Refs:

Spinfoam:

[1] **Haggard, Han, Kaminski, Riello**, "SL(2,C) Chern-Simons theory, a non-planar graph operator, and 4D quantum gravity with a cosmological constant: semiclassical geometry", arxiv: *1412.7546*

[2] **Haggard, Han, Riello**, "Encoding curved tetrahedron in face holonomies: phase space of shapes from group-valued moment maps", arxiv: *1506.03053*

[3] **Haggard, Han, Kaminski, Riello**, "SL(2,C) Chern-Simons theory, flat connections, and four-dimensional quantum geometry", arxiv: *1512.07690*

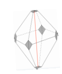
[4] **Han**, "Four-dimensional spinfoam quantum gravity with a cosmological constant: finiteness and semiclassical limit", arxiv: *2109.00034*

[5] **Han, QP**, "Melonic radiative correction in four-dimensional spinfoam model with cosmological constant", arxiv: *2310.04537*

[6] **Han, QP**, "Deficit angles in 4D spinfoam with cosmological constant: (Anti)-de Sitter-ness and more", arxiv: *2401.14643*

SL(2,C) Chern-Simons

[7] **Gaiotto, Moore, Neitzke**, "Wall-crossing, Hitchin systems, and the WKB approximation", arxiv: *0907.3987*



[8] **Dimofte**, "Quantum Riemann surfaces in Chern-Simons theory", arxiv: *1102.4847*

[9] **Dimofte**, "Complex Chern-Simons theory at level k via the 3d-3d correspondence", arxiv: *1409.0857*

Outline

lecture 1: Motivation & scheme

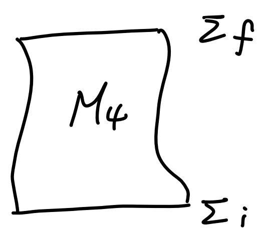
2: SL(2, ℂ) CS theory on "ideal tetrahedron"

3: SL(2, ℂ) CS theory on M_3 → spinfoam amplitude in 4D

§1. Motivation

§1.1 idea of SF

$$\mathcal{Z} = \int_{(h_{ab})_i}^{(h_{ab})_f} \mathcal{D}g_{\mu\nu}\, e^{\frac{i}{\ell_p^2} S_{EH}(g_{\mu\nu})}$$



↓ regularize

SF amplitude $\mathcal{Z}_{T(M_4)}$ s.t. it respects

local SF ansatz: $\mathcal{Z}_{T(M_4)} = \sum_{\{j,\iota\}} \prod_f A_f(j) \prod_t A_t(j,\iota) \prod_\sigma A_\sigma(j,\iota)$

4D SF w/ $\Lambda = 0$ (EPRL) $\xrightarrow{j\to\infty}$ $A_\sigma \sim \exp[i S_{Regge}]$

↖ discretized S_{EH}.

$\mathcal{Z}_{EPRL}(T(M_4))$ diverges

$\sum_{j=0}^{\infty} \to \sum_{j=0}^{J_{max}} \Rightarrow$ finite.

→ **[4D SF w/ $\Lambda \neq 0$ finite]** ex. in 3D Turaev-Viro SF $\Lambda > 0$.

§1.2 EPRL w/ $\Lambda=0$

$\eta = \mathrm{diag}(-1,1,1,1)$

$S_{GR}[e,\mathcal{A}] = \frac{1}{2}\int_{\mathcal{M}_4} [\epsilon^{IJKL}\, e_I \wedge e_J \wedge F_{KL}(\mathcal{A})]$ $\quad 8\pi G = 1.$

e: tetrad $sl(2,\mathbb{C})$ 1-form

$\mathcal{A}$: connection $sl(2,\mathbb{C})$ 1-form.

(1) $S_{Holst}[e,\mathcal{A}] = \frac{1}{2}\int_{\mathcal{M}_4} [\epsilon^{IJKL}\, e_I\wedge e_J \wedge F_{KL}(\mathcal{A}) - \frac{1}{\gamma}\epsilon^{IJKL}(*_\iota(e\wedge e))_{IJ}\wedge F_{KL}(\mathcal{A})]$

γ: Barbero-Immirzi param. $\gamma\in\mathbb{R}$

$(*X)_{IJ} = \frac{1}{2}\epsilon_{IJ}{}^{KL}X_{KL}$, $*^2=-1$

(1) $\Rightarrow$ $S_{HBF}[B,A] = \int_{\mathcal{M}_4} \mathrm{Tr}[(* + \frac{1}{\gamma})B\wedge F(A)]$ w/ simplicity constraint: $B = \pm e\wedge e$.

B: $sl(2,\mathbb{C})$ 2-form $\quad \mathrm{Tr}(XY) := X^{IJ}Y_{IJ}$.

quantize HBF $\rightarrow$ impose s.c. quantumly = generate quantum geometry.

$|BF\rangle = \int d\mathcal{A}\, dB\, e^{\frac{i}{\ell_p^2}S_{HBF}} = \int d\mathcal{A}\,\delta(F(\mathcal{A}))$

consider $\mathcal{M}_4$ w/ bdy $\partial\mathcal{M}_4$

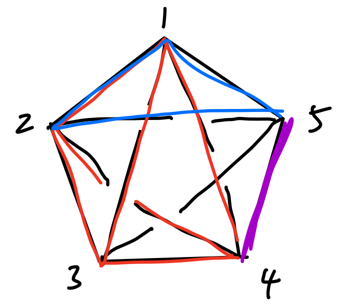
$\psi[\mathcal{A}_\partial]$, $\mathcal{A}_\partial = \mathcal{A}|_{\partial\mathcal{M}_4}$

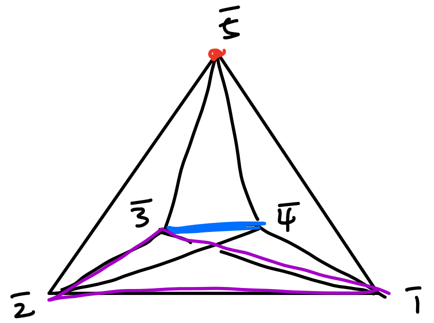
BF amplitude $\langle\psi[\mathcal{A}_\partial]|BF\rangle = \int d\mathcal{A}\,\delta(F(\mathcal{A}))\,\psi[\mathcal{A}_\partial]$.

choose $\psi[\mathcal{A}_\partial] = \psi_\Gamma[\mathcal{A}_\partial] = \psi_\Gamma(G_e[\mathcal{A}_\partial])$: $sl(2,\mathbb{C})$ spin network state.
$\Gamma \in \partial\mathcal{M}_4$

$\mathcal{M}_4 = B^4$, $\partial\mathcal{M}_4 = S^3 \longrightarrow \mathcal{A}_\sigma$

$\downarrow$ $T(B^4) = $ 4-simplex σ $\quad T(S^3) = 5$ tetra $\} \leftrightarrow$ 3D dual $\Gamma = \Gamma_5$





$T(S^3)$	Γ_5
tetra t	node v
triangle f	link e
edge E	face

BF amplitude for σ: $\langle\psi_{\Gamma_5}|BF\rangle \rightarrow$ impose s.c. on ψ_{Γ_5} ~ regularized $B = \pm e\wedge e$.

