Karlsruhe Institute of Technology

Institute for Theoretical Particle Physics

Higher-order calculations for B meson mixing

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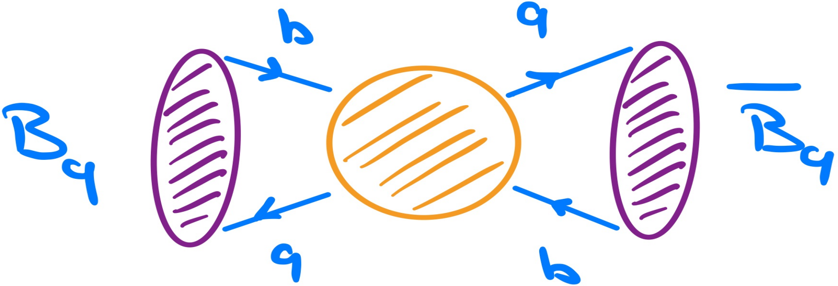
TTP KARLSRUHE



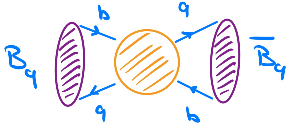
Loops for Meson Mixing

- Introduction
- Framework
- Loop calculations
- Results
- Conclusions

I. Introduction

Lifetime differences

- weak interaction
- $\Delta B = 2$: $B_q \leftrightarrow \overline{B}_q$, $(\bar{b}, q) \leftrightarrow (b, \bar{q})$, $q = d, s$



- time evolution of $(B_q, \overline{B}_q)$ system ($q = d, s$):

$$i\frac{\partial}{\partial t}\begin{pmatrix} B_q \\ \overline{B}_q \end{pmatrix} = \left(M^q - i\frac{\Gamma^q}{2}\right)\begin{pmatrix} B_q \\ \overline{B}_q \end{pmatrix}$$

mass matrix: M^q decay matrix: Γ^q

M_{12}^q: dispersive part of $(M^q - i\Gamma^q/2)_{12}$

$\Gamma_{12}^q/2$: absorptive part of $(M^q - i\Gamma^q/2)_{12}$

- M_{12}^q: dominated by top quarks
  Γ_{12}^q: internal u, c quarks
  interference of $B_q \to f$ and $\bar{B}_q \to f$

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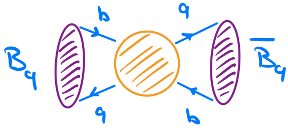

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- diagonalize $M^q - i\Gamma^q/2$ ⇨ eigenvalues: $M_L^q - i\Gamma_L^q/2$, $M_H^q - i\Gamma_H^q/2$
- Define mass and width of B_L and B_H ("light" and "heavy")

  $\Delta M_q = M_H^q - M_L^q$ $\quad\quad$ $\Delta\Gamma_q = \Gamma_L^q - \Gamma_H^q$

  $|B_{q,L}\rangle = p|B_q\rangle + q|\bar{B}_q\rangle$ $\quad\quad$ $|B_{q,H}\rangle = p|B_q\rangle - q|\bar{B}_q\rangle$

Lifetime differences

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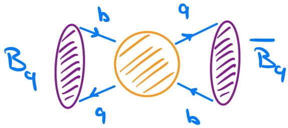

- time evolution of $(B_q, \overline{B}_q)$ system ($q = d, s$):
  M_{12}^q: dispersive part of $(M^q - i\Gamma^q/2)_{12}$
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- Define mass and width of B_L and B_H (“light” and “heavy”)
  $\Delta M_q = M_H^q - M_L^q$ $\quad \Delta\Gamma_q = \Gamma_L^q - \Gamma_H^q$
  $|B_{q,L}\rangle = p|B_q\rangle + q|\bar{B}_q\rangle$ $\quad |B_{q,H}\rangle = p|B_q\rangle - q|\bar{B}_q\rangle$
  ➪
  $\Delta M_q = 2|M_{12}^q|$ $\quad \frac{\Delta\Gamma_q}{\Delta M_q} = -\mathrm{Re}\frac{\Gamma_{12}^q}{M_{12}^q}$ $\quad a_{\mathrm{fs}}^q = \mathrm{Im}\frac{\Gamma_{12}^q}{M_{12}^q}$

CP asymmetry

- CP asymmetry in flavour-specific $B_q \to f$ decays
  (i.e. $\overline{B}_q \to f$ and $B_q \to \bar{f}$ are forbidden)
  standard way to access $a_{\rm fs}^q$: $B_q \to X\ell^-\bar{\nu}_l$
  ("semi-leptonic CP asymmetry"):

$$\Rightarrow \quad a_{\rm fs}^q = \frac{\Gamma(\overline{B}_q(t)\to f) - \Gamma(B_q(t)\to f)}{\Gamma(\overline{B}_q(t)\to f) + \Gamma(B_q(t)\to f)} = \mathrm{Im}\frac{\Gamma_{12}^q}{M_{12}^q} \quad \Rightarrow \quad \text{"small"}$$

- quantifies CP violation in $B - \overline{B}$ mixing
- SM: relative phase between M_{12} and $(-\Gamma_{12})$ is tiny

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$$\Delta M_q = 2|M_{12}^q| \qquad \frac{\Delta\Gamma_q}{\Delta M_q} = -\mathrm{Re}\frac{\Gamma_{12}^q}{M_{12}^q} \qquad a_{\rm fs}^q = \mathrm{Im}\frac{\Gamma_{12}^q}{M_{12}^q}$$

Experiment

[. . . ,CLEO,BABAR,Belle,CDF,D0,ATLAS,CMS,LHCb] [HFLAV'22]

$$\begin{aligned}
\Delta M_s^{\rm exp} &= (17.765 \pm 0.006)\ \text{ps}^{-1} \\
\Delta \Gamma_s^{\rm exp} &= (0.083 \pm 0.005)\ \text{ps}^{-1} \\
a_{\rm fs}^{s,\rm exp} &= -0.0006 \pm 0.0028
\end{aligned}$$

$$\begin{aligned}
\Delta M_d^{\rm exp} &= (0.5065 \pm 0.0019)\ \text{ps}^{-1} \\
\Delta \Gamma_d^{\rm exp} &= (0.001 \pm 0.010)\ \text{ps}^{-1} \\
a_{\rm fs}^{d,\rm exp} &= -0.0021 \pm 0.0017
\end{aligned}$$

➪ $|M_{12}^s|,\ |\Gamma_{12}^s|,\ \arg(-M_{12}^s/\Gamma_{12}^s) \quad |M_{12}^d|,\ |\Gamma_{12}^d|,\ \arg(-M_{12}^d/\Gamma_{12}^d)$

