Subleading colour, amplitude evolution and all that

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At
Parton Showers for Future Event Generators
CERN | 25 April 2023

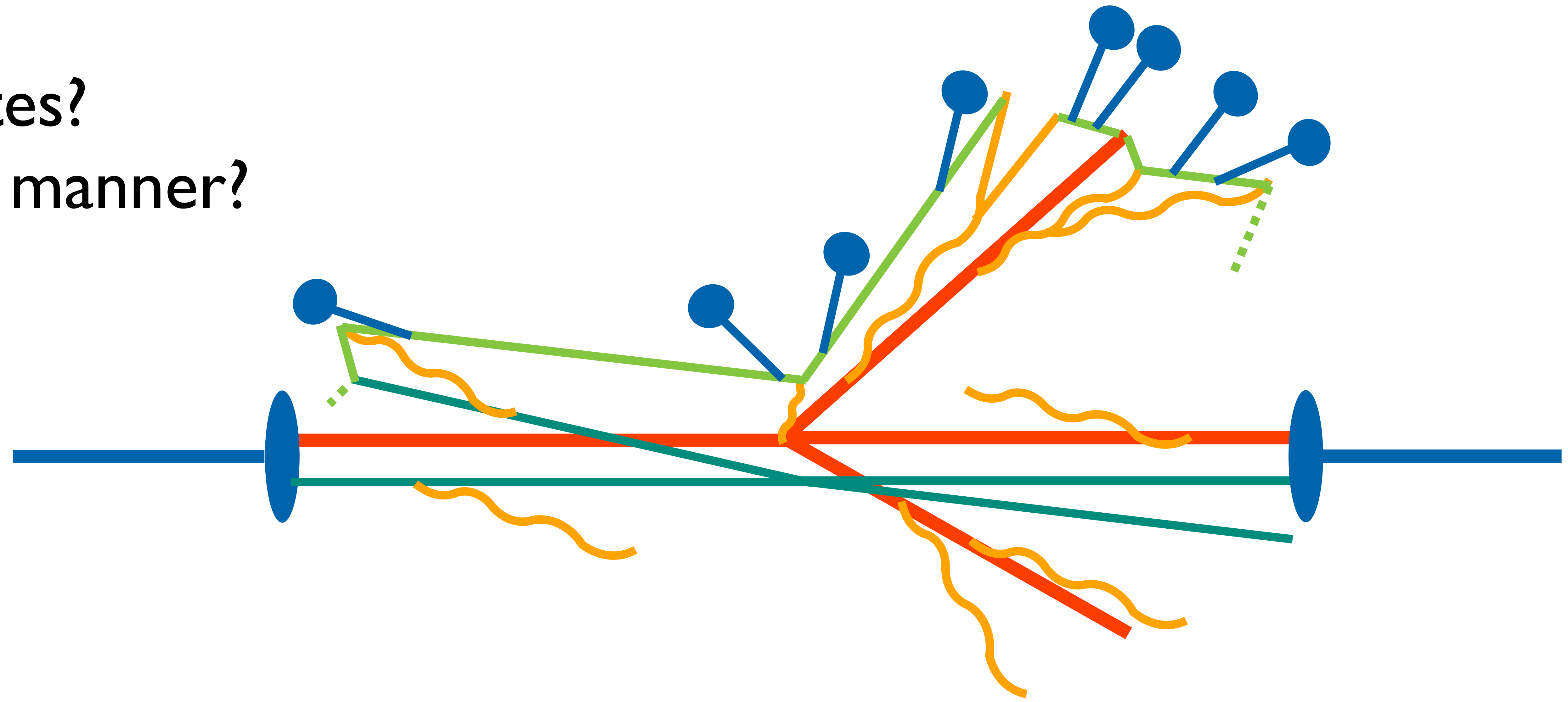
Current challenges

How do we accurately describe details of final states?
How do we quantify precision in a comprehensive manner?

Matching beyond NLO QCD?
Solve shower bottlenecks first?

How to benchmark precision of QCD algorithms?
How to accurately include EW and QED?

