Decoupling of heavy quarks as a tool to determine α_s

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Mattia Dalla Brida, Roman Höllwieser, Francesco Knechtli, Tomasz Korzec, Rainer Sommer, Stefan Sint.

- *Non-perturbative renormalization by decoupling*. [arXiv: 1912.06001]
- *Determination of $\alpha_s(m_Z)$ by the non-perturbative decoupling method*. [arXiv: 2209.14204]

Motivation

Computing the strength of fundamental interactions

- Take some experimental observable $O(\mu; p)$.
- Work hard to get

$$O(\mu; p) = A(p)\alpha_{\overline{\rm MS}}(\mu) + B(p)\alpha^2_{\overline{\rm MS}}(\mu) + \dots$$

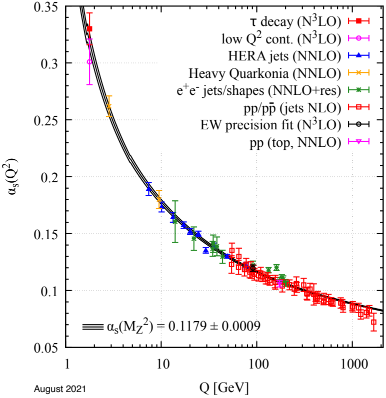
- Determine $\alpha_{\overline{\rm MS}}(\mu)$ by comparing experiment and theory computation

$$\begin{aligned} g_e - 2 : \alpha_{em} &= 7.297\,352\,5698(24) \times 10^{-3} & \tau : \alpha_s(M_Z) &= 0.1198(15) \\ \text{recoil} : \alpha_{em} &= 7.297\,352\,585(48) \times 10^{-3} & e^+e^- : \alpha_s(M_Z) &= 0.1172(37) \end{aligned}$$

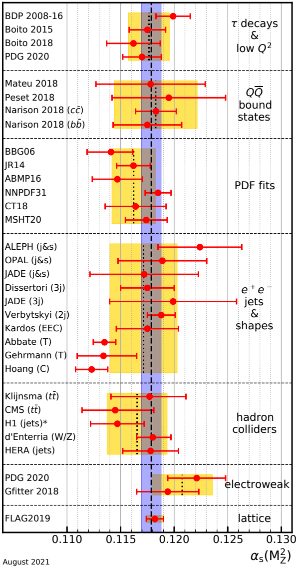
Caveats

- Asymptotic states are not quarks/gluons ("hadronization", ...).
- α_s is larger. Sometimes extracted at a few GeV ($\alpha_s \approx 0.3$!). What about the ...?
  - Perturbative corrections?
  - Non-perturbative corrections?

Determinations of $\alpha_s(m_Z)$ [PDG '21]



- Low energy determinations are more precise (!!?)



Determinations of α_s

All uncertainties from this step ($N \sim 2, 3$)

$$O(Q) \overset{Q\to\infty}{\sim} \alpha_s(Q) + \sum_{n=2}^{N} c_n \alpha_s^n(Q) + \mathcal{O}(\alpha_s^{N+1}(Q)) + \mathcal{O}\left(\frac{\Lambda^p}{Q^p}\right) + \ldots$$

$O(Q)$ (lattice, experiment) $\Longrightarrow \alpha_s(Q)$

No uncertainties here (5-loop running in $\overline{\mathrm{MS}}$)

- Run to a convenient scale (i.e. M_Z)

$$\alpha_s(Q) \longrightarrow \alpha_s(M_Z)$$

- Quote the RGI invariant

$$\alpha_s(Q) \longrightarrow \Lambda_{\overline{\mathrm{MS}}}$$

Uncertainties in $\alpha_s(M_Z)$:

- Non-perturbative uncertainties $\propto \left(\frac{\Lambda}{Q}\right)^p$
- Perturbative uncertainties $\propto \alpha_s^{N+1}(Q)$

The problem: Summary

$$O(Q) \overset{Q\to\infty}{\sim} \alpha_s(Q) + \sum_{n=2} c_n \alpha_s^n(Q) + \mathcal{O}(\alpha_s^{N+1}(Q)) + \mathcal{O}\left(\frac{\Lambda^p}{Q^p}\right) + \dots$$

Non-perturbative corrections

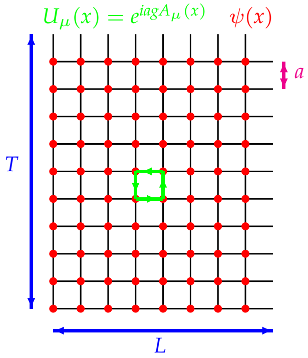
- Difficult to compute (NP physics is difficult!)
- Better use smaller $\alpha \Longrightarrow$ (larger Q)

Perturbative corrections

- Difficult to estimate (i.e. scale variation might fail)
- Main source of uncertainty in most lattice QCD extractions of α_s
- Better use smaller $\alpha \Longrightarrow$ (exponentially larger Q)

Computing path integrals: Lattice field theory

Lattice field theory $\longrightarrow$ Non Perturbative definition of QFT.