$\Delta M_q = 2|M_{12}^q|$ $\quad \frac{\Delta\Gamma_q}{\Delta M_q} = -\text{Re}\frac{\Gamma_{12}^q}{M_{12}^q}$ $\quad a_{\rm fs}^q = \text{Im}\frac{\Gamma_{12}^q}{M_{12}^q}$

SM precision and sensitivity to new physics

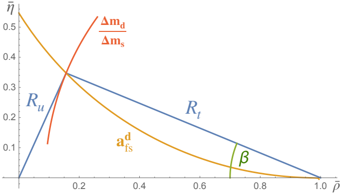
- $\Delta M_s \leftrightarrow |M_{12}^s| \qquad \Delta M_d \leftrightarrow |M_{12}^d|$
- $\Delta\Gamma_s/\Delta M_s$: robust; compare theory and experiment

- ΔM_q sensitive to NP with masses $> \mathcal{O}(100)$ TeV
- $\Delta\Gamma_q$ probes light new particles
- $a_{\rm fs}^d$ and $\Delta\Gamma_d$ “small” ➪ sensitive to NP
- needed: small perturbative and non-perturbative uncertainties

$B - \overline{B}$ mixing and the Unitarity Triangle

- $\Delta M_d / \Delta M_s$ ➪ R_t
- $a_{CP}(B_d(t) \to J/\psi K_S)$ ➪ β
- Independent observable:
  - $R_u \sim |V_{ub}/V_{cb}| \leftrightarrow$ "exclusive vs. inclusive"
  - But: $a_{\rm fs}^d \propto \frac{\sin\beta}{R_t} = \frac{\bar\eta}{\sqrt{(1-\bar\rho)^2+\bar\eta^2}}$

  [Beneke,Buchalla,Lenz,Nierste'02]
- Note: $a_{fs}^d = \mathcal{O}(m_c^2/m_b^2)$
  potentially: NP is not suppressed by m_c^2/m_b^2

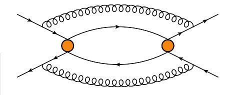


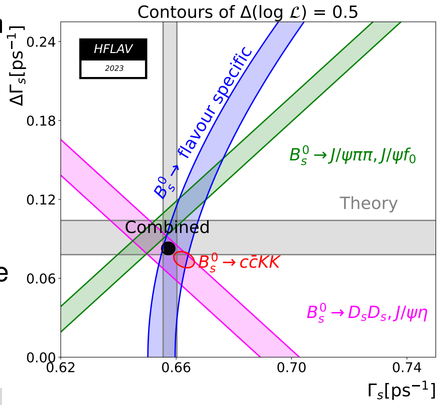
Motivation

- $\Delta\Gamma_s$, $\Delta\Gamma_d$, $a_{\rm fs}^d$, $a_{\rm fs}^s$; $\Delta B = 2$
- $\Delta\Gamma_s$: theory uncertainties $>$ experimental errors
- NLO perturbative uncertainty $\gtrsim$ hadronic uncertainty
- same technique (HQE) and similar local four-quark operators but different sensitivity to new physics
  ➪ simultaneously test formalism
- OME: lattice and sum rules

This talk:

- perturbative corrections
- complete NLO corrections
- NNLO corrections
- results for B system; can also be applied to D system

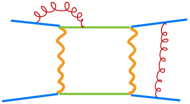


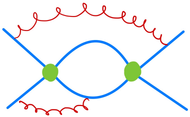


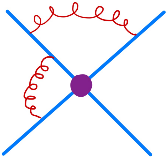
II. Theoretical framework

Effective theories

SM $\rightarrow$ $\mathcal{H}_{\text{eff}}^{|\Delta B|=1}$ $\rightarrow$ $\mathcal{H}_{\text{eff}}^{|\Delta B|=2}$







$\Delta B = 1$

$$\mathcal{H}_{\rm eff}^{|\Delta B|=1} = \frac{4G_F}{\sqrt{2}}\Bigg[-\lambda_t^s\left(\sum_{i=1}^{6} C_i Q_i + C_8 Q_8\right) - \lambda_u^s \sum_{i=1}^{2} C_i(Q_i - Q_i^u)$$
$$+ V_{us}^* V_{cb} \sum_{i=1}^{2} C_i Q_i^{cu} + V_{cs}^* V_{ub} \sum_{i=1}^{2} C_i Q_i^{uc}\Bigg] + \text{h.c.}$$

current-current operators

$$Q_1 = \bar{s}_L \gamma_\mu T^a c_L \; \bar{c}_L \gamma^\mu T^a b_L$$
$$Q_2 = \bar{s}_L \gamma_\mu c_L \; \bar{c}_L \gamma^\mu b_L$$

Penguin operators

$$Q_3 = \bar{s}_L \gamma_\mu b_L \sum_q \bar{q}\gamma^\mu q$$
$$Q_4 = \bar{s}_L \gamma_\mu T^a b_L \sum_q \bar{q}\gamma^\mu T^a q$$
$$Q_5 = \bar{s}_L \gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} b_L \sum_q \bar{q}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} q$$
$$Q_6 = \bar{s}_L \gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} T^a b_L \sum_q \bar{q}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} T^a q$$
$$Q_8 = \frac{g_s}{16\pi^2} m_b \, \bar{s}_L \sigma^{\mu\nu} T^a b_R \, G^a_{\mu\nu}$$

$$\lambda_t^s = V_{ts}^* V_{tb}, \quad \lambda_c^s = V_{cs}^* V_{cb}, \quad \lambda_u^s = V_{us}^* V_{ub}, \quad \lambda_t^s = -\lambda_c^s - \lambda_u^s$$

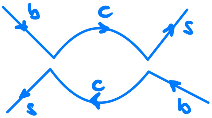
$\Delta B = 1$

$$\mathcal{H}_{\text{eff}}^{|\Delta B|=1} = \frac{4G_F}{\sqrt{2}}\left[-\lambda_t^s\left(\sum_{i=1}^{6} C_i Q_i + C_8 Q_8\right) - \lambda_u^s \sum_{i=1}^{2} C_i (Q_i - Q_i^u) + \ldots\right.$$

- Operator basis [Chetyrkin,Misiak,Münz'98]
- RG-improved Hamiltonian to NNLO

  [Gambino,Gorbahn,Haisch'03; Gorbahn,Haisch'05; Czakon,Haisch,Misiak'06]
- Dirac structures with multiple insertions of γ matrices in $D = 4 - 2\epsilon$ dimensions.
  Example: $(\gamma^\mu\gamma^\nu\gamma^\rho P_L)_{ij} \times (\gamma_\mu\gamma_\nu\gamma_\rho P_L)_{kl}$
- Proper treatment using evanescent operators

