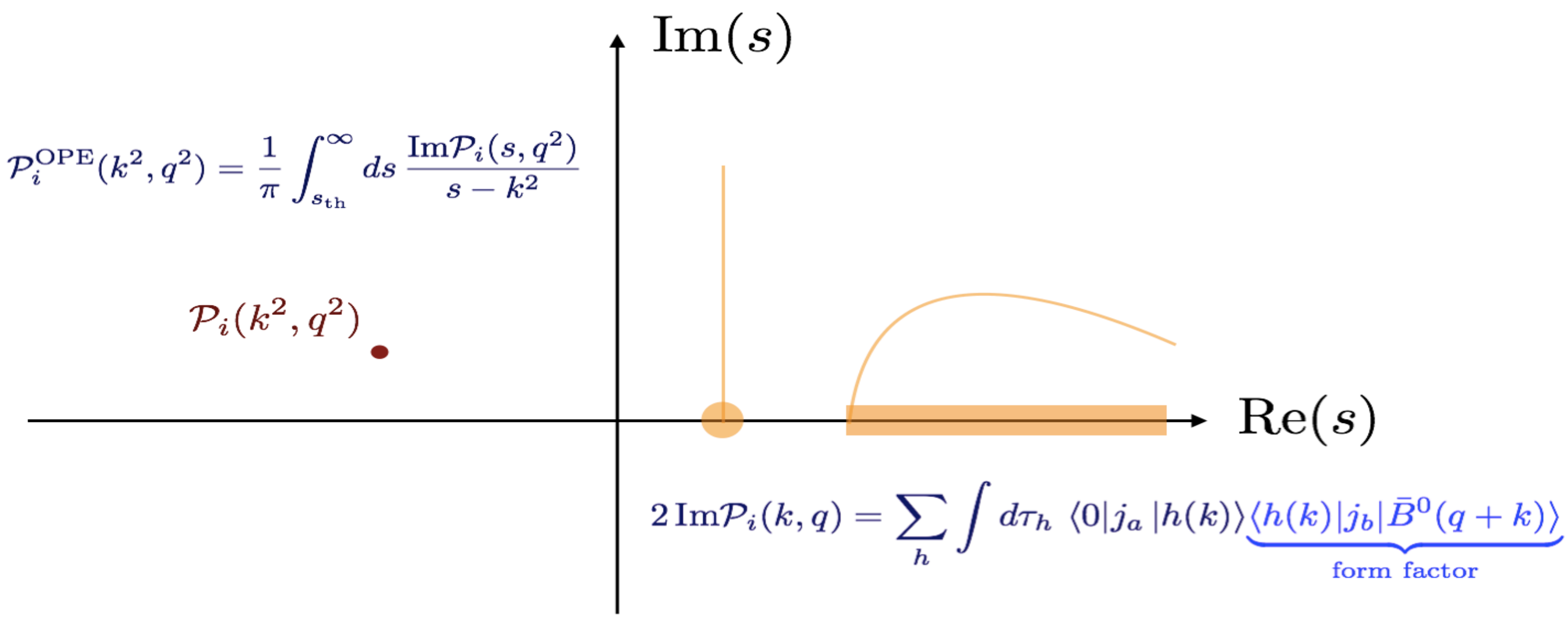
$$\boxed{\sum_{\mathcal{F},k} |\alpha_k^{\mathcal{F}}|^2 < 1}$$

Light-Cone Sum Rules

Light-Cone Sum Rules with B-meson LCDAs

Khodjamirian, Mannel, Offen 2006 [Unitarity+Analyticity+Duality]

Consider a correlation function: $$\mathcal{P}_{ab}(k,q) = i\int d^4x\, e^{ik\cdot x}\langle 0|\mathrm{T}\{j_a(x), j_b(0)\}|\bar{B}^0(q+k)\rangle$$



$$h(k) = K^* + continuum \;\Rightarrow\; 2\,\mathrm{Im}\mathcal{P}_{ab}(k,q) \sim f_K^*\, F^{BK*}\, \delta(k^2 - m_{K^*}) + duality(s_0)$$

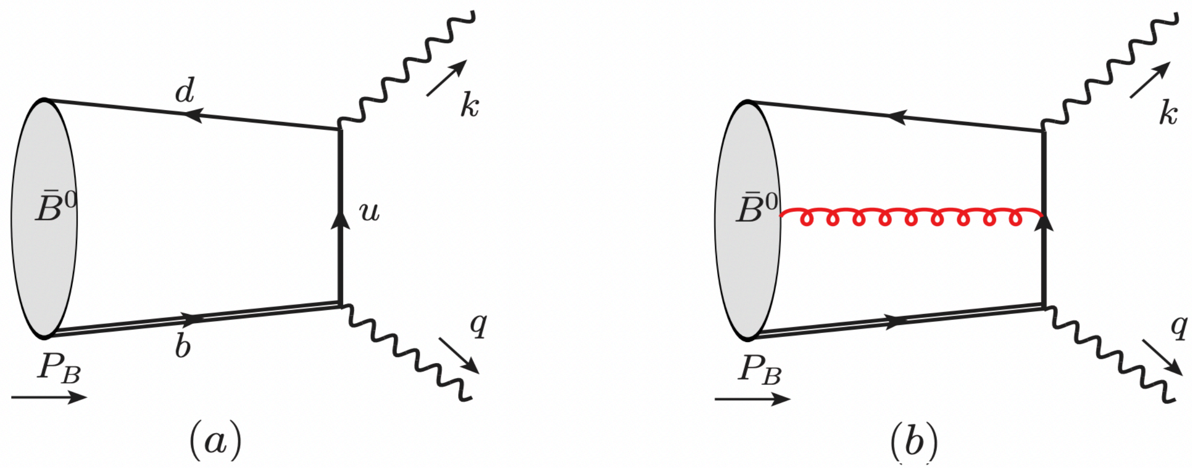
$$F^{BK*}(q^2) = \frac{1}{f_{K^*}\, m_{K^*}} e^{m_{K^*}^2/M^2} \cdot \mathcal{P}^{\mathrm{OPE}}(q^2, s_0, M^2)$$

Light-Cone Sum Rules with B-meson LCDAs: OPE

Kolulu, Gubernari, van Dyk 2018 [Unitarity+Analyticity+Duality]

Descotes-Genon, Khodjamirian, Virto 2019

Consider a correlation function: $\mathcal{P}_{ab}(k,q) = i \int d^4x\, e^{ik\cdot x} \langle 0 | \mathrm{T}\{j_a(x), j_b(0)\} | \bar{B}^0(q+k) \rangle$



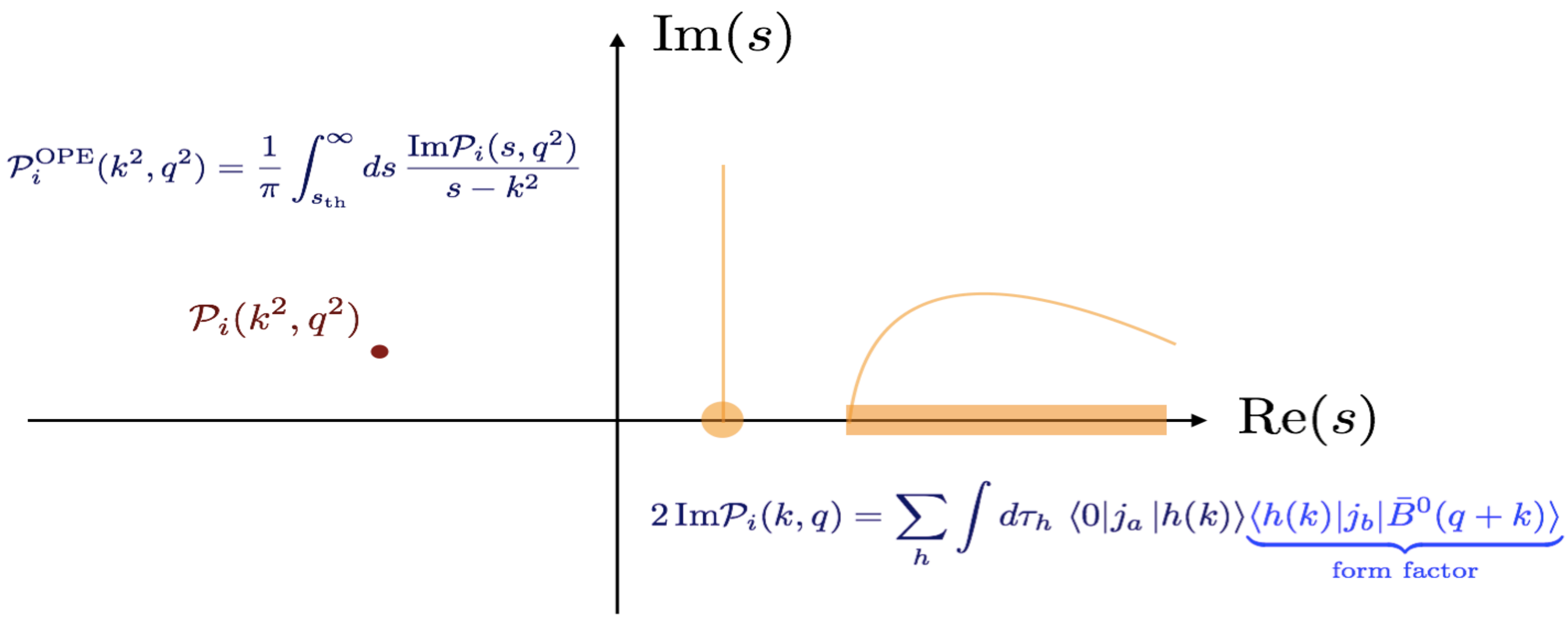
$$\mathcal{P}^{\mathrm{OPE}}(q^2, s_0, M^2) = \sum_{n\geq 0} \frac{f_B m_B}{(M^2)^n} \int_0^{s_0} ds\, e^{-s/M^2} \mathcal{G}_n(s)$$

New calculation includes DAs up to twist-4 Braun, Ji, Manashov 2017

Form Factors: beyond the Narrow Width limit

Cheng, Khodjamirian, Virto 2017; Descotes-Genon, Khodjamirian, Virto 2019

Consider a correlation function: $$\mathcal{P}_{ab}(k,q) = i\int d^4x\, e^{ik\cdot x}\langle 0|\mathrm{T}\{j_a(x), j_b(0)\}|\bar{B}^0(q+k)\rangle$$

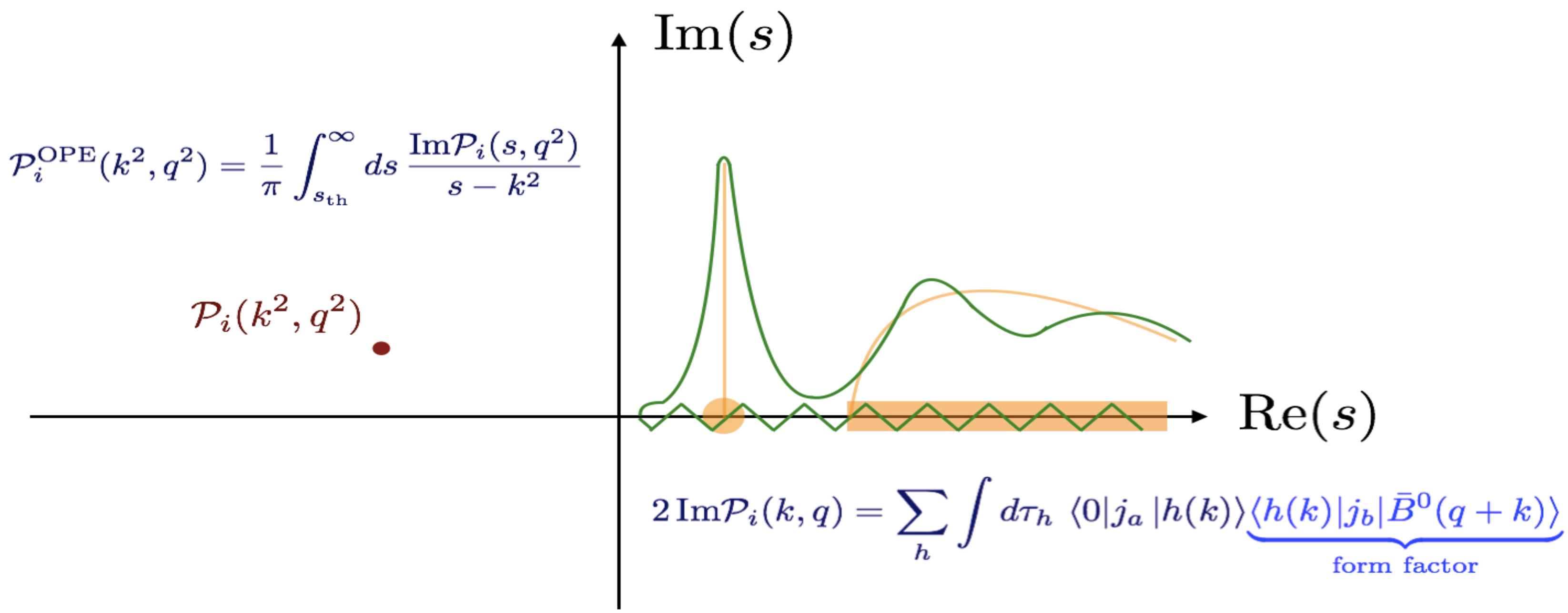


- Traditionally, $h(k) = K^* + continuum \;\Rightarrow\; 2\,\mathrm{Im}\mathcal{P}_{ab}(k,q) \sim f_K^*\, F^{BK*}\,\delta(k^2 - m_{K*}) + \cdots$

Form Factors: beyond the Narrow Width limit

Cheng, Khodjamirian, Virto 2017; Descotes-Genon, Khodjamirian, Virto 2019

Consider a correlation function: $\mathcal{P}_{ab}(k,q) = i\int d^4x\, e^{ik\cdot x}\langle 0|\mathrm{T}\{j_a(x), j_b(0)\}|\bar{B}^0(q+k)\rangle$



- Traditionally, $h(k) = K^* + \mathit{continuum} \;\Rightarrow\; 2\,\mathrm{Im}\mathcal{P}_{ab}(k,q) \sim f_K^* F^{BK*}\delta(k^2 - m_{K^*}) + \cdots$
- Generalization for unstable mesons Cheng, Khodjamirian, Virto 2017 : $h(k) = K\pi + \cdots$

LCSRs with B-meson DAs, natural for this generalization.

P-wave Projector

$$\mathcal{P}_i(k,q) = i \int d^4x\, e^{ik\cdot x} \langle 0 | \mathrm{T}\{\underbrace{\bar{d}(x)\gamma^\mu s(x)}, j_i(0)\} | \bar{B}^0(q+k)\rangle$$

$$\boxed{\int_{s_{\rm th}}^{s_0} ds\, e^{-s/M^2}\ \ \omega_i(s,q^2)\ \ f_+^\star(s)\ \ F_i^{(\ell=1)}(s,q^2) = \mathcal{P}_i^{\rm OPE}(q^2,\sigma_0,M^2)}$$

- s_0 – Effective threshold
- $\omega_i(s,q^2)$ – (known) kinematic factors
- $\langle K^-(k_1)\pi^+(k_2)|\bar{s}\gamma_\mu d|0\rangle = f_+(k^2)\,\bar{k}_\mu + \frac{m_K^2 - m_\pi^2}{k^2} f_0(k^2)\,k_\mu$
- $\mathcal{P}_i^{\rm OPE}$ – OPE result for the correlation function

Descotes-Genon, Khodjamirian, Virto 2019

LCSRs for P-wave $B \to K\pi$ Form Factors

$$\int_{s_{\rm th}}^{s_0} ds\, e^{-s/M^2}\ \omega_i(s,q^2)\ f_+^\star(s)\ F_i^{(\ell=1)}(s,q^2) = \mathcal{P}_i^{\rm OPE}(q^2,\sigma_0,M^2)$$

- Generalize LCSRs in Khodjamirian, Mannel, Offen 2006 beyond the K^*, including LCSRs for A_0, $T_{2,3}$
- Recalculate $\mathcal{P}_i^{\rm OPE}$ including 3-particle contributions, and extended consistently to twist-4 accuracy. Full (numerical) agreement with Gubernari,Kokulu,van Dyk 2018 (not input parameters)
- Revisit s_0 $\Rightarrow$ significantly lower value!! – f_{K^*} is derived quantity
- Study of Narrow-width limit, Finite-Width effects, and effects beyond the K^*
- Applications to $B \to K\pi\ell\ell$

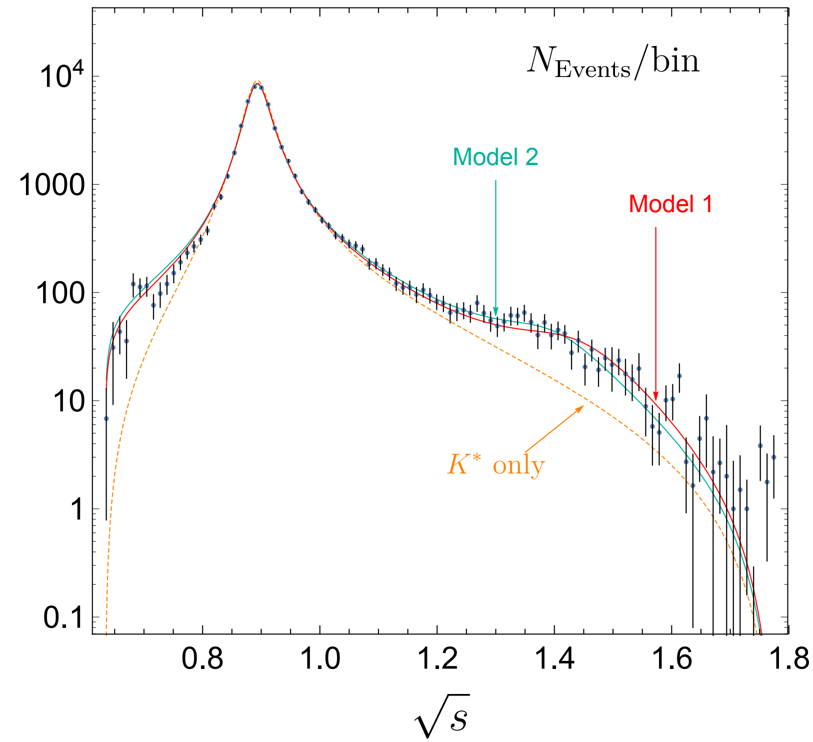
Descotes-Genon, Khodjamirian, Virto 2019

$$\int_{s_{\rm th}}^{s_0} ds\, e^{-s/M^2}\ \omega_i(s,q^2)\ f_+^\star(s)\ F_i^{(\ell=1)}(s,q^2) = \mathcal{P}_i^{\rm OPE}(q^2,\sigma_0,M^2)$$

$$f_+^\star(s)\ F_i^{(\ell=1)}(s,q^2) \longrightarrow f_K^*\ \mathcal{F}_{R,i}(q^2)\ \delta(s-m_K^*)$$

$$f_+(s) = -\sum_R \frac{m_R\, f_R\, g_{RK\pi}\ e^{i\phi_R(s)}}{m_R^2 - s - i\sqrt{s}\,\Gamma_R(s)}$$

$$F_i^{(\ell=1)}(s,q^2) = \sum_R \frac{Y_{R,i}(s,q^2)\, g_{RK\pi}\ \mathcal{F}_{R,i}(q^2)\, e^{i\phi_R(s)}}{m_R^2 - s - i\sqrt{s}\,\Gamma_R(s)}$$