- Discretize space-time in an hyper-cubic lattice (spacing a)
- Path integral $\longrightarrow$ multiple integral (one variable for each field at each point)
- Compute the integral numerically $\rightarrow$ Monte Carlo sampling.

$$\langle O \rangle = \frac{1}{N_{\text{conf}}} \sum_{i=1}^{N_{\text{conf}}} O(U_i) + \mathcal{O}(1/\sqrt{N_{\text{conf}}})$$

Observable computed averaging over samples

- This works both in the perturbative and non-perturbative regimes!

$$S_G[U] = \frac{\beta}{6} \sum_{p \in \text{Plaquettes}} Tr(1 - U_p - U_p^+) \xrightarrow[a \to 0]{} -\frac{1}{2} \int d^4x \, \text{Tr}(F_{\mu\nu} F_{\mu\nu})$$

The problem: α_s extractions are a multi-scale problem

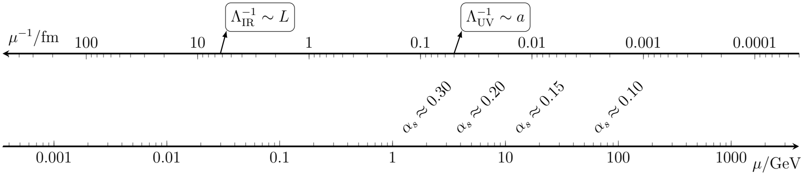
$$O(Q) \overset{Q\to\infty}{\sim} \alpha_s(Q) + \sum_{n=2} c_n \alpha_s^n(Q) + \mathcal{O}(\alpha_s^{N+1}(Q)) + \mathcal{O}\left(\frac{\Lambda^p}{Q^p}\right) + \ldots$$

Why not just use larger Q?

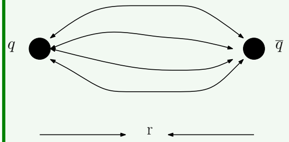
Experimentalist: At large Q the effect you are trying to measure is "weak" $\Longrightarrow$ Larger uncertainties

Latticero: In all simulations $a^{-1} \gg Q \gg L^{-1}$. You need $m_\pi L \approx 4$, so with current computers ($L/a \approx 128$) we have $Q \ll 4$ GeV. In fact:

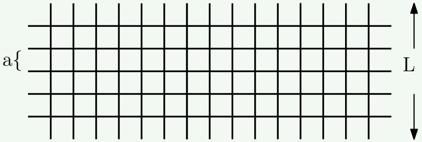
- Computer cost $\propto (L/a)^7$
- Non-perturbative uncertainties $\propto (a/L)^p$
- Perturbative uncertainties $\propto 1/\log(L/a)$



The strength of YM



$$\alpha_{qq}(\mu) = \frac{3r^2}{4} F(r)\Big|_{\mu=1/r}$$



$$a \ll \frac{1}{\mu_{\max}} < \frac{1}{\mu_{\min}} \ll L$$

- Take $O(Q) = \frac{3r^2}{4} F(r)\Big|_{Q=1/r}$
- This defines the "potential scheme". Non-perturbative coupling definition.

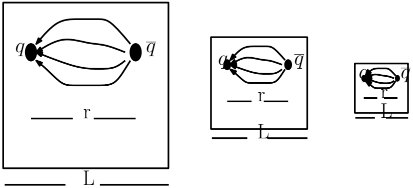
$$\alpha_{qq}(Q) = \frac{3r^2}{4} F(r)\Big|_{Q=1/r} \overset{Q\to\infty}{\sim} \alpha_{\overline{\mathrm{MS}}}(Q) + \ldots$$

- Useful to define convenient scales. i.e. the CERN scale

$$\alpha_{qq}(\mu_{\mathrm{CERN}}) = 12.34/(4\pi)$$

(**NOTE:** Many lattice scales are basically this!: $r_0, t_0, w_0, r_1, \ldots$)

The soution: Finite size scaling [Lüscher, Weisz, Wolff '91]



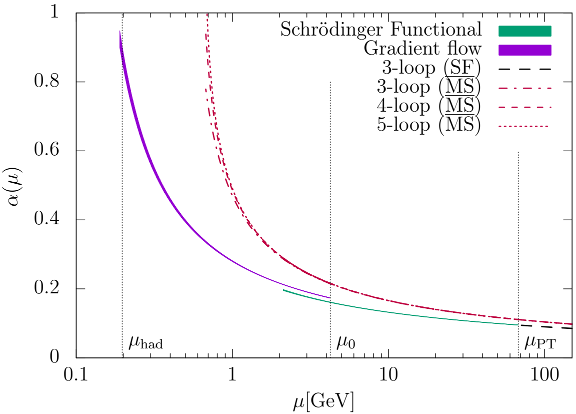
Finite volume renormalization schemes: fix QL = constant

- Coupling $\alpha(Q)$ depends on no other scale but L (Notation: $\alpha(L), \alpha(1/L)$).
- Small $L \Longrightarrow$ small $\alpha(L)$
- $a \ll 1/Q$ easily achieved: $L/a \sim 10 - 40$
- Step scaling function: How much changes the coupling when we change the renormalization scale:

  $$\sigma(u) = g^2(Q/2)\Big|_{g^2(Q)=u}$$

  achieved by simple changing $L/a \to 2L/a$!
- $1/L$ is a IR cutoff $\Rightarrow$ simulate directly $m_q = 0$
- We need dedicated simulations of the **femto-universe**

Results for $\alpha_s(M_Z)$ [ALPHA '17. Phys.Rev.Lett (2017) 119. [arXiv:1706.03821]]



- Non-perturbative running from 200 MeV to 140 GeV
- Many technical improvements:
  - Gradient flow couplings
  - Symanzik analysis of cutoff effects
  - ...

$$\alpha_s(M_Z) = 0.11852(84)\ [0.7\%]\,.$$

Checkpoint

- Extraction of α_s is a very difficult multi-scale problem on the lattice.
- Computational cost grows like $(L/a)^7$
- Perturbative uncertainties decrease as $\log \mu$
- Perturbative uncertainties $\approx 1 - 2\%$ for most large volume approaches **[L. Del Debbio, A. Ramos. Phys.Rep. (2021) 970 [arXiv:2101.04762]]**
- Dedicated approach: step scaling. **Solves** the multi-scale problem.