How to constrain hadronization models?
What is their response to perturbative variations?



$$\mathrm{d}\sigma \sim \mathrm{L} \times \mathrm{d}\sigma_H(Q) \times \mathrm{PS}(Q \to \mu) \times \mathrm{MPI} \times \mathrm{Had}(\mu \to \Lambda) \times \ldots$$

Can we understand this better?

Perturbative precision is far from the last word:

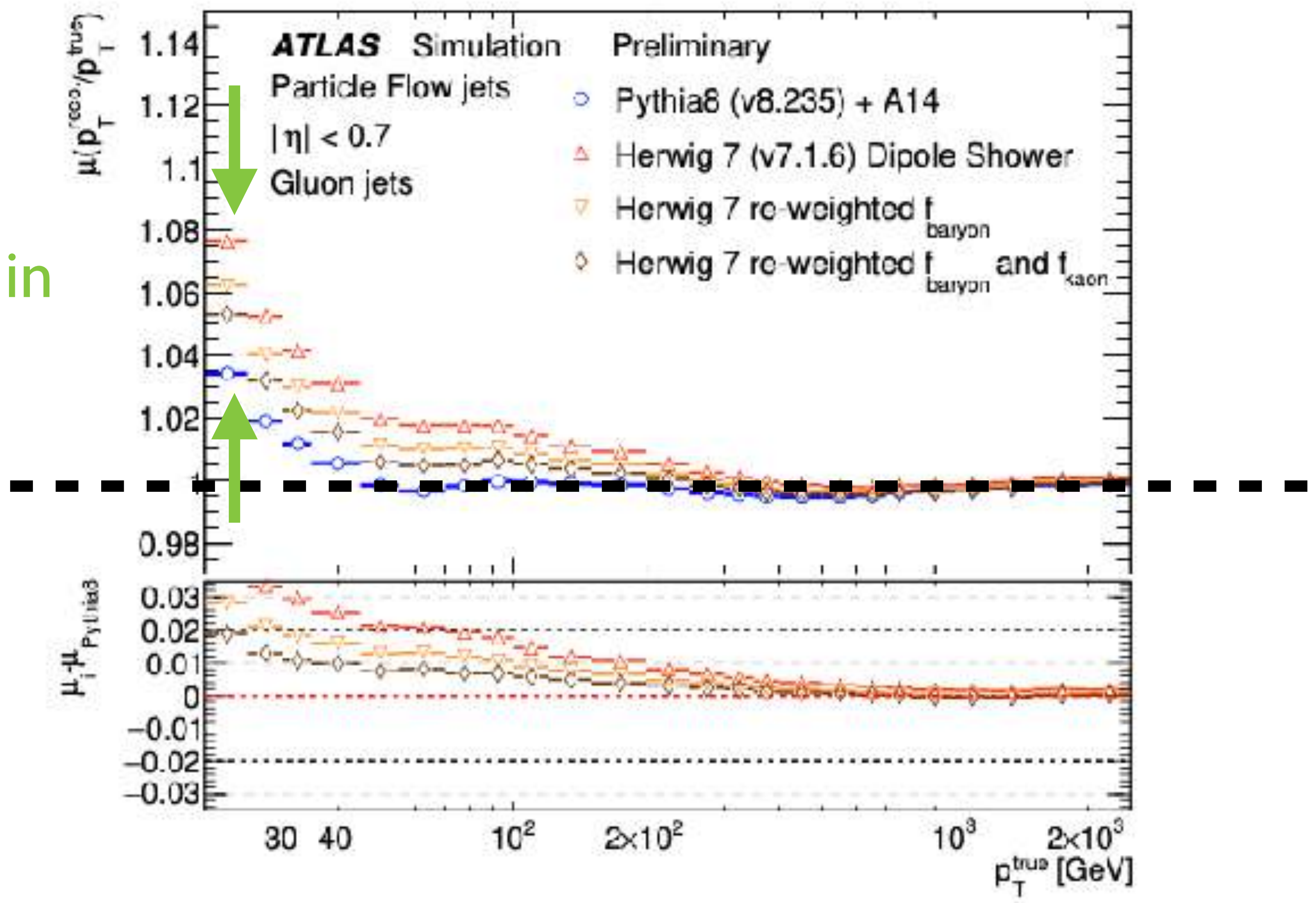
E.g. lack of understanding of baryon production is limiting the power of q/g discrimination.