Descotes-Genon, Khodjamirian, Virto 2019

LCSRs for P-wave $B \to K\pi$ Form Factors: Narrow-Width Limit

		Khodjamirian Mannel Offen 2006	KMPW 2010	Gubernari Kokulu van Dyk 2018	BSZ 2015
Form Factor	This work	Ref. [12]	Ref. [24]	Ref. [15]	Ref. [17]
$\mathcal{F}_{K^*,\perp}(0) = V^{BK^*}(0)$	0.26(15)	0.39(11)	0.36(18)	0.32(11)	0.34(4)
$\mathcal{F}_{K^*,\parallel}(0) = A_1^{BK^*}(0)$	0.20(12)	0.30(8)	0.25(13)	0.26(8)	0.27(3)
$\mathcal{F}_{K^*,-}(0) = A_2^{BK^*}(0)$	0.14(13)	0.26(8)	0.23(15)	0.24(9)	0.23(5)
$\mathcal{F}_{K^*,t}(0) = A_0^{BK^*}(0)$	0.30(7)	–	0.29(8)	0.31(7)	0.36(5)
$\mathcal{F}^T_{K^*,\perp}(0) = T_1^{BK^*}(0)$	0.22(13)	0.33(10)	0.31(14)	0.29(10)	0.28(3)
$\mathcal{F}^T_{K^*,\parallel}(0) = T_2^{BK^*}(0)$	0.22(13)	0.33(10)	0.31(14)	0.29(10)	0.28(3)
$\mathcal{F}^T_{K^*,-}(0) = T_3^{BK^*}(0)$	0.13(12)	–	0.22(14)	0.20(8)	0.18(3)

Table 6: *Results for the form factors at $q^2 = 0$ in the narrow-width limit, compared to corresponding results in the literature. The approach in Ref. [17] is a completely different LCSR approach, in terms of K^* DAs.*

Descotes-Genon, Khodjamirian, Virto 2019

LCSRs for P-wave $B \to K\pi$ Form Factors: Narrow-Width Limit

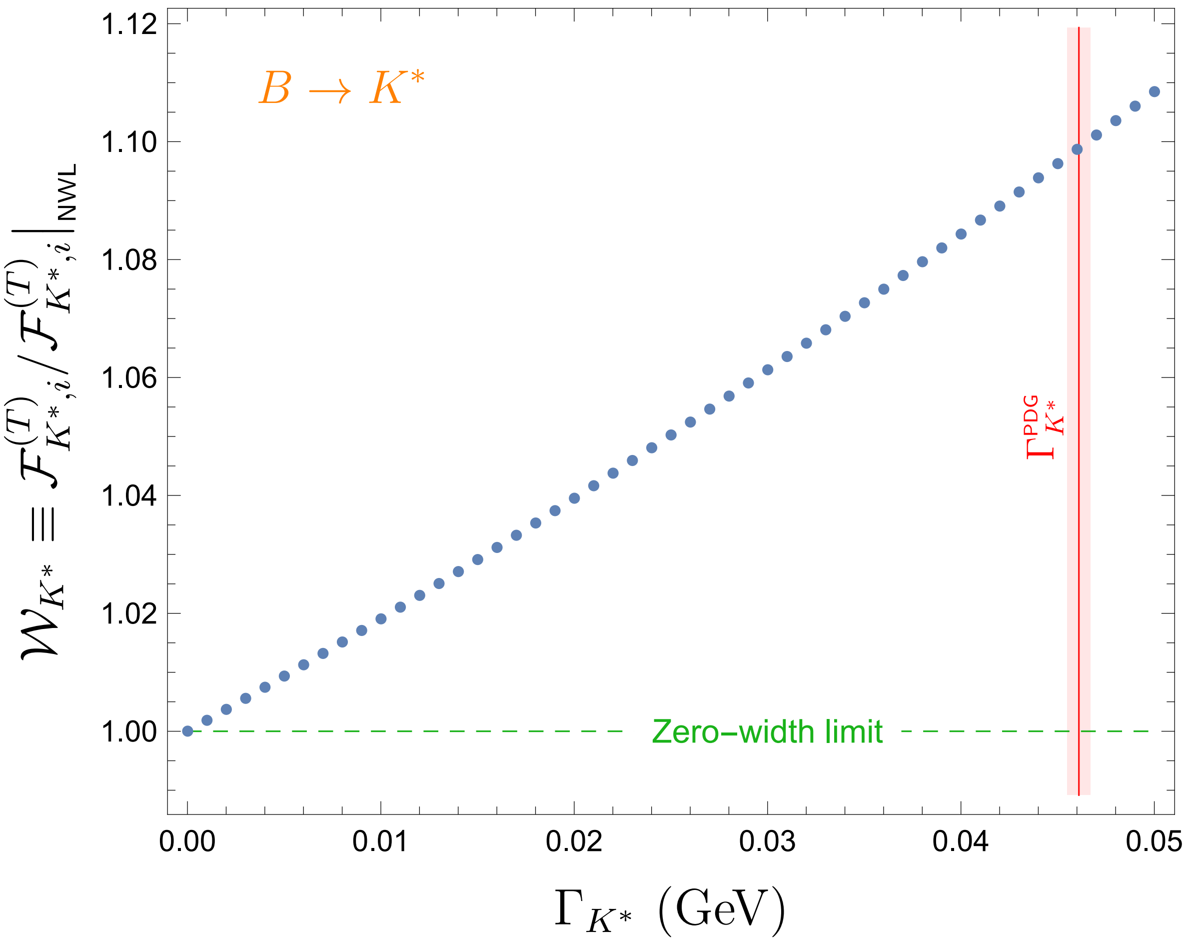
$\mathcal{F}^{BK^*}(q^2=0)$	V^{BK^*}	$A_1^{BK^*}$	$A_2^{BK^*}$	$A_0^{BK^*}$	$T_{1,2}^{BK^*}$	$T_3^{BK^*}$
Ref. [12]	0.39	0.30	0.26	–	0.33	–
Inputs [12], no g_+	0.38	0.29	0.26	0.31	0.33	0.25
Inputs [12], with g_+	0.27	0.21	0.14	0.31	0.24	0.14
Our inputs, but $s_0 = 1.7\,\mathrm{GeV}^2$	0.33	0.26	0.17	0.38	0.29	0.17
Our inputs, our s_0, no g_+	0.36	0.28	0.25	0.30	0.31	0.23
Our inputs, our s_0, with g_+	0.26	0.20	0.14	0.30	0.22	0.13

Table 7: *Deconstruction of the different effects explaining the difference between our results for the form factors at $q^2 = 0$ and those in Ref. [12]. The difference stems mainly from the inclusion of the twist-four two-particle contributions. See the text for more details.*

2pt Twist-4 LCDA

Descotes-Genon, Khodjamirian, Virto 2019

Finite-width effects



$$\mathcal{W}_{K^*} \simeq 1 + 1.9\,\frac{\Gamma_{K^*}}{m_{K^*}}$$

$$\mathcal{W}_{K^*} = 1.09 \pm 0.01$$

$\Rightarrow$ BRs are corrected by a factor $|\mathcal{W}_{K^*}|^2 \simeq 1.2$

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Beyond the $K^*(892)$

Consider the sum rule with $R = \{K^*(892), K^*(1410)\}$:

$$\sum_R \mathcal{F}_{R,i}^{(T)}(q^2)\, d_{R,i}^{(T)}\, I_R(s_0, M^2) = \mathcal{P}_i^{(T),\mathrm{OPE}}(q^2, \sigma_0, M^2)$$

		$M^2 = 1.00\,\mathrm{GeV}^2$	$M^2 = 1.25\,\mathrm{GeV}^2$	$M^2 = 1.50\,\mathrm{GeV}^2$
Model 1	$I_{K^*(892)}$	0.1506(23)	0.1781(16)	0.1992(13)
	$I_{K^*(1410)}$	0.0050(07)	0.0062(07)	0.0072(06)
Model 2	$I_{K^*(892)}$	0.1491(22)	0.1766(20)	0.1975(16)
	$I_{K^*(1410)}$	0.0048(07)	0.0061(06)	0.0070(06)

Table 8: *Values for the quantities I_R for $R = \{K^*(892), K^*(1410)\}$ for the different values of the Borel parameter M^2 and for the two models for the $K\pi$ form factor. The $K^*(1410)$ contribution is very suppressed in the sum rules, with $I_{K^*(1410)}/I_{K^*(892)} \simeq 0.03$ in all cases.*

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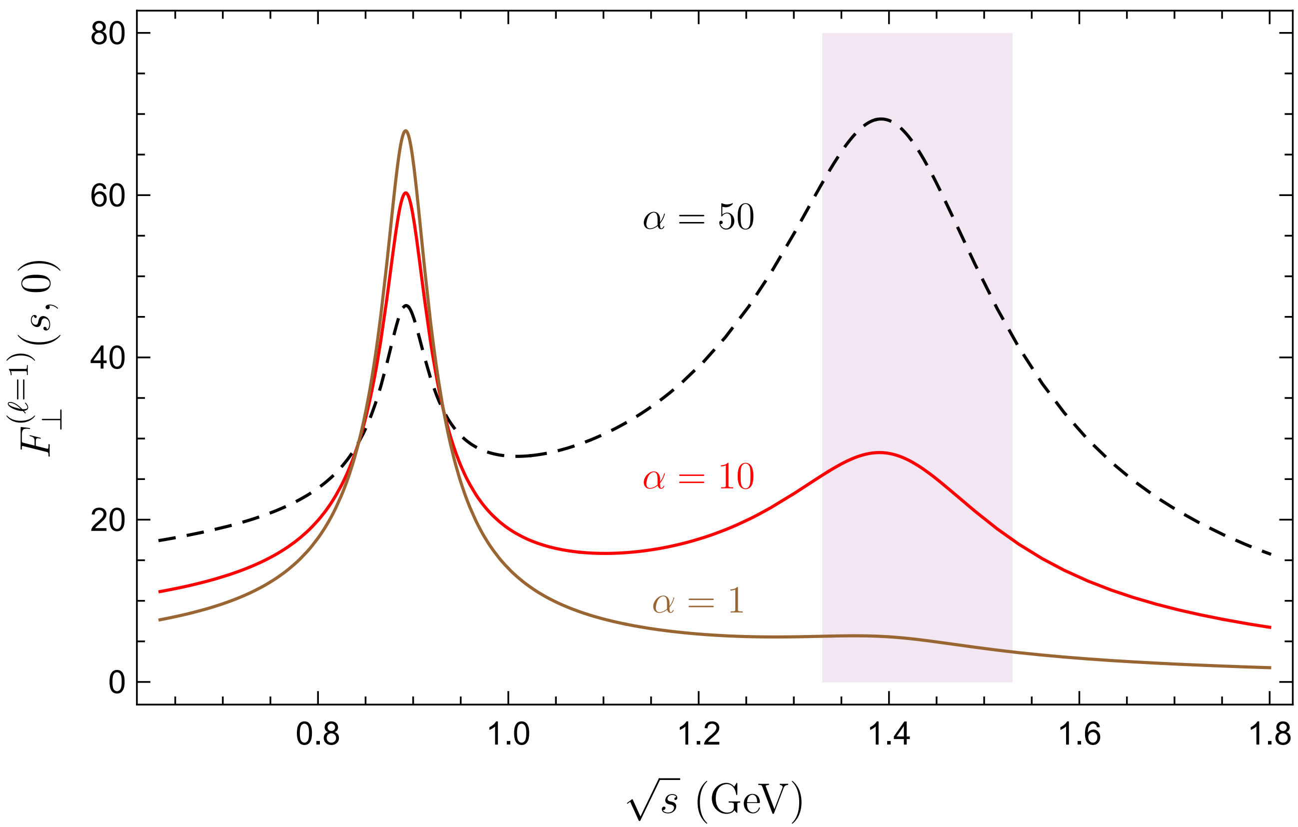
Beyond the $K^*(892)$

Set $\mathcal{F}_{K^*(1410)} = \alpha\, \mathcal{F}_{K^*(892)}$ with α a floating parameter



$\alpha = 1: \ \mathcal{F}_{K^*,\perp}(0) = 0.28\,; \quad \alpha = 10: \ \mathcal{F}_{K^*,\perp}(0) = 0.22\,; \quad \alpha = 50: \ \mathcal{F}_{K^*,\perp}(0) = 0.11\,.$

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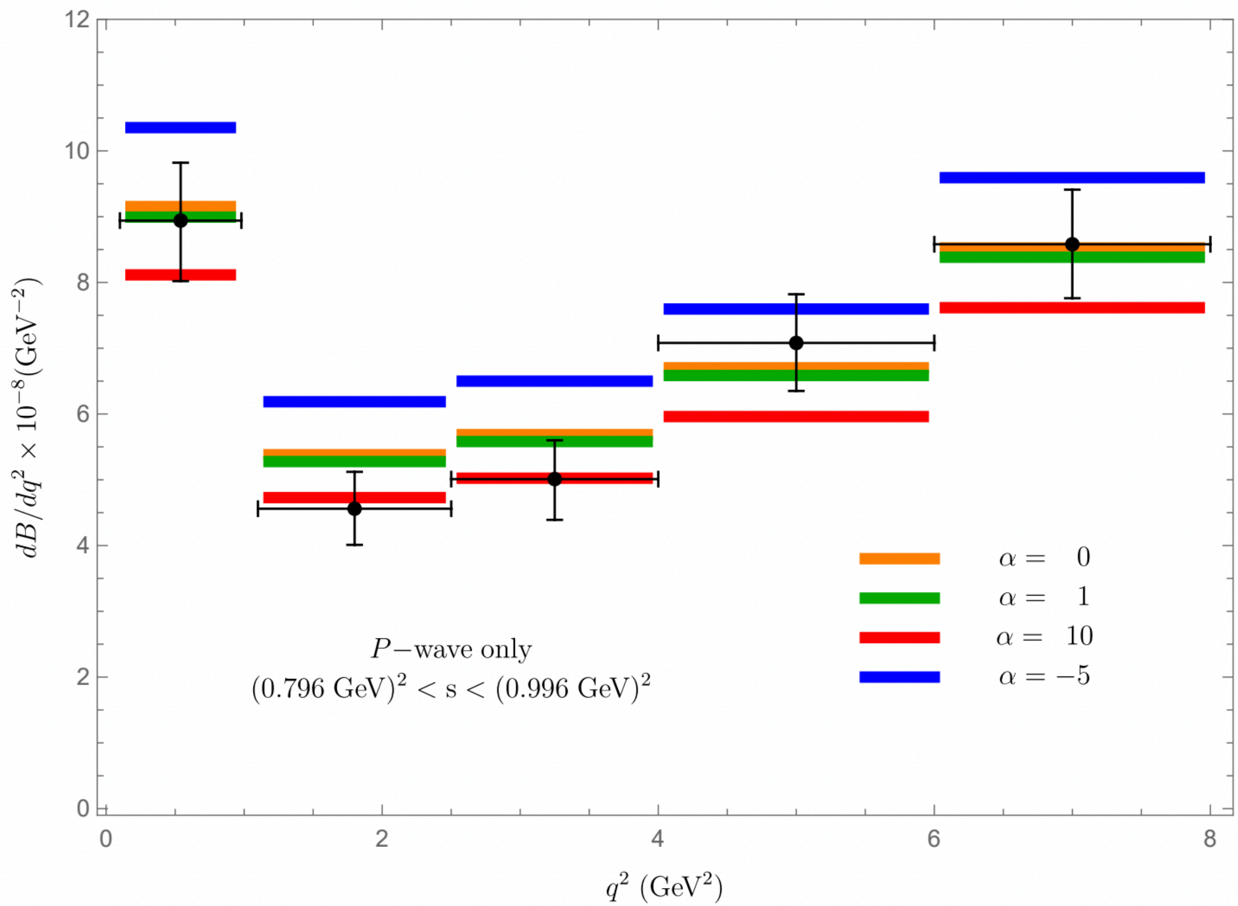


Figure 8: *Theory predictions for the $B \to (K\pi)_P \ell^+\ell^-$ branching ratio within the $K\pi$ invariant mass bin $(0.796\ \mathrm{GeV})^2 < s < (0.996\ \mathrm{GeV})^2$, for different values of α, compared to the LHCb measurements of $B \to K^*\mu^+\mu^-$ in Ref. [13].*

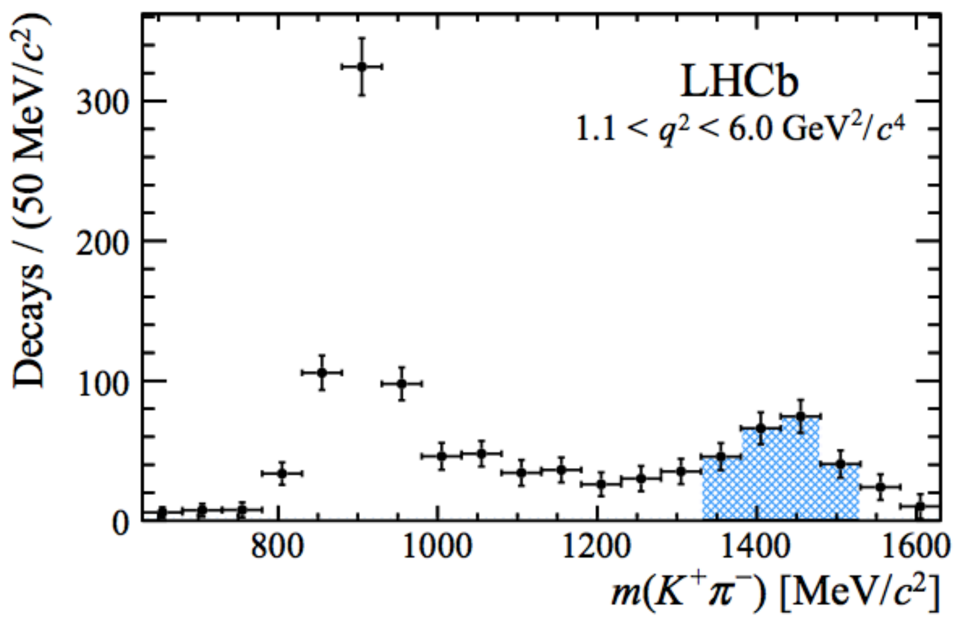
LHCb arXiv: 1609.04736

Differential decay rate including S,P,D waves – – [$d\Omega = d\cos\theta_\ell\, d\cos\theta_K\, d\phi$]

$$\frac{d\Gamma}{dq^2 dk^2 d\Omega} = \frac{1}{4\pi}\sum_{i=1}^{41} f_i(\Omega)\, \tilde{\Gamma}_i(q^2,k^2)$$