$$\alpha_s(M_Z) = 0.11852(84)\ [0.7\%]\,.$$

Massless renormalization schemes: Tremendous advantages

- Renormalization group functions are mass independent

$$\mu \frac{\mathrm{d}\bar{g}^2(\mu)}{\mathrm{d}\mu} = \beta(\bar{g}, \cancel{m}) \,.$$

- RGI invariants that characterize the running (i.e $\Lambda, M, B_K, \dots$) **only** exists in massless schemes

$$\Lambda_s = \mu \left[b_0 \bar{g}_s^2(\mu) \right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0 \bar{g}_s^2(\mu)}} \exp\left\{ -\int_0^{\bar{g}_s(\mu)} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x} \right] \right\}$$

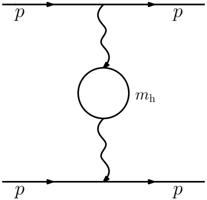
- Precision: high loop computations available in perturbation theory

$$\beta_{\overline{\mathrm{MS}}}(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3 (b_0 + b_1 \bar{g}^2 + b_2^{\overline{\mathrm{MS}}} \bar{g}^4 + b_3^{\overline{\mathrm{MS}}} \bar{g}^6 + b_4^{\overline{\mathrm{MS}}} \bar{g}^8 + \text{unknown})$$

  Always universal but universal only in massless schemes

- In LQCD: easier to define the chiral point ($m_q = 0$) than the physical point ($m_q =$??)

Decoupling of heavy quarks: Perturbation theory



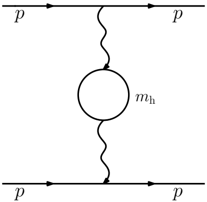
Quark-Quark scattering with N_l light and one heavy

$$T = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\left\{ T_1(p,m) + \frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2} + \mathrm{c}\right\} + \mathcal{O}(\alpha^3)$$

Five Stages of understanding: (I) Denial

- If I choose $\mu \approx m_h(\mu)$ the $T_1(p,m)$ gets large...
- The computation has to be wrong, because this heavy quark cannot break perturbation theory

Decoupling of heavy quarks: Perturbation theory



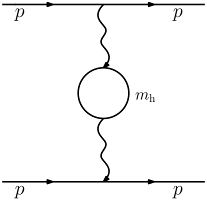
Quark-Quark scattering with $N_{\rm l}$ light and one heavy

$$T = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\left\{ T_1(p,m) + \frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2} + {\rm c}\right\} + \mathcal{O}(\alpha^3)$$

Five Stages of understanding: (II) Anger

- So the existence of a quark with $m_{\rm h} \sim 2000$ TeV is breaking perturbation theory at scale $p \approx \mu \approx 20$ GeV.
- Nonsense!!!!!!
- Nothing works!!!!!

Decoupling of heavy quarks: Perturbation theory



Quark-Quark scattering with N_l light and one heavy

$$T = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\left\{T_1(p,m) + \frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2} + \mathrm{c}\right\} + \mathcal{O}(\alpha^3)$$

Five Stages of understanding: (III) Bargaining

ALICE: Look, If I only could say that

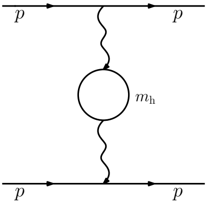
$$\frac{\alpha'(\mu)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2}$$

Then everything would make sense:

$$T = \frac{\alpha'(\mu)}{\pi} + \frac{\alpha'^2(\mu)}{\pi^2}\left[T_1(p,m) + \mathrm{c}\right] + \mathcal{O}(\alpha^3)$$

But then the coupling would depend on m_h!

Decoupling of heavy quarks: Perturbation theory



Quark-Quark scattering with N_l light and one heavy

$$T = \frac{\alpha_{\overline{\mathrm{MS}}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\mathrm{MS}}}(\mu)}{\pi^2}\left\{ T_1(p,m) + \frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2} + \mathrm{c}\right\} + \mathcal{O}(\alpha^3)$$

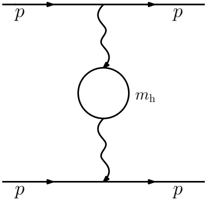
Five Stages of understanding: (IV) The right question

BOB: And this coupling of yours...

$$\frac{\alpha'(\mu)}{\pi} = \frac{\alpha_{\overline{\mathrm{MS}}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\mathrm{MS}}}(\mu)}{\pi^2}\frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2}$$

How would it run?

DECOUPLING OF HEAVY QUARKS: PERTURBATION THEORY



Quark-Quark scattering with N_l light and one heavy

$$T = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\left\{ T_1(p,m) + \frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2} + {\rm c}\right\} + \mathcal{O}(\alpha^3)$$

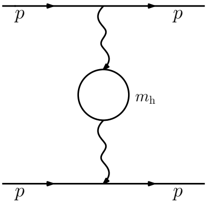
Five Stages of understanding: (V) All fits nicely

$$\frac{\alpha'(\mu)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2}$$

And determine

$$\beta' = \mu^2\frac{\rm d}{{\rm d}\mu^2}\alpha'(\mu) = \left(\mu^2\frac{\partial}{\partial\mu^2} + \beta\frac{\partial}{\partial\alpha} + \gamma\frac{\partial}{\partial m_h}\right)\left[\alpha'_{\overline{\rm MS}}(\mu) + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi}\frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2}\right]$$