$\begin{cases} \text{bulk of } \sigma : \text{ topological} & \sim \text{ Regge calculus.} \\ \partial\sigma : \text{ geometry} \end{cases}$

$B = \pm e \wedge e$ 2-form $\Rightarrow$ $B_f^{IJ}(t) := \int_f B^{IJ}(t)$

$\Rightarrow B_f^{IJ}(t) = \pm e^I(t) \wedge e^J(t) \sim f$

$\hookrightarrow \in$ Cartian coordinate patch covering t.

$\Leftrightarrow$ ① $\exists N^J \in \mathbb{R}^{1,3}$ s.t. $N^J B_f^{IJ}(t) = 0, \ \forall f \in t$

② $\sum_{f\in t} B_f^{IJ}(t) = 0$ closure condition $\Leftrightarrow \sum_{f\in t} a_f n_f^I = 0$

$\updownarrow$ Minkowski. thm

convex tetra.

$a_f n_f^I := \frac{1}{2}\epsilon^I{}_{JKL} N^J B_f^{KL}(t)$

$\Rightarrow$ quantize ...

§1.3. path-integral of 4D gravity w/ $\Lambda \neq 0$

$$S_{H\wedge BF}[B, \mathcal{A}] = \int_{\mathcal{M}_4} \mathrm{Tr}\left[\left(\star + \tfrac{1}{\gamma}\right) B \wedge \left(\mathcal{F}(\mathcal{A}) - \tfrac{|\Lambda|}{6} B\right)\right]$$

B: $sl(2,\mathbb{C})$ 2-form

$\mathcal{A}$: $sl(2,\mathbb{C})$ 1-form

$\downarrow$ s.c. $B = \mathrm{sgn}(\Lambda)\, e \wedge e$

$S_{Holst}[e, \mathcal{A}]$ ✓

$$\frac{\partial S_{H\wedge BF}}{\partial B_{IJ}} = 0 \Rightarrow \mathcal{F} = \frac{|\Lambda|}{3} B \xrightarrow{\text{s.c.}} \boxed{\mathcal{F} = \frac{\Lambda}{3}\, e \wedge e}$$

field strength ↙ ↘ geometry

$$\int d\mathcal{A}\, dB \exp\left[\frac{i}{\ell_p^2} \int_{\mathcal{M}_4} \mathrm{Tr}\left[\left(\star + \tfrac{1}{\gamma}\right) B \wedge \left(\mathcal{F}(\mathcal{A}) - \tfrac{|\Lambda|}{6} B\right)\right]\right]$$

$\mathcal{F} = \frac{|\Lambda|}{3} B$ $\downarrow$ $B = \frac{3}{|\Lambda|}\mathcal{F}$

$$= \int d\mathcal{A} \exp\left[\frac{3i}{2|\Lambda|\ell_p^2} \int_{\mathcal{M}_4} \mathrm{Tr}\left[\left(\star + \tfrac{1}{\gamma}\right)\mathcal{F} \wedge \mathcal{F}\right]\right] \qquad \mathcal{A} \to (A, \bar{A})$$

$\downarrow$ $\mathcal{F} = F + \bar{F}$ $\begin{cases} F := \frac{1}{2}(1 - i\star)\mathcal{F} & \star F = iF, \quad F(A) \\ \bar{F} := \frac{1}{2}(1 + i\star)\mathcal{F} & \star\bar{F} = -i\bar{F}, \quad \bar{F}(\bar{A}) \end{cases}$

$$= \int dA\, d\bar{A} \exp\left[-\frac{3}{2\ell_p^2|\Lambda|} \int_{\mathcal{M}_4} \left(1 - \tfrac{i}{\gamma}\right) \mathrm{Tr}(F \wedge F) - \left(1 + \tfrac{i}{\gamma}\right) \mathrm{Tr}[\bar{F} \wedge \bar{F}]\right].$$

$\downarrow$ $\mathcal{M}_4$ trivial topo.

$$|CS\rangle = \int dA_\partial\, d\bar{A}_\partial \exp\left[-i S_{CS}[A] - i S_{CS}[\bar{A}]\right] \qquad -i S_{CS}^t[A, \bar{A}]$$

w/. $S_{CS}[A] := \frac{t}{8\pi} \int_{\partial\mathcal{M}_4} \mathrm{Tr}\left[A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right]$

$S_{CS}[\bar{A}] := \frac{\bar{t}}{8\pi} \int_{\partial\mathcal{M}_4} \mathrm{Tr}\left[\bar{A} \wedge d\bar{A} + \tfrac{2}{3} \bar{A} \wedge \bar{A} \wedge \bar{A}\right]$

$t = k + is$, $\bar{t} = k - is$. $k = \frac{12\pi}{\ell_p^2 \gamma |\Lambda|} \in \mathbb{Z}$, $s = \gamma k \in \mathbb{R}$.

Consider gauge transf. $A \to g^{-1} A g + g^{-1} dg$, $g \in sl(2, \mathbb{C})$.

$$\omega(g) = \frac{1}{24\pi^2} \mathrm{Tr}\,(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg)$$

winding number $\int_{\mathcal{M}_3} \omega(g) = N \in \mathbb{Z}$ $\mathcal{M}_3$: closed manif. w/o bdy.

Under a large gauge transf.: $S_{CS}[A] \to S_{CS}[A] + 2\pi k N$.

$e^{i S_{CS}[A]}$ gauge-invariant $\Rightarrow k \in \mathbb{Z}$.

[Witten '91] $(k \in \mathbb{Z}, s \in \mathbb{R}) \Rightarrow$ unitary quantum CS.

$$\langle \psi[A, \bar{A}] | CS \rangle = \int dA\, d\bar{A}\; e^{-i S^t_{CS}[A, \bar{A}]}\; \underline{\psi[A, \bar{A}]}.$$

quantum s.c.