The 41 moments $\tilde{\Gamma}_i(q^2,k^2)$ have been measured by LHCb (arXiv: 1609.04736) in the bins

$$\sqrt{k^2} \in [1.33, 1.53]\text{GeV}\ , \quad q^2 \in [1.1, 6]\text{GeV}^2$$



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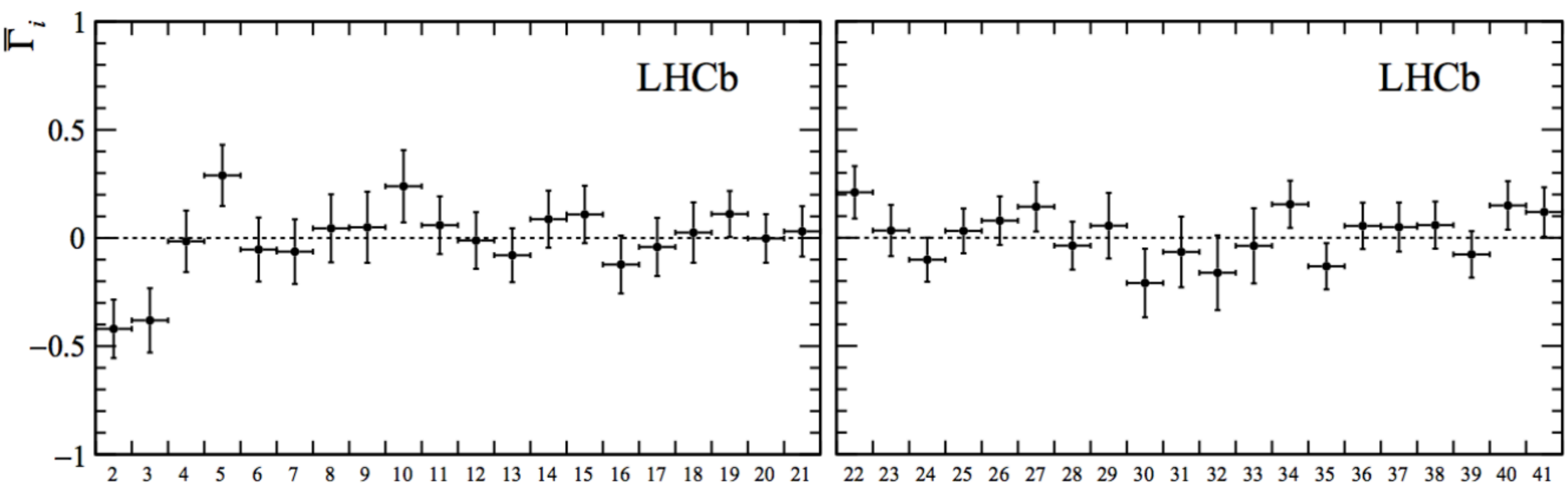
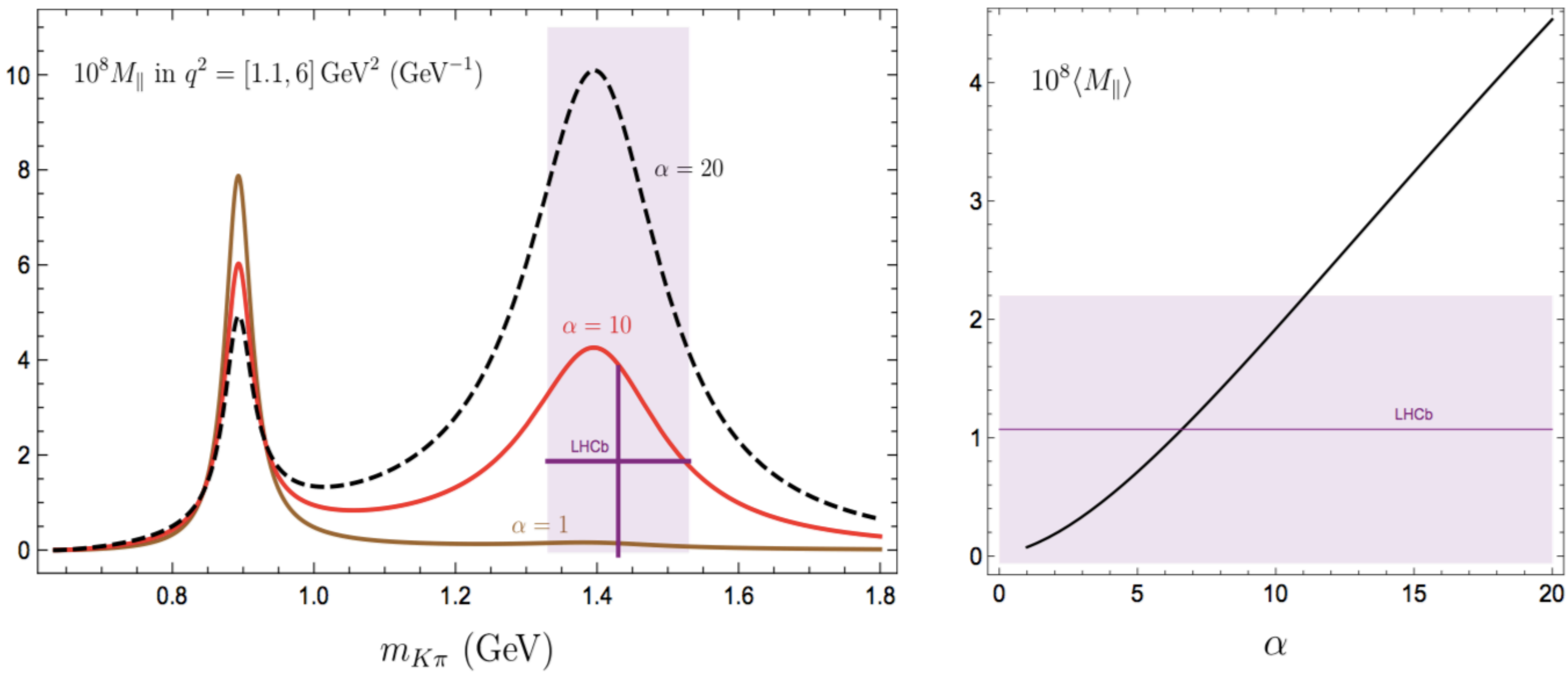


High $K\pi$-Mass Moments in $B \to K\pi\ell\ell$

Example: $\langle M_{\|} \rangle \equiv \tau_B \, \langle |\widehat{A}_{\|}^L|^2 + |\widehat{A}_{\|}^R|^2 \rangle = \frac{\tau_B}{36} \langle 5\tilde{\Gamma}_1 - 7\sqrt{5}\tilde{\Gamma}_3 + 5\sqrt{5}\tilde{\Gamma}_6 - 35\tilde{\Gamma}_8 - 5\sqrt{15}\tilde{\Gamma}_{19} + 35\sqrt{3}\tilde{\Gamma}_{21} \rangle$



Bounds: From $\langle M_{\|} \rangle$: $\alpha \lesssim 11$; From $\langle M_{\perp} \rangle$: $\alpha \lesssim 17$; From $\langle M_{\rm re} \rangle$: $\alpha \lesssim 18$.

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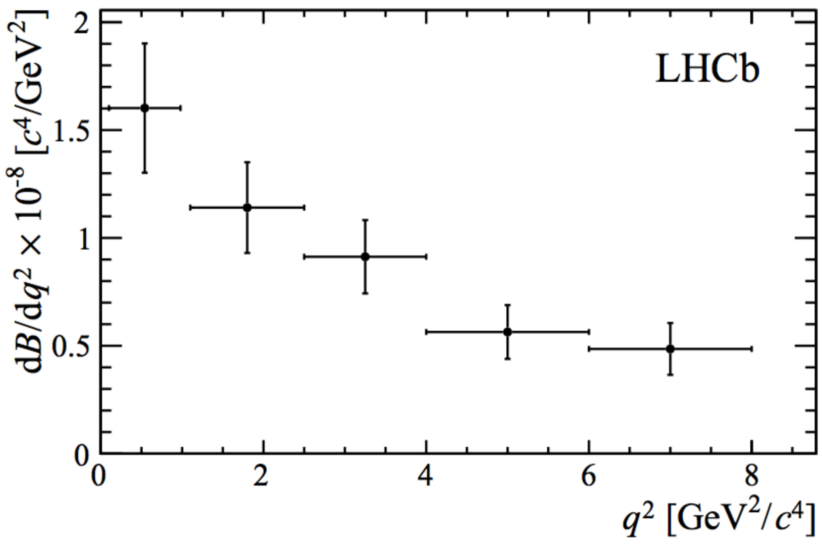
High $K\pi$-Mass Moments in $B \to K\pi\ell\ell$

Upper bounds on P-wave from differential BR:

$$\frac{d\Gamma}{dq^2 dk^2} = \tilde{\Gamma}_1 = |\widehat{A}_{\|}^L|^2 + |\widehat{A}_{\|}^R|^2 + |\widehat{A}_{\perp}^L|^2 + |\widehat{A}_{\perp}^R|^2 + |\widehat{A}_0^L|^2 + |\widehat{A}_0^R|^2 + \dots$$



$$\begin{aligned}
10^8 \cdot \langle \mathcal{B} \rangle_{[0.10,0.98]} &= 1.41 \pm 0.27 \to \ \alpha \lesssim 5 \\
10^8 \cdot \langle \mathcal{B} \rangle_{[1.10,2.50]} &= 1.60 \pm 0.29 \to \ \alpha \lesssim 6 \\
10^8 \cdot \langle \mathcal{B} \rangle_{[2.50,4.00]} &= 1.37 \pm 0.26 \to \ \alpha \lesssim 5 \\
10^8 \cdot \langle \mathcal{B} \rangle_{[4.00,6.00]} &= 1.12 \pm 0.26 \to \ \alpha \lesssim 4 \\
10^8 \cdot \langle \mathcal{B} \rangle_{[6.00,8.00]} &= 0.98 \pm 0.23 \to \ \alpha \lesssim 3
\end{aligned}$$

Bounds are easily improved with some info on S-wave form factors.

LCSRs for S-wave $B \to K\pi$ Form Factors

Descotes-Genon, Khodjamirian, Virto, Vos 2023

S-wave Projector

$$\mathcal{S}_i(k,q) = i\int d^4x\, e^{ik\cdot x}\langle 0|\mathrm{T}\{\overbrace{\bar{d}(x)s(x)}, j_i(0)\}|\bar{B}^0(q+k)\rangle$$

$$\boxed{\int_{s_{\rm th}}^{s_0} ds\, e^{-s/M^2}\ \omega_i(s,q^2)\ f_0^\star(s)\ F_i^{(\ell=0)}(s,q^2) = \mathcal{S}_i^{\rm OPE}(q^2,\sigma_0,M^2)}$$

- s_0 – Effective threshold
- $\omega_i(s,q^2)$ – (known) kinematic factors
- $\langle K^-(k_1)\pi^+(k_2)|\bar{s}d|0\rangle = (m_K^2 - m_\pi^2) f_0(k^2)$
- $\mathcal{S}_i^{\rm OPE}$ – OPE result for the correlation function

Descotes-Genon, Khodjamirian, Virto, Vos 2023

Modelling S-wave spectrum much more challenging

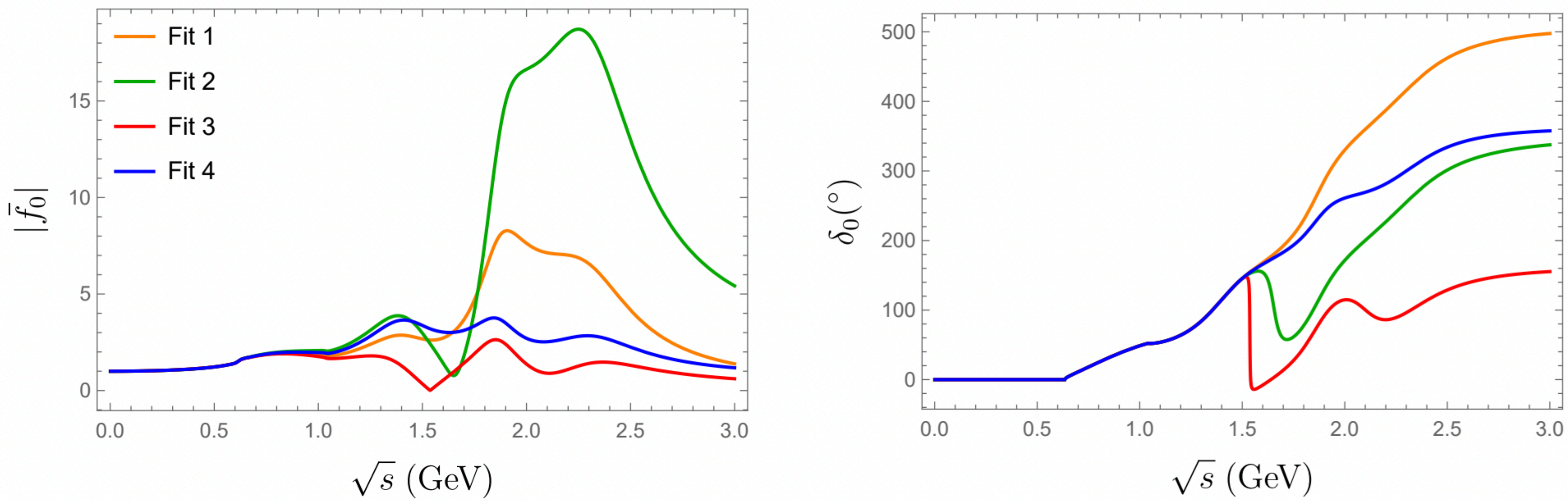
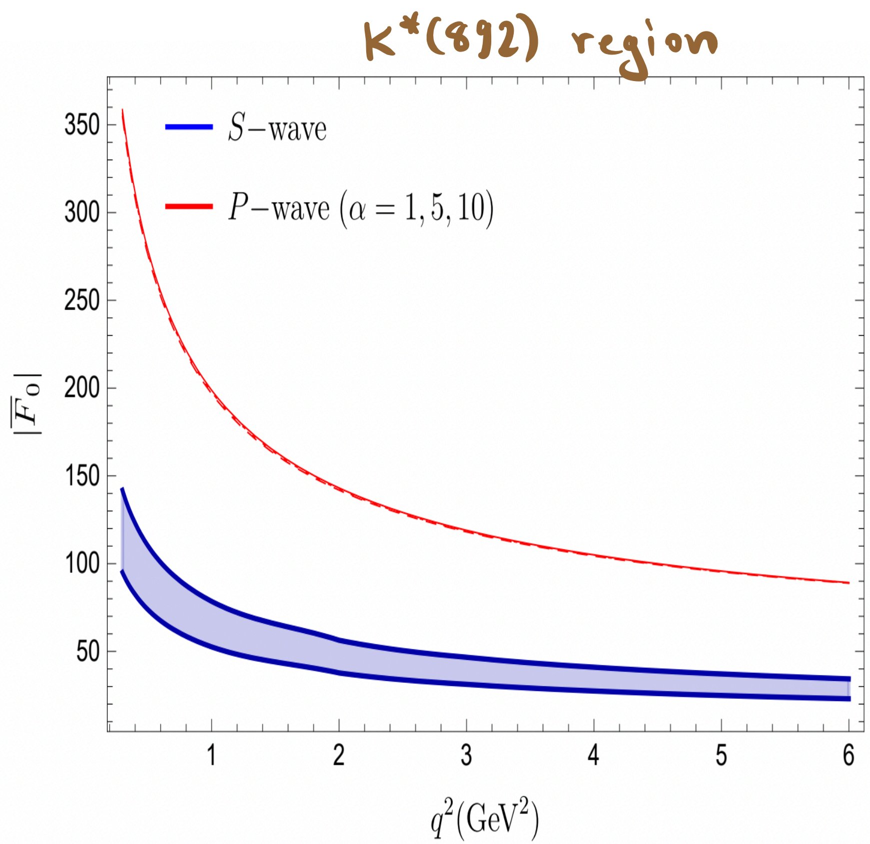


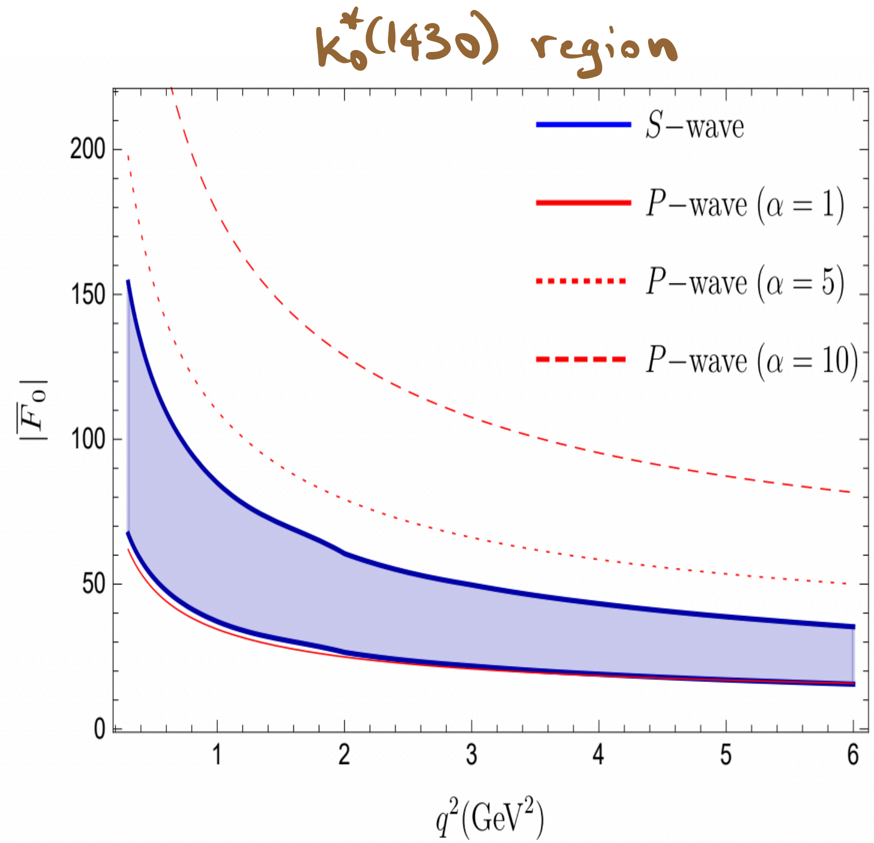
Figure 1: *Modulus of the normalized scalar form factor* $|\bar{f}_0|$ *and its strong phase* δ_0 *obtained from the four different fit scenarios of Ref. [30].*

Two channel $K\pi - K\eta'$ model from Von Detten, Noel, Hanhart, Hoferichter, Kubis 2021