Decoupling of heavy quarks: Perturbation theory



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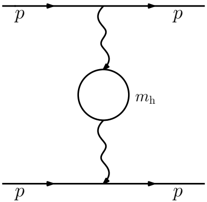
And determine

$$\beta' = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\alpha'(\mu) \overset{\alpha'\to 0}{\sim} \beta - \frac{\alpha'^2(\mu)}{6\pi} + \mathcal{O}(\alpha^3)$$

Decoupling of heavy quarks: Perturbation theory



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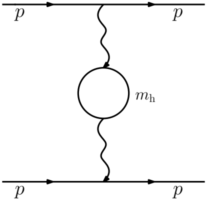
And determine

$$\beta' = \mu^2\frac{{\rm d}}{{\rm d}\mu^2}\alpha'(\mu) \overset{\alpha'\to 0}{\sim} -\frac{\alpha'^2(\mu)}{\pi}\left(\frac{11}{4} - \frac{1}{6}N_{\rm f} + \frac{1}{6}\right) + \mathcal{O}(\alpha^3)$$

Decoupling of heavy quarks: Perturbation theory



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And determine

$$\beta' = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\alpha'(\mu) \overset{\alpha'\to 0}{\sim} -\frac{\alpha'^2(\mu)}{\pi}\left[\frac{11}{4} - \frac{1}{6}(N_f - 1)\right] + \mathcal{O}(\alpha^3)$$

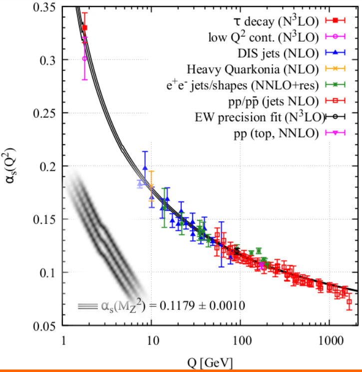
$\alpha'(\mu)$ is the running coupling with $N_l = N_f - 1$ flavors!

Decoupling of heavy quarks in massless schemes

Matching between theories

- At energy scales Q just forget about all quarks with $m > Q$
- "Nice" perturbative expressions if you only use **active** quarks
- Matching between effective theory (with **active quarks**) and fundamental theory (with **active** and heavy quarks)

$$\alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f}-1)}(\mu) = \alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(\mu) \times \left\{1 + a_1(m_h/\mu)\alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(\mu) + \dots\right\}$$



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$$\alpha_{\overline{\rm MS}}^{(N_{\rm f}-1)}(\mu) = \alpha_{\overline{\rm MS}}^{(N_{\rm f})}(\mu) \times \left\{1 + a_1(m_h/\mu)\alpha_{\overline{\rm MS}}^{(N_{\rm f})}(\mu) + \dots\right\}$$

Abuse of language: A single $\alpha_{\overline{\rm MS}}(\mu)$ that "jumps" at quark thresholds

- $\alpha_{\overline{\rm MS}}(4\,{\rm GeV})$: This is the four flavor coupling
- $\alpha_{\overline{\rm MS}}(10\,{\rm GeV})$: This is the five flavor coupling
- $\alpha_{\overline{\rm MS}}(M_Z)$: This is the five flavor coupling

Caveats

Power corrections are neglected (more later)

Decoupling of heavy quarks in massless schemes

$$\Lambda_{\overline{\rm MS}}^{(N_{\rm f})} \xrightarrow{P(M/\Lambda)} \Lambda_{\overline{\rm MS}}^{(N'_{\rm f})}$$

Relation between Λ parameters

If you happen to know $\Lambda_{\overline{\rm MS}}^{(6)}$, then

1. Determine $\alpha_{\overline{\rm MS}}^{(6)}(\mu) = \bar{g}^2_{\overline{\rm MS}}(\mu)/(4\pi)$ at some scale $\mu \approx m_{\rm t}$

$$\frac{\Lambda_{\overline{\rm MS}}^{(6)}}{\mu} = \left[b_0 \bar{g}^2_{\overline{\rm MS}}(\mu)\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2_{\overline{\rm MS}}(\mu)}} \exp\left\{-\int_0^{\bar{g}_{\overline{\rm MS}}(\mu)} dx \left[\frac{1}{\beta_{\overline{\rm MS}}^{(6)}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$

2. Match across the top threshold (4 loops known!)

$$\frac{\bar{g}'^2(\mu)}{4\pi} = \alpha_{\overline{\rm MS}}^{(5)}(\mu) = \alpha_{\overline{\rm MS}}^{(6)}(\mu) \times \left\{1 + a_1(m_{\rm t}/\mu)\alpha_{\overline{\rm MS}}^{(6)}(\mu) + \dots\right\}$$

3. Determine the Λ parameter of the 5 flavor theory

$$\frac{\Lambda_{\overline{\rm MS}}^{(5)}}{\mu} = \left[b_0 \bar{g}'^2_{\overline{\rm MS}}(\mu)\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}'^2_{\overline{\rm MS}}(\mu)}} \exp\left\{-\int_0^{\bar{g}'_{\overline{\rm MS}}(\mu)} dx \left[\frac{1}{\beta_{\overline{\rm MS}}^{(5)}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$

Decoupling of heavy quarks in massless schemes

$$\frac{\Lambda_{\overline{\rm MS}}^{(N_{\rm f})}}{\mu} = \left[b_0 \bar{g}^2_{\overline{\rm MS}}(\mu)\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0 \bar{g}^2_{\overline{\rm MS}}(\mu)}} \exp\left\{-\int_0^{\bar{g}_{\overline{\rm MS}}(\mu)} dx \left[\frac{1}{\beta^{(N_{\rm f})}_{\overline{\rm MS}}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$