[see also Gieseke's talk]

Personal selection of some recent topics:
Parton showers, hadronization and their interface.
And new algorithms.

deviation of reconstructed pt

O(1) variation in the correction



simulated pt

[ATLAS-PUB-2022-021]

$$d\sigma \sim \mathrm{L} \times d\sigma_H(Q) \times \mathrm{PS}(Q \to \mu) \times \mathrm{MPI} \times \mathrm{Had}(\mu \to \Lambda) \times \ldots$$

Subleading colour corrections

Colour reconnection and hadronization is about subleading-N.
So are shower accuracy and interference terms.

Colour factor algorithms

Coherent, NLL-accurate dipole showers

[Gustafson] [PanScales '21]
[Forshaw, Holguin, Plätzer '21]

Colour ME corrections

Colour-exact real emissions as far as possible

[Plätzer, Sjödahl '12, '18]
[Höche, Reichelt '20]

Full amplitude evolution

Colour-exact real and virtual corrections

[Forshaw, Plätzer, Sjödahl, Holguin + … '13 …]
[Nagy, Soper '12 …]

$$\mathrm{d}\sigma \sim \mathrm{L} \times \mathrm{d}\sigma_H(Q) \times \mathrm{PS}(Q \to \mu) \times \mathrm{MPI} \times \mathrm{Had}(\mu \to \Lambda) \times \ldots$$