  [Dugan,Grinstein'91; Herrlich,Nierste'95]
- Example: $E_2^{(1)} = (\bar{b}_i\gamma^\mu\gamma^\nu\gamma^\rho P_L s_j)(\bar{b}_j\gamma_\mu\gamma_\nu\gamma_\rho P_L s_i) - (16 - 4\epsilon)\widetilde{Q}$
- Note: $\langle E_i^{(j)}\rangle/\epsilon \to$ finite
- NLO: up to 9 γ matrices
  NNLO: up to 11 γ matrices



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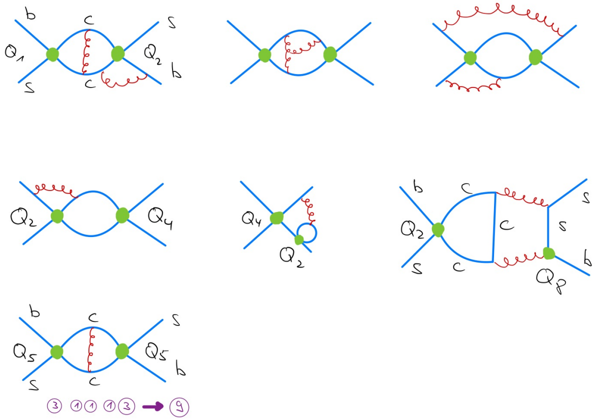
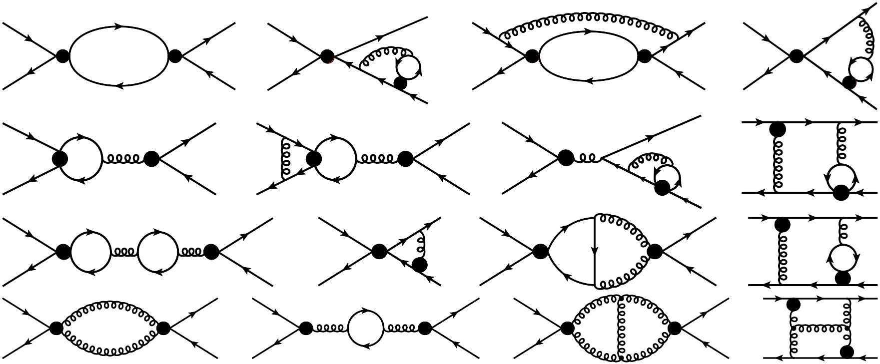


2-loop $\Delta B = 1$ Feynman diagrams



$\Delta B = 2$

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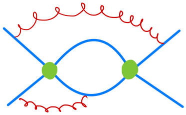
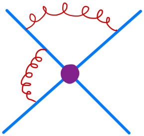
- Heavy Quark Expansion [Khoze,Shifman'83; . . . ; Manohar,Wise'94]

$$\Gamma_{12}^{s} = \frac{1}{2M_{B_s}} \text{Abs}\langle B_s | i \int \mathrm{d}^4x \; T \; \mathcal{H}_{\text{eff}}^{\Delta B=1}(x)\mathcal{H}_{\text{eff}}^{\Delta B=1}(0)|\bar{B}_s\rangle$$



- $\Delta\Gamma_s$ in terms of $|\Delta B| = 2$ operators [Beneke,Buchalla,Greub,Lenz,Nierste'99; . . .]



$$\Gamma_{12}^{s} = -(\lambda_c^s)^2\Gamma_{12}^{cc} - 2\lambda_c^s\lambda_u^s\Gamma_{12}^{uc} - (\lambda_u^s)^2\Gamma_{12}^{uu}$$

$$\Gamma_{12}^{ab} = \frac{G_F^2 m_b^2}{24\pi M_{B_s}} \left[H^{ab}(z)\langle B_s|Q|\bar{B}_s\rangle + \widetilde{H}_S^{ab}(z)\langle B_s|\widetilde{Q}_S|\bar{B}_s\rangle\right] + \mathcal{O}(\Lambda_{\text{QCD}}/m_b)$$

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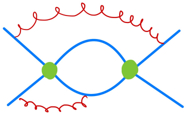

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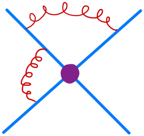

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- $\Delta\Gamma_s$: $|\lambda_u^s| = |V_{us}^\star V_{ub}| \ll |V_{cs}^\star V_{cb}| = |\lambda_c^s|$ ➪ Γ_{12}^{cc} most important
- $\Delta\Gamma_d$ and a_{fs}^d: also Γ_{12}^{uc} und Γ_{12}^{uu} needed

$\Delta B = 2$

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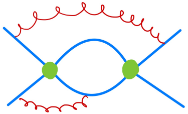
- Heavy Quark Expansion [Khoze,Shifman'83; ...; Manohar,Wise'94]

$$\Gamma_{12}^s = \frac{1}{2M_{B_s}} \text{Abs}\langle B_s | i \int \mathrm{d}^4x \; T \; \mathcal{H}_{\text{eff}}^{\Delta B=1}(x) \mathcal{H}_{\text{eff}}^{\Delta B=1}(0) | \bar{B}_s \rangle$$



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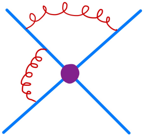

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$$\Gamma_{12}^{ab} = \frac{G_F^2 m_b^2}{24\pi M_{B_s}} \left[H^{ab}(z) \langle B_s | Q | \bar{B}_s \rangle + \widetilde{H}_S^{ab}(z) \langle B_s | \widetilde{Q}_S | \bar{B}_s \rangle \right] + \mathcal{O}(\Lambda_{\text{QCD}}/m_b)$$

$$Q = \bar{s}_i \gamma^\mu (1 - \gamma^5) b_i \; \bar{s}_j \gamma_\mu (1 - \gamma^5) b_j \qquad \widetilde{Q}_S = \bar{s}_i (1 + \gamma^5) b_j \; \bar{s}_j (1 + \gamma^5) b_i$$

-

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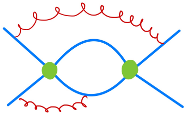

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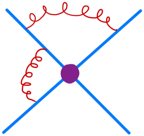

$$\Gamma_{12}^{ab} = \frac{G_F^2 m_b^2}{24\pi M_{B_s}} \left[H^{ab}(z) \langle B_s | Q | \bar{B}_s \rangle + \widetilde{H}_S^{ab}(z) \langle B_s | \widetilde{Q}_S | \bar{B}_s \rangle \right] + \mathcal{O}(\Lambda_{\mathrm{QCD}}/m_b)$$

- $Q = \bar{s}_i \gamma^\mu (1-\gamma^5) b_i \; \bar{s}_j \gamma_\mu (1-\gamma^5) b_j$ $\quad$ $\widetilde{Q}_S = \bar{s}_i (1+\gamma^5) b_j \; \bar{s}_j (1+\gamma^5) b_i$
  $\widetilde{Q} = \bar{s}_i \gamma^\mu (1-\gamma^5) b_j \; \bar{s}_j \gamma_\mu (1-\gamma^5) b_i$ $\quad$ $Q_S = \bar{s}_i (1+\gamma^5) b_i \; \bar{s}_j (1+\gamma^5) b_j$
- + 7 $\quad 1/m_b$ suppressed operators