Some numerical examples

- Start with $\Lambda_{\overline{\rm MS}}^{(6)} \approx 91.1$ MeV
- Determine $\alpha_{\overline{\rm MS}}^{(6)}(m_{\rm t}) \Longrightarrow \alpha_{\overline{\rm MS}}^{(5)}(m_{\rm t})$
- Get $\Lambda_{\overline{\rm MS}}^{(5)} \approx 215$ MeV
- Determine $\alpha_{\overline{\rm MS}}^{(5)}(m_{\rm b}) \Longrightarrow \alpha_{\overline{\rm MS}}^{(4)}(m_{\rm b})$
- Get $\Lambda_{\overline{\rm MS}}^{(4)} \approx 298$ MeV
- Determine $\alpha_{\overline{\rm MS}}^{(4)}(m_{\rm c}) \Longrightarrow \alpha_{\overline{\rm MS}}^{(3)}(m_{\rm c})$
- Get $\Lambda_{\overline{\rm MS}}^{(3)} \approx 312$ MeV
- We cannot get $\Lambda_{\overline{\rm MS}}^{(2)}$: No valid perturbative matching at $\mu \approx m_{\rm s} < \Lambda$

Perturbative uncertainties **ridiculously small** in this game! [ALPHA '18]

Decoupling of heavy quarks: Non-perturbatively

- Large coefficients in PT **is a problem of PT**
- In Lattice QCD we can use as many (heavy) flavors as we want
- Sometimes useful to consider massive schemes:

$$\alpha_{qq}(\mu, M_u^{\text{phys}}, M_d^{\text{phys}}, M_s^{\text{phys}}, M_c^{\text{phys}})$$

- But simulating heavy quarks is challenging:
  - m_{h} is large
  - am_{h} **has to be** small

  Requires large computational resources!

Multi-scale problem → Large PT coefficients

Multi-scale problem → Large computational cost

Checkpoint

- Massless schemes are needed for precision.
- One should use perturbative expressions with only the number of **active** quarks
- Matching between theories

$$\alpha^{(3)}_{\overline{\mathrm{MS}}} \to \alpha^{(4)}_{\overline{\mathrm{MS}}} \to \alpha^{(5)}_{\overline{\mathrm{MS}}} \to \alpha^{(6)}_{\overline{\mathrm{MS}}}\,.$$

- Non perturbatively one can use massless or massive schemes.

3M: A universe with three heavy degenerate quarks $(M \gg \Lambda)$

Alice uses fundamental theory

$$S_{\text{fund}}[A_\mu, \psi, \bar{\psi}] = \int \mathrm{d}^4x \, \left\{ -\frac{1}{2g^2} \mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu}) + \sum_{i=1}^{3} \bar{\psi}_i(\gamma_\mu D_\mu + M)\psi_i \right\}$$

Bob uses effective theory

$$S_{\text{eff}}[A_\mu] = -\frac{1}{2g_{\text{eff}}^2} \int \mathrm{d}^4x \, \{\mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu})\} + \frac{1}{M^2} \sum_k \omega_k \int \mathrm{d}^4x \, \mathcal{L}_k^{(6)} + \dots$$

3M: A universe with three heavy degenerate quarks $(M \gg \Lambda)$

Alice uses fundamental theory

$$S_{\text{fund}}[A_\mu, \psi, \bar{\psi}] = \int \mathrm{d}^4x \left\{ -\frac{1}{2g^2} \mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu}) + \sum_{i=1}^{3} \bar{\psi}_i(\gamma_\mu D_\mu + M)\psi_i \right\}$$

Bob uses effective theory

$$S_{\text{eff}}[A_\mu] = -\frac{1}{2g_{\text{eff}}^2} \int \mathrm{d}^4x \, \{\mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu})\} + \cancel{\frac{1}{M^2} \sum_k \omega_k \int \mathrm{d}^4x \, \mathcal{L}_k^{(6)}} + \ldots$$

Decoupling

- Dimensionless "low energy quantities" $\sqrt{t_0}/r_0, w_0/\sqrt{8t_0}, r_0/w_0, \ldots$ from effective theory

$$\frac{\mu_1^{\text{fund}}(M)}{\mu_2^{\text{fund}}(M)} = \frac{\mu_1^{\text{eff}}}{\mu_2^{\text{eff}}} + \mathcal{O}\left(\frac{\mu^2}{M^2}\right)$$

Renormalization in 3M: Alice determines the strong coupling

$$\frac{\Lambda}{\mu} = \left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}.$$

- Determine non-perturbatively the β-function in the fundamental ($N_f = 3$) theory, mass-less scheme.
- Integral up to $\bar{g}^{(3)}(\mu_{\rm dec}) = \text{value}$ (in a mass-less scheme!) gives:

$$\frac{\Lambda^{(3)}}{\mu_{\rm dec}}$$

- Turn on quark masses and relate $\mu_{\rm dec}$ with its massive version ($\bar{g}^{(3)}(\mu_{\rm dec}(M), M) = \text{value}$)

$$\frac{\mu_{\rm dec}(M)}{\mu_{\rm dec}}$$

- Result

$$\frac{\Lambda^{(3)}}{\mu_{\rm dec}(M)} = \frac{\Lambda^{(3)}}{\mu_{\rm dec}} \times \frac{\mu_{\rm dec}(M)}{\mu_{\rm dec}}$$

