Regge calculus and applications

Loops School 2024

- Regge calculus basics — Today
- Lorentzian Regge calculus — (Tomorrow)

Path integral for Quantum Gravity

$$Z \sim \int \underbrace{\mathcal{D}\,\mathrm{geom}}\ \exp(i\,S)$$

Very hard to compute.

$\exp(i\,S) \longrightarrow \exp(-S_{Eu})$

Easier to compute, but
- conformal factor problem
- Relation to $\exp(i\,S)$

Many ways to define this.

- Regge calculus
- spin foams
- effective spin foams

Even with $\exp(iS)$ factor: can sum over Euclidean or Lorentzian metrics.

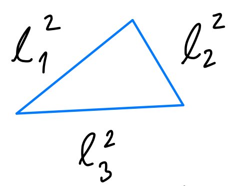
Piecewise flat spaces

Consider piecewise flat geometries.

Here: Using simplices, but can be easily generalized.

NB: Can also use homogeneously curved simplices. [Bahr, BD 2009]

Simplices: 2D

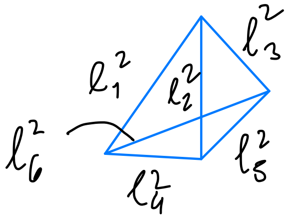


triangle: geometry uniquely specified by lengths (squared) of edges.

Can be embedded into (Euclidean or Minkowskian) flat space, iff appropriate triangle inequalities are satisfied.

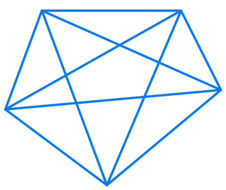
Piecewise flat spaces

3D



- Edge lengths (squared) uniquely specify geometry
- Can be embedded into (E or M) flat space if $\underbrace{\text{generalized triangle inequalities}}_{\text{GTI}}$ are satisfied

4D



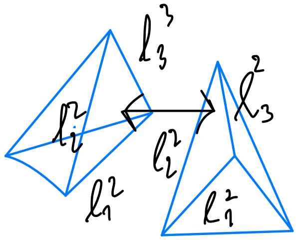
- As above.
- includes 10 edges, 10 triangles, 5 tetrahedra

For a given simplex: Using embedding into flat space, can construct (Cartesian) coordinate system.

A different choice: barycentric coordinates

[Sorkin 70's, BD, Fridel, Speziale '07]

Gluing two simplices

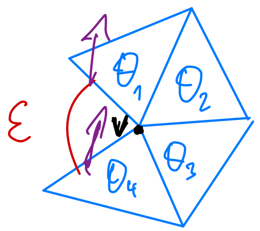


- Can always embed a pair of glued simplices into flat space (if GT1 satisfied)
- Can find common Cartesian coordinate system
- This defines parallel transport between simplices.

- Can go from simplex to simplex around a "bone" = co-dimension-2 simplex

Curvature (Euclidean)

2D: (Euclidean)



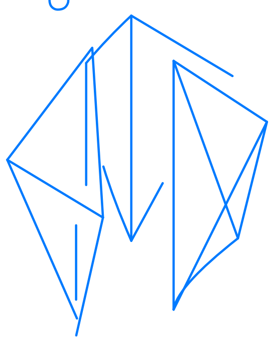
Parallel transport vector around v:

- Vector will be rotated by deficit angle ε.

$$\varepsilon = 2\pi - \sum_i \theta_i$$

$\Rightarrow$ Curvature has delta function like support on vertices.

Higher dimension:



- Consider a bone shared by (top-dim.) simplices.
- Project each such top-dim. simplex onto plane orthogonal to the bone.
- Result: A chain of triangles around a vertex.
- Angle at vertex = dihedral angle

$$\varepsilon_{bone} = 2\pi - \sum_{\sigma \supset bone} \theta_{\sigma, bone}$$

$\Rightarrow$ Curvature has delta function like support on bones.

2023
Borissova, BD

The Regge action (Euclidean)

- Curvature characterized by ε_b → constant along given bone.
- Integrate curvature over mfd:

$$S_{Regge} = \sum_{\substack{\text{bones} \\ (\text{bulk})}} Vol_b \cdot \varepsilon_b \quad + \quad \text{boundary term}$$

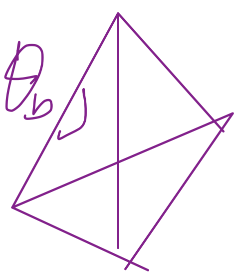
boundary term : $\displaystyle\sum_{\substack{\text{bones} \\ (\text{bdry})}} Vol_b \left(k_b \cdot \pi - \sum_{\sigma \supset b} \theta_{\sigma, b} \right)$

(fixed bdry length)

$k_b = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ Choice.

Can be adjusted to expected type of bdry:

$$S_{3D} = -\sum_b l_b \cdot \theta_b$$



Choice of k_b does not matter for EOM:

$$\sum_{\text{bdry-}b} Vol_b \, k_b \cdot \pi = \text{const.}$$

Equation of Motion

$$S = \sum_{\text{bones}} Vol_b \cdot \varepsilon_b$$ (boundary or bulk deficit angle)

$$\frac{\delta S}{\delta l_e} = \sum_{\text{bones} \supset e} \frac{\partial Vol_b}{\partial l_e} \cdot \varepsilon_b - \underbrace{\sum_{\text{bones}} Vol_b \sum_{\sigma \supset \text{bones}} \frac{\partial \theta_{\sigma,b}}{\partial l_b}}_{\sum_{\sigma} \sum_{b \subset \sigma} Vol_b \frac{\partial \theta_{\sigma,b}}{\partial l_b} = 0}$$

Schläfli identity

Note: · Importance of bdry term.
· Similar mechanism in contin.

$$\Rightarrow \boxed{\sum_{\text{bones} \supset e} \frac{\partial Vol_b}{\partial l_e} \varepsilon_b = 0}$$

Regge EOM = discretized Einstein equations

- 2D: Regge action is a topological invariant. EOM empty.
- 3D: $\varepsilon_b = 0$
- 4D: $\sum_{\text{triang} \supset e} \frac{\partial A_t}{\partial l_e} \varepsilon_t = 0$

- $\varepsilon_b = 0$ solution, might not be allowed by bdry cond. in 4D.

Schläfli identity

$$0 = \sum_f V_f \, \hat{n}_f \quad \Rightarrow \quad 0 = \sum_f V_f \, \underbrace{\hat{n}_f \cdot \hat{n}_{f'}}_{-\cos \theta_{ff'}}$$

$$\Rightarrow \quad 0 = \sum_f V_f \cos \theta_{ff'} \qquad (\text{with } \cos \theta_{ff} := -1)$$

$$\Rightarrow \quad 0 = \sum_f (\delta V_f) \cos \theta_{ff'} - \sum_f V_f \sin \theta_{ff'} \, \delta \theta_{ff'}$$

Contract with $V_{f'}$: $\quad \sum_{f'} V_{f'} \cos \theta_{ff'} = 0$

$$\Rightarrow \quad 0 = \sum_{f,f'} \underbrace{V_f V_{f'} \sin \theta_{ff'}}_{= \frac{d}{(d-1)} V_\sigma V_b} \, \delta \theta_{ff'}$$

$$\Rightarrow \quad \boxed{0 = \sum_b V_b \, \delta \theta_b = 0}$$

$$\varepsilon_{bulk} = 2\pi - \sum \theta_{\sigma,b}$$

$$\varepsilon_{bdry} = k\pi - \sum \theta_{\sigma,b}$$

$$S_\sigma = - \sum_t A_t \, \theta_t$$

$$\Pi_e = \frac{\partial S}{\partial l_e} = - \sum_t \frac{\partial A_t}{\partial l_e} \theta_t$$

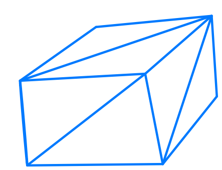
$$\{ l_e, \Pi_{e'} \} = 1$$

$$\to A_t(l_e), \quad \Pi_t \qquad \{ A_t, \Pi_t \} = 1$$

$$\to A_t, \quad \Pi_t = -\theta_t$$

Questions ?

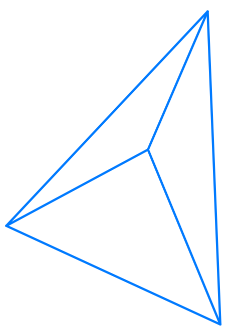
- Some unusual configurations: Spikes
- Coupling of Matter — Sorkin 70's
- Lattice Continuum limit
  Rocek, Williams 80's, BD, Freidel, Speziale '07

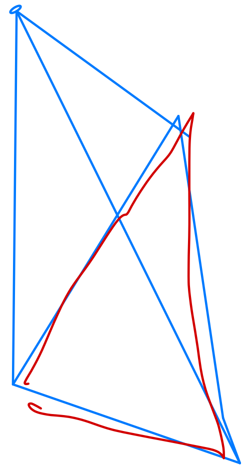


+ body diagonal
=> 6 tetrah.