Subleading colour corrections

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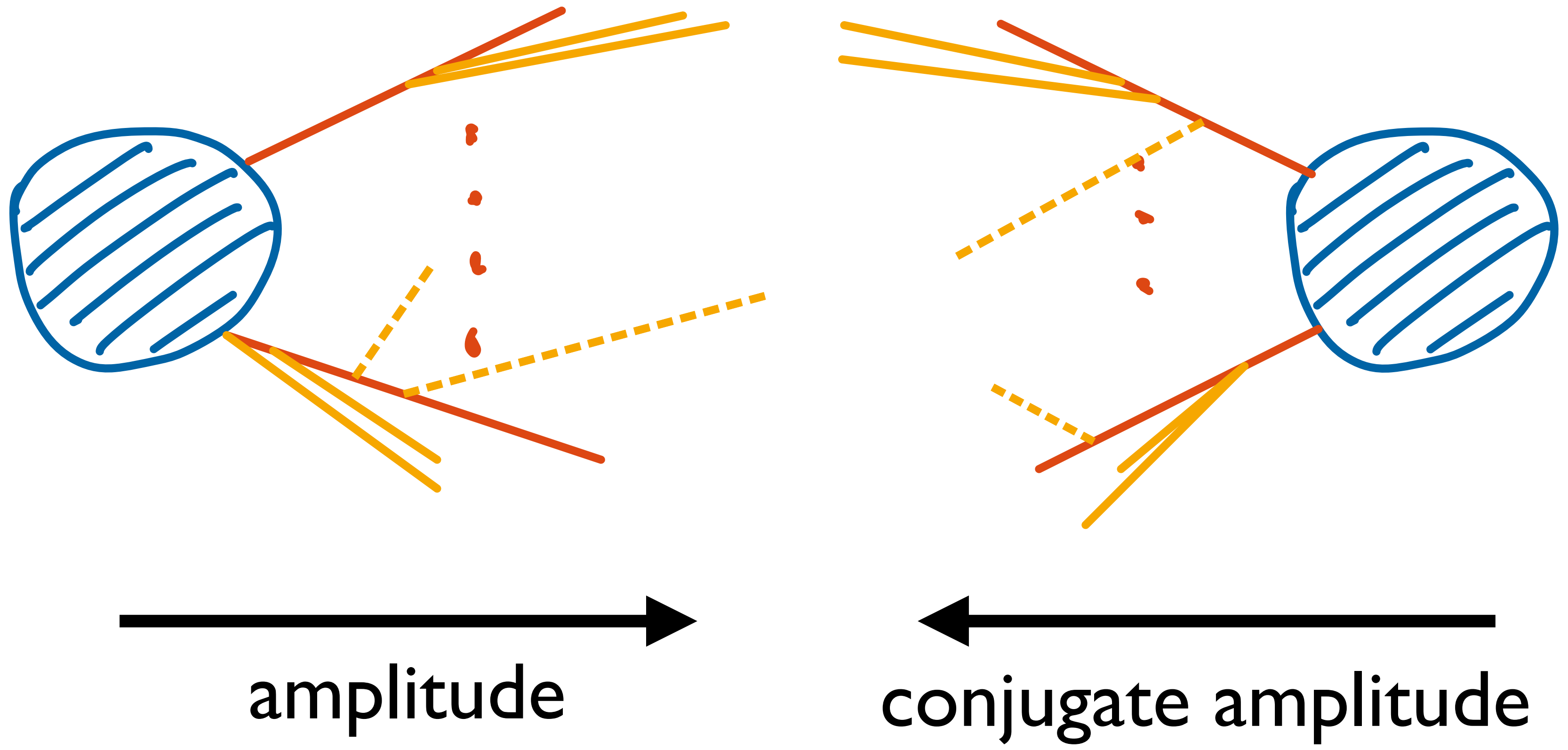
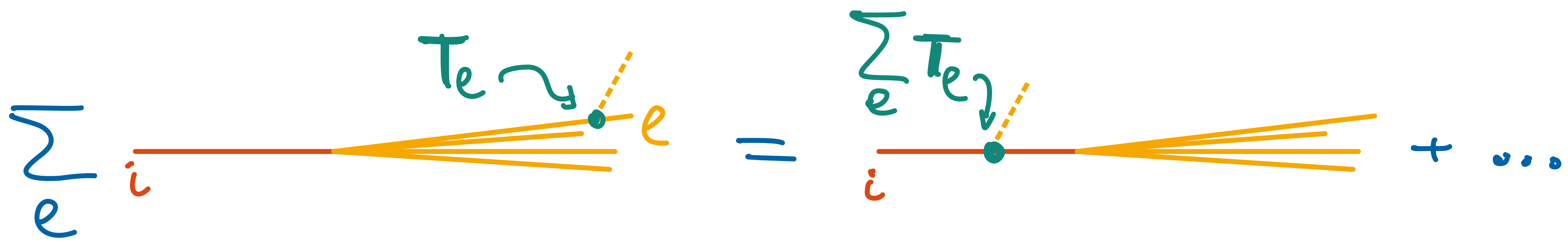
Colour factor algorithms	Colour ME corrections	Full amplitude evolution
Coherent, NLL-accurate dipole showers	Colour-exact real emissions as far as possible	Colour-exact real and virtual corrections
[Gustafson] [PanScales '21] [Forshaw, Holguin, Plätzer '21]	[Plätzer, Sjödahl '12, '18] [Höche, Reichelt '20]	[Forshaw, Plätzer, Sjödahl, Holguin + … '13 …] [Nagy, Soper '12 …]

$$\mathrm{d}\sigma \sim \mathrm{Tr}\left[\mathbf{PS}(Q \to \mu)\mathrm{d}\mathbf{H}(Q)\mathbf{PS}^{\dagger}(Q \to \mu)\mathbf{Had}(\mu \to \Lambda)\right]$$