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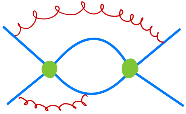

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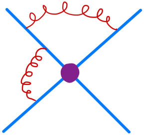

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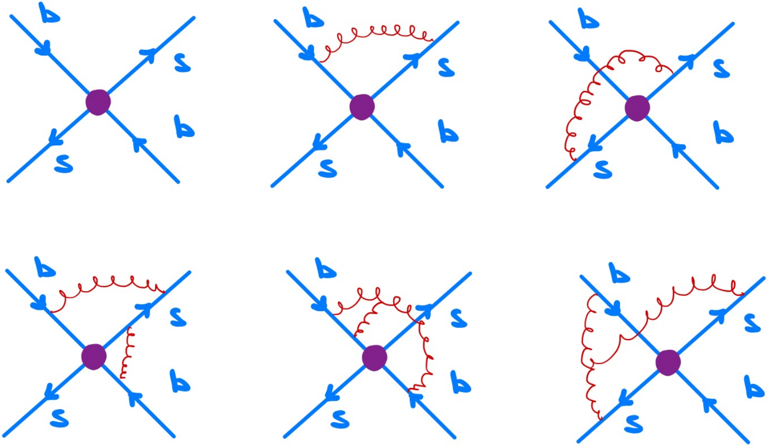
$$\Gamma_{12}^{ab} = \frac{G_F^2 m_b^2}{24\pi M_{B_s}} \left[H^{ab}(z)\langle B_s|Q|\bar{B}_s\rangle + \widetilde{H}_S^{ab}(z)\langle B_s|\widetilde{Q}_S|\bar{B}_s\rangle \right] + \mathcal{O}(\Lambda_{\rm QCD}/m_b)$$

- Nonperturbative MEs from lattice or sum rules [...; Kirk,Lenz,Rauh'17; King,Lenz,Rauh'19'21; Bazavov et al.'16; Dowdall,Davies,Horgan,Lepage,Monahan,et al.'19; Di Luzio,Kirk,Lenz,Rauh'19]
- $H^{ab}(z)$, $\widetilde{H}_S^{ab}(z)$: Wilson coefficients from matching of $\Delta B = 1$ to $\Delta B = 2$ theory

$\Delta B = 2$ Feynman diagrams



R_0

- Q, $\widetilde{Q}_S$ sufficient — $\widetilde{Q}$ and Q_S introduced for convenience

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  with

  $$\alpha_i = 1 + \frac{\alpha_s}{4\pi}\alpha_i^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \alpha_i^{(2)} + \ldots .$$

  Finite UV renormalization which guarantees
  the $1/m_b$ suppression of $\langle R_0 \rangle$

R_0

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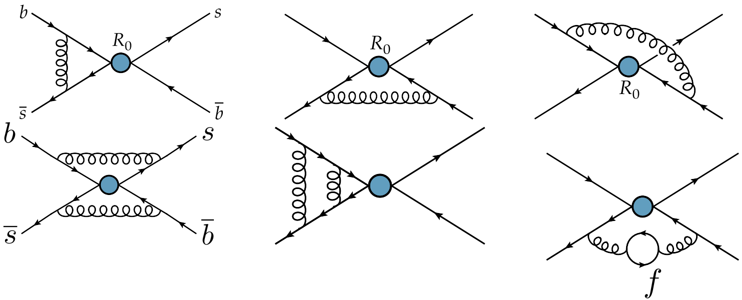
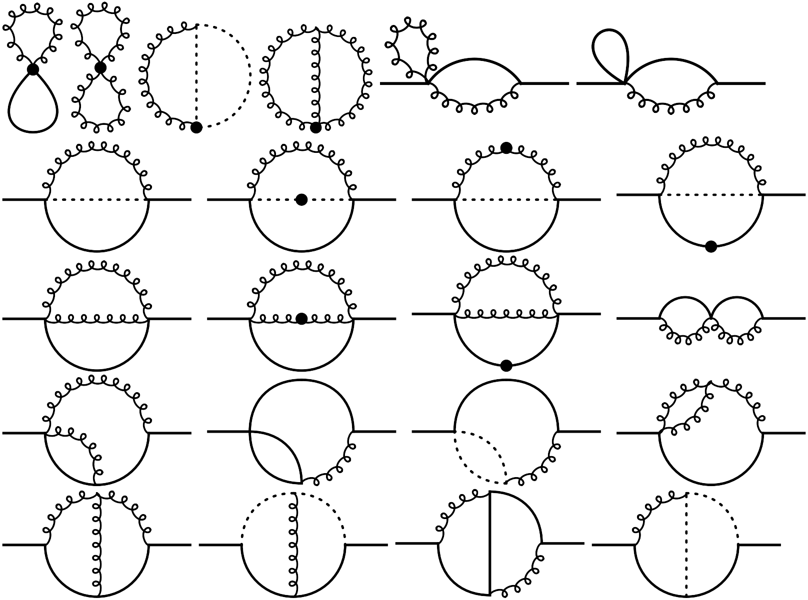
  $$\langle R_0 \rangle = \frac{1}{2}\alpha_1 \langle Q \rangle + \alpha_2 \langle Q_S \rangle + \langle \widetilde{Q}_S \rangle$$
- 1 loop: $\alpha_1^{(1)}$, $\alpha_2^{(1)}$ [Beneke,Buchalla,Greub,Lenz,Nierste'98]
  2 loops n_f: [Asatrian,Hovhannisyan,Nierste,Yeghiazaryan'17]
  Complete 2 loop $\alpha_1^{(2)}$ and $\alpha_2^{(2)}$ [Gerlach,Shtabovenko,Nierste,Steinhauser'22]

R_0



- Important: distinguish UV and IR divergences.
- Introduce gluon mass m_g
  asymptotic expansion for $m_g \ll m_b$
- $\alpha_1^{(1)} = 1 + \frac{\alpha_s(\mu_2)}{4\pi} C_F \left(6 + 12 \log \frac{\mu_2}{m_b}\right)$

Master integrals, $m_g \ll m_b$



Matching

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- $\epsilon \equiv \epsilon_{UV} \equiv \epsilon_{IR}$
- Matching coefficients of evanescent operators needed (in intermediate steps)

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- Matching coefficients of evanescent operators needed (in intermediate steps)
- Example: NLO

$$C = f_0^{(0)} + \epsilon f_1^{(0)} + \frac{\alpha_s}{\pi} f_0^{(1)} \qquad C_E = f_{E,0}^{(0)} + \epsilon f_{E,1}^{(0)} + \frac{\alpha_s}{\pi} f_{E,0}^{(1)}$$