- Diffeomorphism symmetry, Triangulation invariance
  BD 08, Bahr, BD 09
- Canonical Analysis — BD, Höhn 2010 +
  Pachner Moves as time evolution — BD, Höhn 2010 +
- Path integral measure — BD, Steinhaus 2011; BD, Borissova 2023

Spikes

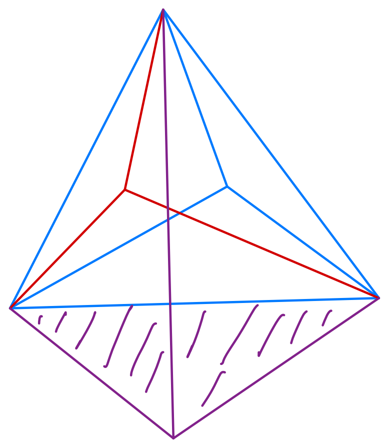




- arbitrary long bulk edge length
- "small" boundary edge length
- represents conformal factor ⇝ kills Euclidean approach

→ Even more possibilities for Lorentzian signature

Diffeomorphism symmetry



- All examples describe the same (flat) geometry but have different edge length.
- Subdivide flatly a flat simplex into $(d+1)$ simplices

$\Rightarrow$ Provides family of gauge-equivalent solutions.

$$\Rightarrow \quad S|_{sol} = \text{const. on this family}; \quad \left.\frac{\partial S}{\partial l_e \partial l_{e'}}\right|_{sol} = 0 .$$

- What about solutions with curvature?

$$\frac{\partial S}{\partial l_e \partial l_{e'}} \sim \varepsilon_b^2 , \text{ very small.}$$

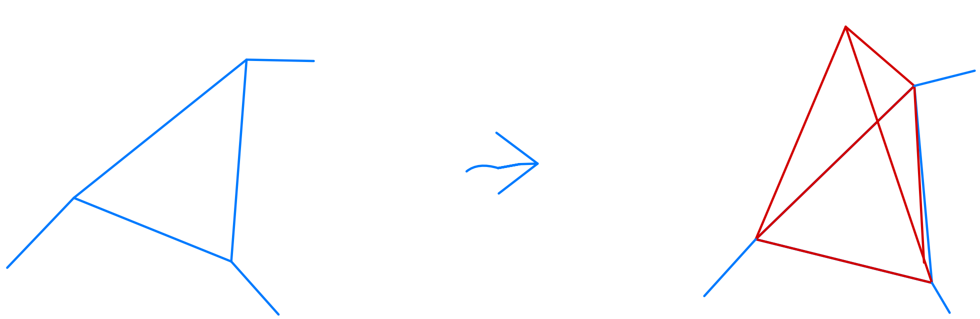
$\Rightarrow$ Broken symmetry.

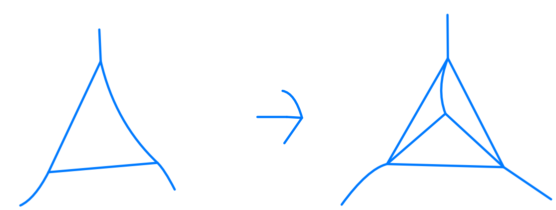
Confirmed by

- Degree of freedom count
- Canonical Analysis

Time evolution

- discrete: always possible, even if # dof change
  [BD, Höhn, 2010+]
- continuous: if diff-symmetry is there
- or in time-continuum limit for symmetry reduced configurations [BD, Gielen, Schander 2021]
- Discrete Evolution steps 3D:





Time evolution

- Discrete time evolution with Pachner moves in hypersurface
- Tent moves: Combination of Pachner moves which do not change the triangulation.
- Phase space can be constructed
- Generating function: Regge action for glued simplex.

BD, Höhn 2012, 2013. Pre- & Post Constraints

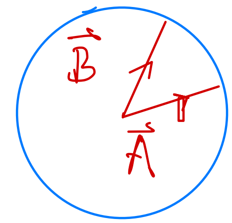
Lorentzian Regge calculus

More surprises.

Lorentzian angles

Euclidean angle

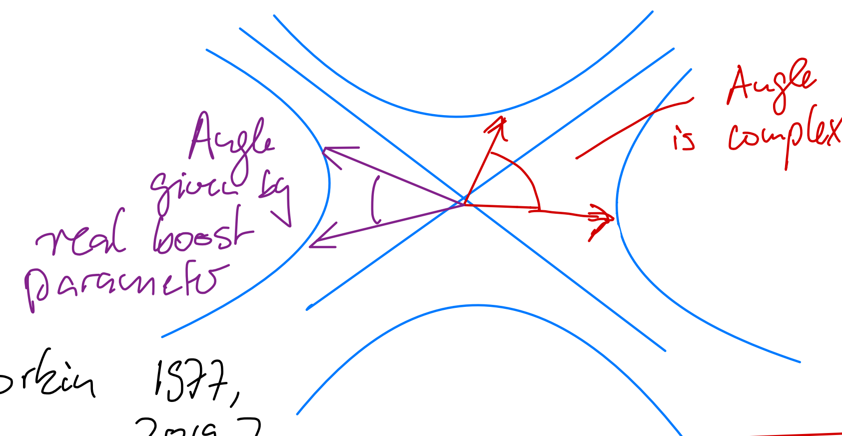
$$\theta = \arccos \frac{\vec{a}\cdot\vec{b}}{\sqrt{\vec{a}\cdot\vec{a}}\,\sqrt{\vec{b}\cdot\vec{b}}}$$



$\theta \in [0, \pi]$

Lorentzian angle

What should we do?



- Allow angles w/ imaginary parts [Sorkin 1977, 2019]
- Indeed: boost with $\zeta = \pm i \frac{\pi}{2} \Rightarrow \begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix}$

Each light ray crossing contributes $\mp i\,\pi/2$.

- More straightforward: Define angles via analytical continuation.

[Asante, BD, Padua-Argüelles '21]

Lorentzian angles

Generalized Wick transform in $(1+1)D$: $\vec{a} * \vec{b} = e^{i\phi} a_0 b_0 + a_1 b_1$

$\phi = 0, 2\pi$: Euclidean

$\phi = \pm\pi$: Minkowskian

$$\theta = \arccos \frac{\vec{a} * \vec{b}}{\sqrt{\vec{a} * \vec{a}}\,\sqrt{\vec{b} * \vec{b}}}$$

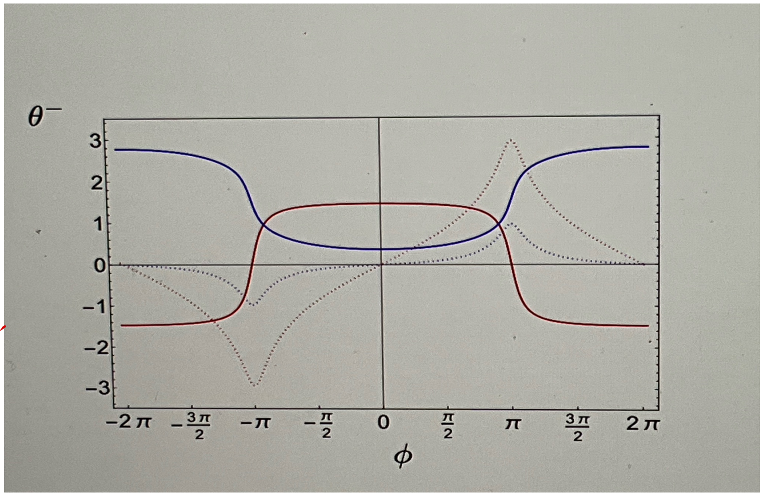
$$\theta^{(-)} = -i \log_{+} \frac{\vec{a} * \vec{b} + i \sqrt_{-}{(\vec{a} * \vec{a})(\vec{b} * \vec{b}) - (\vec{a} * \vec{b})^2}}{\sqrt_{-}{\vec{a} * \vec{a}}\,\sqrt_{-}{\vec{b} * \vec{b}}}$$

$\log_{+}(-1) = +i\pi$

$\sqrt_{-}{-1} = -i$

Analytical continuation in ϕ:

Natural extension to $(-2\pi, 2\pi)$!



Red: Both in Qu I.

Blue: One in Qu I, one in Qu II

Solid: Real part

Dotted: Imaginary part

Lorentzian angles

$$a * b = e^{i\phi} a_0 b_0 + a_1 b_1$$

Complex angle:

$$\theta^{(-)} = -i \log_{+} \frac{\vec{a}*\vec{b} + i\sqrt{(\vec{a}*\vec{a})(\vec{b}*\vec{b}) - (\vec{a}*\vec{b})^2}_{-}}{\sqrt{\vec{a}*\vec{a}}_{-}\ \sqrt{\vec{b}*\vec{b}}_{-}}$$

Analytical contin. from $(-\pi, \pi)$ to $[-2\pi, 2\pi]$:

$\Theta(\phi = 0) = + \psi_E$ Eucl. dihedral angle

$\Theta(\phi = +\pi) = + i \psi_{L+}$ Lor. dihedral angle

$\Theta(\phi = -\pi) = - i \psi_{L-}$

$\Theta(\phi = 2\pi) = - \psi_E + \pi N_c$ # of light ray crossings

Proof in [Asante, BD, Padua-Argüelles '21]

$\psi_{L\pm}$ = real boost parameter $\mp N_c \cdot i \frac{\pi}{2}$.