Canonical analysis

local $\partial \mathcal{M}_4 = \Sigma \times \mathbb{R}$

$$S^t_{CS} = \frac{t}{8\pi} S_{CS}[A^I_0, A^I_i] + \frac{\bar{t}}{8\pi} S_{CS}[\bar{A}^I_0, \bar{A}^I_i] . \quad i = 1, 2 .$$

$$\omega_{CS} = \frac{t}{4\pi} \int_\Sigma \mathrm{Tr}[\delta A \wedge \delta A] + \frac{\bar{t}}{4\pi} \int_\Sigma \mathrm{Tr}[\delta \bar{A} \wedge \delta \bar{A}]$$

$$\{ A^I_i(\vec{x}), A^J_j(\vec{y}) \} = \frac{4\pi}{t} \varepsilon_{ij}\, \delta^{(2)}(\vec{x} - \vec{y})$$

$$\{ \bar{A}^I_i(\vec{x}), \bar{A}^J_j(\vec{y}) \} = \frac{4\pi}{\bar{t}} \varepsilon_{ij}\, \delta^{(2)}(\vec{x} - \vec{y}) \qquad \vec{x}, \vec{y} \in \Sigma .$$

$$\{ A^I_i(\vec{x}), \bar{A}^J_j(\vec{y}) \} = 0$$

$$\frac{\partial S}{\partial A^I_0} = \frac{\partial S}{\partial \bar{A}^I_0} = 0 \Rightarrow F^I_{ij} = \bar{F}^I_{ij} = 0$$

{ CS eom $\Rightarrow \mathcal{F} = 0$

s.c. $\Rightarrow \underline{\mathcal{F} = \frac{\Lambda}{3} e \wedge e}$ $\to$ introduce defects on $\partial \mathcal{M}_4$.

↙ magnetic flux ↘ geometry.

☆ **4D QG w/ $\Lambda \neq 0$ on B^4 $\sim$ quantum CS w/ $t, \bar{t}$ on S^3 + defects on S^3**

$\mathcal{M}_4 = B^4$, $\partial \mathcal{M}_4 = S^3$

$\downarrow$ $\downarrow$

$T(B^4) = \sigma$ $T(S^3) = \{ 5 \text{ tetra} \}$.

$$\mathcal{F}^{IJ}_f(t) := \int_f F^{IJ}(t)$$

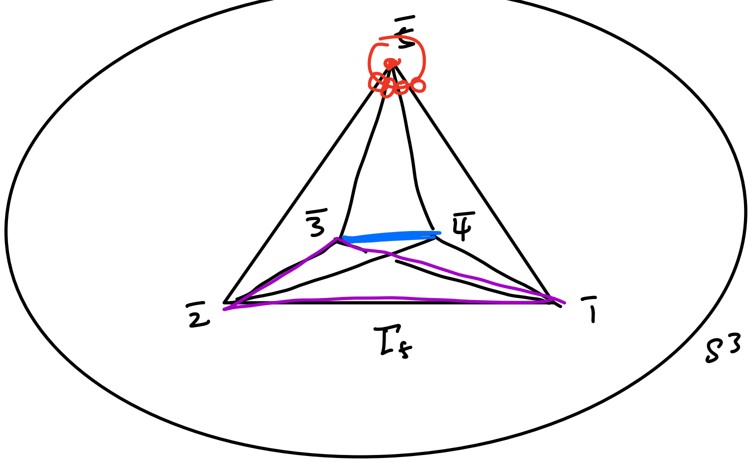
regularized s.c. $\Rightarrow \mathcal{F}^{IJ}_f(t) = \frac{\Lambda}{3} e^I(t) \wedge e^J(t)$

$\sim$ defects on $T(S^3)$

face defect?

$\Gamma_5 \hookrightarrow S^3 \Rightarrow$ impose s.c. on links.

$F^{IJ}_f(t) \to \hat{F}^{IJ}_f(t)$ insert to Z_σ



☆ operator inserted along a graph
||
☆ remove the graph & tubular open neighbourhood + impose bdy condition on the graph complement

$S^3 \setminus N(\Gamma_5) \equiv S^3 \setminus \Gamma_5$

$\partial(S^3\setminus\Gamma_5) = \begin{cases} \text{4-hole spheres } \Sigma_{0,4} \\ \text{annuli} \end{cases} \Rightarrow \begin{matrix} \# = 5 \\ \# = 10 \end{matrix}$

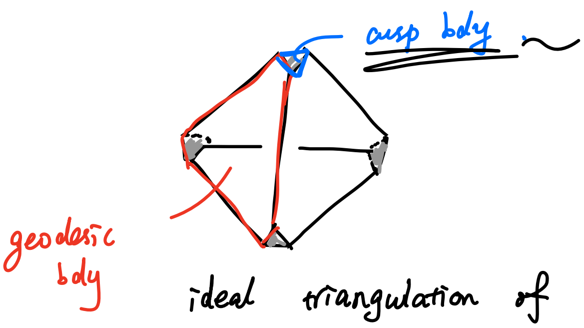
☆ + ☆ $\Rightarrow$ 4D QG w/ $\Lambda \neq 0$ on B^4 $\sim$ CS on $S^3\setminus\Gamma_5$ w/ s.c. imposed on $\partial(S^3\setminus\Gamma_5)$

Task I: CS partition function on $S^3\setminus\Gamma_5$ $\to$ lecture 2

II: Impose s.c. on $\partial(S^3\setminus\Gamma_5)$ $\to$ lecture 3.

§2. CS partition function on an ideal tetrahedron

Def. ideal tetra Δ = tetra whose vertices are at ∞ $\sim$ vetex-truncated tetra.



cusp bdy $\sim$ annuli.

ideal triangulation of $\mathcal{M}_3 \setminus \Gamma = \{\Delta_i\}_i$

ideal tri of $\mathcal{M}_3$ $\neq$ tri of $\mathcal{M}_4$
(w/ $\partial\mathcal{M}_3 \neq \phi$) $\neq$ tri of $\mathcal{M}_3$.

$T_{id}(S^3\setminus\Gamma_5) = \{\Delta_i\}_{i=1}^{20}$ [c.f. fig. 4].