LO matching ➪ $f_0^{(0)}, f_1^{(0)}, f_{E,0}^{(0)}, f_{E,1}^{(0)}$

Matching

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LO matching ➪ $f_0^{(0)}, f_1^{(0)}, f_{E,0}^{(0)}, f_{E,1}^{(0)}$

➪ $A_{\rm ren}^{|\Delta B|=1} - A_{\rm ren}^{|\Delta B|=2}$ finite ➪ $f_0^{(1)}$

  - Only $f_0^{(0)}$ and $f_0^{(1)}$ enter physical quantities
  - $f_{E,0}^{(1)}$ needed for NNLO calculation

III. 2- and 3-loop calculations

Setup

- matching on-shell: $p_b^2 = m_b^2$; $p_s = 0$
- only imaginary part of $|\Delta B| = 1$ diagrams
- asymptotic expansion in $z = m_c^2/m_b^2$
  good approximation after including $\mathcal{O}(z)$ term
- dimensional regularization for UV and IR divergences
- $\Delta B = 1$:

LO (1 loops):	all contributions
NLO (2 loops):	all contributions
NNLO (3 loops):	$Q_{1,2} \times Q_{1,2}$

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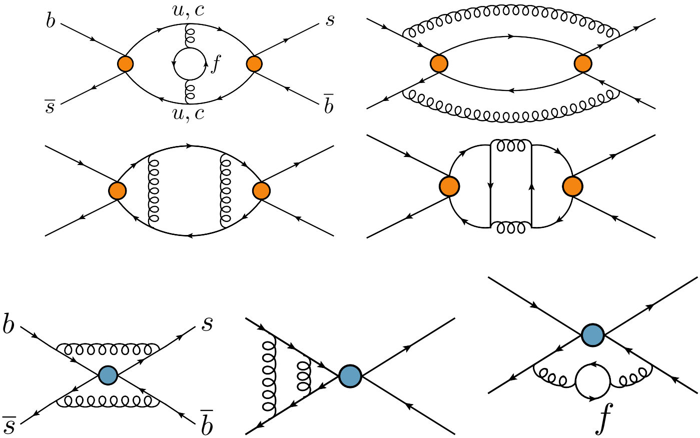
NNLO (3 loops):	$Q_{1,2} \times Q_{1,2}$

qgraf [Nogueira'93] q2e/exp [Seidensticker'99; Harlander,Seidensticker,Steinhauser'99]
tapir [Gerlach,Herren,Lang'22] FORM [Ruijl,Ueda,Vermaseren'17] FIRE [Smirnov,Chuharev'20]
LiteRed [Lee'13'14] FeynArts [Hahn'01] FeynRules [Christensen,Duhr'09; Alloul,Christensen,Degrande,Duhr,Fuks'14] FeynCalc [Mertig,Böhm,Denner'91; Shtabovenko,Mertig,Orellana'16'20] HyperInt [Panzer'15] HyperlogProcedures [Schnetz]
PolyLogTools [Duhr,Dulat'19] FIESTA [Smirnov,Shapurov,Vysotsky'21]
pySecDec [Borowka,Heinrich,Jahn,Jones,Kerner,...'18]


b
u, c
s
f
s̄
u, c
b̄
b
s
s̄
b̄
f

Known results

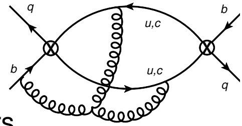
- LO [. . . , Beneke,Buchalla,Greub,Lenz,Nierste'99; Beneke,Buchalla,Dunietz'96]
- NLO Beneke,Buchalla,Greub,Lenz,Nierste'99; Ciuchini,Franco,Lubicz,Mescia,Tarantino'03;

  Beneke,Buchalla,Lenz,Ulrich'03; Lenz,Nierste'06; Asatrian,Asatryan,Hovhannisyan,Nierste,Tumasyan,Yeghiazaryan'20;

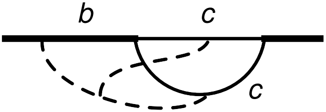
  Gerlach,Nierste,Shtabovenko,Steinhauser'21'22]
- NNLO n_f part: [Asatrian, Hovhannisyan,Nierste,Yeghiazaryan'17]

  full $Q_{1,2} \times Q_{1,2}$: [Gerlach,Nierste,Shtabovenko,Steinhauser'22]
- N^{3}LO $Q_8 \times Q_8$ [Gerlach,Nierste,Shtabovenko,Steinhauser'22]