$\Rightarrow$ $\boxed{\text{The complex angle reproduces:}}$

- Euclidean angle for Euclidean data
- Both versions of the Lorentzian angle for Lorentzian data

- Null edges are special: Logarithmic branch point.

Complex deficit angle

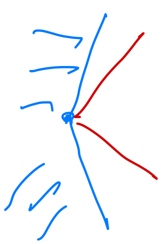
Bulk: $\delta(\phi) = 2\pi - \sum_{\sigma>b} \theta_{\sigma,b}(\phi)$

Bdry $\delta(\phi) = k_b \cdot \frac{\pi}{2} - \sum_{\sigma>b} \theta_{\sigma,b}(\phi)$

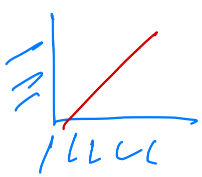
$$\left.\begin{aligned} \delta(0) &= \epsilon_E \\ \delta(+\pi) &= +i\,\epsilon_{L+} \\ \delta(-\pi) &= -i\,\epsilon_{L-} \end{aligned}\right.$$

$$\delta(2\pi) = -\epsilon_E + (4-N_c)\pi$$

- For each $\theta_{\sigma,b}$ we have associated $N_c^{\sigma,b}$ = # lightrays in wedge.



$N_c = 2$

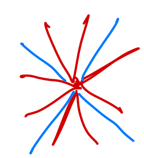


$N_c = 1$

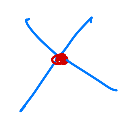
- If $N_c = \sum_{\sigma>b} N_c^{\sigma,b} \neq 4 \Rightarrow \delta(\pm\pi)$ has a real part.

($\epsilon_{L\pm}$ has an imaginary part)

$\Rightarrow$ $\boxed{\text{Light cone irregular configuration:}}$



8 light rays



0 lightrays

Complex Regge action

$$iS := \sum_b \sqrt{_+ V_b^s}\ \delta_b \quad - \quad \Lambda \sum_\sigma \sqrt{_+ V_\sigma^s}$$

Consistency check: * Eucl. triang.: $\delta_b = \epsilon_b \Rightarrow iS = -S_E$ ✓

$(S_{Regge} = -S_E)$

* Lorentz. triang.:

Timelike bone $\Rightarrow$ Euclidean angle

$\Rightarrow iS_b = i\sqrt{|V_b^s|} \cdot \epsilon_E$ ✓

Spacelike bone $\Rightarrow$ Lorentzian angle

$\Rightarrow iS_b = \sqrt{|V_b^s|} \cdot (\pm i)\, \epsilon_{L\pm}$ ✓

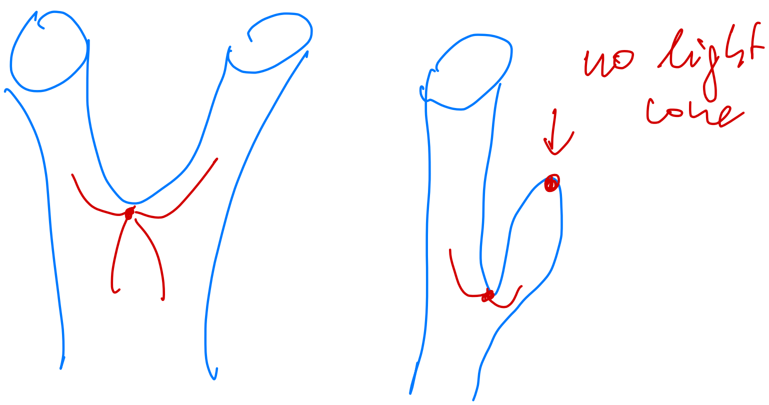
Null bone $\Rightarrow$ do not contribute to action

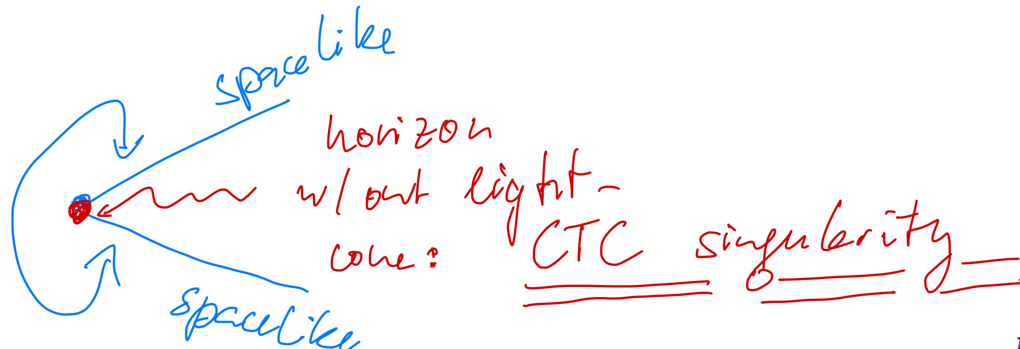
Light cone irregular configurations:
- iS has real parts
- $\exp(iS)$ either enhancing or suppr.

Question:

Are these light cone irregular configurations relevant?

$\boxed{\text{Yes.}}$





- But also appear in cosmology!

Essential for computing Entropie!!

[Marolf 2022] [BD, Jacobson, Padua-Argüelles '24]

Question:

- I learned a different definition for Lorentzian angles in spin foams.

[Barrett, Foxon 90's : thin & thick angles]

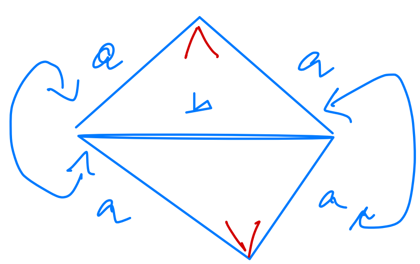
No imaginary parts: $\epsilon = \sum_{thin} \tilde{\theta}_b + \sum_{thick} \tilde{\theta}_{thick}$

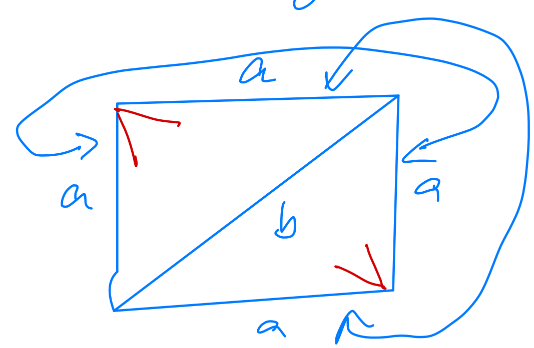
This definition does <u>not</u> satisfy the (Lorentzian) Gauß-Bonnet theorem.

For sphere and torus triangulation we would get the same action:

a, b spacelike

$b > 2a$

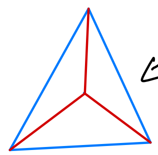




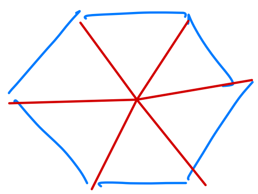
A cosmological example

[BD, Gielen, Schreiber '21]
[Asante, BD, Padua-Argüelles '21]

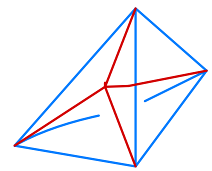
2D:



equal time hypersurf.



3D



· subdivided icosahedron

4D: · subdivided 4-simplex

· subdivided 16-cell

· subdivided 600-cell

blue edges : $\cdot$ $s_a > 0$ (squared edge length)

red edges : $s_b \gtrless 0$

Lorentzian triangle cond.

4D: $s_b < \frac{3}{8} s_a$

Introduce height of top-dim. simplices :

4D $\quad s_h = s_b - \frac{3}{8} s_a$

Lorentzian condition: $s_h < 0$

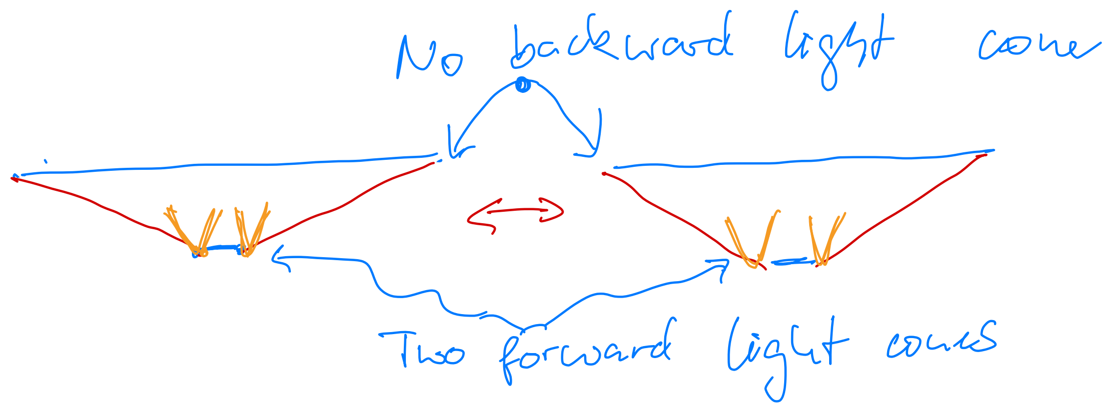
$s_h \sim \text{Lapse}^2$

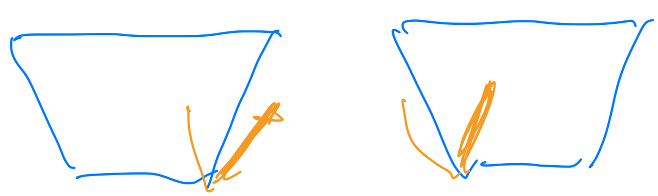
· Note that bulk sub-simplices might be spacelike.