Goal: CS partition fn on $S^3\setminus\Gamma_5$

$\mathcal{Z}(\Delta)$ is known!

$$\Rightarrow \mathcal{Z}(S^3\setminus \Gamma_5) = \prod_{i=1}^{20} \mathcal{Z}(\Delta)\Big|_{\{C_i=0\}}.$$

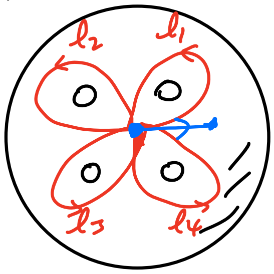
§2.1. CS phase space for $\partial\Delta$

2-manifold Σ: phase space of CS w/ G on Σ:

Poisson manifold $\Rightarrow \mathcal{M}_{flat}(\Sigma, G) := \{\text{flat } \mathfrak{g}\text{-valued connection on } \Sigma\}/G$

$= \underbrace{\mathrm{Hom}(\pi_1(\Sigma), G)}_{\text{holonomy}}/G$ ← gauge.

ex. $\Sigma_{0,4}$



$\Rightarrow l_4\circ l_3\circ l_4\circ l_1 = 1$ $l_i \in \pi_1(\Sigma_{0,4})$

holonomies $H_i \in G$

$H_4H_3H_2H_1 = \mathbb{1}_G$

$H_i \mapsto gH_ig^{-1}, \quad \forall g \in G.$

$\Sigma = \partial M_3$. $P_{\partial M_3} = \mathcal{M}_{flat}(\partial M_3, G) \rightarrow$ symplectic manifold.

$F=0$ on $M_3 \Rightarrow \mathcal{L}_{M_3} =$ Lagrangian submanif. of $P_{\partial M_3}$.

def. A Lagrangian submanif. L of a symplectic manifold (M, Ω) is $L\subset M$ s.t.

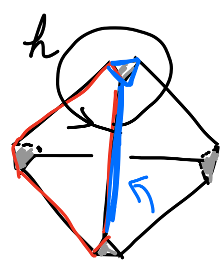
$$\Omega\big|_L = 0, \quad \dim L = \tfrac{1}{2}\dim M$$

ex. $M=\mathbb{R}^2$. $\Omega = dp\wedge dx$. def: L: $p=0$ $L\cong\mathbb{R}$

$\Rightarrow$ choose x as the polarization $\rightarrow \psi(x)$

Our case: $G = SL(2,\mathbb{C})$. $M_3 = \Delta$, $\partial M_3 = \partial\Delta$, $P_{\partial\Delta}$, $\mathcal{L}_\Delta$.

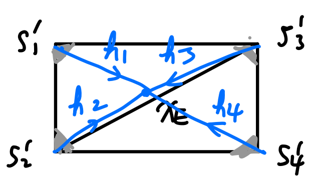
ADD a framing flag s to each cusp body of Δ



$h\in SL(2,\mathbb{C})$. $hs = \lambda s$

s: eigen vector of h around the cusp body up to a complex scaling $\Rightarrow s\in \mathbb{CP}^1$.

Fock-Goncharov (FG) coordinate of $P_{\partial\Delta}$ → gauge-invariant
→ simple Poisson bracket.



$s_i = h_i s_i' \quad \forall i=1,\dots,4$

$$x_E := \frac{\langle s_1 \wedge s_2 \rangle \langle s_3 \wedge s_4 \rangle}{\langle s_1 \wedge s_3 \rangle \langle s_2 \wedge s_4 \rangle} \qquad s = \begin{pmatrix} s^0 \\ s^1 \end{pmatrix} \in \mathbb{CP}^1 \subset \mathbb{C}^2$$

$$\langle s_i \wedge s_j \rangle := s_i^0 s_j^1 - s_i^1 s_j^0$$

$$\Rightarrow \langle s_i \wedge s_j \rangle = - \langle s_j \wedge s_i \rangle$$

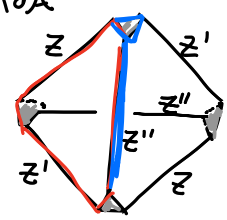
$$\langle s_i \wedge s_j \rangle = \langle g s_i \wedge g s_j \rangle \quad \forall g \in SL(2, \mathbb{C})$$

$$x_E = e^{\chi_E} \qquad \{\chi_E, \chi_{E'}\} = \begin{cases} +1, & E \wedge E' \\ -1, & E' \wedge E \\ 0, & E \mid \quad \_\_ E' \end{cases}$$

on $\partial\Delta$: 6 edge $\lambda = 1 \quad \forall i = 1, \cdots, 4 \Rightarrow \dim(\mathcal{P}_{\partial\Delta}) = 2$

holo. kin $\mathcal{P}_{\partial\Delta}$ spanned by (z, z'') $\quad \Omega = \frac{dz''}{z''} \wedge \frac{dz}{z}$

anti-holo. $(\bar{z}, \bar{z}'')$ $\quad \bar{\Omega} = \frac{d\bar{z}''}{\bar{z}''} \wedge \frac{d\bar{z}}{\bar{z}}$ $\quad \sim \int_{\partial\Delta} \delta A \wedge \delta A$



$$z' = -\frac{1}{z z''}$$

$$h(A) \to s(A) \to z(A), z''(A) \Rightarrow h(z, z'')$$

$$\Rightarrow \omega_{CS} = \frac{t}{4\pi} \Omega + \frac{\bar{t}}{4\pi} \bar{\Omega}, \quad t, \bar{t} \in \mathbb{C}.$$

$F = 0 \qquad \mathcal{L}_\Delta \qquad h = \mathbb{1}$

"snake rule" $\to$ [(40) on pg 11].

Result on $\partial\Delta$

$$\mathbb{1} = h = \begin{pmatrix} 1 & 0 \\ -\frac{1}{z''}(z'' + z^{-1} - 1) & 1 \end{pmatrix} \in SL(2, \mathbb{C})$$

$= 0$

$$\mathcal{L}_\Delta := \{ (z, z''; \bar{z}, \bar{z}'') \mid z'' + z^{-1} - 1 = 0, \quad \bar{z}'' + \bar{z}^{-1} - 1 = 0 \}$$

$$\omega = c \sum_i dp_i \wedge dq_i \qquad c \in \mathbb{R} \text{ const.} \qquad q_i \to \hat{q}_i, \quad p_i \to \frac{i}{\hbar c} \partial_{q_i}$$

param. $z(\mu, m) = \exp\left[\frac{2\pi i}{k}(-ib\mu - m)\right]$, $z''(\nu, n) = \exp\left[\frac{2\pi i}{k}(-ib\nu - n)\right]$ $\quad b^2 = \frac{1 - i\gamma}{1 + i\gamma}$,

$\bar{z}(\mu, m) = \exp\left[\frac{2\pi i}{k}(-ib^{-1}\mu + m)\right]$, $\bar{z}''(\nu, n) = \exp\left[\frac{2\pi i}{k}(-ib^{-1}\nu + n)\right]$ $\quad \mathrm{Re}\, b > 0$, $\mathrm{Im}\, b \neq 0$

$\bar{b} = b^{-1} \Rightarrow b = e^{i\theta}$.