Descotes-Genon, Khodjamirian, Virto, Vos 2023

Relative size of S- and P-wave contributions





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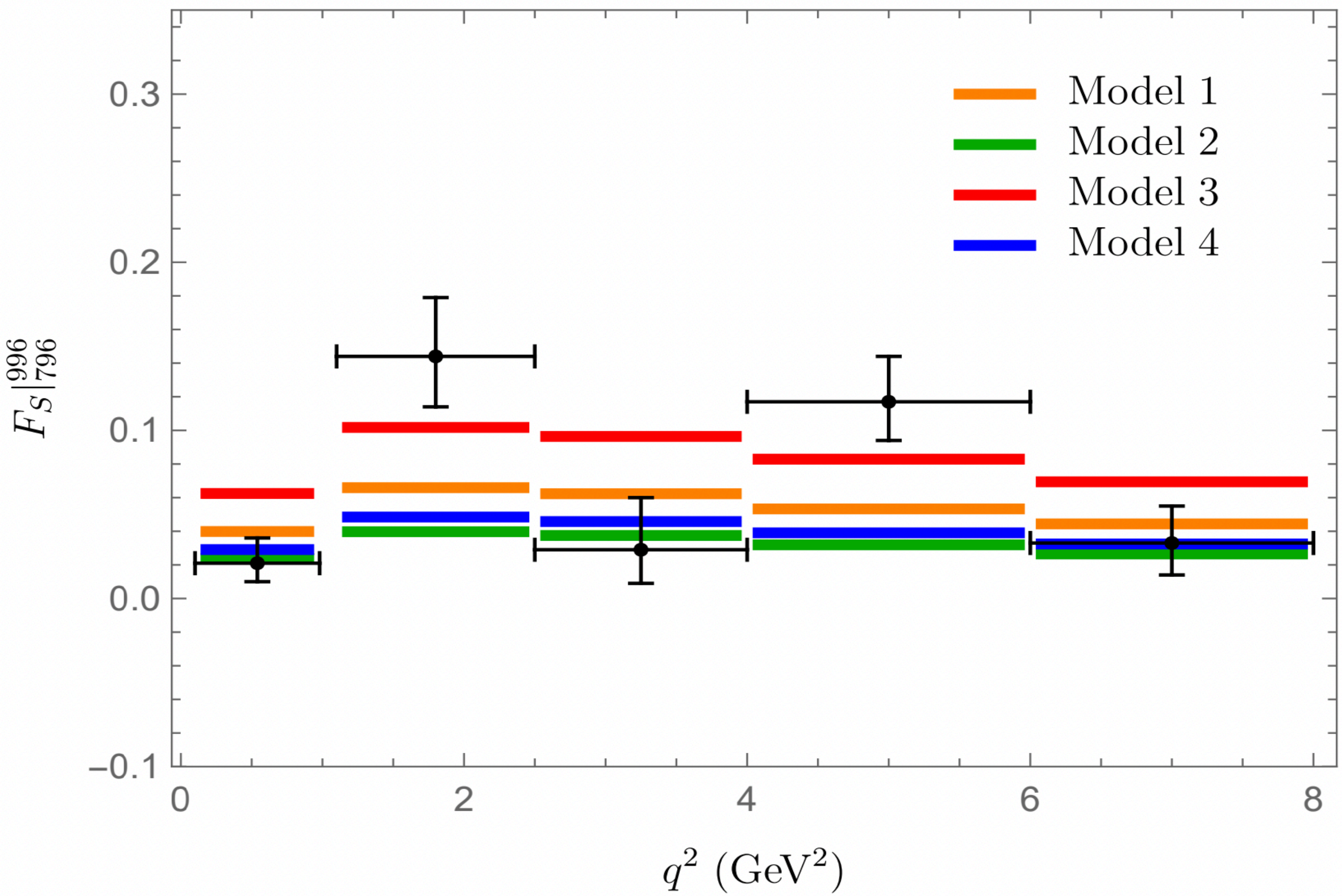
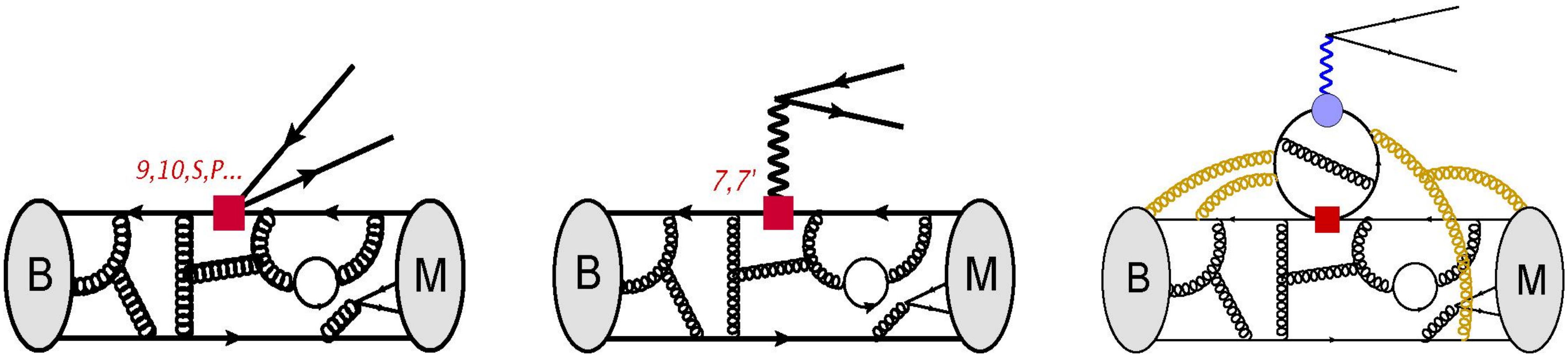
S-wave fraction $F_S = BR(S\text{-wave})/BR(S+P\text{-wave})$



Non-Local Form Factors

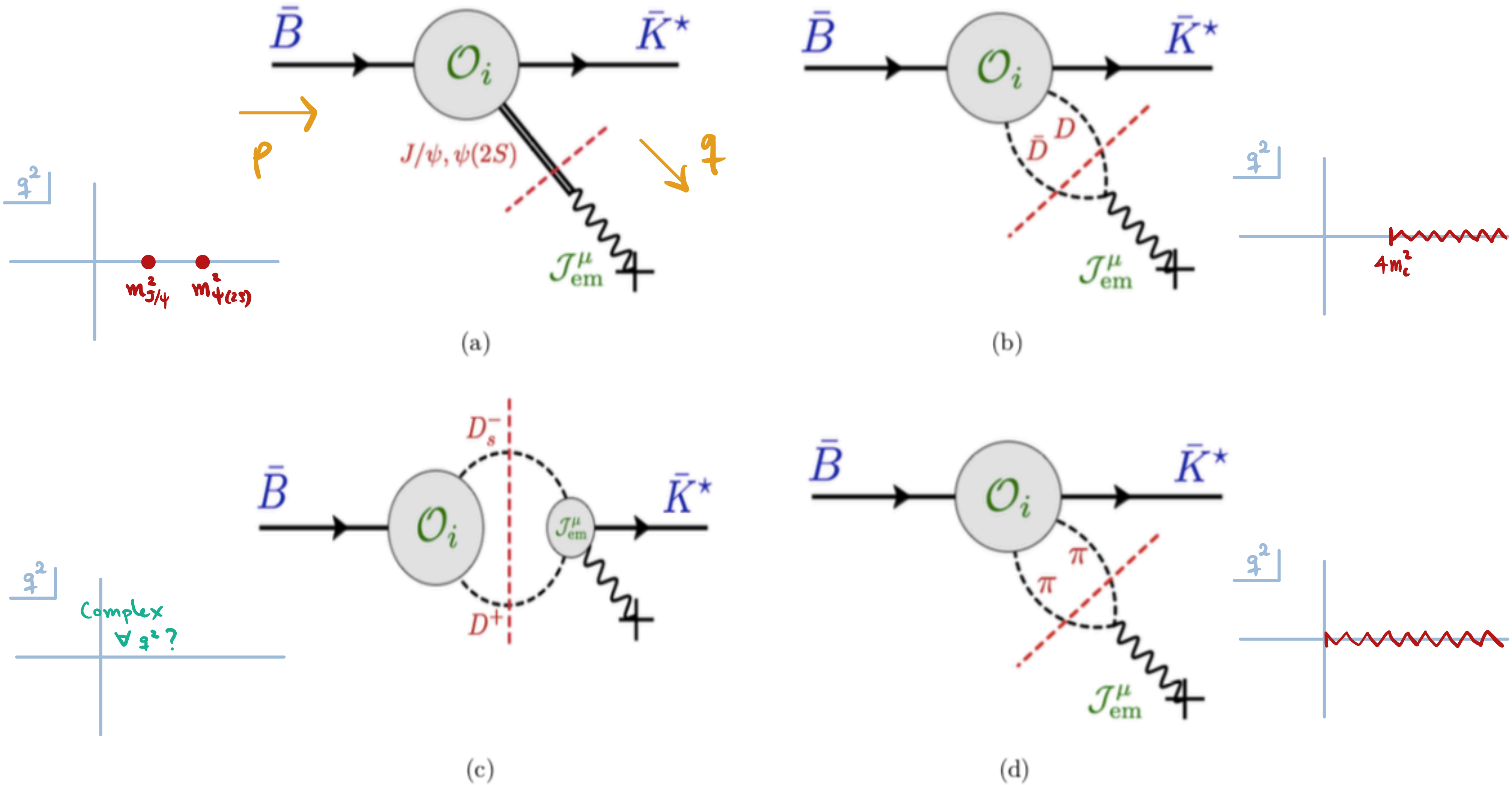
Non-Local Form Factors



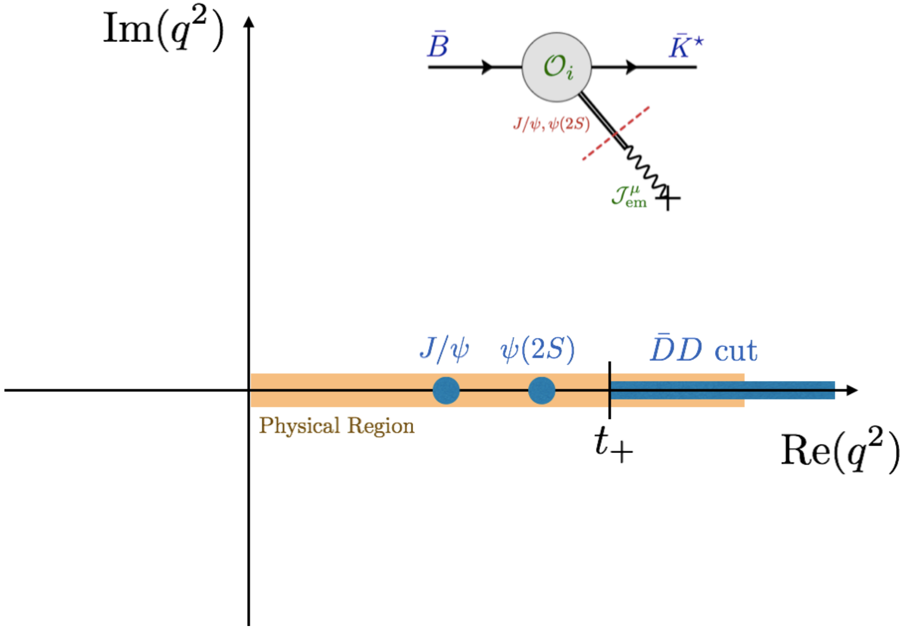
$$\mathcal{A}_\lambda^{L,R} = \mathcal{N}_\lambda \left\{ (C_9 \mp C_{10}) \mathcal{F}_\lambda(q^2) + \frac{2 m_b M_B}{q^2} \left[C_7 \mathcal{F}_\lambda^T(q^2) - 16\pi^2 \frac{M_B}{m_b} \mathcal{H}_\lambda(q^2) \right] \right\}$$

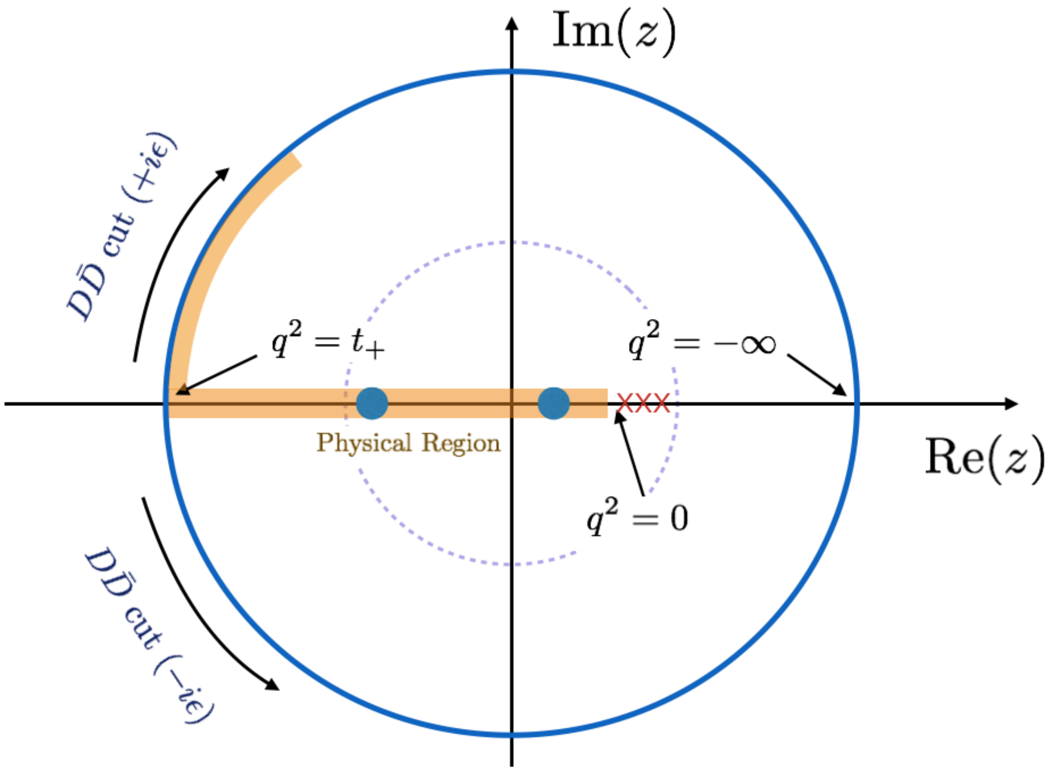
- Local (Form Factors) : $\mathcal{F}_\lambda^{(T)}(q^2) = \langle \bar{M}_\lambda(k) | \, \bar{s} \, \Gamma_\lambda^{(T)} \, b \, | \bar{B}(k+q) \rangle$
- Non-Local : $\mathcal{H}_\lambda(q^2) = i \, \mathcal{P}_\mu^\lambda \int d^4x \, e^{iq\cdot x} \, \langle \bar{M}_\lambda(k) | \mathcal{T}\{ \mathcal{J}_{\rm em}^\mu(x), \mathcal{C}_i \, \mathcal{O}_i(0) \} | \bar{B}(q+k) \rangle$

Non-Local Form Factors: Analytic structure

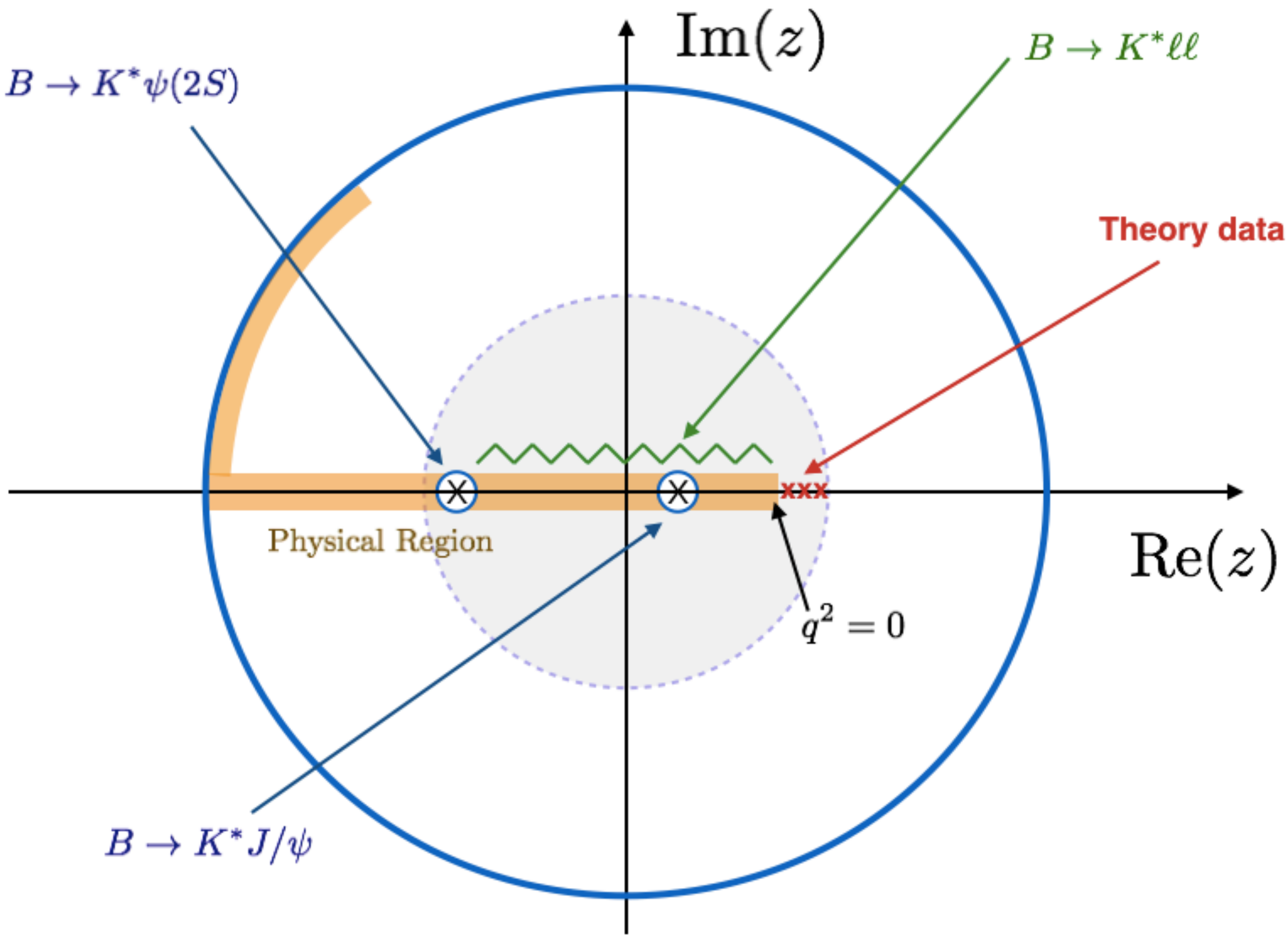


z-parametrisation for $\mathcal{H}_\lambda(q^2)$





- $\hat{\mathcal{H}}_\lambda(q^2(z)) = (q^2 - M^2_{J/\psi})(q^2 - M^2_{\psi(2S)})\,\mathcal{H}_\lambda(q^2)$ is **analytic in** $|z| < 1$
- Taylor expand $\hat{\mathcal{H}}_\lambda(z)$ around $z = 0$: $\hat{\mathcal{H}}_\lambda(z) = \left[\sum_{k=0}^{K} \alpha_k^{(\lambda)} z^k\right] \mathcal{H}_\lambda(z)$
- Expansion needed for $|z| < 0.52$ $\left(-7\,\mathrm{GeV}^2 \leq q^2 \leq 14\mathrm{GeV}^2 \right)$



$\mathrm{Im}(z)$
$B \to K^* \psi(2S)$
$B \to K^* \ell\ell$
Theory data
Physical Region
$\mathrm{Re}(z)$
$q^2 = 0$
$B \to K^* J/\psi$

Non-local form factors: Operator Product Expansion

$$\mathcal{H}^\mu(q,k) = i\int d^4x\, e^{iq\cdot x}\, \langle \bar{M}_\lambda(k)|\mathcal{T}\{\mathcal{J}^\mu_{\rm em}(x), \mathcal{C}_i\, \mathcal{O}_i(0)\}|\bar{B}(q+k)\rangle$$