A cosmological example

4D:
- bulk triangles timelike: light cone regular
- bulk triangles spacelike, bulk tetrah. timelike: lc irreg.
- bulk triangles spacelike, bulk tetrah. spacelike: lc irreg.

- These light cone irregular config. also appear when evolving from a finite vol. hypersurface to a finite vol. hypersurface:

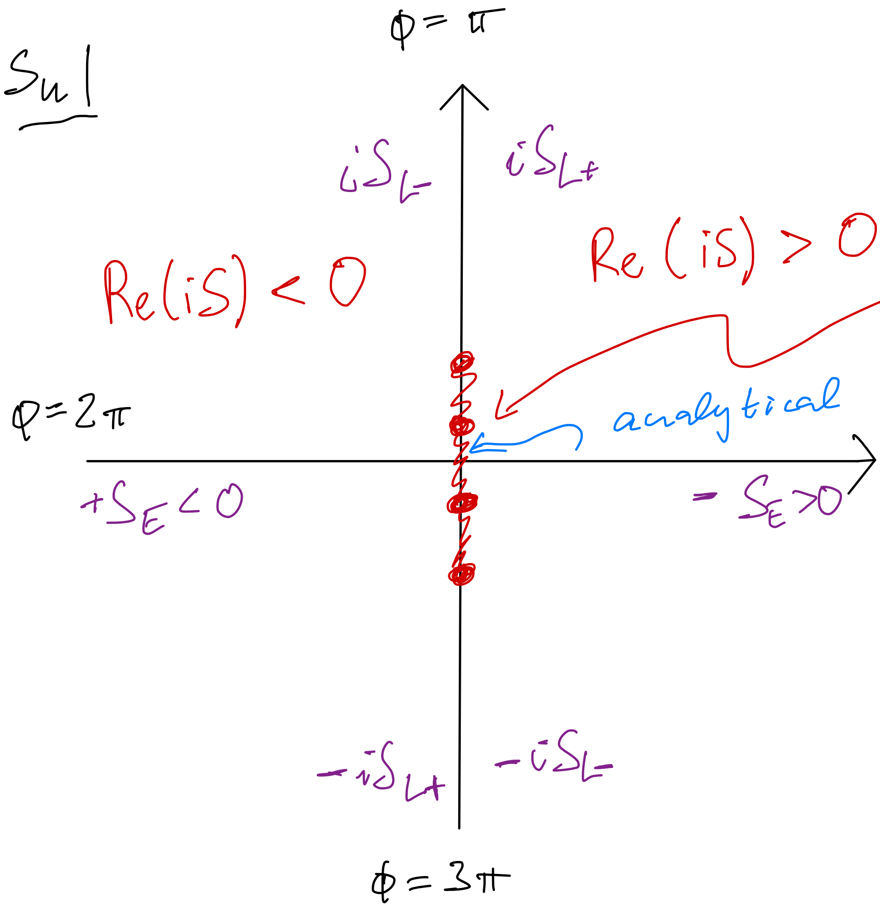




Complex Regge action for cosmology

- $s_h \sim \text{lapse}^2$ $\Rightarrow$ Complexify this variable
- $0 \leq s_h \leq -\frac{1}{8} s_a$ lc irregular configuration

$$s_h = e^{i\phi}\, r_h \quad (r_h > 0)$$



$\phi = 0$

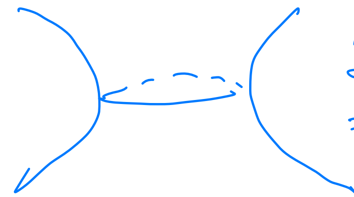
branch cut for lc irreg config.

analytical continuation leads to a different Riemann sheet.
($W = iS$ changes by multiples of π.)

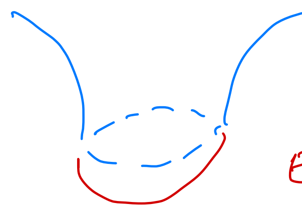
$\Rightarrow$ Explanation for N_c dependence of $\delta(\phi = 2\pi)$.

No-boundary wave-function

- De Sitter cosmology: $\Lambda > 0$

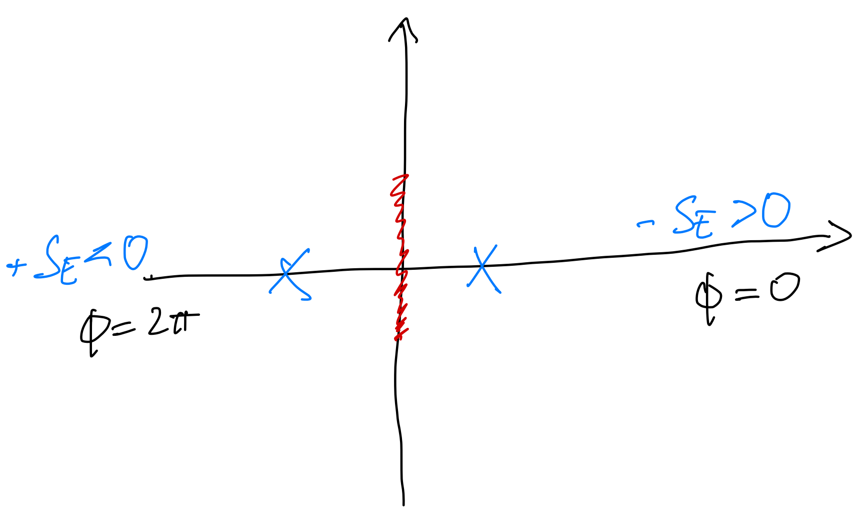


- Euclidean "phase"



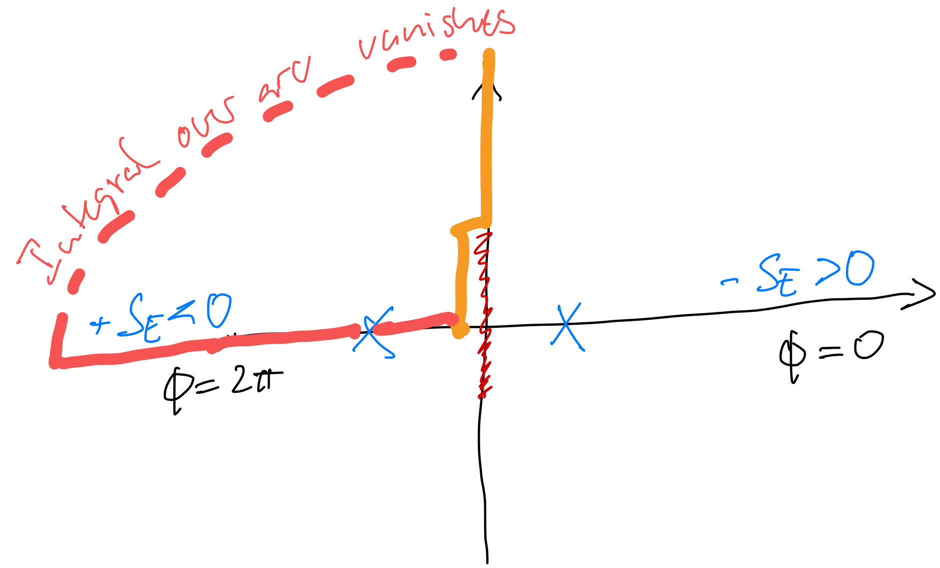
In ~~our~~ discrete case: Path integral = integral over s_h.

$\Rightarrow$ As long as $s_a \leq S_{crit}$ there are only Euclidean saddle points at $\phi = 0, 2\pi$.



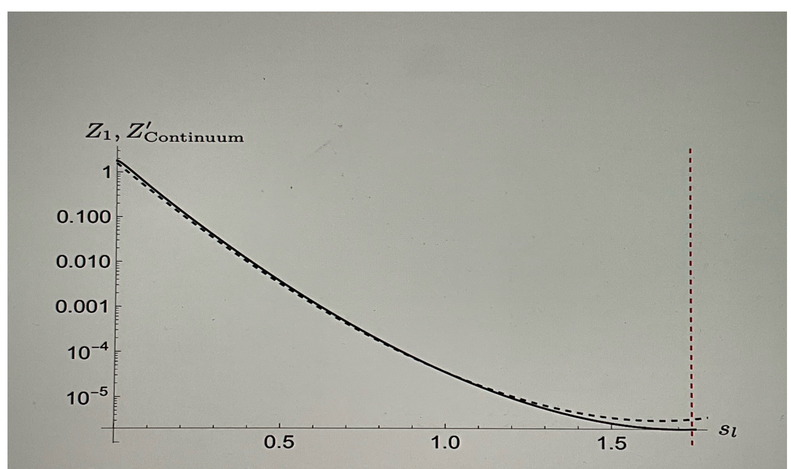
- Need to decide on Lorentzian contour:
  On the left or the right of branch cut.
- Decides on saddle point!