Renormalization in 3M: Bob determines the strong coupling

$$\frac{\Lambda}{\mu} = \left[b_0 \bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}.$$

- Determine non-perturbatively the β-function in the effective $(N_f = 0)$ theory.
- Integral up to $\bar{g}^{(0)}(\mu'_{\rm dec}) =$ value gives:

$$\frac{\Lambda^{(0)}}{\mu'_{\rm dec}}$$

- Match across quark threshold to convert to $\Lambda^{(3)}$ (using perturbation theory)

$$\frac{\Lambda^{(3)}}{\mu'_{\rm dec}} = \frac{\Lambda^{(0)}}{\mu'_{\rm dec}} \times \frac{1}{P(\Lambda/M)}.$$

Relation between Alice and Bob computation

$$\left.\begin{aligned} \bar{g}^{(3)}(\mu_{\rm dec}(M), M) &= \text{value} \\ \bar{g}^{(0)}(\mu'_{\rm dec}) &= \text{value} \end{aligned}\right\} \Longrightarrow \frac{\mu_{\rm dec}(M)}{\mu'_{\rm dec}} = 1 + \mathcal{O}(\mu_{\rm dec}^2/M^2)$$

Relation between Alice and Bob computations

$$\frac{\Lambda^{(3)}}{\mu_{\rm dec}(M)} = \frac{\Lambda^{(0)}}{\mu'_{\rm dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

Bob is telling us that $\Lambda^{(3)}$ can be computed from $\Lambda^{(0)}$

$$\Lambda^{(3)} = \lim_{M\to\infty} \mu_{\rm dec}(M) \times \frac{\Lambda^{(0)}}{\mu'_{\rm dec}} \times \frac{1}{P(\Lambda/M)}$$

We need

- Running in pure gauge: $\Lambda^{(0)}/\mu'_{\rm dec}$
- A scale in a world with degenerate massive quarks: $\mu_{\rm dec}(M)$ in fm/MeV.

Lattice QCD can simulate *unphysical* worlds

$$\mu_{\rm dec}(M) = M_p \times \frac{\mu_{\rm dec}(M)}{M_p} = M_p^{\rm PDG} \lim_{a\to 0} \frac{a\mu_{\rm dec}(M)}{aM_p}$$

Matching worlds

All lattice simulations depends only on dimensionless input: $g_0, am_i, L/a$. No dimensionfull output possible!

$$W1(\text{"our" world}) \quad : \quad \frac{M_\pi}{M_p} = 0.14; \quad \frac{M_K}{M_p} = 0.37 \,.$$
$$W2 \quad : \quad \frac{M_\pi}{M_p} = 0.5; \quad \frac{M_K}{M_p} = 0.5 \,.$$

How much changes the proton mass between W1 and W2?

- Choose one g_0, tune $am_i \ll 1$ to match LCP of W1, W2
- Repeat for several values g_0 and perform continuum limit:

$$\frac{M_p(W2)}{M_p(W1)} = \lim_{aM_p \to 0} \frac{aM(W2)}{aM(W1)} \,.$$

- Since W1 is "our" world:

$$M_p(W2) = M_p^{\text{exp}} \times \lim_{aM_p \to 0} \frac{aM(W2)}{aM(W1)} \,.$$

Our setup: Choices optimized to be able to simulate heavy quarks

$$\Lambda^{(3)} = \mu_{\rm dec}(M) \times \frac{\Lambda^{(0)}}{\mu_{\rm dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_{\rm dec}}{M}\right) + \mathcal{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

- Work in <u>finite volume</u> schemes with Schrödinger Functional boundary conditions: $T \times L^3$ with Dirichlet bcs. in time. ($\mu \sim 1/L$): "Only" two scales.
- Use Gradient Flow couplings

$$\bar{g}^2(\mu) = \mathcal{N}^{-1}(c, a/L)\, t^2 \langle E(t) \rangle \Big|_{\mu^{-1}=\sqrt{8t}=cL}.$$

- Fix $\bar{g}^2(\mu_{\rm dec})|\Big|_{N_{\rm f}=3, M=0, T=L} = 3.95$. This defines $\mu_{\rm dec} = 1/L \sim 800$ MeV
- Small volume $\Longrightarrow$ We can simulate heavy quarks (i.e. $a \sim 30 - 50\,{\rm GeV}^{-1}$)
- Matching condition ($\{N_{\rm f} = 3, M\} \leftrightarrow \{N_{\rm f} = 0\}$) between massive scheme and effective theory

$$\bar{g}^2(\mu_{\rm dec}(M))\Big|_{N_{\rm f}=3, M, T=2L} = \bar{g}^2(\mu_{\rm dec})\Big|_{N_{\rm f}=0, T=2L}.$$

  Matching: QCD in a finite volume!
- Convenient variable: $z = M/\mu_{\rm dec}$

Our setup: Choices optimized to be able to simulate heavy quarks

$$\Lambda^{(3)} = \mu_{\rm dec}(M) \times \frac{\Lambda^{(0)}}{\mu_{\rm dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_{\rm dec}}{M}\right) + \mathcal{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

We only need to fill in a table!

$\mu_{\rm dec}(M)$ [MeV]	$M/\mu_{\rm dec}(M)$	$\bar{g}_z^2$	$\Lambda^{(0)}/\mu_{\rm ref}$	$\Lambda_{\rm eff}^{(3)}$
789(15)	1.972	-	-	-
789(15)	4	-	-	-
789(15)	6	-	-	-
789(15)	8	-	-	-
789(15)	10	-	-	-
789(15)	12	-	-	-

- Difficult continuum extrapolations to determine $\bar{g}_z^2 = \bar{g}^2(\mu_{\rm dec}(M))\Big|_{N_{\rm f}=3,M,T=2L}$
- Use combined Heavy-Quark / Symanzik effective theories.