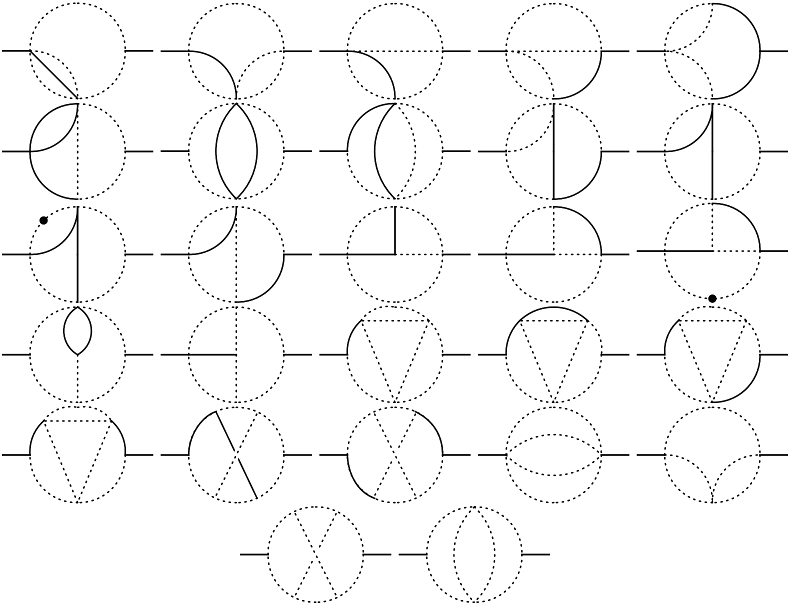
Technical details



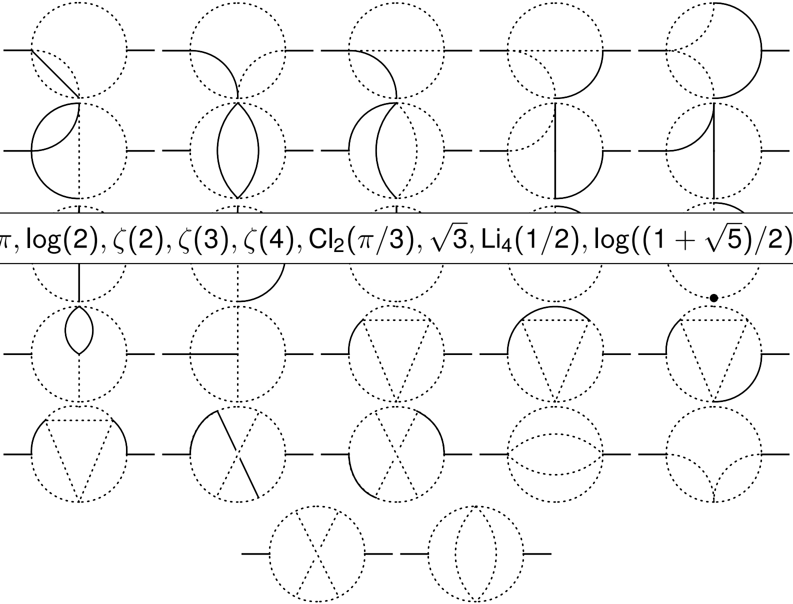
- $\mathcal{O}(20,000)$ diagrams
- $\Delta B = 2$: 3 physical and 17 evanescent operators
- 3-loop integrals, $q_{\text{light quark}} \to 0$
  ➪ 2-point function
  ➪ 2 masses: m_b and m_c
  consider $m_c^2 \ll m_b^2$
- redcution to MIs: `FIRE6` [Smirnov,Chuharev'20], `LiteRed` [Lee'13'14]
- "phase-space MIs"



3-loop MIs

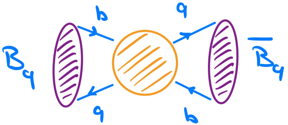


3-loop MIs



$\pi, \log(2), \zeta(2), \zeta(3), \zeta(4), \mathrm{Cl}_2(\pi/3), \sqrt{3}, \mathrm{Li}_4(1/2), \log((1+\sqrt{5})/2)$

Projectors and tensor integrals



➪ 2 spin lines

$$\mathcal{M}(b\bar{s} \to \bar{b}s) \;=\; \sum_{m,n} X^{(m,n)} \Sigma^{(m)}_{i_1 i_2 i_3 i_4} \Gamma^{(n)}_{s_1 s_2 s_3 s_4}\; s^{i_1}_{s_1}\; b^{i_2}_{s_2}\; s^{i_3}_{s_3}\; b^{i_4}_{s_4}$$

$\Sigma^{(m)}$: colour; $\Gamma^{(n)}$: spinor basis elements

Compute scalar coefficients: $X^{(m,n)}$

$$\Gamma^{(n)}_{s_1 s_2 s_3 s_4} \in \{1 \otimes 1,\; 1 \otimes \not{q},\; \ldots \gamma^{\mu_1}\gamma^{\mu_2}\not{q} \otimes \gamma_{\mu_1}\gamma_{\mu_2}\not{q},\; \ldots\}$$

Tensor integrals	Projectors
→ rank 10	→ long traces (evanes. Operators!) sometimes too long for `FORM`
→ system of linear equations (745 × 745)	→ NLO-penguin-penguin: $\mathrm{Tr}(\gamma^{\mu_1} \cdots \gamma^{\mu_{18}})\, \mathrm{Tr}(\gamma_{\mu_1} \cdots \gamma_{\mu_{18}})$
→ large expressions	

IV. Results

Analytic result

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$$
\begin{aligned}
p_{11}^{cc,(2)}(z) = \Bigg(\Bigg(& \frac{4761107}{34992} + \frac{5\pi}{972\sqrt{3}} - \frac{5\pi^2}{4} + N_L\left(-\frac{545}{108} + \frac{5\pi^2}{162}\right) + N_V\left(-\frac{545}{108} + \frac{5\pi^2}{162}\right) \\
& + N_H\left(-\frac{14615}{2916} - \frac{5\pi}{162\sqrt{3}} + \frac{5\pi^2}{162}\right) + N_H\left(\frac{5537}{324} - \frac{\pi^2}{81}\right)z + N_L\left(\frac{5557}{324} - \frac{\pi^2}{81}\right)z \\
& + N_V\left(\frac{1859}{108} - \frac{\pi^2}{81}\right)z + \left(-\frac{676349}{1944} + \frac{7\pi^2}{6}\right)z\Bigg)L_1 + \Bigg(\frac{50783}{1944} - \frac{1139N_H}{972} - \frac{7N_L}{6} - \frac{7N_V}{6} \\
& - \frac{2279z}{27} + \frac{25N_Hz}{9} + \frac{25N_Lz}{9} + \frac{25N_Vz}{9}\Bigg)L_1^2 + \Bigg(-\frac{20549}{972} + \frac{130N_H}{81} + \frac{130N_L}{81} + \frac{130N_V}{81} \\
& - \frac{4\pi^2}{27} + \left(-\frac{2318}{27} - \frac{8\pi^2}{27}\right)z\Bigg)L_2 + \frac{404}{81}L_1L_2 + \left(-\frac{145}{27} + \frac{10N_H}{27} + \frac{10N_L}{27} + \frac{10N_V}{27}\right)L_2^2 \\
& - \frac{20N_V}{27}\Bigg)L_2\log\left(\frac{\mu_2^2}{\mu_1^2}\right) + \left(-228z + \frac{88N_Hz}{9} + \frac{88N_Lz}{9} + \frac{88N_Vz}{9}\right)L_1\log(z) \\
& - \frac{176}{3}z\log\left(\frac{\mu_2}{\mu_1}\right)\log(z)\Bigg) + \Bigg(-\frac{88N_H\pi^2}{27z} + \frac{88N_H\pi^2}{9\sqrt{z}} + \frac{855371N_V\sqrt{z}}{180000} + \frac{88}{9}N_H\pi^2\sqrt{z} \\
& - \frac{71}{972}\pi^2\log(2) - \frac{169}{81}N_H\pi^2\log\left(\frac{1+\sqrt{5}}{2}\right) + \frac{88}{9}N_H\log(z) + \frac{572}{27}N_Hz\log(z) + \frac{572}{27}N_Lz\log(z \\
& + \frac{422}{27}N_Vz\log(z) + \left(-\frac{16607}{27} + \frac{4\pi^2}{27}\right)z\log(z) - \frac{88N_H\log\left(1-\frac{1}{\sqrt{z}}\right)\log(z)}{9z}
\end{aligned}
$$

Input parameters

[PDG; Bazavov et al.'17 (Fermilab-MILC); Dowdall et al.'19 (HPQCD); Chetyrkin et al.'17]