Choice 1:



$\Rightarrow$ This reproduces surprisingly well the continuum no-bdy tunneling wave fct from continuum mini-super space (e.g. [Feldbrugge, Lehners, Turok 2016]).

But we <u>need</u> to include lc-irreg. configurations.



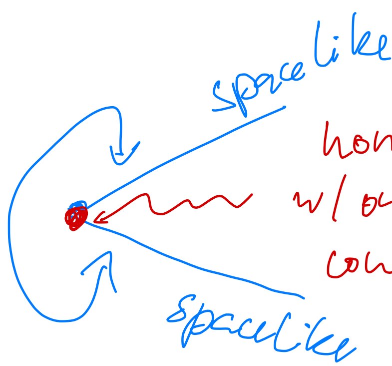
Tunneling amplitude

Lc irreg. configurations are essential:

[Marolf 2022, Jacobson, Visser '23]

Entropy from Lorentzian path integral

Periodically identify spacetime with horizon:

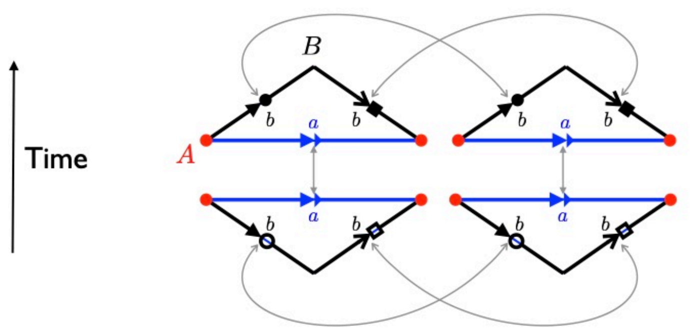


horizon w/out light-cone: CTC singularity

Entropy from simplicial Lorentzian path integral

[BD, Jacobson, Padua-Argüelles 2024]

→ Path integral over Lorentzian metrics on a sphere.

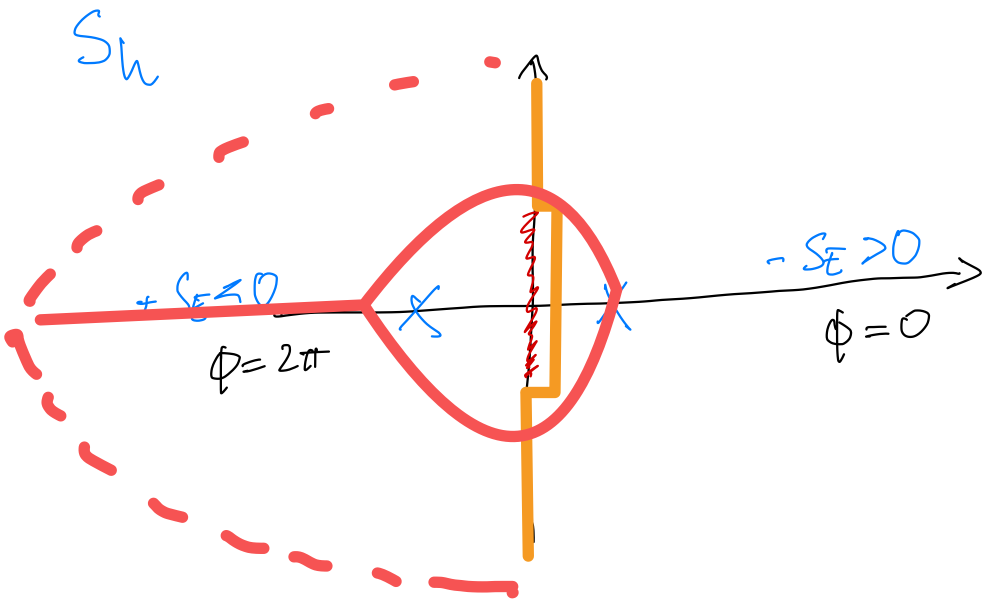


- 2D analogue of our 3D (or 4D) triangulation
- Integrate over s_a and $s_b \to s_h$ (height variable)

For spacelike b:
- A represents a CTC singularity and lc irregularity.

Action(s_a, s_h): has the same structure as before.

Choice 2:



- Entropy ~ # phys. dof
  $\Rightarrow$ Need to integrate over positive and negative lapse to project onto Hamiltonian constraint

- We obtain exponentially enhanced result

$$Z \sim \exp(1.4 \times S_{dS_3})$$

discretization artifact

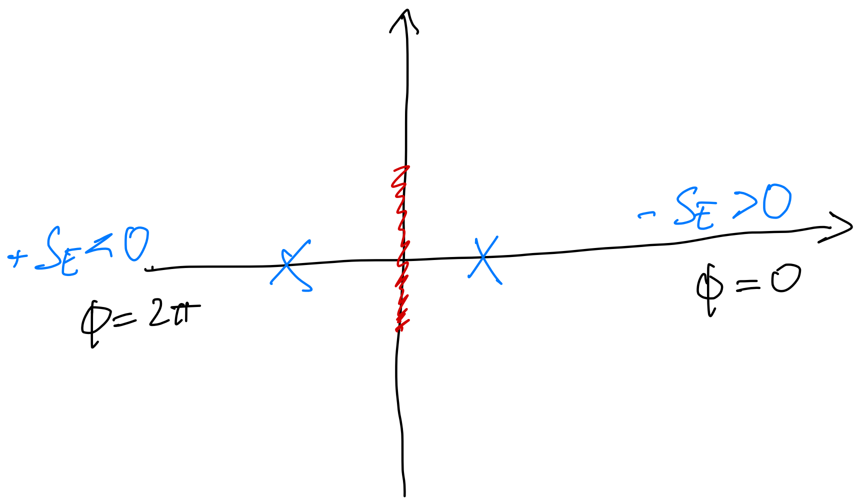
$$S_{dS_3} = \frac{4\pi^2}{\sqrt{\Lambda}}$$

GH - entropy

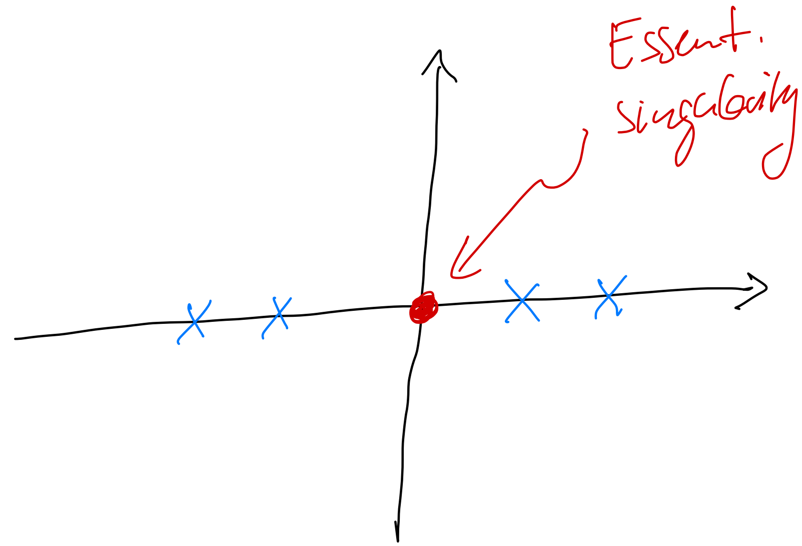
$\Rightarrow$ Lc irregularities are essential for this result

Compare to continuum

Discrete:



- Ambiguity for Lorentzian contour
- Physical interpretation for ambiguity



- Same ambiguity!
- No obvious interpretation
- Debate about contour
  [Feldbrugge, Lehners, Turok]
  vs
  [Diaz Dorronsoro, Halliwell, Hartle, Hertog, Janssen]

- Fluctuation convergence criterion and conformal mode problem

Application:

- Effective spin foam cosmology

Length $\rightsquigarrow$ discrete Areas

How does the replacement of integral by sum change the result?

$\Rightarrow$ **Beyond saddle point approx.**

- Allows to consider time evolution

How to deal with oscillating infinite sum?