Continuum extrapolation ansatze

Quadratic dependence on lattice spacing (a) via $a\mu_{\rm dec}$ and aM

For large enough masses, effective theory applies:

$$\bar{g}^2(z_i, a) = C_i + p_1[\alpha_{\overline{\rm MS}}(a^{-1})]^{\hat{\Gamma}}(a\mu_{\rm dec})^2 + p_2[\alpha_{\overline{\rm MS}}(a^{-1})]^{\hat{\Gamma}'}(aM_i)^2\,.$$

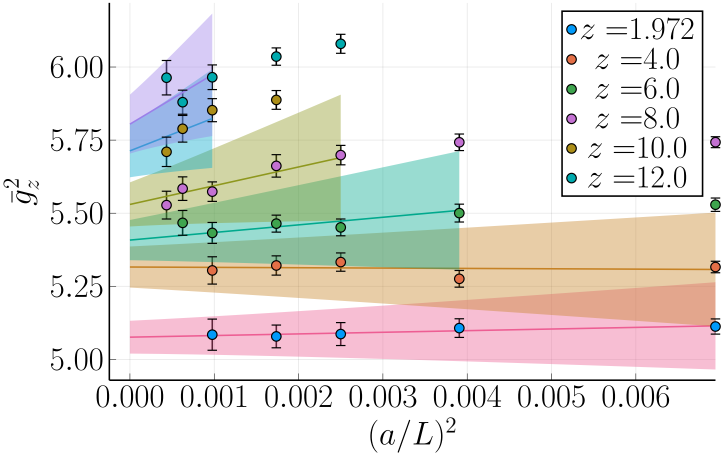
- Continuum values (our target quantity)
- Mass independent cutoff effects
- Mass dependent cutoff effects
- Loop corrections in effective theory: $-1 \leq \hat{\Gamma} \leq 1$ and $-1/9 \leq \Gamma' \leq 1$

Additional assumptions about $\mathcal{O}(aM)$ effects

Partial knowledge based on PT: **Propagate difference between last known orders as additional uncertainty**

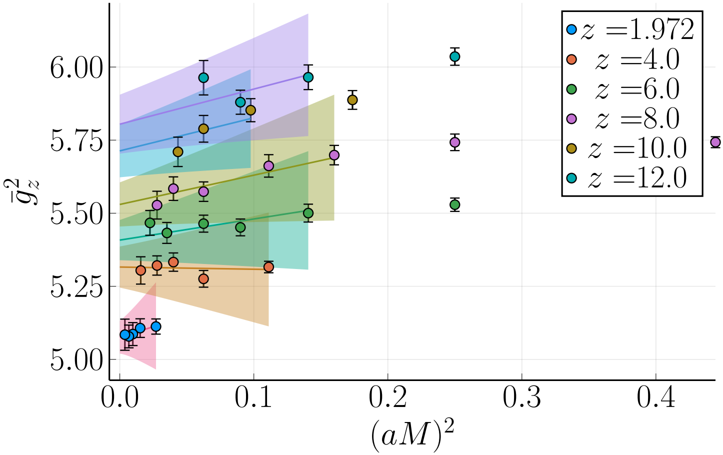
- Schrödinger functional boundaries: Small (negligible to our level of precision). Explicit computation.
- Quark mass improvement: $b_m, b_A, b_P, \ldots$. **Very** small effect.
- Improved bare coupling: b_g. Large effect at large masses (comparable to statistical uncertainties). Decreases as $aM \to 0$.

Continuum extrapolations



Continuum extrapolations with $L/a = 12, 16, 20, 24, 32, , 40, 48$

Continuum extrapolations



Continuum extrapolations with $L/a = 12, 16, 20, 24, 32, , 40, 48$

Table can be filled

$$\Lambda^{(3)} = \mu_{\rm dec}(M) \times \frac{\Lambda^{(0)}}{\mu_{\rm dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_{\rm dec}}{M}\right) + \mathcal{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

$\mu_{\rm dec}(M)$ [MeV]	$M/\mu_{\rm dec}(M)$	$\bar{g}_z^2$	$\Lambda^{(0)}/\mu_{\rm ref}$	$\Lambda_{\rm eff}^{(3)}$ [MeV]
789(15)	1.972	5.076(56)	0.540(14)	426(14)
789(15)	4	5.316(70)	0.492(14)	388(13)
789(15)	6	5.408(69)	0.460(12)	363(12)
789(15)	8	5.530(76)	0.445(12)	351(12)
789(15)	10	5.713(90)	0.443(13)	349(12)
789(15)	12	5.80(10)	0.434(13)	343(12)