- Large-q^2: Dominated by $x \sim 0$ (short-distance dominance - OPE)

  Grinstein, Pirjol; Beylich, Buchalla, Feldmann

- Low-q^2: Dominated by $x^2 \sim 0$ (light-cone dominance - LCOPE)

  Khodjamirian, Mannel, Pivovarov, Wang

We write

$$\mathcal{H}^\mu(q,k) = \langle \bar{M}_\lambda(k)|\mathcal{K}^\mu(q)|\bar{B}(q+k)\rangle$$

With the operator $\mathcal{K}^\mu(q)$ given by

$$\mathcal{K}^\mu(q) = i\int d^4x\, e^{iq\cdot x}\,\mathcal{T}\{\mathcal{J}^\mu_{\rm em}(x), \mathcal{C}_i\,\mathcal{O}_i(0)\}$$

It turns out that: **Leading-order OPE = Leading order LCOPE**

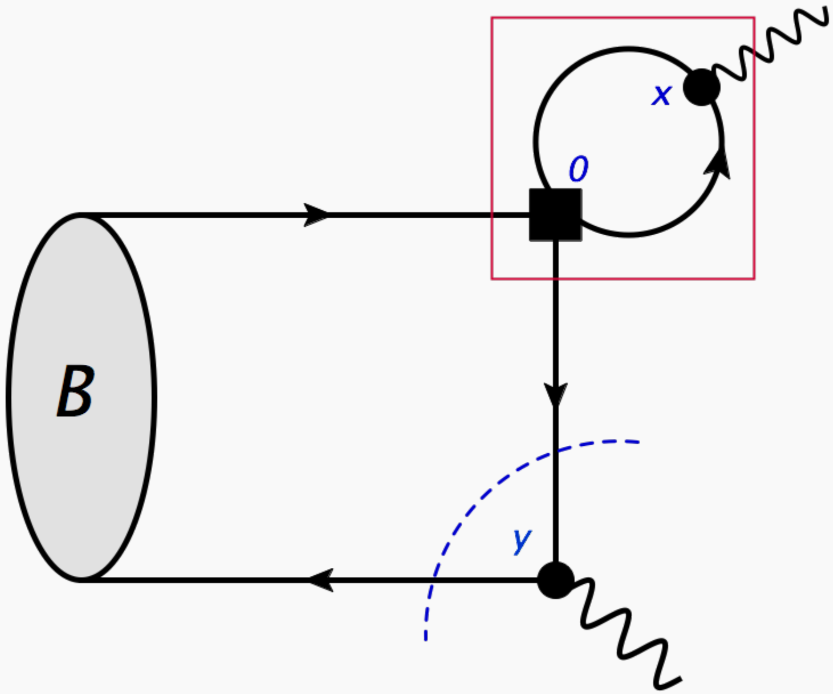
$$\mathcal{K}^\mu_{\rm OPE}(q) = \Delta C_9(q^2)\big(q^\mu q^\nu - q^2 g^{\mu\nu}\big)\,\bar{s}\gamma_\nu P_L b + \Delta C_7(q^2)\,2im_b\,\bar{s}\sigma^{\mu\nu}q_\nu P_R b + \cdots$$

With this we have:

$$\mathcal{H}^\mu_{\rm OPE}(q,k) = \Delta C_9(q^2)\big(q^\mu q^\nu - q^2 g^{\mu\nu}\big)\mathcal{F}_\nu + 2im_b\,\Delta C_7(q^2)\mathcal{F}^{T\mu} + \cdots$$

LCOPE very low q^2

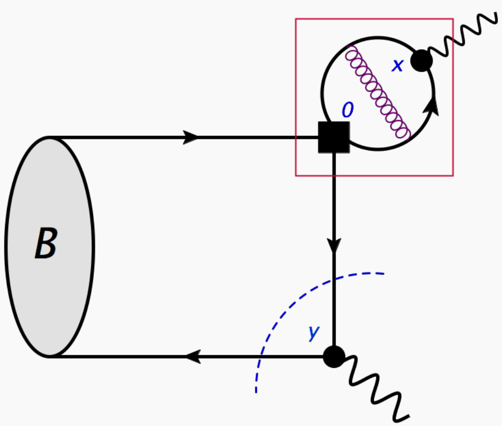
- LCSRs with B-meson DAs Khodjamirian, Mannel, Pivovarov, Wang

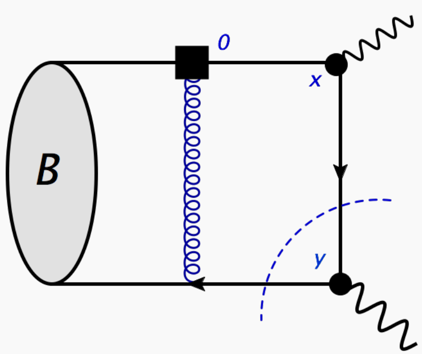


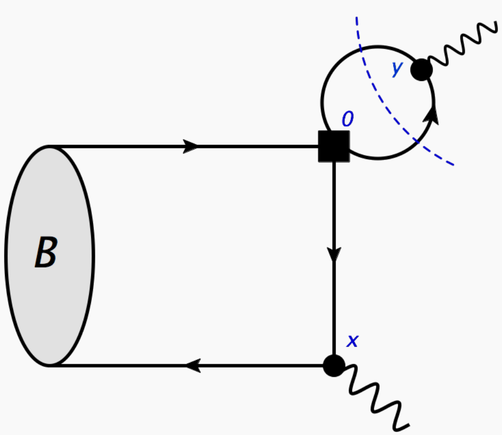
LC exp. of charm prop. Balitsky, Braun 1989

$$\xrightarrow{q^2 \ll 4m_c^2} \underbrace{\left(\frac{C_1}{3} + C_2\right) g(m_c^2, q^2)}_{\text{matching coeff}} [\bar{s}\,\Gamma\, b] + \cdots$$

$$\Rightarrow \; \mathcal{H}_\lambda = (\text{matching coeff}) \times \mathcal{F}_\lambda^{LCSR}$$

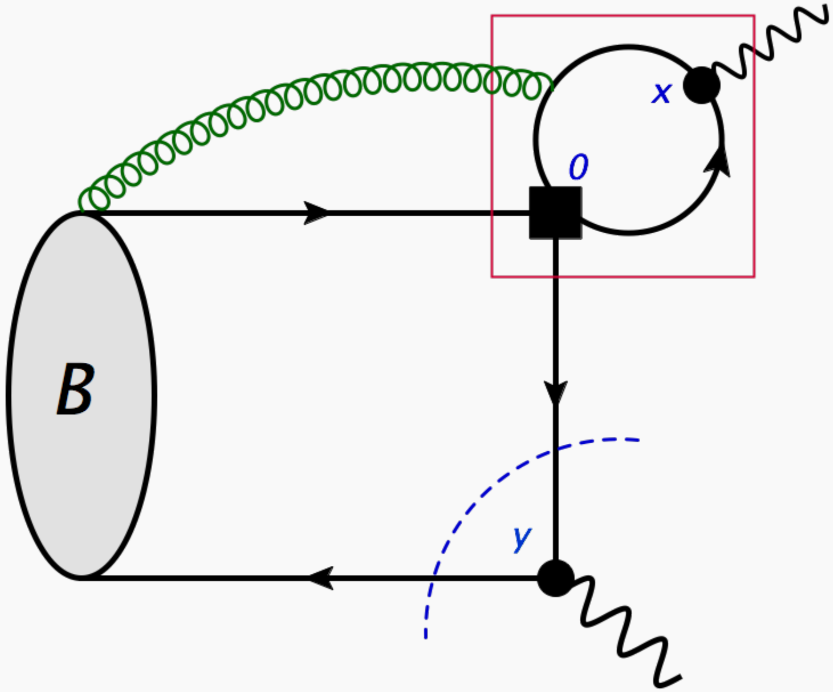






LCOPE at very low q^2 – Subleading power

- LCSRs with B-meson DAs   Khodjamirian, Mannel, Pivovarov, Wang

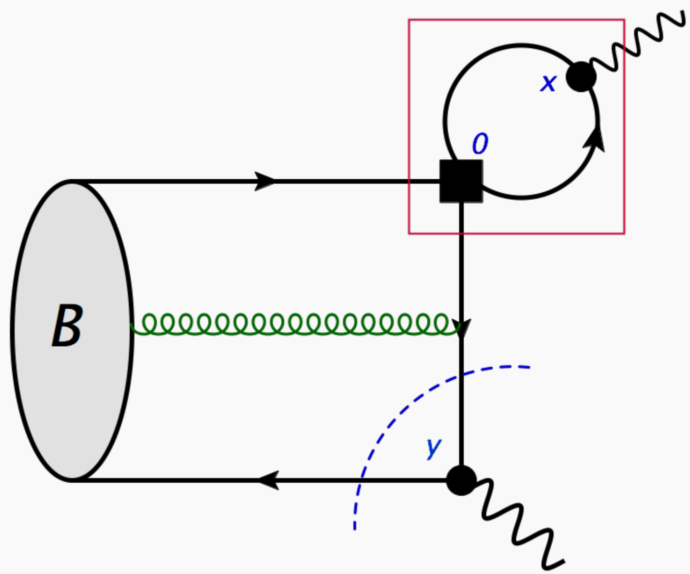


LC exp. of charm prop. Balitsky, Braun 1989

$$\xrightarrow{q^2 \ll 4m_c^2} \underbrace{\left(\frac{C_1}{3} + C_2\right) g(m_c^2, q^2)}_{\text{matching coeff}} [\bar{s}\,\Gamma\, b] +$$

$$+ \text{(coeff)} \times [\bar{s}_L \gamma^\alpha (in_+ \cdot \mathcal{D})^n \tilde{G}_{\beta\gamma} b_L\,] \;+ \cdots$$

3-particle correction to $\mathcal{F}_\lambda \longrightarrow$



LCOPE at very low q^2 – Subleading power

Recalculation of charm-loop effect Gubernari, van Dyk, Virto, 2011.09813

Transition	$\tilde{\mathcal{V}}(q^2 = 1\,\text{GeV}^2)$	This work	Ref. [11]
$B \to K$	$\tilde{\mathcal{A}}$	$(+4.9 \pm 2.8) \cdot 10^{-7}$	$(-1.3^{+1.0}_{-0.7}) \cdot 10^{-4}$
$B \to K^*$	$\tilde{\mathcal{V}}_1$	$(-4.4 \pm 3.6) \cdot 10^{-7}\,\text{GeV}$	$(-1.5^{+1.5}_{-2.5}) \cdot 10^{-4}\,\text{GeV}$
	$\tilde{\mathcal{V}}_2$	$(+3.3 \pm 2.0) \cdot 10^{-7}\,\text{GeV}$	$(+7.3^{+14}_{-7.9}) \cdot 10^{-5}\,\text{GeV}$
	$\tilde{\mathcal{V}}_3$	$(+1.1 \pm 1.0) \cdot 10^{-6}\,\text{GeV}$	$(+2.4^{+5.6}_{-2.7}) \cdot 10^{-4}\,\text{GeV}$
$B_s \to \phi$	$\tilde{\mathcal{V}}_1$	$(-4.4 \pm 5.6) \cdot 10^{-7}\,\text{GeV}$	—
	$\tilde{\mathcal{V}}_2$	$(+4.3 \pm 3.1) \cdot 10^{-7}\,\text{GeV}$	—
	$\tilde{\mathcal{V}}_3$	$(+1.7 \pm 2.0) \cdot 10^{-6}\,\text{GeV}$	—

- We reproduce the result of KMPW'2010
- We incude complete set of 3-particle LCDAs Braun,Li,Manashov 2017
- Cancellations + Parametric lead to a reduction of the effect of **two orders of magnitude**

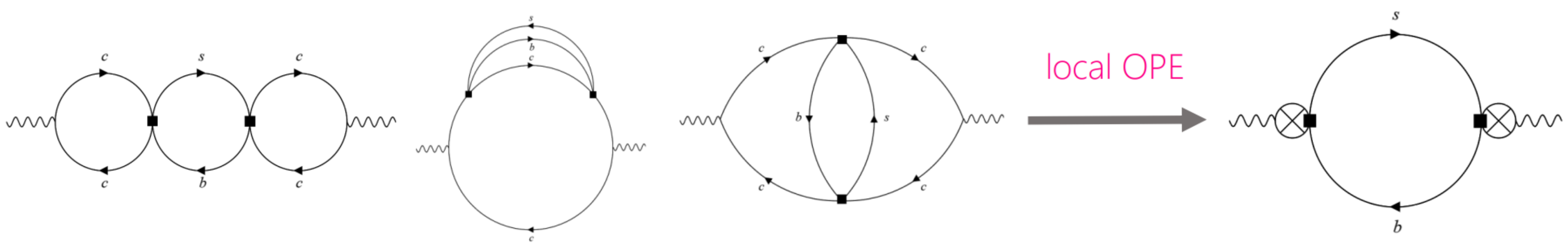
1. Consider the correlation function

$$\Pi(q) \equiv i \int d^4x\, e^{iq\cdot x} \langle 0| T\left\{ O^{\mu}(q;x), O^{\mu,\dagger}(q;0) \right\} |0\rangle$$

where

$$O^{\mu}(q;x) = -i \int d^4y\, e^{+iq\cdot y}\; T\{ j^{\mu}_{\text{em}}(x+y), (C_1\mathcal{O}_1 + C_2\mathcal{O}_2)(x) \}$$

2. Calculate in OPE region

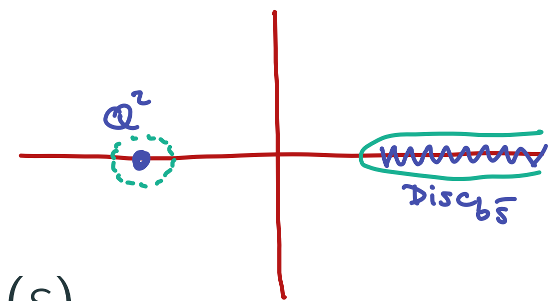


$$\chi^{\text{OPE}}(-m_b^2) = (1.81 \pm 0.02) \times 10^4 \text{GeV}^{-2}$$

Dispersive bound

Gubernari, van Dyk, Virto 2020

3. Twice-subtracted dispersion relation:



$$\chi^{\text{OPE}}(Q^2) \equiv \frac{1}{2i\pi} \int_0^\infty ds \, \frac{\text{Disc}_{b\bar{s}} \Pi^{\text{had}}(s)}{(s - Q^2)^3}$$

$$\begin{aligned}
\frac{3}{32 i \pi^3} \text{Disc}_{b\bar{s}} \Pi^{\text{had}}(s) = {} & \frac{2 M_B^4 \, \lambda^{3/2}(M_B^2, M_K^2, s)}{s^4} \left| \mathcal{H}_0^{B\to K}(s) \right|^2 \theta(s - s_{BK}) \\
& + \frac{2 M_B^6 \sqrt{\lambda(M_B^2, M_{K^*}^2, s)}}{s^3} \left(\left| \mathcal{H}_\perp^{B\to K^*}(s) \right|^2 + \left| \mathcal{H}_\parallel^{B\to K^*}(s) \right|^2 + \frac{M_B^2}{s} \left| \mathcal{H}_0^{B\to K^*}(s) \right|^2 \right) \theta(s - s_{BK^*}) \\
& + \frac{M_B^6 \sqrt{\lambda(M_{B_s}^2, M_\phi^2, s)}}{s^3} \left(\left| \mathcal{H}_\perp^{B_s\to \phi}(s) \right|^2 + \left| \mathcal{H}_\parallel^{B_s\to \phi}(s) \right|^2 + \frac{M_{B_s}^2}{s} \left| \mathcal{H}_0^{B_s\to \phi}(s) \right|^2 \right) \theta(s - s_{B_s\phi}) \\
& + \text{further positive terms}
\end{aligned}$$

(e.g. $\Lambda_b \to \Lambda$, $B \to K\pi\pi\pi\pi\pi\pi$, ...)

Gubernari, van Dyk, Virto 2020

Redefine $\mathcal{H}_i$ as before:

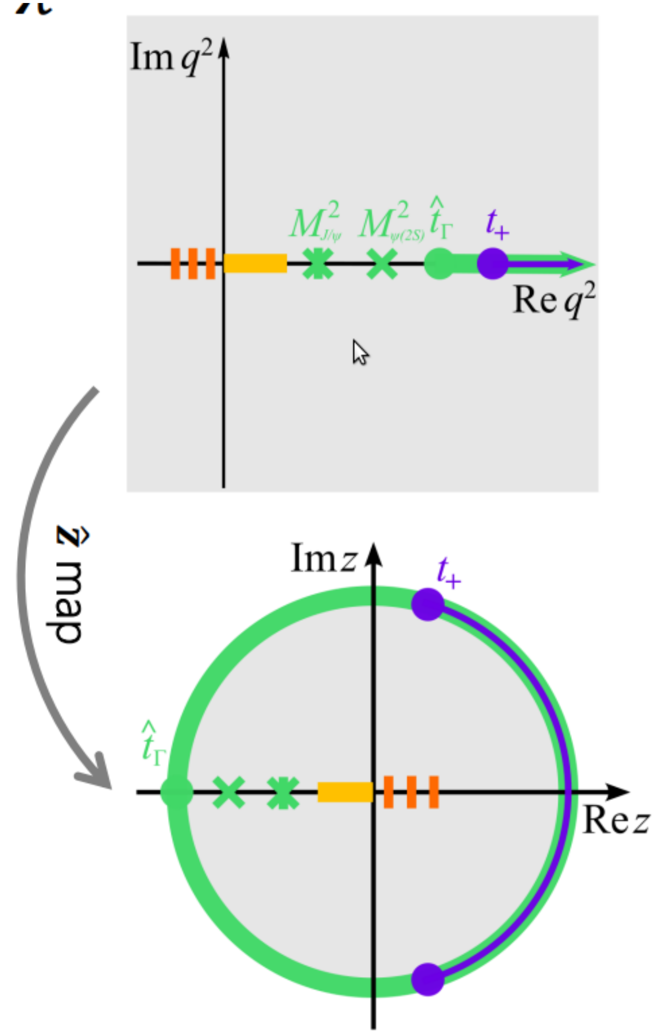
$$\hat{\mathcal{H}}_\lambda^{B\to M}(z) \equiv \phi_\lambda^{B\to M}(z)\,\mathcal{P}(z)\,\mathcal{H}_\lambda^{B\to M}(z)\,,$$

Expand in ortogonal polynomials in **arc**:

$$\hat{\mathcal{H}}_\lambda^{B\to M}(z) = \sum_{n=0}^{\infty} a_{\lambda,n}^{B\to M} p_n^{B\to M}(z)$$

The dispersive bound then takes the simple form

$$\sum_{n=0}^{\infty}\left\{2\left|a_{0,n}^{B\to K}\right|^2 + \sum_{\lambda=\perp,\parallel,0}\left[2\left|a_{\lambda,n}^{B\to K^*}\right|^2 + \left|a_{\lambda,n}^{B_s\to\phi}\right|^2\right]\right\} < 1\,.$$