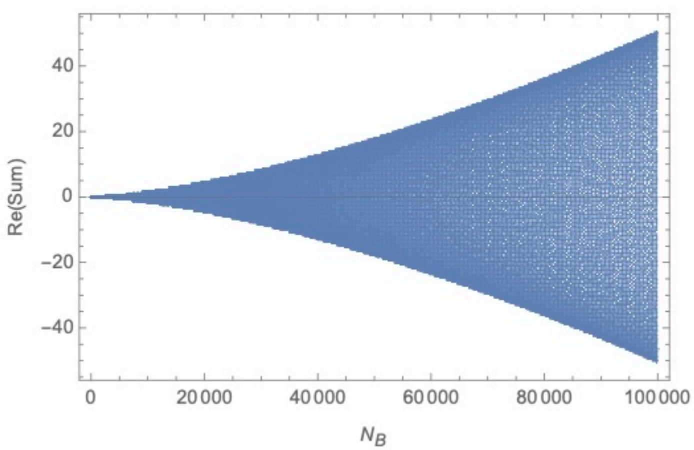
$\Rightarrow$ · Shank transform and Wynn algorithm

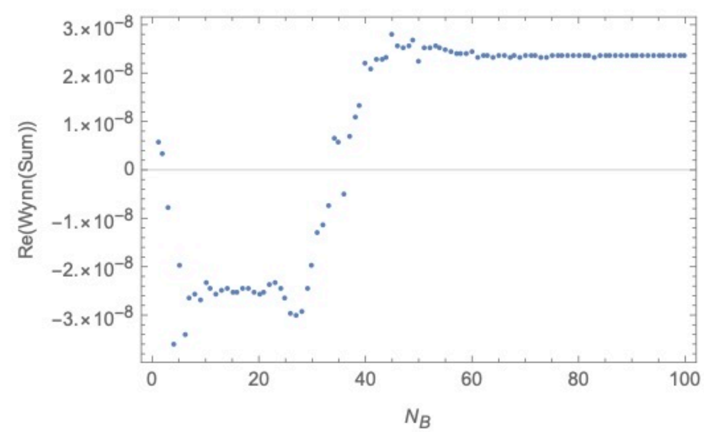
(explained in [BD, Padua-Argüelles 23])

· Works very well for path integrals / state sums

e.g. for $A_n \underset{n\ \text{large}}{\sim} \exp(i\, n \cdot c)$

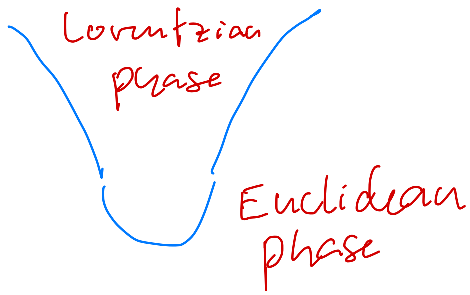
For computation of exp. value:



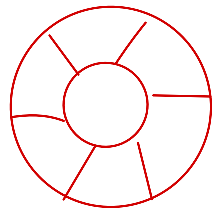


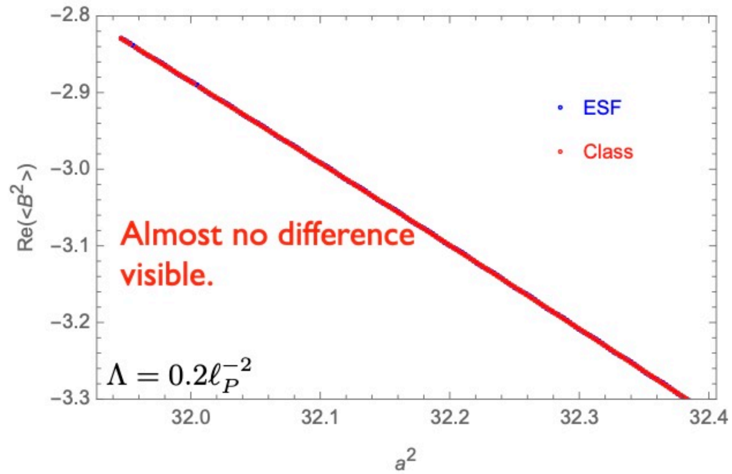
Rel. error $\sim 10^{-8}$

Compare Regge integral with Eff. SF sum



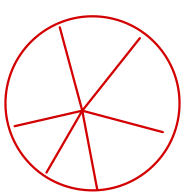
Lorentzian phase:

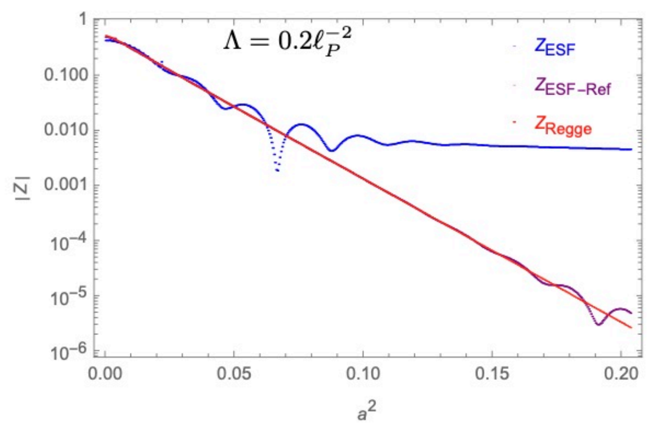




Good! Consistency check.

Euclidean phase





Tunneling amplitude increased for larger scale factors.

General effect?

(Should be confirmed using more time steps.)

Lorentzian (Regge) path integral

- Lc irregular configurations: physically important role

  To appear: Lc irregularities in the regime of large edge lengths [Borissova, BD, Dou, Schiffer]

- Lead to imaginary contributions to the action

- Applications to cosmology & entropy calc.

- Lattice continuum limit for Lorentzian effective spin foams To appear [Asante, BD]

Effective spin foams

- Both Euclidean and Lorentzian signatures possible
- Lorentzian case: no restriction on spacelike/timelike nature of sub-building blocks

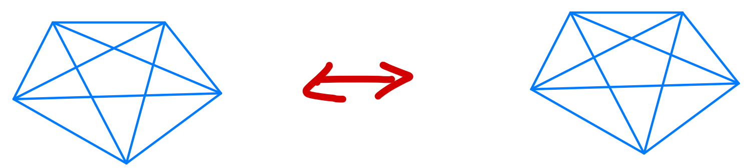
Key ingredients of spin foams

a) discrete areas [Rovelli, Smolin, Ashtekar, Lewandowski, ...]

b) extension of configuration space from: length metric to area metric [BD, Ryan 08, Freidel, Speziale 10, BD, Ryan 11, ..., BD, Borissova '22, BD, Padua-Arguelles '23]

Remarkably b) follows from a).

Area - Length constraints



- 10 lengths
- 10 areas

Shared tetrah.

- 10 lengths
- 10 areas

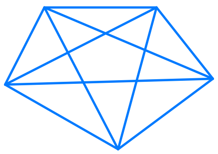
$20 - 6 = 14$ lengths
$20 - 4 = 16$ areas

⇒ There are two area length constraints.
(Requiring that two dihedral angles in shared tetrah. agree, if computed from either 4-simplex data)

With discrete area values:
- Constraints lead to diophantine eqs.
- Very few solutions, preventing suitable semiclassical regime [Asante, BD, Haggard '20]

⇒ Need to enlarge configuration space.

Twisted simplex geometry and Area metrics



In LQG

→ 10 areas

→ Also 2 dihedral angles in each of the two tetrahedra

Overall $\boxed{20}$ variables

$$\{\varphi_e^\tau, \varphi_{e'}^\tau\} \sim \gamma$$

Barbero - Immirzi Parameter [BD, Ryan '08]

$\varphi_e^\tau \sim \partial_t \cdot \partial_{t'}$

"Twisted Simplex" [Freidel, Speziale '10]

[BD, Padua-Argüelles '23]:

The 20 variables associated to the twisted simplex define an Area-metric.

Area metrics

$$\underbrace{\underbrace{G_{\mu\nu\rho\sigma}}_{\text{symmetric}}}_{\text{antisymmetric}} \quad \text{and} \quad G_{\mu\nu\rho\sigma}\,\epsilon^{\mu\nu\rho\sigma} = 0$$

$\Rightarrow$ same algebraic symmetries as Riemann tensor

$\Rightarrow$ 20 components

Prescribes areas of parallelograms & dihedral angles

Each length metric defines an area metric:

$$G_{\mu\nu\rho\sigma} = g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho}$$

$\Rightarrow$ Area metrics extend configuration space of length metric

Area metrics

Microscopic: LQG simplex data $\Rightarrow$ Area metric [BD, Padua-Arguelles]

Mesoscopic: Effective spin foams on regular lattice $\xrightarrow[\text{limit}]{\text{Continuum}}$ Action as a fct. of Area metr. [BD '21, BD, Kogios '22]

$\Updownarrow$ consistent

Continuum: Modified Plebanski [Krasnov, Freidel, Speziale 06+] $\Rightarrow$ Area metric action [Borissova, BD 22]

Area metric action

Linearized action obtained from modified Plebanski:

$G_{\mu\nu\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + a_{\mu\nu\rho\sigma}$, $a_{\mu\nu\rho\sigma} \longrightarrow h_{\mu\nu}, \chi^{\pm}_{\mu\nu}$

$$\mathcal{L} = \mathcal{L}_{EH}(h_{\mu\nu}) + \frac{\gamma_+}{2} h_{\mu\nu} p^2 \chi_+^{\mu\nu} + \frac{\gamma_-}{2} h_{\mu\nu} p^2 \chi_-^{\mu\nu}$$
$$+ \frac{(p^2 + M^2)}{4}\left(\gamma_+ \chi^+_{\mu\nu} \chi_+^{\mu\nu} + \gamma_- \chi^-_{\mu\nu} \chi_-^{\mu\nu}\right)$$

with $\gamma_+ = 1 + \frac{1}{\gamma}$, $\gamma_- = 1 - \frac{1}{\gamma}$ Barbero-Immirzi parameter

[Borissova, BD, Krasnov 2024]:
- γ affects classical EOM in area metric gravity
- Leads to mixing of cross and plus polarization of transverse-traceless modes.