$(\mu, \nu) \in \mathbb{R}^2$, $(m, n) \in (\mathbb{Z}/k\mathbb{Z})^2$. $\quad m, n = 0, 1, 2, \cdots, k-1, \ k \in \mathbb{Z}_+$

$$\omega_{CS} = \frac{2\pi}{k}(d\nu \wedge d\mu - dn \wedge dm) \Rightarrow \{\mu, \nu\} = \{n, m\} = \frac{k}{2\pi}.$$

§2.2. Quantization of CS on Δ

quantum param. $q = e^{h}$, $\tilde{q} = e^{\tilde{h}}$

$$\hbar = \frac{4\pi i}{t} = \frac{2\pi i}{k}(1+b^2) \qquad \tilde{\hbar} = \frac{4\pi i}{\tilde t} = \frac{2\pi i}{k}(1+b^{-2})$$

$$k = \frac{12\pi}{\gamma \ell_p^2 |\Lambda|} \qquad \ell_p(\hbar) \to 0 \Rightarrow k \to \infty \Rightarrow \hbar \to 0, \quad q, \tilde q \to 1$$

$$\{\mu, \nu\}, \ \{m, n\} \to \quad [\hat\mu, \hat\nu] = [\hat n, \hat m] = \frac{k}{2\pi i}$$

kin $P_{\partial\Delta}$ $\to$ kin Hilbert space $\quad \mathcal{H}^{kin} = L^2(\mathbb{R}) \otimes V^k$, $\quad L^2(\mathbb{R}) \leftarrow \mu$, $\quad V^k = \mathbb{C}^k$

$$\langle f, g \rangle := \int d\mu \sum_{m \in (\mathbb{Z}/k\mathbb{Z})} \bar f(\mu, m)\, g(\mu, m) \qquad \forall f, g \in \mathcal{H}^{kin}.$$

$$\begin{cases} \hat\mu f(\mu, m) = \mu f(\mu, m) & \hat\nu f(\mu, m) = -\frac{k}{2\pi i}\partial_\mu f(\mu, m) \\ e^{\frac{2\pi i}{k}\hat m} f(\mu, m) = e^{\frac{2\pi i}{k} m} f(\mu, m) & e^{\frac{2\pi i}{k}\hat n} f(\mu, m) = f(\mu, m+1). \end{cases}$$

$$\hat z = \exp\left[\frac{2\pi i}{k}(-ib\hat\mu - \hat m)\right], \qquad \hat z'' = \exp\left[\frac{2\pi i}{k}(-ib\hat\nu - \hat n)\right]$$

$$\hat z f(z, \bar z) = z f(z, \bar z), \qquad \hat z'' f(z, \bar z) = f(qz, \bar z)$$

$$\hat{\bar z} f(z, \bar z) = \bar z f(z, \bar z), \qquad \hat{\bar z}'' f(z, \bar z) = f(z, \tilde q \bar z)$$

$\mathcal{L}_\Delta \to$

$$\Rightarrow \begin{cases} (\hat z'' + \hat z^{-1} - 1) f(z, \bar z) = 0 \\ (\hat{\bar z}'' + \hat{\bar z}^{-1} - 1) f(z, \bar z) = 0 \end{cases}$$

sol. quantum dilogarithm function $\quad \Psi_\Delta(\mu, m) = \prod_{j=0}^{\infty} \frac{1 - \tilde q^{j+1} \bar z^{-1}}{1 - q^{-j} z^{-1}}$

$q, \tilde q \to 1$

$$\Psi_\Delta = \exp\left[-\frac{ik}{2\pi(1+b^2)} \mathrm{Li}_2(z^{-1}) - \frac{ik}{2\pi(1+b^{-2})} \mathrm{Li}_2(\bar z^{-1})\right]\left[1 + O\left(\tfrac{1}{k}\right)\right].$$

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \text{for } |z| < 1.$$

$$-\ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad \text{for } |z| < 1.$$

$\Psi_\Delta \sim$ 6j symbol of Weyl algebra $\sim$ Borel subalgebra of $U_q(sl(2,\mathbb{C}))$
[Kashaev '94].

$$\hat z'' \Psi_\Delta(z, \bar z) = \Psi_\Delta(qz, \bar z) = (1 - z^{-1}) \Psi_\Delta(z, \bar z)$$

$$\hat{\bar z}'' \Psi_\Delta(z, \bar z) = \Psi_\Delta(z, \tilde q \bar z) = (1 - \bar z^{-1}) \Psi_\Delta(z, \bar z)$$

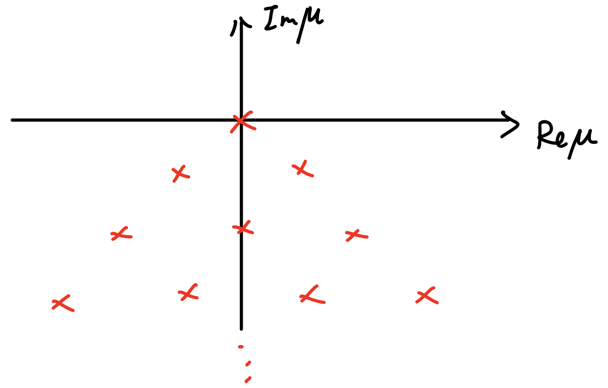
$$\mathcal{Z}_\Delta = \Psi_\Delta(\mu, m) \quad \rightsquigarrow \quad \mathcal{Z}(S^3 \setminus \Gamma_5)$$

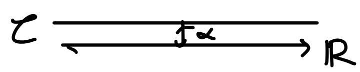
Ψ_Δ : singularity at $\mu = 0$

analytic cont. of $\mu \in \mathbb{C}$, $\nu \in \mathbb{C}$

$$\int_{\mathbb{R}+i\alpha} d\mu \, \mathcal{I}_\Delta(\mu, m)$$

$\mathcal{I}_\Delta(\mu,m)$: meromorphic function of μ $\begin{cases} \text{holomorphic in } \mathrm{Im}\,\mu > 0 \\ \text{poles in } \mathrm{Im}\,\mu \leq 0 \end{cases}$





$\alpha = \mathrm{Im}\,\mu$, $\beta = \mathrm{Im}\,\nu$

$(\alpha, \beta) \in \mathcal{P}$: "positive angle structure" [Pg 14-15]

exercise : ideal octahedron (Oct)

$F = 0 \Rightarrow \quad c = xyzw = 1 \Rightarrow \quad w(x,y,z)$

$\tilde{c} = \tilde{x}\tilde{y}\tilde{z}\tilde{w} = 1$

$$\mathcal{P}_{\partial(oct)} = \left(\bigotimes_{i=1}^{4} \mathcal{P}_{\partial\Delta_i}\right) // (c, \tilde{c})$$