Dispersive bound

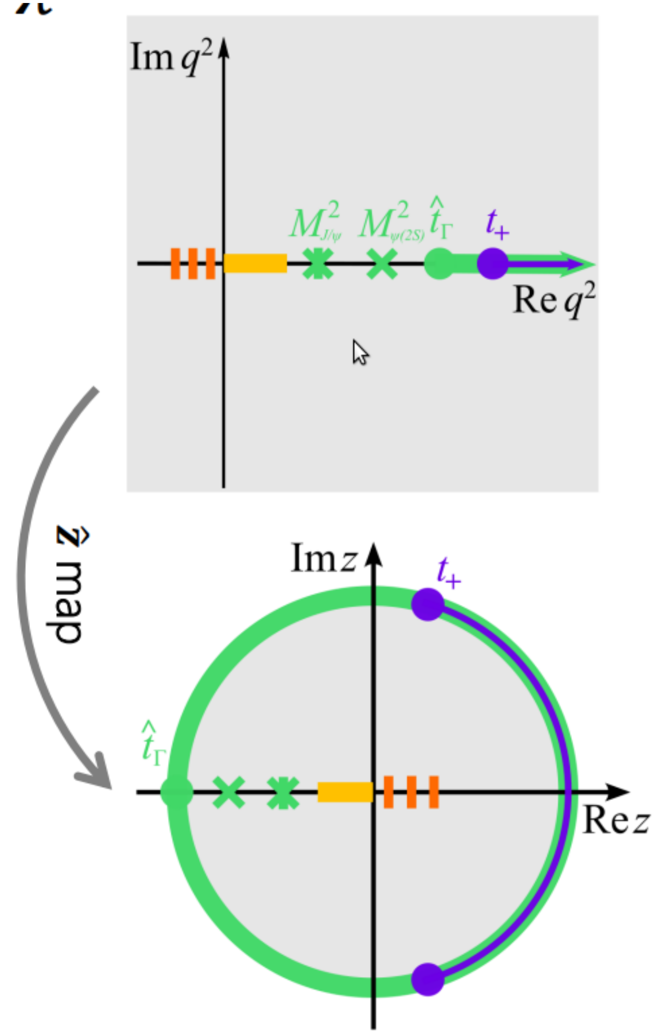
FOR FIT RESULTS SEE TALK BY MÉRIL REBOUD

Redefine $\mathcal{H}_i$ as before:

$$\hat{\mathcal{H}}_\lambda^{B\to M}(z) \equiv \phi_\lambda^{B\to M}(z)\,\mathcal{P}(z)\,\mathcal{H}_\lambda^{B\to M}(z)\,,$$

Expand in ortogonal polynomials in **arc**:

$$\hat{\mathcal{H}}_\lambda^{B\to M}(z) = \sum_{n=0}^{\infty} a_{\lambda,n}^{B\to M}\, p_n^{B\to M}(z)$$

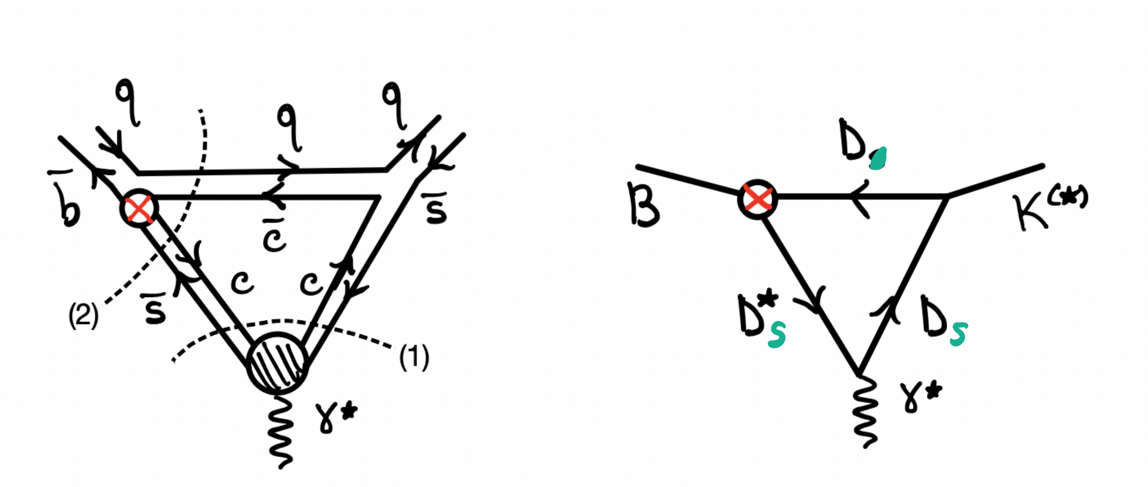


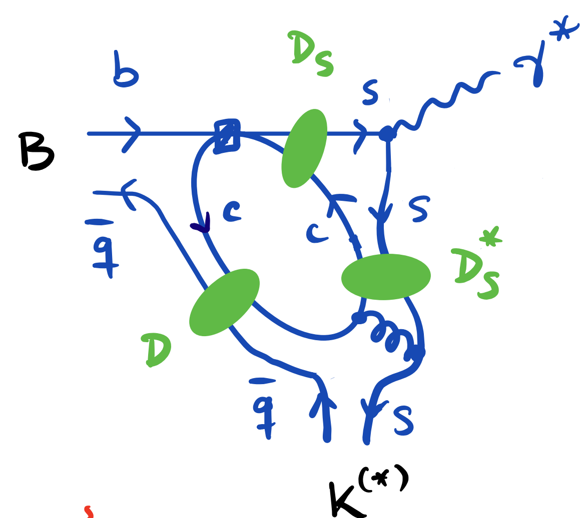
The dispersive bound then takes the simple form

$$\sum_{n=0}^{\infty}\left\{2\left|a_{0,n}^{B\to K}\right|^2 + \sum_{\lambda=\perp,\|,0}\left[2\left|a_{\lambda,n}^{B\to K^*}\right|^2 + \left|a_{\lambda,n}^{B_s\to\phi}\right|^2\right]\right\} < 1\,.$$

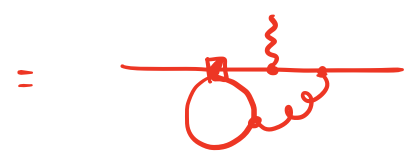
Non-Local Form Factors: Issues

- **No LQCD calculation**: need (?) LCSRs
- Leading term is given by Local Form Factors ✓
- $B \to \psi(nS)M \longrightarrow$ important model-independent input (**)
- z-parametrization $\longrightarrow$ **Analytic structure**



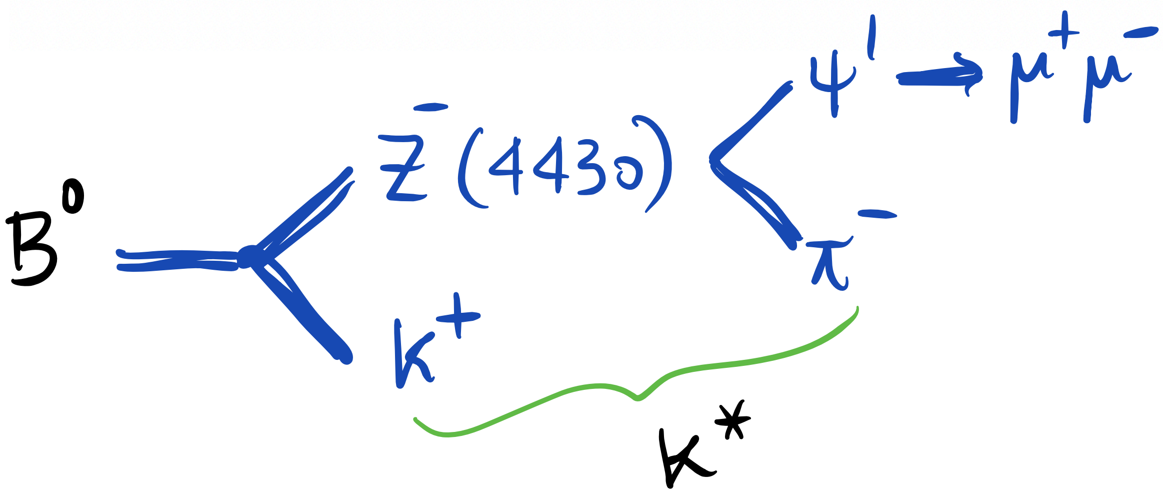


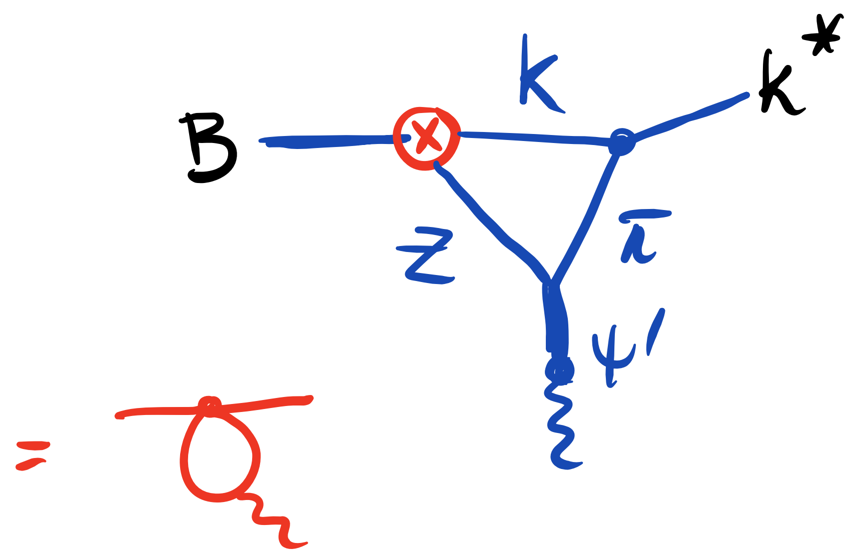
(Ciuchini, Fedele, Franco, Paul, Silvestrini, Valli 2022)



Non-Local Form Factors: Issues

- **No LQCD calculation**: need (?) LCSRs
- Leading term is given by Local Form Factors ✓
- $B \to \psi(nS)M \longrightarrow$ important model-independent input (**)
- z-parametrization $\longrightarrow$ **Analytic structure**





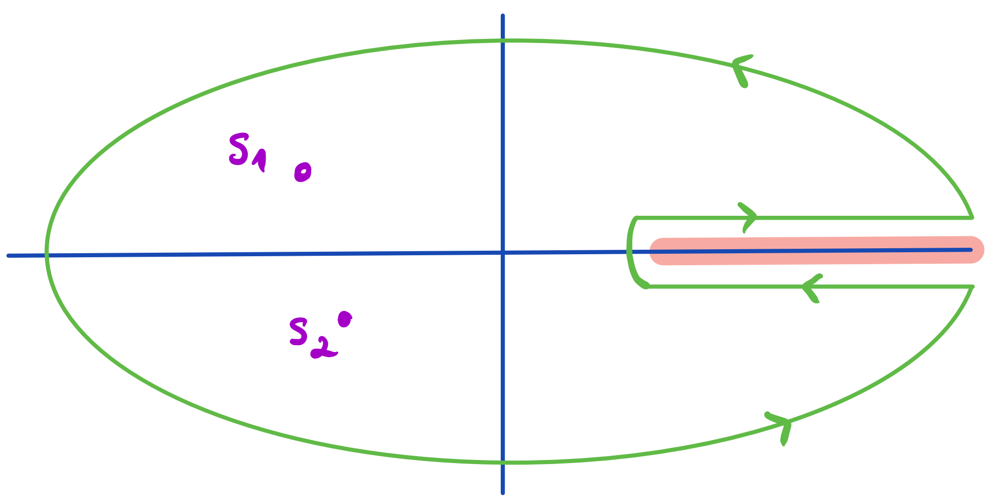
Direct check of analytic structure at two loops: Asatian, Greub, Virto 2019

$$F(s_1) - F(s_2) = \frac{s_1 - s_2}{2\pi i} \int_{s_{\rm th}}^{\infty} dt \frac{F(t+i0) - F(t-i0)}{(t-s_1)(t-s_0)}$$

Example:

$$F^{(7)}_{2,(b)}(-3+i) - F^{(7)}_{2,(b)}(-1-2i) = 0.0894864 - 0.160827\, i\,,$$

$$\frac{-2+3i}{2\pi i} \int_4^{\infty} dt\, \frac{\text{Disc}\, F^{(7)}_{2,(b)}(t)}{(t+3-i)(t+1+2i)} = 0.0894966 - 0.160839\, i\,.$$



Summary

The understanding of rare B decay measurements requires the knowledge of local and non-local form factors

LCSRs with B-meson LCDAs **suitable for Global Analyses:**

- Valid for

$$B \to K\,,\; B \to K^*\,,\; B_s \to \phi\,,$$
$$B \to \pi\,,\; B \to \rho\,,\; B \to \gamma\ell\nu\,,\; \ldots$$

- Beyond the Narrow-Width Limit + non-resonant:

$$B \to \pi\pi\,,\; B \to K\pi\; \ldots$$

- Non-local Form factors at negative q^2

“I summon the spirits of long-distance enhancement”

12 **Beyond the Flavour Anomalies IV, Barcelona, 21 April 2023** **Ulrich Nierste**

BACK-UP

Based on

Bobeth, Chrzaszcz, van Dyk, Virto, 1707.07305

Descotes-Genon, Khodjamirian, Virto, 1908.02267

Asatrian, Greub, Virto, 1912.09099

Gubernari, van Dyk, Virto, 2011.09813

Gubernari, Reboud van Dyk, Virto, 2206.03797

Descotes-Genon, Khodjamirian, Virto, Vos, 2304.02973

Gubernari, Reboud van Dyk, Virto, w.i.p

Bobeth, Chrzaszcz, van Dyk, Virto 2017

Experimental constraints :

- The residues of the poles are given by $B \to K^* \psi_n$:

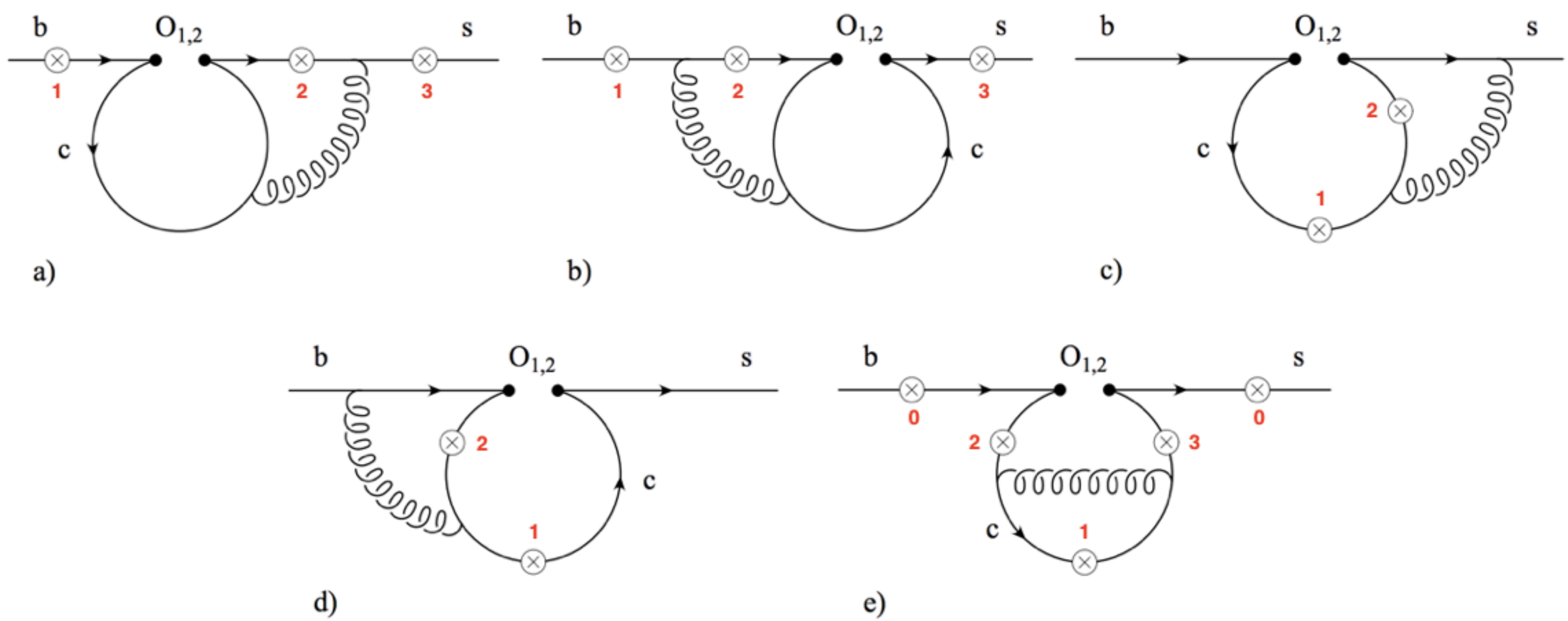
$$\mathcal{H}_\lambda(q^2 \to M_{\psi_n}^2) \sim \frac{M_{\psi_n} f_{\psi_n}^* \mathcal{A}_\lambda^{\psi_n}}{M_B^2 (q^2 - M_{\psi_n}^2)} + \cdots$$

- Angular analyses Belle, Babar, LHCb determine :

$$|r_\perp^{\psi_n}|,\ |r_{||}^{\psi_n}|,\ |r_0^{\psi_n}|,\ \arg\{r_\perp^{\psi_n} r_0^{\psi_n *}\},\ \arg\{r_{||}^{\psi_n} r_0^{\psi_n *}\},$$

where $\quad r_\lambda^{\psi_n} \equiv \underset{q^2 \to M_{\psi_n}^2}{\text{Res}} \frac{\mathcal{H}_\lambda(q^2)}{\mathcal{F}_\lambda(q^2)} \sim \frac{M_{\psi_n} f_{\psi_n}^* \mathcal{A}_\lambda^{\psi_n}}{M_B^2\, \mathcal{F}_\lambda(M_{\psi_n}^2)}$

Objective: Fully analytical calculation in two variables: q^2 and m_c.