- One finds the action, if looking for general diff-inv. area-metric actions

Area metric action → Effective length metric action

Integrate out χ- fields:

$$\mathcal{L}_{eff}(h) = \mathcal{L}_{EH}(h) - \underbrace{{}^{(1)}C_{\mu\nu\rho\sigma}(h)\,\frac{1}{p^2+M^2}\,{}^{(1)}C_{\mu\nu\rho\sigma}(h)}_{\text{non-local Weyl}^2\text{-term}}$$

(linearized) Weyl tensor

Does not add further poles: $(\mathrm{Prop})^{TT} = 2\left(\frac{1}{p^2} + \frac{1}{M^2}\right)$

Very promising !

Later: Caveat for Lorentzian signature.

Area metric actions

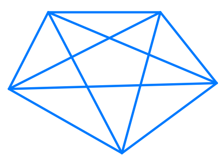
- First candidate effective actions describing the continuum limit of spin foams

Evidence:

- continuum limit of effective spin foams [BD 2021, BD, Koslos '22]
- derivation from modified Plebanski framework [Borissova, BD '22]
- Extension of spin foam config. space to area metrics and constructing diff.-invariant area-metric actions [BD, Padua-Arguelles '23] [Borissova, BD, Krasnov '23]

Area-angle variables in spin-foams

[BD, Speziale '08]



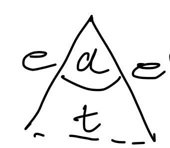
- 10 areas
- 2 dihedral angles φ_e^τ in each tetrahedron

- Can invert areas $\rightarrow$ length
- Compute Dihedral angles as functions of areas

$\Rightarrow$ Shape matching / Gluing constraints

$$C_e^\tau = \varphi_e^\tau - \Phi_e^\tau(A)$$

- But φ_e^τ do not commute! $\hbar\{\varphi_e^\tau, \varphi_{e'}^\tau\} = \gamma\, \ell_p^2 \frac{\sin\alpha}{a_t}$



$\Rightarrow$ Constraints C_e^τ are second class.
Cannot be imposed sharply, need to be imposed "weakly".
[EPRL, FK]

How weakly? As strongly as allowed by commutator!

Effective spin foams

[Asante, BD, Haggard '20]

$$Z = \sum_{\{j_t\}} \mu(j_t) \int \prod_{\tau} d\mu[\phi^{\tau}] \prod_t \mathcal{A}_t(j) \prod_{\sigma} \mathcal{A}_\sigma(j) \prod_{(\tau\sigma)} K_\tau(\phi^\tau ; \Phi^{\tau,\sigma}(j))$$

face weight

simplex amplitude

coherent state in angle variables, peaked on Dihedral angles

$\Rightarrow$ integrate out angle variables / sum over intertwiners

$\Rightarrow$ **Effective model:**

$$Z = \sum_{\{j_t\}} \mu(j_t) \underbrace{\prod_t \mathcal{A}_t(j) \prod_\sigma \mathcal{A}_\sigma(j)} \prod_\tau G_\tau^{\sigma\sigma'}(j)$$

gluing function

$\exp(i\, S_{\text{Area-Regge}}(j))$

Recoupling symbol in Higher Gauge theory

[Baratin, Freidel; Asante, BD, Girelli, Tsimiklis, Riello]

$$G_\tau^{\sigma\sigma'} \sim \exp\left(- \frac{\sum_e (C_e^\tau)^2}{4\, \Sigma^2} \right)$$

$$\Sigma^2 \sim \ell_p^2\, \gamma\, |V_\sigma^s|$$

Effective spin foams

- Computations require much much less resources.
  So far only explicit calculation of Z for inner edge: **Successful test of EOM.**
  (Seconds - Minutes on a laptop) [Asante, BD, Haggard '20]
- Direct geometric interpretation / less variables.
  Allows for (perturbative) continuum limit.
  [BD '21; BD, Kogios '22]
- Lorentzian signature: Timelike & spacelike subbuilding blocks allowed; no vector geometries
  [Asante, BD, Padua-Arguelles '21]

Area Regge action

$$S_{\text{Length-Regge}} = \sum_t A_t(\ell)\,\epsilon_t(\ell)$$

$$= \sum_t 2\pi A_t(\ell) - \sum_\sigma \sum_{t\in\sigma} A_t(\ell)\,\theta_t^\sigma(\ell)$$

$\Rightarrow$ Invert in <u>each</u> simplex length for areas

$\theta_t^\sigma(\ell) \longmapsto \theta_t^\sigma(A)$ areas in given simplex

$$\Rightarrow S_{\text{Area-Regge}} = \sum_t 2\pi A_t - \sum_\sigma \sum_{t\in\sigma} A_t\,\theta_t^\sigma(A)$$

$$= \sum_t A_t \cdot \tilde{\epsilon}_t(A)$$

Area Regge action

- appears in semiclassical limit of spin foams

The Flatness Problem (or not)

$$S_{\text{Area-Regge}} = \sum_t A_t \, \tilde{\epsilon}_t(A)$$

EOM:
$$\frac{\delta S_{\text{Area-Regge}}}{\delta A_t} \underset{\substack{\text{Schläfli}\\\text{identity}}}{=} \tilde{\epsilon}_t(A) \overset{!}{=} 0$$

Are these EOM demanding flatness?

This is not the case. $\tilde{\epsilon}_t(A)$ is a combination of curvature and shape mismatch.
In the continuum limit (because of different scalings) one does obtain the Length-Regge EOM.

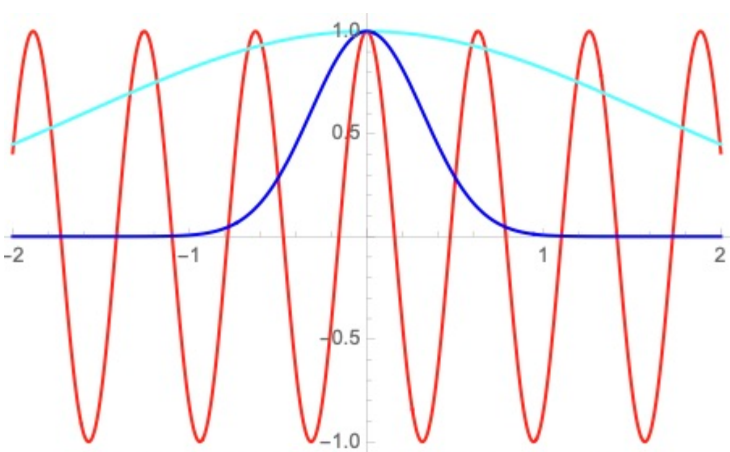
[BD 2021]

The flatness problem

$$Z = \sum_{A_t\text{-values}} e^{iS_{\mathrm{AR}}} \prod_{\tau} G^{\sigma,\sigma'}$$

Oscillating factor with Area Regge action

Gaussian factor peaked on constraints.



Oscillating and Gaussian factor for two different $\gamma < \gamma$.

In $\hbar \to 0$ limit:

Stronger and stronger oscillations wash out Gaussians/Constraints

Thus γ has to be small:

$$\sqrt{\gamma}\;\frac{\sqrt{\text{Area}}}{\ell_p}\;\underbrace{\frac{\delta S_{AR}}{\delta\,\text{Area}}}_{\epsilon} \leq \mathcal{O}(1)$$

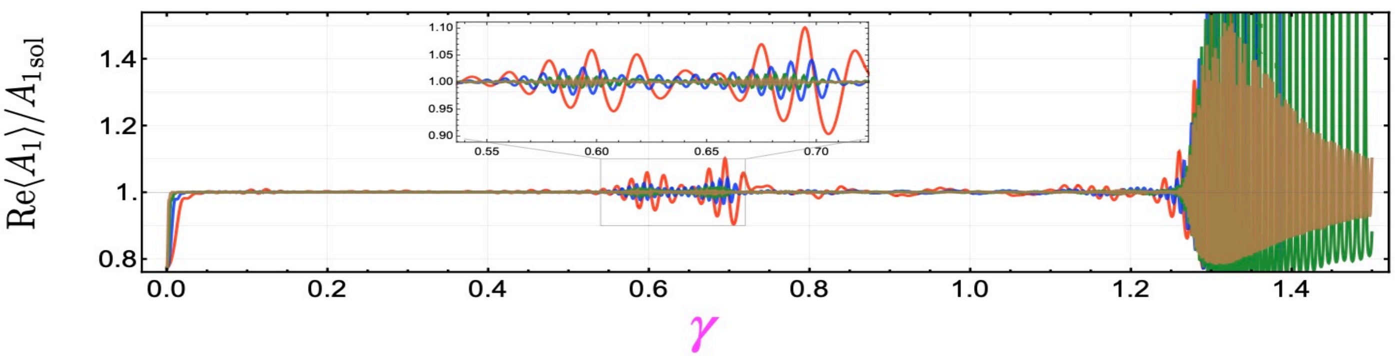
[Han 13, Asante, BD, Haggard 20]

Explicit test ?

Test of (discrete) EOM

[Asante, BD, Haggard '20]

- Compute expectation value (bulk areas) by directly evaluating the path integral
- As a function of γ. (Computation can be done for general γ!)
- Expectation: Reproduce Regge for small values of γ, small curvatures.