$\alpha_s(M_Z)$	=	0.1179 ± 0.001	$m_c(3\text{ GeV})$	=	0.993 ± 0.008 GeV
m_t^{pole}	=	172.9 ± 0.4 GeV	$m_b(m_b)$	=	4.163 ± 0.016 GeV
M_{B_s}	=	5366.88 MeV	M_{B_d}	=	5279.64 MeV
B_{B_s}	=	0.813 ± 0.034	B_{B_d}	=	0.806 ± 0.041 MeV
$\widetilde{B}'_{S,B_s}$	=	1.31 ± 0.09	$\widetilde{B}'_{S,B_d}$	=	1.20 ± 0.09 MeV
f_{B_s}	=	0.2307 ± 0.0013 GeV	f_{B_d}	=	0.1905 ± 0.0013 MeV

$$\langle B_s|Q(\mu_2)|\overline{B}_s\rangle = \frac{8}{3} M_{B_s}^2\, f_{B_s}^2 B_{B_s}(\mu_2)$$
$$\langle B_s|\widetilde{Q}_S(\mu_2)|\overline{B}_s\rangle = \frac{1}{3} M_{B_s}^2\, f_{B_s}^2 \widetilde{B}'_{S,B_s}(\mu_2)$$

[Davies et al.'19; Dowdall et al.'19]

$$\langle B_s|R_0|\bar{B}_s\rangle = -(0.43 \pm 0.17) f_{B_s}^2 M_{B_s}^2 \,,$$
$$\langle B_s|R_1|\bar{B}_s\rangle = (0.07 \pm 0.00) f_{B_s}^2 M_{B_s}^2 \,, \qquad \langle B_s|\widetilde{R}_1|\bar{B}_s\rangle = (0.04 \pm 0.00) f_{B_s}^2 M_{B_s}^2 \,,$$
$$\langle B_s|R_2|\bar{B}_s\rangle = -(0.18 \pm 0.07) f_{B_s}^2 M_{B_s}^2 \,, \qquad \langle B_s|\widetilde{R}_2|\bar{B}_s\rangle = (0.18 \pm 0.07) f_{B_s}^2 M_{B_s}^2 \,,$$
$$\langle B_s|R_3|\bar{B}_s\rangle = (0.38 \pm 0.13) f_{B_s}^2 M_{B_s}^2 \,, \qquad \langle B_s|\widetilde{R}_3|\bar{B}_s\rangle = (0.29 \pm 0.10) f_{B_s}^2 M_{B_s}^2$$

[Kirk,Lenz,Rauh'17; King,Lenz,Rauh'19'21; . . .]

Physical quantities

Compute Γ_{12}^q:

- $\frac{\Delta\Gamma_q}{\Delta M_q} = -\mathrm{Re}\frac{\Gamma_{12}^q}{M_{12}^q}$
- $\Delta\Gamma_q = \frac{\Delta\Gamma_q}{\Delta M_q}\Delta M_q^{\mathrm{exp}}$
- $a_{\mathrm{fs}}^q = \mathrm{Im}\frac{\Gamma_{12}^q}{M_{12}^q}$

⇨

- λ_t^q, f_{B_q}, M_{B_q} cancel
- bag parameters cancel to large extend

$$\langle B_s|Q|\overline{B}_s\rangle = \frac{8}{3}M_{B_s}^2 f_{B_s}^2 B_{B_s},$$

$$M_{12}^s = (\lambda_t^s)^2 \frac{G_F^2 M_{B_s}}{12\pi^2}\hat{\eta}_B S_0\left(\frac{m_t^2}{m_W^2}\right) f_{B_s}^2 B_{B_s}$$ [Buras,Jamin,Weisz'90]

Renormalization schemes

- m_c, $\overline{\mathrm{MS}}$: $m_c(\mu_c)$; $m_c(3\text{ GeV}) = 0.993\text{ GeV}$
- m_b:
  - $\overline{\mathrm{MS}}$: $m_b(m_b) = 4.163\text{ GeV}$
  - pole: $m_b^{\mathrm{pole}} = 4.76\text{ GeV}$ (2 loops)
  - PS

    $m^{\mathrm{PS}}(\mu_f) = m^{\mathrm{OS}} - \delta m(\mu_f)$ [Beneke'98]

    $\delta m(\mu_f) = -\frac{1}{2}\int_{|\vec{q}\,|<\mu_f} \frac{\mathrm{d}^3 q}{(2\pi)^3} V(\vec{q}\,)$ $V(\vec{q}\,)$: static potential

    $m_b^{\mathrm{PS}}(\mu_f = 2\text{ GeV}) = 4.163 + 0.207 + 0.080 + 0.032 - 0.0004$

- $\frac{\Delta\Gamma_q}{\Delta M_q} \sim \boxed{m_b^2} \times f(z = m_c^2(\mu_c)/m_b^2(\mu_b))$

Numerical results for $\Delta\Gamma_s$

[Gerlach,Nierste,Shtabovenko,Steinhauser'22]

$$\frac{\Delta\Gamma_s}{\Delta M_s} = \left(3.79^{+0.53}_{-0.58}{}_{\rm scale}{}^{+0.09}_{-0.19}{}_{\rm scale,\ 1/m_b} \pm 0.11_{B\tilde{B}_S} \pm 0.78_{1/m_b} \pm 0.05_{\rm input}\right) \times 10^{-3}\ (\text{pole})$$

$$\frac{\Delta\Gamma_s}{\Delta M_s} = \left(4.33^{+0.23}_{-0.44}{}_{\rm scale}{}^{+0.09}_{-0.19}{}_{\rm scale,\ 1/m_b} \pm 0.12_{B\tilde{B}_S} \pm 0.78_{1/m_b} \pm 0.05_{\rm input}\right) \times 10^{-3}\ (\overline{\rm MS})$$

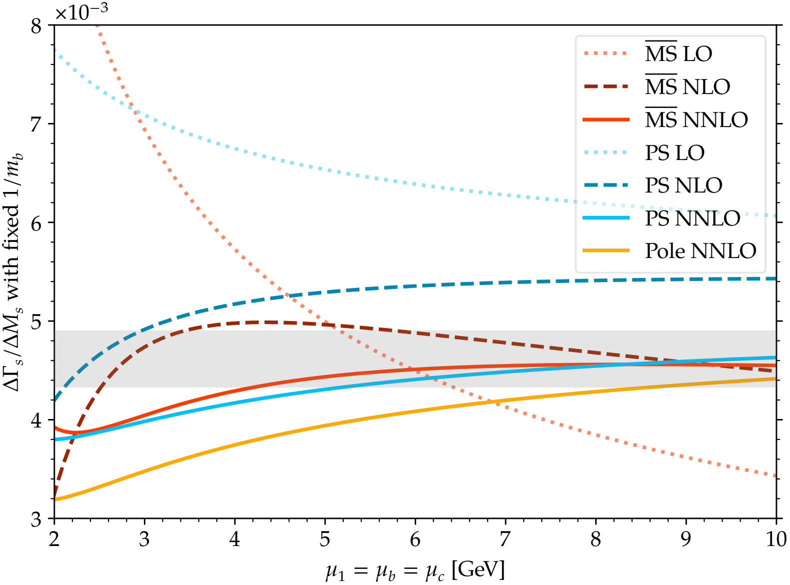
$$\frac{\Delta\Gamma_s}{\Delta M_s} = \left(4.20^{+0.36}_{-0.39}{}_{\rm scale}{}^{+0.09}_{-0.19}{}_{\rm scale,\ 1/m_b} \pm 0.12_{B\tilde{B}_S} \pm 0.78_{1/m_b} \pm 0.05_{\rm input}\right) \times 10^{-3}\ (\text{PS})$$