Two-loop Master Integrals

$$J_i(q^2, m_c) = (2\pi)^{-2d} \int \frac{(m_b^2)^{N_i-4} (\tilde{\mu}^2)^{2\epsilon}\, d^d\ell\, d^d r}{P_{i_1}^{n_{i_1}} P_{i_2}^{n_{i_2}} P_{i_3}^{n_{i_3}} P_{i_4}^{n_{i_4}} P_{i_5}^{n_{i_5}} P_{i_6}^{n_{i_6}} P_{i_7}^{n_{i_7}}}$$

$$\begin{aligned}
P_1 &= (\ell+q)^2 - m_c^2 & P_5 &= (r+p-q)^2 & P_9 &= \ell\cdot q \\
P_2 &= \ell^2 - m_c^2 & P_6 &= r\cdot q & P_{10} &= (r+p-q)^2 - m_b^2 \\
P_3 &= (\ell+r)^2 - m_c^2 & P_7 &= \ell\cdot(p-q) & P_{11} &= (r+p)^2 - m_b^2 \\
P_4 &= r^2 & P_8 &= (r+p)^2 & P_{12} &= (\ell+r+q)^2 - m_c^2 \\
P_{13} &= r\cdot(p-q) & & & &
\end{aligned}$$

Differential Equations in Canonical Form

Henn 2013

$$J_i(q^2, m_c) = (2\pi)^{-2d} \int \frac{(m_b^2)^{N_i-4} (\tilde{\mu}^2)^{2\epsilon}\, d^d\ell\, d^d r}{P_{i_1}^{n_{i_1}} P_{i_2}^{n_{i_2}} P_{i_3}^{n_{i_3}} P_{i_4}^{n_{i_4}} P_{i_5}^{n_{i_5}} P_{i_6}^{n_{i_6}} P_{i_7}^{n_{i_7}}}$$

$$\partial_x J_{i,k}(\epsilon, x, y) = a_{i,x}^{k\ell}(\epsilon, x, y)\, J_{i,\ell}(\epsilon, x, y)\ , \quad \partial_y J_{i,k}(\epsilon, x, y) = a_{i,y}^{k\ell}(\epsilon, x, y)\, J_{i,\ell}(\epsilon, x, y)\ ,$$

$\to$ Transformation to “Canonical” Basis: $\vec{M}(x, y) = T(\epsilon, x, y) \cdot \vec{J}(x, y)$

$$\partial_x \vec{M}(\epsilon, x, y) = \epsilon\, A_x(x, y)\, \vec{M}(\epsilon, x, y) \quad ; \quad \partial_y \vec{M}(\epsilon, x, y) = \epsilon\, A_y(x, y)\, \vec{M}(\epsilon, x, y)$$

Asatrian, Greub, Virto 2019

Iterative solution of DEs

$$\partial_x \vec{M}(\epsilon, x, y) = \epsilon\, A_x(x,y)\, \vec{M}(\epsilon, x, y) \quad ; \quad \partial_y \vec{M}(\epsilon, x, y) = \epsilon\, A_y(x,y)\, \vec{M}(\epsilon, x, y)$$

$$\vec{M}(\epsilon, x, y) = \sum_{n=0}^{\infty} \epsilon^n\, \vec{M}_n(x,y)$$

$$\partial_{x,y} \vec{M}_n(x,y) = A_{x,y}(x,y) \vec{M}_{n-1}(x,y)$$

Asatrian, Greub, Virto 2019

Iterative solution of DEs First y dependence, then x:

$$\begin{aligned}
\vec{M}_0(x,y) &= \vec{C}_0(x)\ , \\
\vec{M}_1(x,y) &= \sum_{j_1} \left[A_y^{j_1}\, G(w_{j_1}(x);y)\right] \vec{C}_0(x) + \vec{C}_1(x)\ , \\
\vec{M}_2(x,y) &= \sum_{j_2,j_1} \left[A_y^{j_2}\, A_y^{j_1}\, G(w_{j_2}(x), w_{j_1}(x);y)\right] \vec{C}_0(x) \\
&\quad + \sum_{j_2} \left[A_y^{j_2}\, G(w_{j_2}(x);y)\right] \vec{C}_1(x) + \vec{C}_2(x)\ , \\
\vec{M}_3(x,y) &= \cdots
\end{aligned} \tag{1}$$

Iterative solution of DEs

Solutions in terms of **Generalized Polylogarithms (GPLs)** Goncharov 1998

$$G(w_1,\dots,w_n;y) = \int_0^y \frac{dt}{t-w_1} G(w_2,\dots,w_n;t)\ ; \quad G(;y)=1\ ; \quad G(\vec{0}_n;x) = \frac{\log^n x}{n!}$$

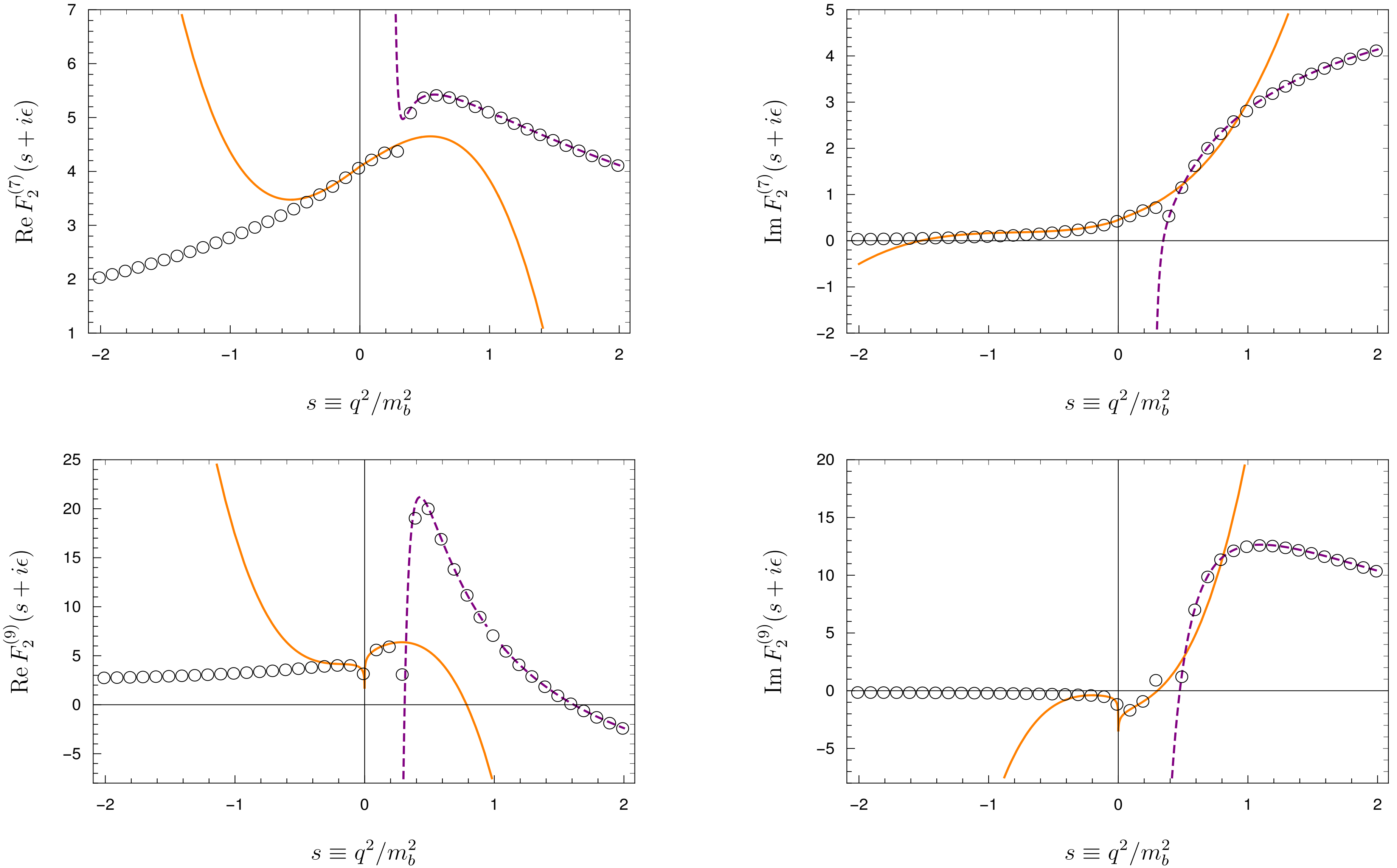
i.e

$$G(1;x) = \log(1-x)\ , \quad G(0,1;x) = -Li_2(x)\ , \quad G(0,0,1;x) = -Li_3(x)\dots$$

Fast numerical evaluation of general GPLs in the complex plane available (C++, python, matlab, ...)

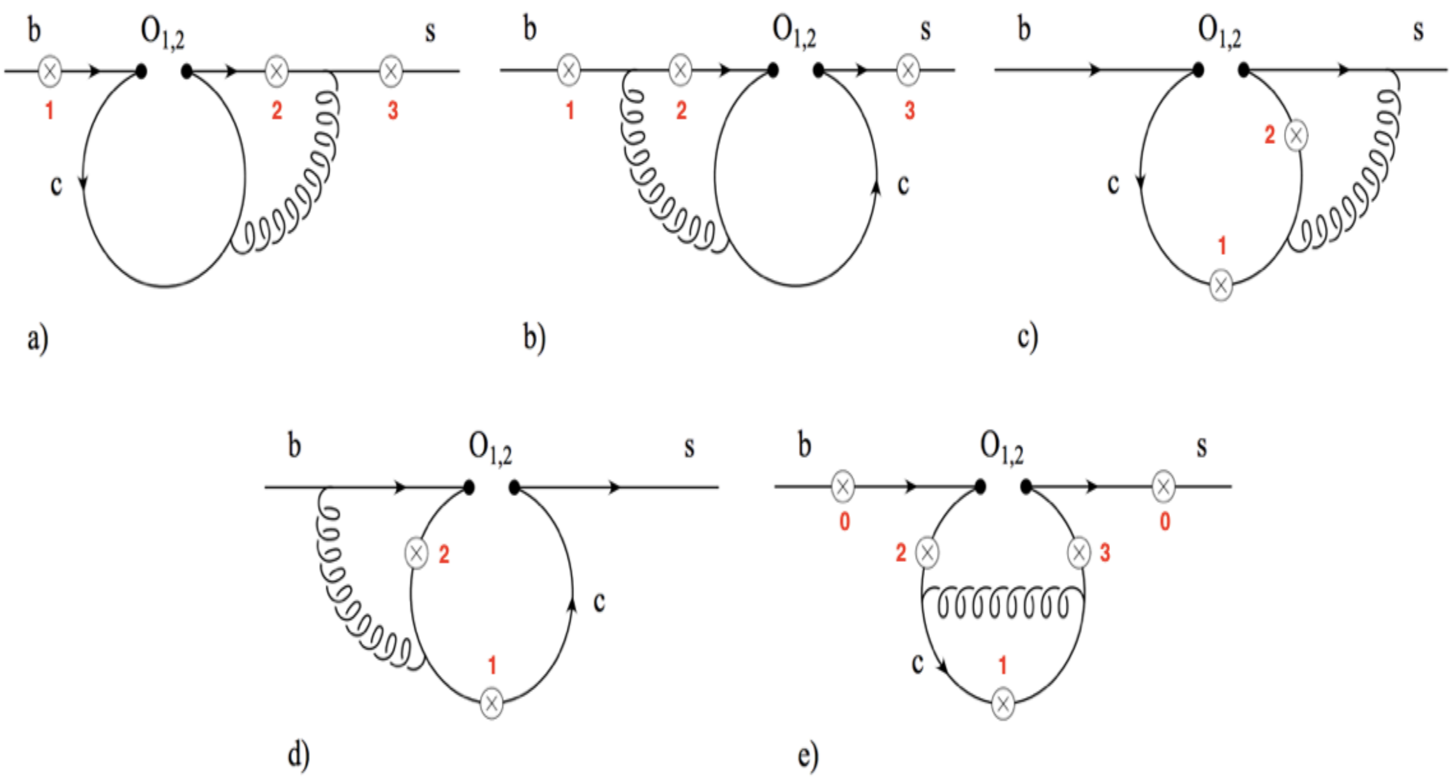
Asatrian, Greub, Virto 2019

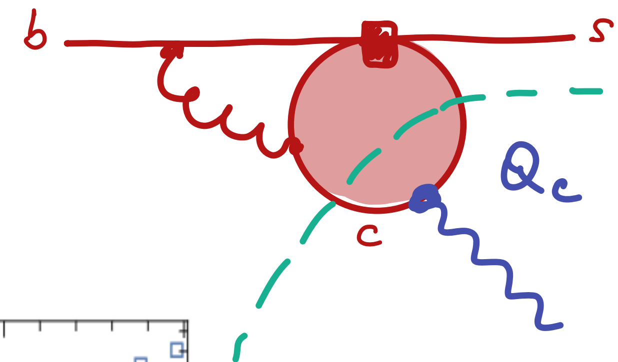
Results: Comparison to previous calculations:



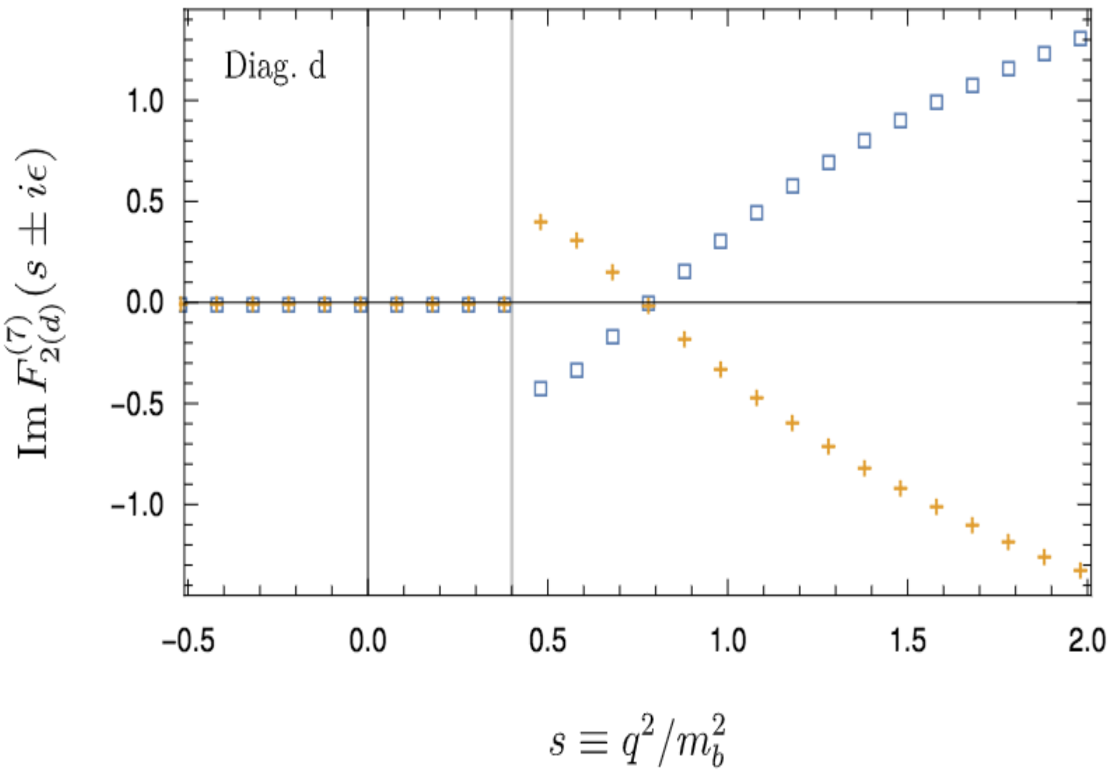
Asatrian, Greub, Virto 2019

Analytic structure in q^2 plane



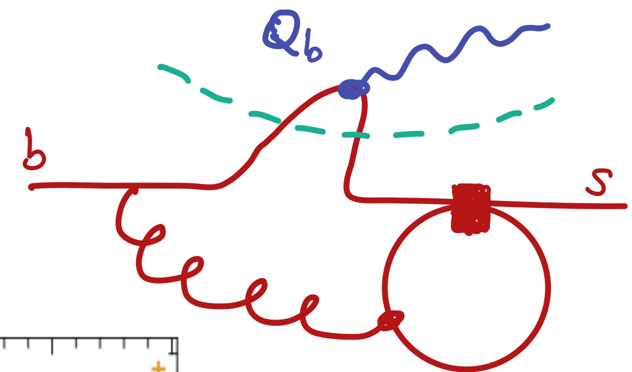


Checking analytic structure of $\mathcal{H}(q^2)$

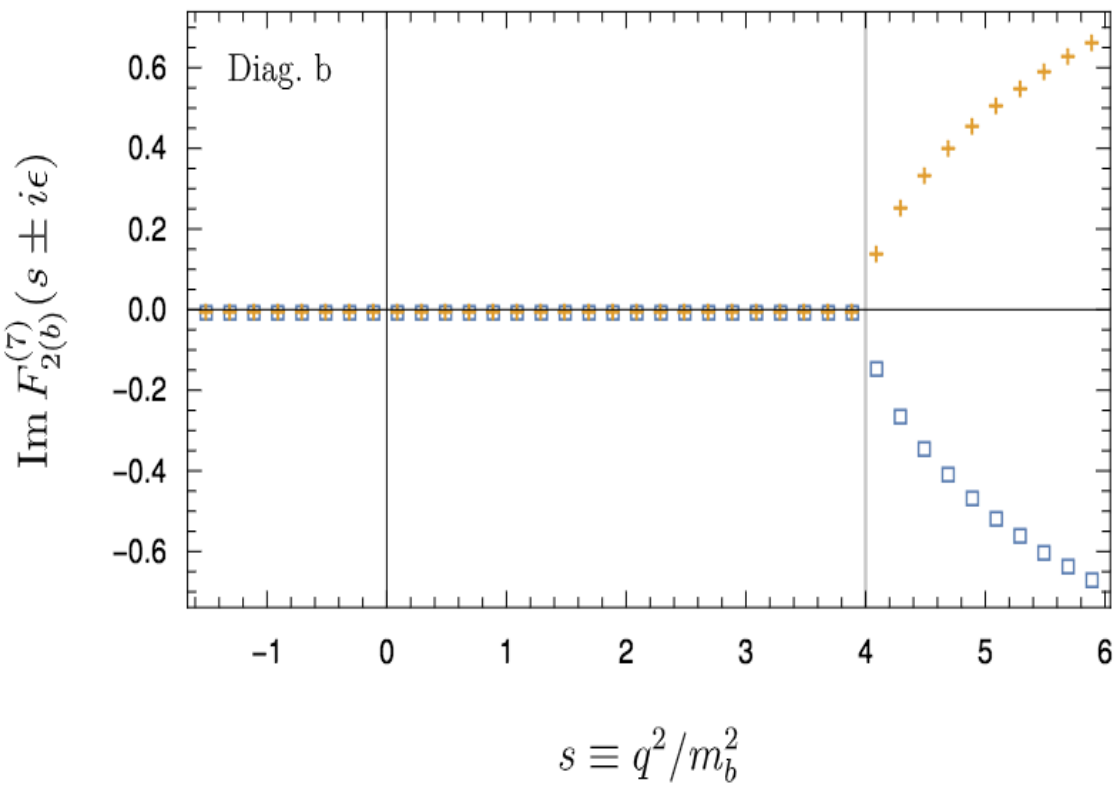


Matching calculation at NLO

Asatrian, Greub, Virto 2019

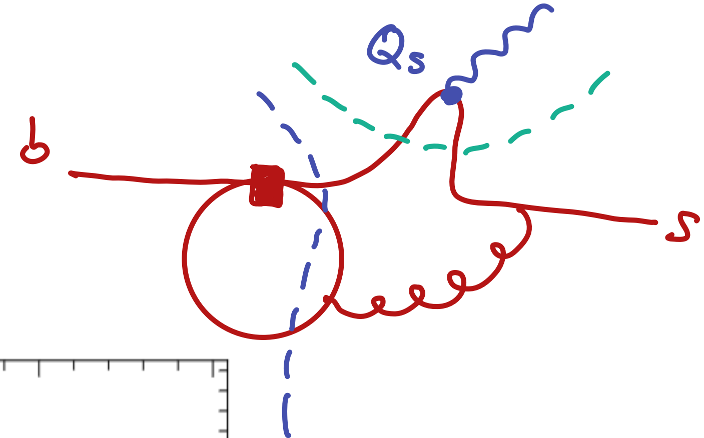


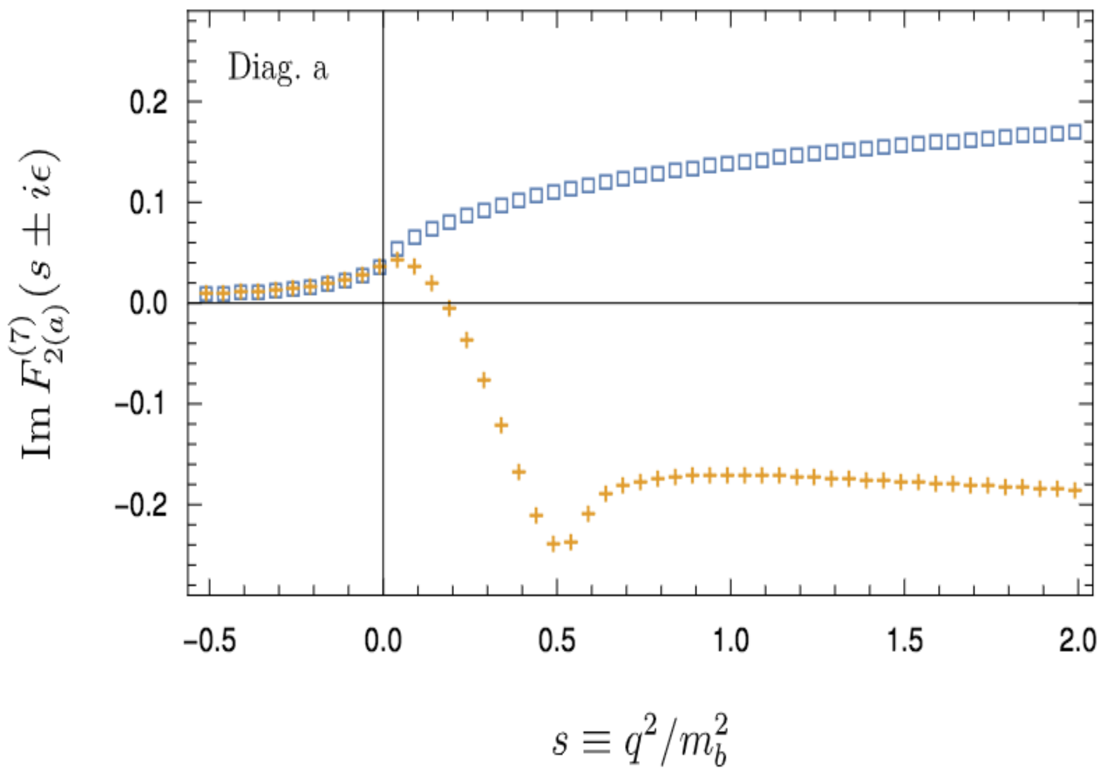
Checking analytic structure of $\mathcal{H}(q^2)$