- Expectations are even exceeded.
- Check paper for different curvature regimes.

Test of (discrete) EOM

[Asante, BD, Haggard '20]

- Effective spin foam action has imaginary parts from $G_\tau^{\sigma\sigma'}$ - factors.

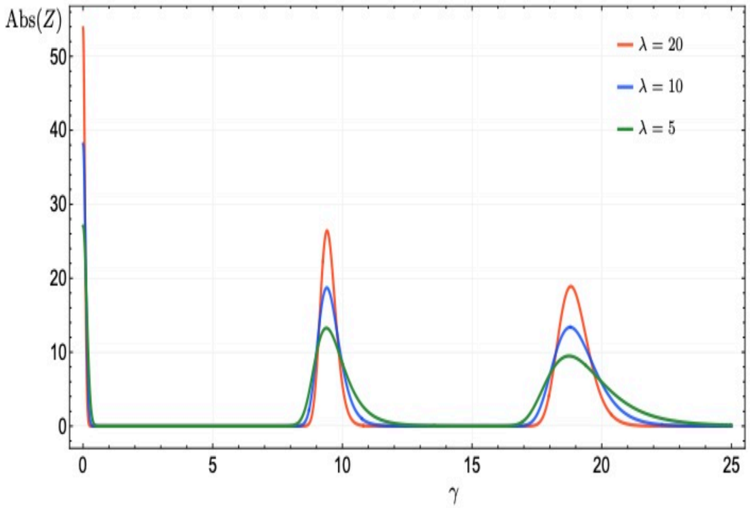
$\Rightarrow$ Saddle points are in the complex plane.

Also the case for EPRL [Han, Huang, Liu, Qu 21]

$\Rightarrow$ One other aspect of resolution of the flatness problem.

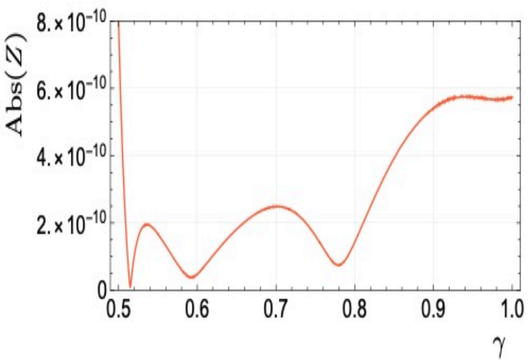
Additional aspect:

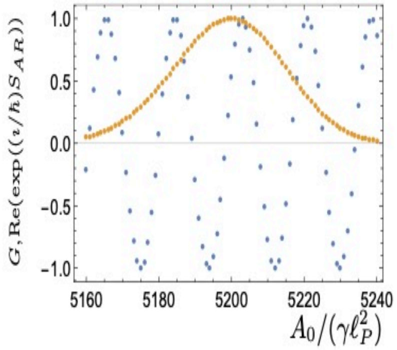
Beyond stationary phase effects

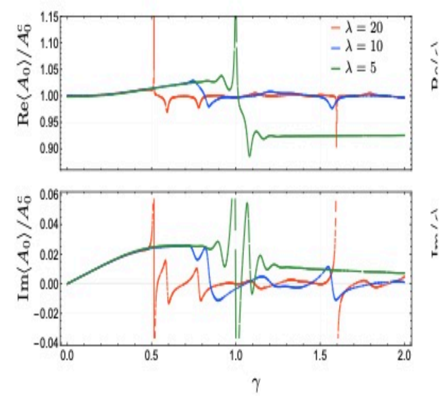


There is a strong suppression of the partition function with growing γ.
This has been interpreted as proof of flatness problem.
But for fixed γ parameter, absolute value of Z does not necessarily matter, if computing expectation values.

$$\langle \mathcal{O} \rangle = \frac{\sum_{a_t} \mathcal{O}(a_t)\mathcal{A}(a_t)}{\sum_{a_t} \mathcal{A}(a_t)}$$

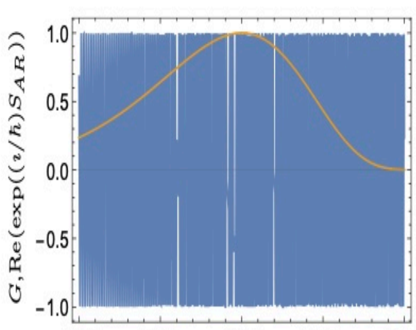


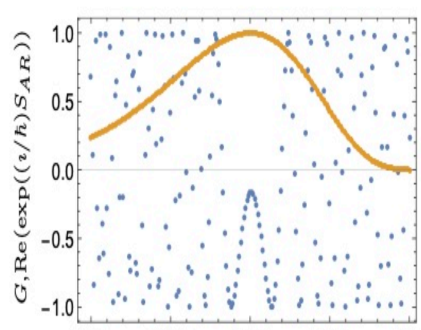




But what does lead to instabilities in the expectation values are γ-values where Abs Z goes more sharply to almost zero.
This is caused by destructive interference.

(Not visible with stationary phase method.)





Maximal in Abs Z caused by "pseudo stationary points" resulting from discreteness of areas.
Occur for unreasonable large values of γ.

Strongest case: Continuum limit

- Effective action = Area-Regge + Constraint terms
- Can be put on hypercubical lattice, where each cube is subdivided into X simplices ($X = 24, 48, \dots$).
- Expand around flat (shape-matching) configuration

$\Rightarrow$ Compute Hessian (k)

Lattice momentum

Huge matrix with k-dependent entries

Continuum limit

Analyze this Hessian.

Gauge modes = null modes for any k

Massless modes = null modes for $k = 0$

	Length Regge	Area Regge	Massless — gauge modes
Standard [BD 2021]	15 variables	(singular) 50 variables	6 [(Regge)-Williams 2000]
+hyper-cube vertices [BD, Kogios 2022]	30 variables	100 variables	6
+cube vertices	66 variables	204 variables	6
+square vertices	132 variables	(singular) 408 variables	6

[to appear]

- There are always 6 massless ≠ gauge modes and <u>at least</u> 4 gauge modes → length d.o.f.

Integrating out all other dof's: Linearized GR at leading order.

Continuum limit

⇒ Get GR on family of different lattices.

Surprise: Even Area-Regge without constraint terms leads to GR in continuum limit!

Barrett-Crane model can lead to GR.

Leading order correction

- Can identify blocks with different scaling behaviour (in k) in the Hessian (might require an insightful variable transform.)

- These contribute to different order in the length-effective action

$$H \sim \frac{\partial \epsilon_t \cdot \partial A}{\partial A_t \cdot \delta l} \sim \underline{B} \cdot \epsilon \epsilon$$

- Variables which contribute to correction: Area metric dof (in addition to length dof)

- Lead to Weyl - squared correction

Consistent with modified Plebanski approach.

Effective spin foams & flatness problem

- Addressed flatness problem
  - * in the discrete by explicit computation of path integral / expectation value.

    Reproduce discrete EOM.
    Allowed regime larger than expected.

  - * in the (perturbative) continuum limit:

    Surprise: · Constraints are actually <u>not</u> needed to get GR at leading order.
    · Do influence corrections

Area metric actions

- Find that leading order correction comes from area metrics
- Motivates to look for diff.-invariant area metric actions:

Area metric actions

$$\gamma_\pm = 1 \pm \frac{1}{\gamma}$$

$$\mathcal{L} = \mathcal{L}_{EH}(h_{\mu\nu}) + \frac{\gamma_+}{2} h_{\mu\nu}\, p^2 \chi_+^{\mu\nu} + \frac{\gamma_-}{2} h_{\mu\nu}\, p^2 \chi_-^{\mu\nu}$$

$$+ \frac{(p^2 + M^2)}{4} \left(\gamma_+ \chi^+_{\mu\nu} \chi_+^{\mu\nu} + \gamma_- \chi^-_{\mu\nu} \chi_-^{\mu\nu} \right)$$

This is actually for the Euclidean theory!

Lorentzian case:
- $\chi_\pm$ are complex (self-dual decomp.) but conjugated to each other
- introduce real and imaginary parts
- But then

$$(\chi_+^2 + \chi_-^2) \rightsquigarrow (\chi_1^2 - \chi_2^2)$$

Euclidean

Lorentzian: Dangerous!

Lorentzian Area Metric actions

- Well known effect:

$\mathcal{L}_{EM} = E^2 + B^2 \longrightarrow \mathcal{L}_{EM} = E^2 - B^2$

Euclidean — Lorentzian

- Kodama state: Euclidean: normalizable [Freidel, Smolin]
  Lorentzian: non-normalizable [Witten]

- Dangerous: Could lead to unstable behaviour.

Second order: • position and negative energy modes decouple ⇒ Dynamics is stable

[Borissova, BD, Krasnov '23]

⇒ Analyze Lorentzian Area Metric actions to higher order.

Effective Spin foams: Applications

- Effective spin foam cosmology:
  - Effect of discrete spectra on time evolution

→ More time steps and inhomogeneities

- Area metric actions → Effective **continuum action** describing spin foam dynamics

→ Higher order terms, renormalization flow, area metric phenomenology, Lorentzian sign.