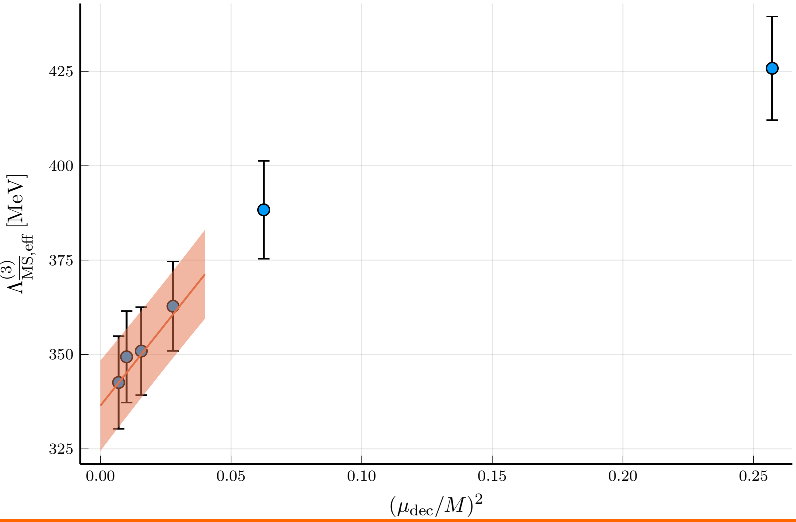
Perturbative uncertainties

$$\mathcal{O}(\alpha^4(m^\star))$$

Completely negligible!. (Take difference between 4-loops and 2-loops as estimate)

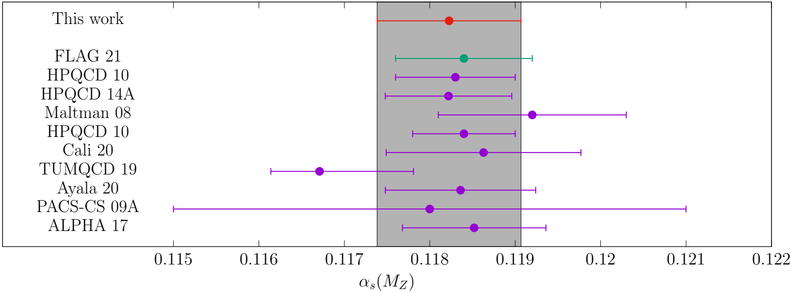
Determination of $\Lambda^{(3)}$ from decoupling: $\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 336(12)$ MeV

$$\Lambda^{(3)}_{\mathrm{eff}} = \Lambda^{(3)} + \frac{B}{z^2}\left[\alpha(m_\star)\right]^{\Gamma_m}$$



Result

$$\alpha_s(m_Z) = 0.11823(69)(42)_{b_g}(20)_{\Gamma_m}(6)_{3\to5,\mathrm{PT}}(7)_{3\to5,\mathrm{NP}} = 0.11823(84)\,.$$



Several variations of analysis

Continuum extrapolation

- $(aM)^2 < \mathbf{0.16}, 0.25$
- Various values of $\hat{\Gamma}, \Gamma'$
- Several coupling definitions (statistically correlated) labeled by $c = 0.30, 0.33, \mathbf{0.36}, 0.39, 0.42$

$M \to \infty$

- Several values of Γ_m
- Several values of c
- $z4, \mathbf{6}, 8$

Example

$z \geq 4$			$z \geq 6$			$z \geq 8$		
c	$\Lambda^{(3)}_{\overline{\text{MS}}}$	Q [%]	c	$\Lambda^{(3)}_{\overline{\text{MS}}}$	Q [%]	c	$\Lambda^{(3)}_{\overline{\text{MS}}}$	Q [%]
0.30	349(11)	2	0.30	340(12)	11	0.30	338(13)	4
0.33	345(11)	8	0.33	338(12)	13	0.33	338(13)	4
0.36	342(11)	16	0.36	336(12)	16	0.36	338(13)	6
0.39	339(11)	21	0.39	335(12)	16	0.39	338(13)	7
0.42	336(11)	23	0.42	333(12)	15	0.42	337(13)	7

Conclusions

- Extraction of α_s is a **very hard** multi-scale problem
  - Computational cost $\Longrightarrow (L/a)^7$
  - Perturbative uncertainties $\Longrightarrow \log(L/a)^{\#}$
- Perturbative uncertainties hard to estimated with data in a limited range of scales
- One should take "non-perturbative" limit seriously (i.e. $\alpha \to 0$)
- Perturbative uncertainties using scale variation are a guide: Common framework to all approaches? **[L. Del Debbio, A. Ramos Phys.Rep.(2021)190]**
- One **real solution**: Step scaling
  - Non-perturbative running from 200 MeV to 140 GeV: $\alpha_s(M_Z) = 0.1185(8)$
- *Exponential* improvement (still a multi-scale problem): Decoupling of heavy quarks
  - Perturbative uncertainties negligible ($M \approx 10$ GeV)
  - Non-perturbative corrections can be extrapolated
  - Relies on *pure gauge determinations of* $\Lambda^{(0)}$
  - Precise result: $\alpha_s(M_Z) = 0.1182(8)$
- $\delta\alpha_s(M_Z) \approx 0.4\%$ certainly possible (uncertainties dominated by pure gauge (!!) and low energy running (!)).
- $\delta\alpha_s(M_Z) < 0.3\%$ requires serious thinking.
- Potential for **other lattice approaches**: How difficult to simulate high M?.

Conclusions

- Extraction of α_s is a **very hard** multi-scale problem
  - Computational cost $\Longrightarrow (L/a)^7$
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- Perturbative uncertainties hard to estimated with data in a limited range of scales
- One should take "non-perturbative" limit seriously (i.e. $\alpha \to 0$)
- ...ainties using scale variation are a guide: Common

Personal opinion

Future belongs to **dedicated** approaches, **not** to beating an exponentially hard problem with your machines

Many thanks!

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- $\delta\alpha_s(M_Z) \approx 0.4\%$ certainly possible (uncertainties dominated by pure gauge (!!) and low energy running (!)).
- $\delta\alpha_s(M_Z) < 0.3\%$ requires serious thinking.
- Potential for **other lattice approaches**: How difficult to simulate high M?.