Asatrian, Greub, Virto 2019

Checking analytic structure of $\mathcal{H}(q^2)$





Differential decay rate of $\tau \to K\pi\nu_\tau$:

$$\frac{d\Gamma}{ds} = \frac{N_\tau}{s^3}\left(1-\frac{s}{m_\tau^2}\right)^2\left(1+2\frac{s}{m_\tau^2}\right)\lambda_{K\pi}^{3/2}\,|\widetilde{f}_+(s)|^2\left\{1+\frac{3(\Delta m^2)^2}{(1+2s/m_\tau^2)\,\lambda_{K\pi}}\,|\widetilde{f}_0(s)|^2\right\}$$

with the normalization [Total BR will give $|f_+(0)|^2 = 0.99$, consistent with $f_+^{LQCD}(0) = 0.97$]

$$N_\tau = \frac{G_F^2\,|V_{us}|^2\,|f_+(0)|^2\,m_\tau^3}{1536\pi^3}S_{EW}^{\rm had}$$

Belle fits to models: [This gives $f_{K^*} \simeq 205$ MeV, compared to $f_{K^*} = 217(5)$ MeV (NWL)]

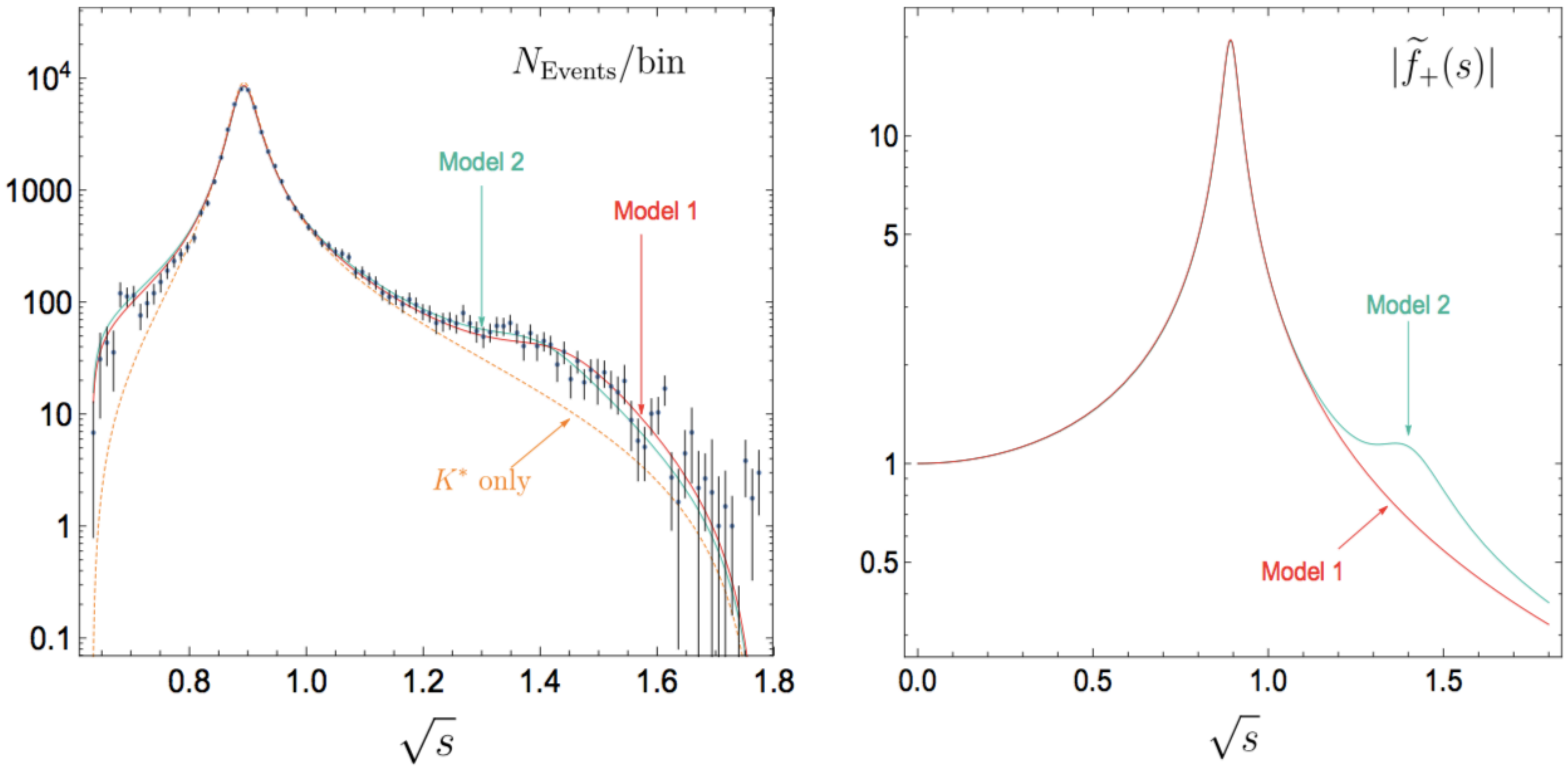
$$\widetilde{f}_+(s) = \sum_R \frac{\xi_R\,m_R^2}{m_R^2 - s - i\sqrt{s}\,\Gamma_R(s)}\ ,\quad f_0(s) = f_+(0)\cdot\sum_{R_0}\frac{\xi_{R_0}\,s}{m_{R_0}^2 - s - i\sqrt{s}\,\Gamma_{R_0}(s)}\ ,$$

$$\textbf{Model 1}: \qquad \xi_{K^*(892)} = 1\,,\ \xi_{K_0^*(800)} = 1.27\,,\ \xi_{K_0^*(1430)} = 0.954\,e^{i\,0.62}$$

$$\textbf{Model 2}: \qquad \xi_{K^*(892)} = 0.988\,e^{-i\,0.07}\,,\ \xi_{K^*(1410)} = 0.074\,e^{i\,1.37}\,,\ \xi_{K_0^*(800)} = 1.57$$

$K\pi$ form factor $f_+(s)$ from $\tau \to K\pi\nu_\tau$

Data from Belle, arXiv:0706.2231 [hep-ex]



Effective threshold: 2-point SVZ sum rule

Knowing $|f_+(s)|$ we can extract s_0 from a QCD sum rule:

$$
\begin{aligned}
\Pi_{\mu\nu}(k) &= i\int d^4x e^{ikx}\langle 0|\mathrm{T}\{\bar{d}(x)\gamma_\mu s(x), \bar{s}(0)\gamma_\nu d(0)|0\rangle \\
&= (k_\mu k_\nu - k^2 g_{\mu\nu})\,\Pi(k^2) + k_\mu k_\nu\,\widetilde{\Pi}(k^2)
\end{aligned}
$$

$$
\Pi(M^2, s_0) \equiv \frac{1}{\pi}\int_{s_{\rm th}}^{s_0} ds\, e^{-s/M^2}\,\mathrm{Im}\Pi(s) = \int_{s_{\rm th}}^{s_0} ds\, e^{-s/M^2}\,\frac{\lambda_{K\pi}^{3/2}(s)}{32\pi^2 s^3}\,|f_+(s)|^2
$$

$$
\begin{aligned}
\Pi^{\rm OPE}(M^2, s_0) &= \frac{1}{8\pi^2}\int_{m_s^2}^{s_0} ds\, e^{-s/M^2}\,\frac{(s-m_s^2)^2(2s+m_s^2)}{s^3} \\
&\quad + \frac{\alpha_s(M)}{\pi}\,\frac{M^2}{4\pi^2}\left(1 - e^{-s_0/M^2}\right) + \frac{v_4}{M^2} + \frac{v_6}{2M^4}
\end{aligned}
$$

Effective threshold: 2-point SVZ sum rule

Borel parameter M^2	Effective threshold s_0	
1.00 GeV^2	1.28 ± 0.18 GeV^2 (Model 1) 1.25 ± 0.18 GeV^2 (Model 2)	1.26 ± 0.18 GeV^2 (Average)
1.25 GeV^2	1.33 ± 0.12 GeV^2 (Model 1) 1.31 ± 0.12 GeV^2 (Model 2)	1.31 ± 0.12 GeV^2 (Average)
1.50 GeV^2	1.36 ± 0.09 GeV^2 (Model 1) 1.34 ± 0.09 GeV^2 (Model 2)	1.35 ± 0.09 GeV^2 (Average)

Table 3: *Values for the effective threshold s_0 extracted from the SVZ sum rules.*

Significantly low value compared to the usual $s_0^{K^*} \simeq 1.7$ GeV$^2 \sim (\sqrt{s_0^\rho} + m_s)^2$

Differential decay rate including S,P,D waves – – [$d\Omega = d\cos\theta_\ell \, d\cos\theta_K \, d\phi$]

$$\frac{d\Gamma}{dq^2 dk^2 d\Omega} = \frac{1}{4\pi} \sum_{i=1}^{41} f_i(\Omega) \, \tilde{\Gamma}_i(q^2, k^2)$$

The 41 moments $\tilde{\Gamma}_i(q^2, k^2)$ depend on S, P, D-wave amplitudes:

i	$f_i(\Omega)$	$\Gamma_i^{L,\mathrm{tr}}(q^2)/\mathbf{k}q^2$	$\eta_i^{L\to R}$
1	$P_0^0 Y_0^0$	$\left[\lvert H_0^L\rvert^2 + \lvert H_\parallel^L\rvert^2 + \lvert H_\perp^L\rvert^2 + \lvert S^L\rvert^2 + \lvert D_0^L\rvert^2 + \lvert D_\parallel^L\rvert^2 + \lvert D_\perp^L\rvert^2\right]$	$+1$
2	$P_1^0 Y_0^0$	$2\left[\frac{2}{\sqrt{5}} Re(H_0^L D_0^{L*}) + Re(S^L H_0^{L*}) + \sqrt{\frac{3}{5}}\, Re(H_\parallel^L D_\parallel^{L*} + H_\perp^L D_\perp^{L*})\right]$	$+1$
3	$P_2^0 Y_0^0$	$\frac{\sqrt{5}}{7}\,(\lvert D_\parallel^L\rvert^2 + \lvert D_\perp^L\rvert^2)$ - $\frac{1}{\sqrt{5}}\,(\lvert H_\parallel^L\rvert^2 + \lvert H_\perp^L\rvert^2) + \frac{2}{\sqrt{5}}\,\lvert H_0^L\rvert^2 + \frac{10}{7\sqrt{5}}\,\lvert D_0^L\rvert^2 + 2\, Re(S^L D_0^{L*})$	$+1$
4	$P_3^0 Y_0^0$	$\frac{6}{\sqrt{35}}\left[- Re(H_\parallel^L D_\parallel^{L*} + H_\perp^L D_\perp^{L*}) + \sqrt{3} Re(H_0^L D_0^{L*})\right]$	$+1$
5	$P_4^0 Y_0^0$	$\frac{2}{7}\left[-2(\lvert D_\parallel^L\rvert^2 + \lvert D_\perp^L\rvert^2) + 3\lvert D_0^L\rvert^2\right]$	$+1$
6	$P_0^0 Y_2^0$	$\frac{1}{2\sqrt{5}}\left[(\lvert D_\parallel^L\rvert^2 + \lvert D_\perp^L\rvert^2) + (\lvert H_\parallel^L\rvert^2 + \lvert H_\perp^L\rvert^2) - 2\lvert S^L\rvert^2 - 2\lvert D_0^L\rvert^2 - 2\lvert H_0^L\rvert^2\right]$	$+1$
7	$P_1^0 Y_2^0$	$\left[\frac{\sqrt{3}}{5}\, Re(H_\parallel^L D_\parallel^{L*} + H_\perp^L D_\perp^{L*}) - \frac{2}{\sqrt{5}}\, Re(S^L H_0^{L*}) - \frac{4}{5}\, Re(H_0^L D_0^{L*})\right]$	$+1$
8	$P_2^0 Y_2^0$	$\left[\frac{1}{14}(\lvert D_\parallel^L\rvert^2 + \lvert D_\perp^L\rvert^2) - \frac{2}{7}\lvert D_0^L\rvert^2 - \frac{1}{10}(\lvert H_\parallel^L\rvert^2 + \lvert H_\perp^L\rvert^2) - \frac{2}{5}\lvert H_0^L\rvert^2 - \frac{2}{\sqrt{5}} Re(S^L D_0^{L*})\right]$	$+1$
9	$P_3^0 Y_2^0$	$-\frac{3}{5\sqrt{7}}\left[Re(H_\parallel^L D_\parallel^{L*} + H_\perp^L D_\perp^{L*}) + 2\sqrt{3}\, Re(H_0^L D_0^{L*})\right]$	$+1$
10	$P_4^0 Y_2^0$	$-\frac{2}{7\sqrt{5}}\left[\lvert D_\parallel^L\rvert^2 + \lvert D_\perp^L\rvert^2 + 3\lvert D_0^L\rvert^2\right]$	$+1$
11	$P_1^1 \sqrt{2}\, Re(Y_2^1)$	$-\frac{3}{\sqrt{10}}\left[\sqrt{\frac{2}{3}}\, Re(H_\parallel^L S^{L*}) - \sqrt{\frac{2}{15}}\, Re(H_\parallel^L D_0^{L*}) + \sqrt{\frac{2}{5}}\, Re(D_\parallel^L H_0^{L*})\right]$	$+1$
12	$P_2^1 \sqrt{2}\, Re(Y_2^1)$	$-\frac{3}{[illegible]}\left[Re(H_\parallel^L H_0^{L*}) + \sqrt{\frac{5}{[illegible]}}\, Re(D_\parallel^L S^{L*}) + \frac{5}{[illegible]}\, Re(D_\parallel^L D_0^{L*})\right]$	$+1$

Combinations of moments depending only on P-wave:

$$|\widehat{A}^L_{\|}|^2 + |\widehat{A}^R_{\|}|^2 = \frac{1}{36}(5\tilde{\Gamma}_1 - 7\sqrt{5}\tilde{\Gamma}_3 + 5\sqrt{5}\tilde{\Gamma}_6 - 35\tilde{\Gamma}_8 - 5\sqrt{15}\tilde{\Gamma}_{19} + 35\sqrt{3}\tilde{\Gamma}_{21})$$

$$|\widehat{A}^L_{\perp}|^2 + |\widehat{A}^R_{\perp}|^2 = \frac{1}{36}(5\tilde{\Gamma}_1 - 7\sqrt{5}\tilde{\Gamma}_3 + 5\sqrt{5}\tilde{\Gamma}_6 - 35\tilde{\Gamma}_8 + 5\sqrt{15}\tilde{\Gamma}_{19} - 35\sqrt{3}\tilde{\Gamma}_{21})$$

$$\mathrm{Im}(\widehat{A}^L_{\perp}\widehat{A}^{L*}_{\|} + \widehat{A}^R_{\perp}\widehat{A}^{R*}_{\|}) = \frac{5}{36}(\sqrt{15}\tilde{\Gamma}_{24} - 7\sqrt{3}\tilde{\Gamma}_{26})$$

$$\mathrm{Re}(\widehat{A}^L_{\perp}\widehat{A}^{L*}_{\|} - \widehat{A}^R_{\perp}\widehat{A}^{R*}_{\|}) = \frac{1}{36}(-5\sqrt{3}\tilde{\Gamma}_{29} + 7\sqrt{15}\tilde{\Gamma}_{31})$$

Binned LHCb results (arXiv: 1609.04736) imply:

$$\tau_B\,\langle|\widehat{A}^L_{\|}|^2 + |\widehat{A}^R_{\|}|^2\rangle \equiv \langle M_{\|}\rangle = (1.07 \pm 1.13) \times 10^{-8}$$

$$\tau_B\,\langle|\widehat{A}^L_{\perp}|^2 + |\widehat{A}^R_{\perp}|^2\rangle \equiv \langle M_{\perp}\rangle = (0.94 \pm 1.06) \times 10^{-8}$$

$$\tau_B\,\langle\mathrm{Im}(\widehat{A}^L_{\perp}\widehat{A}^{L*}_{\|} + \widehat{A}^R_{\perp}\widehat{A}^{R*}_{\|})\rangle \equiv \langle M_{\mathrm{im}}\rangle = (-0.75 \pm 0.79) \times 10^{-8}$$

$$\tau_B\,\langle\mathrm{Re}(\widehat{A}^L_{\perp}\widehat{A}^{L*}_{\|} - \widehat{A}^R_{\perp}\widehat{A}^{R*}_{\|})\rangle \equiv \langle M_{\mathrm{re}}\rangle = (0.27 \pm 0.50) \times 10^{-8}$$