$$\Delta\Gamma_s^{\rm theo} = (7.6 \pm 1.7) \times 10^{-2}{\rm ps}^{-1}$$
$$\Delta\Gamma_s^{\rm exp} = (8.3 \pm 0.5) \times 10^{-2}{\rm ps}^{-1}$$

- $\overline{\rm MS}$ + PS
- $\mu_1 = \mu_c = \mu_b \in \{2.1, 8.4\}$ GeV
- NLO$\rightarrow$ NNLO: scale dependence reduced by factor 2
- uncertainty dominated by $1/m_b$ correction: NLO needed
- pole scheme inadequate

μ dependence

[Gerlach,Nierste,Shtabovenko,Steinhauser'22]



Numerical results for $\Delta\Gamma_d$ and $a_{\rm fs}^d$ PRELIMINARY

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$$\frac{\Delta\Gamma_d}{\Delta M_d} = \left(3.64^{+0.52}_{-0.58}{}_{\rm scale}{}^{+0.12}_{-0.20}{}_{{\rm scale},\,1/m_b} \pm 0.11_{B\tilde{B}_S} \pm 0.79_{1/m_b} \pm 0.05_{\rm input}\right) \times 10^{-3}\ ({\rm pole})$$

$$\frac{\Delta\Gamma_d}{\Delta M_d} = \left(4.17^{+0.24}_{-0.45}{}_{\rm scale}{}^{+0.12}_{-0.20}{}_{{\rm scale},\,1/m_b} \pm 0.12_{B\tilde{B}_S} \pm 0.79_{1/m_b} \pm 0.05_{\rm input}\right) \times 10^{-3}\ (\overline{\rm MS})$$

$$\frac{\Delta\Gamma_d}{\Delta M_d} = \left(4.05^{+0.36}_{-0.40}{}_{\rm scale}{}^{+0.12}_{-0.20}{}_{{\rm scale},\,1/m_b} \pm 0.12_{B\tilde{B}_S} \pm 0.79_{1/m_b} \pm 0.05_{\rm input}\right) \times 10^{-3}\ ({\rm PS})$$

$$\begin{aligned} \Delta\Gamma_d^{\rm theo} &= (2.10 \pm 0.45) \times 10^{-3}{\rm ps}^{-1} \\ \Delta\Gamma_d^{\rm exp} &= (\ 1 \pm 10) \times 10^{-3}{\rm ps}^{-1} \end{aligned}$$

$$a_{\rm fs}^d = \left(-5.05^{+0.12}_{-0.00}{}_{\rm scale}{}^{+0.01}_{-0.03}{}_{{\rm scale},\,1/m_b} \pm 0.02_{B\tilde{B}_S} \pm 0.14_{1/m_b} \pm 0.11_{\rm input}\right) \times 10^{-4}\ ({\rm pole})$$

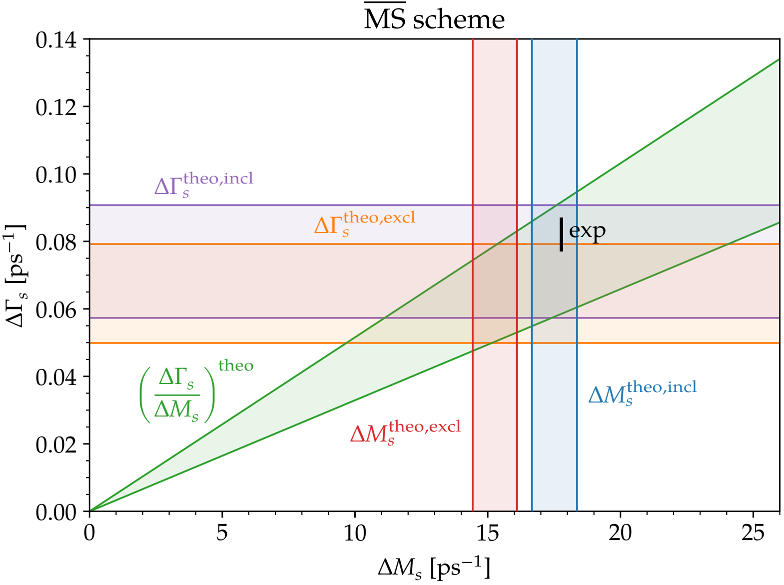
$$a_{\rm fs}^d = \left(-4.96^{+0.40}_{-0.14}{}_{\rm scale}{}^{+0.01}_{-0.03}{}_{{\rm scale},\,1/m_b} \pm 0.02_{B\tilde{B}_S} \pm 0.14_{1/m_b} \pm 0.11_{\rm input}\right) \times 10^{-4}\ (\overline{\rm MS})$$

$$a_{\rm fs}^d = \left(-5.10^{+0.14}_{-0.02}{}_{\rm scale}{}^{+0.01}_{-0.03}{}_{{\rm scale},\,1/m_b} \pm 0.02_{B\tilde{B}_S} \pm 0.14_{1/m_b} \pm 0.11_{\rm input}\right) \times 10^{-4}\ ({\rm PS})$$

$$\begin{aligned} 10^4 \times a_{\rm fs}^{d,\rm theo} &= -5.0 \pm 0.3 \\ 10^4 \times a_{\rm fs}^{d,\rm exp} &= -21 \pm 17 \end{aligned}$$

$\Delta\Gamma_s$ vs. ΔM_s

[Gerlach,Nierste,Shtabovenko,Steinhauser'22]



V. Conclusions and Outlook

Conclusions and outlook

- NNLO corrections to $\Delta\Gamma_s$, $\Delta\Gamma_d$, $a_{\rm fs}^s$, $a_{\rm fs}^d$
- $(m_c^2/m_b^2)^0$ and $(m_c^2/m_b^2)^1$
- reduced μ dependence and theory uncertainty
- dominant uncertainty: $1/m_b$ terms [w.i.p. U. Nierste et al.]

- NNLO contributions from penguin operators
- $a_{\rm fs} \sim m_c^2/m_b^2$ ➪ deeper expansion needed