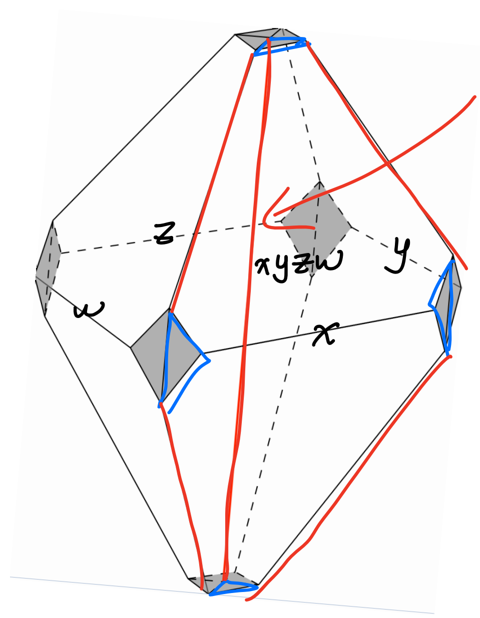
↑ gluing constraint

$x = e^X$, $y = e^Y$ ...

→ $\dim \mathcal{P}_{\partial(oct)} = 2\times 4 - 2 = 6$ ↖

→ X, Y, Z, P_x, P_y, P_z

$\omega_{\partial(oct)} \Rightarrow \{ \ , \ \}$



$$\mathcal{Z}_{oct}(x,y,z,P_x,P_y,P_z) = \mathcal{I}_\Delta(x,\tilde{x})\,\mathcal{I}_\Delta(y,\tilde{y})\,\mathcal{I}_\Delta(z,\tilde{z})\,\mathcal{I}(w(x,y,z),\ \tilde{w}(\tilde{x},\tilde{y},\tilde{z}))$$

§2.3. CS phase space on $\partial(S^3\backslash\Gamma_5)$ & $\mathcal{Z}_{S^3\backslash\Gamma_5}$

$\mathrm{Tid}(S^3\backslash\Gamma_5) = \{\Delta_i\}_{i=1}^{20} = \{Oct(j)\}_{j=1}^{5}$

$$\mathcal{P}_{\partial(S^3\backslash\Gamma_5)} = \bigotimes_{j=1}^{5} \mathcal{P}_{\partial(oct)}$$

$\dim = 30 = 6\times 5$

→ symplectic transf ⇒ better coord. for s.c.

→ $\mathcal{H}^{kin}_{\partial(S^3\backslash\Gamma_5)} \ni f(\vec{\mu}, \vec{m})$

$\hat{\mu}, \hat{m}, \hat{\nu}, \hat{n} \Leftarrow \vartheta$

→ unitary transf. of $f(\vec{\mu}, \vec{m})$ $\quad \vartheta$

$\vec{\Phi} = (X_1, Y_1, Z_1, X_2 \cdots, X_5, Y_5, Z_5)$

$\vec{\Pi} = (P_{X_1}, P_{Y_1}, P_{Z_1}, \ldots P_{X_5}, P_{Y_5}, P_{Z_5})$

$$\begin{matrix} \#15 & \in \\ \#15 & \in \end{matrix} \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} = M \begin{pmatrix} \vec{\Phi} \\ \vec{\Pi} \end{pmatrix} + i\pi \vec{t}$$

$M \in 30\times 30$ matrices s.t.

$\Rightarrow M^T \Omega M = \Omega, \quad \Omega = \begin{pmatrix} 0 & -\mathbb{1}_{15} \\ \mathbb{1}_{15} & 0 \end{pmatrix}$

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} \mathbb{1}_{15} & 0 \\ \hline DB & \mathbb{1}_{15} \end{array}\right) \left(\begin{array}{c|c} 0 & -\mathbb{1}_{15} \\ \hline \mathbb{1}_{15} & 0 \end{array}\right) \left(\begin{array}{c|c} \mathbb{1}_{15} & 0 \\ \hline AB^T & \mathbb{1}_{15} \end{array}\right) \left(\begin{array}{c|c} -(B^{-1})^T & 0 \\ \hline 0 & -B \end{array}\right)$$

$\det B \neq 0$ $\qquad T(DB) \qquad S \qquad T(AB^T) \qquad U(-(B^{-1})^T)$

Quantum

- U-type: $\left(U(-(B^{-1})^T) \circ f\right)(\vec{\mu}, \vec{m}) = \sqrt{\det(-B)}\, f(-B^T\vec{\mu}, -B^T\vec{m})$ $\rightsquigarrow$ "rotation"
- T-type: $(T(AB^T) \circ f)(\vec{\mu}, \vec{m}) = (-1)^{\vec{m}\cdot AB^T\cdot\vec{m}}\, e^{\frac{i\pi}{k}(-\vec{\mu}\cdot AB^T\cdot\vec{\mu} + \vec{m}\cdot AB^T\cdot\vec{m})} f(\vec{\mu}, \vec{m})$ "change momentum"
- S-type: $(S\circ f)(\vec{\mu}, \vec{m}) = \frac{1}{k^{15}} \sum_{\vec{n}\in(\mathbb{Z}/k\mathbb{Z})^{15}} \int d^{15}\vec{\nu}\; e^{\frac{2\pi i}{k}(-\vec{\mu}\cdot\vec{\nu} + \vec{m}\cdot\vec{n})} f(\vec{\nu}, \vec{n})$ "Fourier transf".
- $\sigma_{\vec{t}}$-translation: $(\sigma_{\vec{t}} \circ f)(\vec{\mu}, \vec{m}) = f(\vec{\mu} - \frac{iQ}{2}\vec{t}, \vec{m})$, $Q = b + b^{-1} = 2\,\mathrm{Re}\, b$

[Pg. 20-22].

$(S^3 \setminus \Gamma_5)$ $\qquad D = 0$

$$M = \left(\begin{array}{c|c} A & B \\ \hline -(B^{-1})^T & 0 \end{array}\right)$$

$$\mathcal{Z}_{S^3\setminus\Gamma_5}(\vec{\mu}(\vec{Q}), \vec{m}(\vec{Q})) = \left((\sigma_{\vec{t}} \circ S \circ T \circ U) \circ \underbrace{\prod_{a=1}^{5} \mathcal{Z}_{oct}}_{\mathcal{Z}_X}\right)$$

$$\frac{4i}{k^{15}} \sum_{\vec{n}\in(\mathbb{Z}/k\mathbb{Z})^{15}} \int_{\mathbb{R}^{15}+i\vec{\beta}} d^{15}\vec{\nu}\; (-1)^{\vec{n}\cdot AB^T\cdot\vec{n}}\, e^{\frac{i\pi}{k}(-\vec{\nu}\cdot AB^T\cdot\vec{\nu} + \vec{n}\cdot AB^T\cdot\vec{n})}\, e^{\frac{2\pi i}{k}(-\vec{\nu}\cdot(\vec{\mu} - \frac{iQ}{2}\vec{t}) + \vec{n}\cdot\vec{m})}\, \mathcal{Z}_X(\vec{\nu}, \vec{n})$$

$i\vec{\beta}$ ← positive angle structure

finite

15-dim integral, 15-dim sum } dim = 30 $\qquad$ EPRL : 44 dim$^{\ell}$

$a, b = 1, \ldots, 5$

$\vec{Q} = \{\{2L_{ab}\}_{a<b}, x_1, \cdots, x_5\}$

$\vec{P} = \{\{T_{ab}\}_{a<b}, y_1, \cdots, y_5\}$

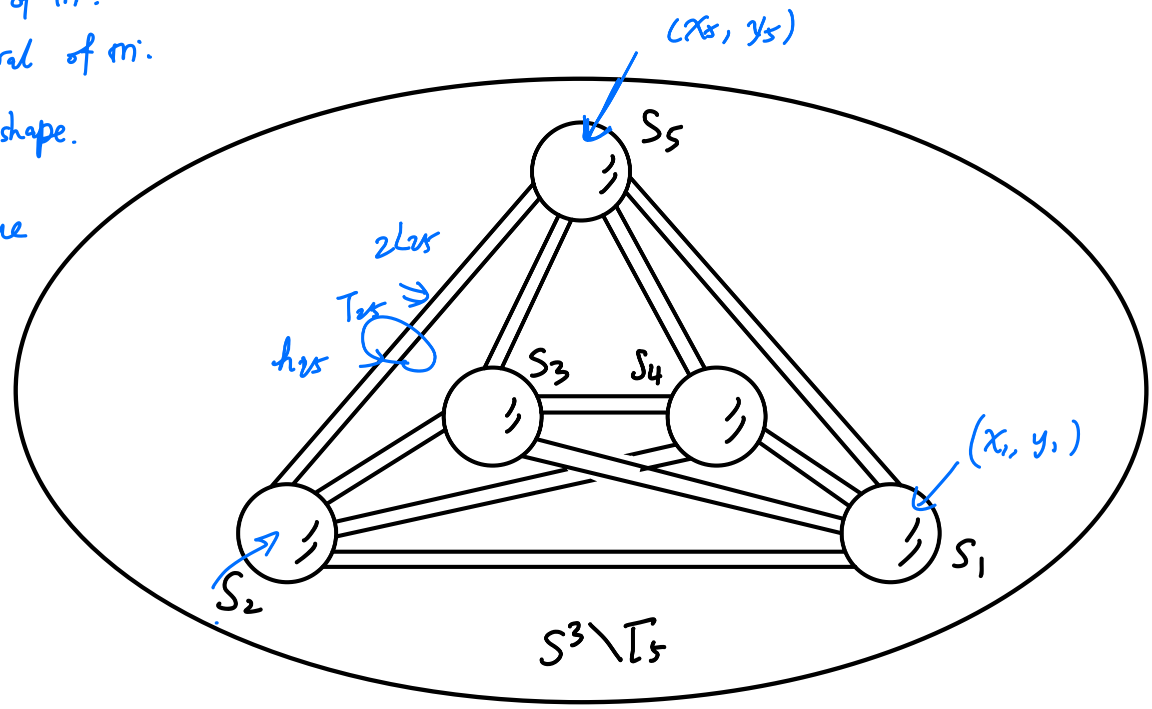
$2L \sim$ area of tri.

$T \sim$ dihedral of tri.

$(x, y) \to$ shape.

$e^{2L_{ab}} \sim$ eigenvalue of h_{ab}

$(x_a, y_a) \to$ FG coord. on S_a



§3. Impose the s.c. on $Z_{S^3 \setminus \Gamma_5}$

Goal: make sense of $\mathcal{F} = \frac{\Lambda}{3} e \wedge e$ on $\partial(S^3 \setminus \Gamma_5)$

In EPRL: $B_f^{IJ}(t) := \int_f B^{IJ}(t)$

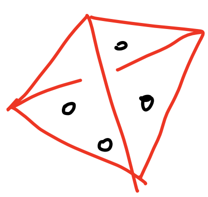
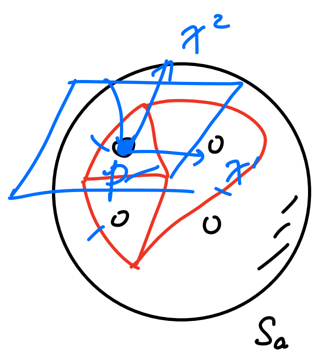
(1) $\exists N^J$ s.t. $N^J B_f^{IJ}(t) = 0 \quad \forall f \in t$

(2) $\sum_{f \in t} B_f^{IJ}(t) = 0$

→ ~~$\mathcal{F}_f^{IJ}(t) = \int_f \mathcal{F}^{IJ}(t)$~~

$\partial(S^3 \setminus \Gamma_5)$ — $\{\Sigma_{0,4}\}$, annuli.

$T(\Sigma_{0,4}) = \partial(\text{tetra})$

$\tau_a = T(S_a)$

$\mathcal{F}_p^{IJ}(S_a) := \frac{|\Lambda|}{3} B_{f_p}(\tau_a)\, \delta^{(2)}(\vec{x})\, dx^1 \wedge dx^2$

(1)' $\exists N^J \in \mathbb{R}^{1,3}$ s.t. $N^J \mathcal{F}_p^{IJ}(S_a) = 0 \quad \forall p$ of S_a

⇒ $\mathcal{F}_p^{IJ}(S_a) \in sl(2, \mathbb{C})$

n.b. stokes' thm → $O_p(S_a) = e^{\frac{|\Lambda|}{3} B_{f_p}(\tau_a)} \in SL(2, \mathbb{C})$

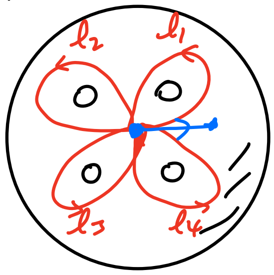
(1)'' $\exists N^J \in \mathbb{R}^{1,3}$ s.t. $O_p(S_a)^I{}_J N^J = N^I \quad \forall p$ of S_a.

Let N^J be time direction. $O_p \in SU(2)$

(1)''' $\Rightarrow$ $\{O_p(S_a)\}_p$ are in a common $SU(2)$ subgp of $SL(2,\mathbb{C})$

s.c. $\Rightarrow$ $\mathcal{M}_{flat}(S_a, SL(2,\mathbb{C})) \xrightarrow{\text{restrict}} \mathcal{M}_{flat}(S_a, SU(2))$

ex. $\Sigma_{0,4}$



$\Rightarrow l_4 \circ l_3 \circ l_4 \circ l_1 = 1$ $\quad l_i \in \pi_1(\Sigma_{0,4})$

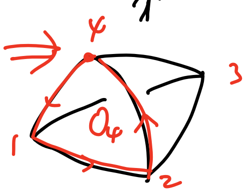
holonomies $H_i \in G$

$H_4 H_3 H_2 H_1 = \mathbb{1}_G$ $\rightarrow$ closure cond.

$H_i \to g H_i g^{-1}, \quad \forall g \in G.$

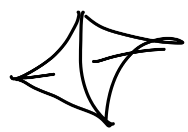
(2)' $\rightarrow$ $O_1, O_2, O_3, O_4 \in SU(2)$, $O_4 O_3 O_2 O_1 = \mathbb{1}_{SU(2)}$ $\xrightarrow[\text{thm.}]{\text{curved Minkowski}}$ convex. non-deg. constantly curved tetra

[Thm. V.1. pg 27]



$\Lambda > 0$

$\subset S^3$

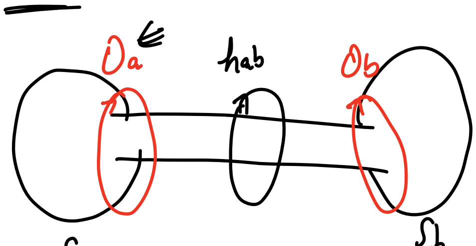


$\Lambda < 0$

$\subset \mathbb{H}^3$

$$O_f = \exp\left(\frac{\Lambda}{3} a_f \hat{n}_f(4) \cdot \vec{\tau}\right) \qquad \vec{\tau} = \frac{\vec{\sigma}}{2i}$$

$\Rightarrow$ $\mathcal{Z}(\{2L_{ab}\}_{a<b}, \{x_a\}_a)$.



$h_{ab} = G O_a G^{-1}, \qquad G \in SL(2,\mathbb{C})$

$O_a \in SU(2)$

$e^{2L_{ab}} = \lambda_{ab} = e^{i\theta}, \; \theta \in \mathbb{R}.$

$\Rightarrow$ $2L_{ab} \in i\mathbb{R}$ $\longrightarrow$ $\mathcal{Z}(\{2L_{ab}\}, \{x_a\})$

↑ 1st

shape of t: (ϕ, ψ)

$$\exp\left[\frac{\Delta}{3} q_{ab}\, \hat{n}_{ab}\cdot\vec{\tau}\right] = O_{ab} = M \begin{pmatrix} \lambda_{ab} & \\ & \lambda_{ab}^{-1} \end{pmatrix} M^{-1} \in SU(2)$$

$$M(\xi) = \begin{pmatrix} \xi^0 & -\bar{\xi}^1 \\ \xi^1 & \bar{\xi}^0 \end{pmatrix} \in SU(2) \qquad \xi = \begin{pmatrix} \xi^0 \\ \xi^1 \end{pmatrix} \in \mathbb{C}^2$$

$$\hat{n}_{ab} = (\bar{\xi}^0, \bar{\xi}^1)\, \vec{\sigma} \begin{pmatrix} \xi^0 \\ \xi^1 \end{pmatrix}$$

$$x_a = \frac{\langle \xi_1 \wedge \xi_2 \rangle \langle \xi_3 \wedge \xi_4 \rangle}{\langle \xi_1 \wedge \xi_3 \rangle \langle \xi_2 \wedge \xi_4 \rangle} \qquad y_a = \ldots \qquad \in \mathcal{M}_{flat}(S_a, SU(2))$$

⇐ 2nd.

⇒

1-class → strongly

2-class → weakly

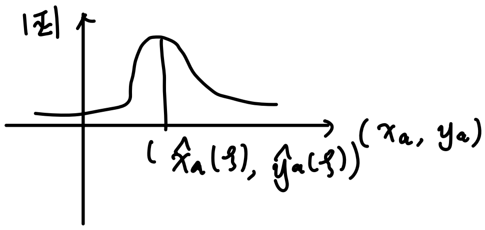
$$2L_{ab} = \frac{2\pi i}{k}(-ib\mu_{ab} - m_{ab}) \in i\mathbb{R} \Rightarrow \mu_{ab} = 0$$

$$\Rightarrow 2L_{ab} = -\frac{2\pi i}{k} m_{ab}. \qquad m_{ab} \in \mathbb{Z}/k\mathbb{Z}$$

spin: $j_{ab} = \frac{m_{ab}}{2} = 0, \frac{1}{2}, \ldots, \frac{k-1}{2}$

$$\rightarrow \hat{\mu}_{ab} \mathcal{Z}(\mu_{ab}, \cdots) = 0 \Rightarrow \mathcal{Z}(\mu_{ab} = 0, \ldots)$$

coherent state Ψ



$$\mathcal{A}_\sigma = \langle \prod_{a=1}^{5} \Psi_{(\hat{x}_a, \hat{y}_a)}(x_a) | CS \rangle = \left[\prod_{a=1}^{5} \int dx_a\right] \mathcal{Z}_{S^3 \setminus \Gamma_5}(\mu_{ab} = 0, x_a) \prod_{a=1}^{5} \Psi_{(\hat{x}_a, \hat{y}_a)}(x_a)$$

↗ **bounded**

ideas of EPRL + SL(2, ℂ) CS theory

§4. Classical limit $\mathcal{A}_\sigma$

classical limit := $\ell_p \to 0 \Rightarrow$ $\boxed{k = \frac{12\pi}{\ell_p^2 |\Lambda| \gamma} \to \infty}$

keep geometrical quantities finite $\Rightarrow a_{ab} \propto \frac{j_{ab}}{k}$ finite $\Rightarrow$ $\boxed{j_{ab} \to \infty}$

large-k limit $\mathcal{A}_\sigma \xrightarrow{k\to\infty} \int d\mu\, e^{kS(x)}$

saddle pt approx. [sect. VI]

$$\mathcal{A}_\sigma \xrightarrow{k\to\infty} \mathcal{N}_+ e^{i S_{Regge,\Lambda}} + \mathcal{N}_- e^{-i S_{Regge,\Lambda}}$$

4D: $S_{Regge,\Lambda} = \frac{\gamma \Lambda}{12\pi} \left(\sum_{a<b} \Theta_{ab} a_{ab} - \Lambda |V_4| \right)$ for constantly curved 4-simplex!

4D SF w/ $\Lambda \neq 0$
- → finite
- → QG
- → computable