Coherent branching parton showers

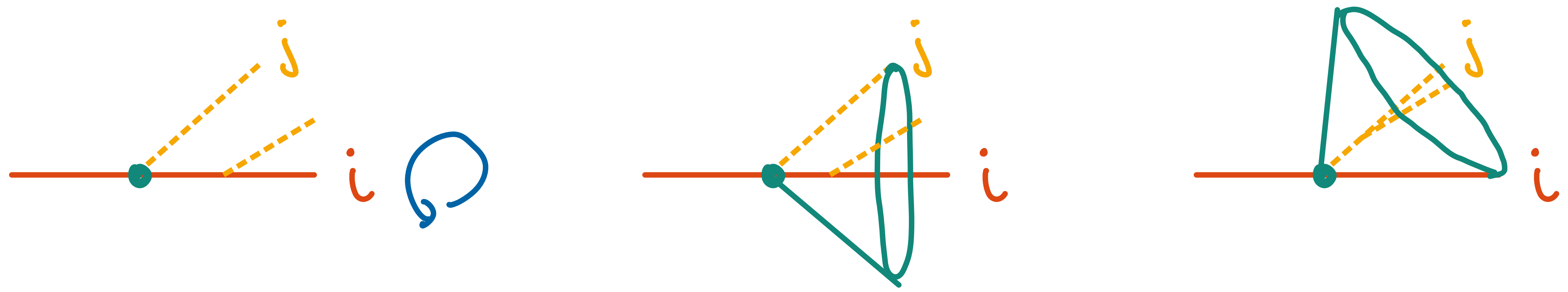
$$d\sigma \sim L \times d\sigma_H(Q) \times PS(Q \to \mu) \times MPI \times Had(\mu \to \Lambda) \times \ldots$$



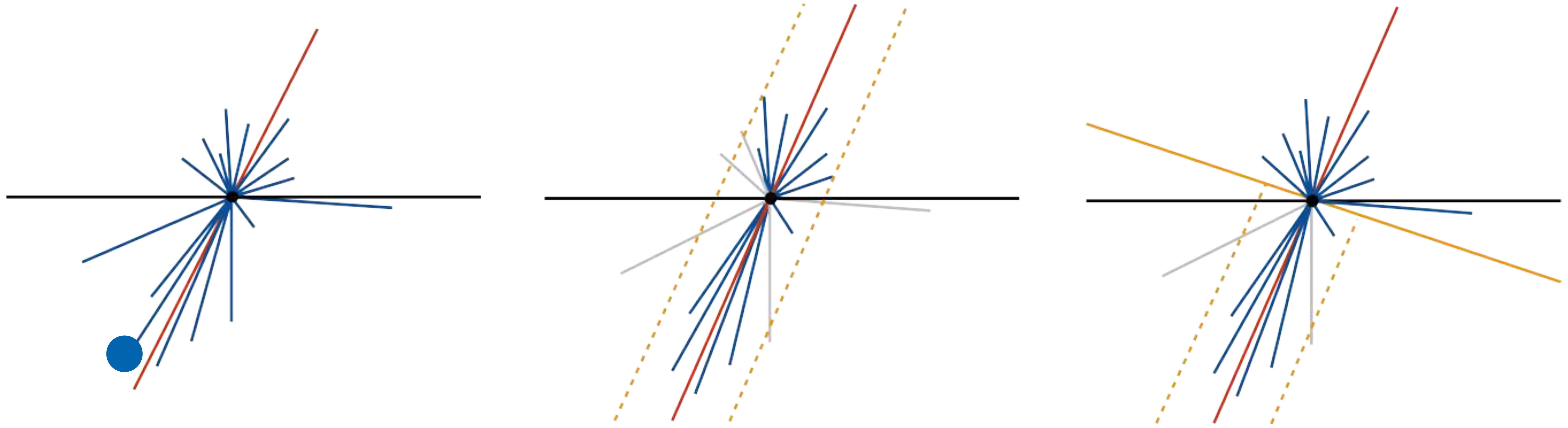


Move soft colour charges towards hard process and use angular ordering for azimuthal average around jet axes:

$$T_j T_l T_i \cdot T_i T_m T_j = C_i T_j T_l \cdot T_m T_j$$



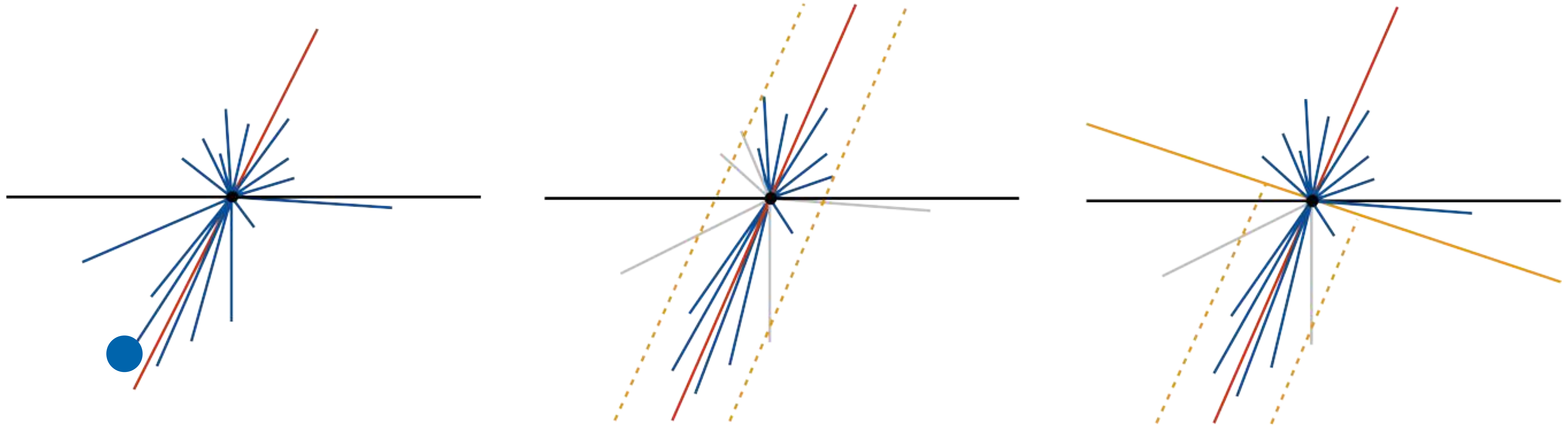
Accuracy of Parton Showers



Fragmentation is fine if we get
collinear physics right.

Accuracy of Parton Showers

[Catani, Trentadue, Webber, Marchesini ….]



Fragmentation is fine if we get collinear physics right.

Global event shapes from coherent branching — for two jets.

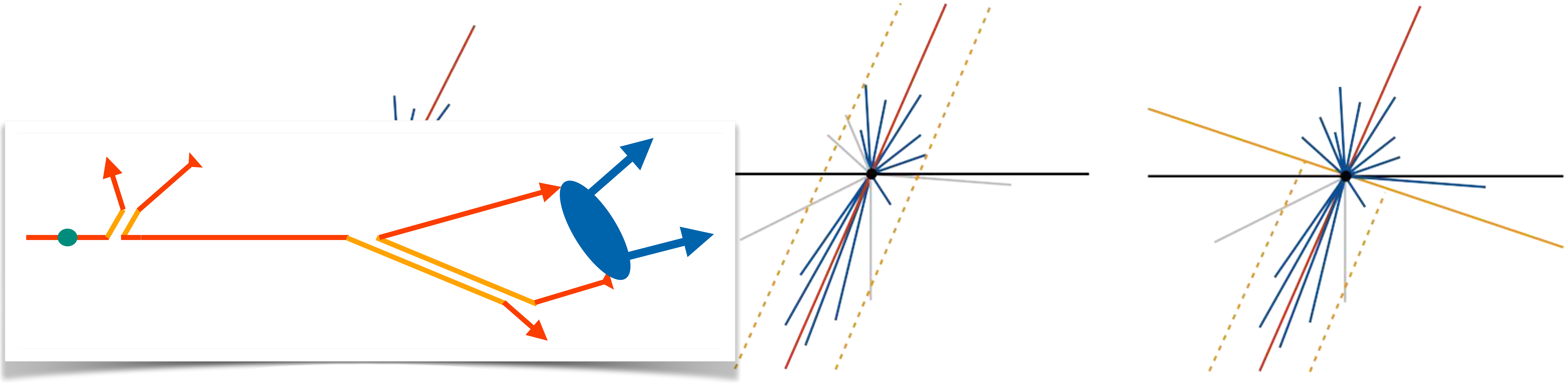
$$H(\alpha_s) \times \exp\left(Lg_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \ldots\right)$$

LL — qualitative　NLL — quantitative　NNLL — precision

$\alpha_s L \sim 1$

Accuracy of Parton Showers

[Catani, Trentadue, Webber, Marchesini ….]



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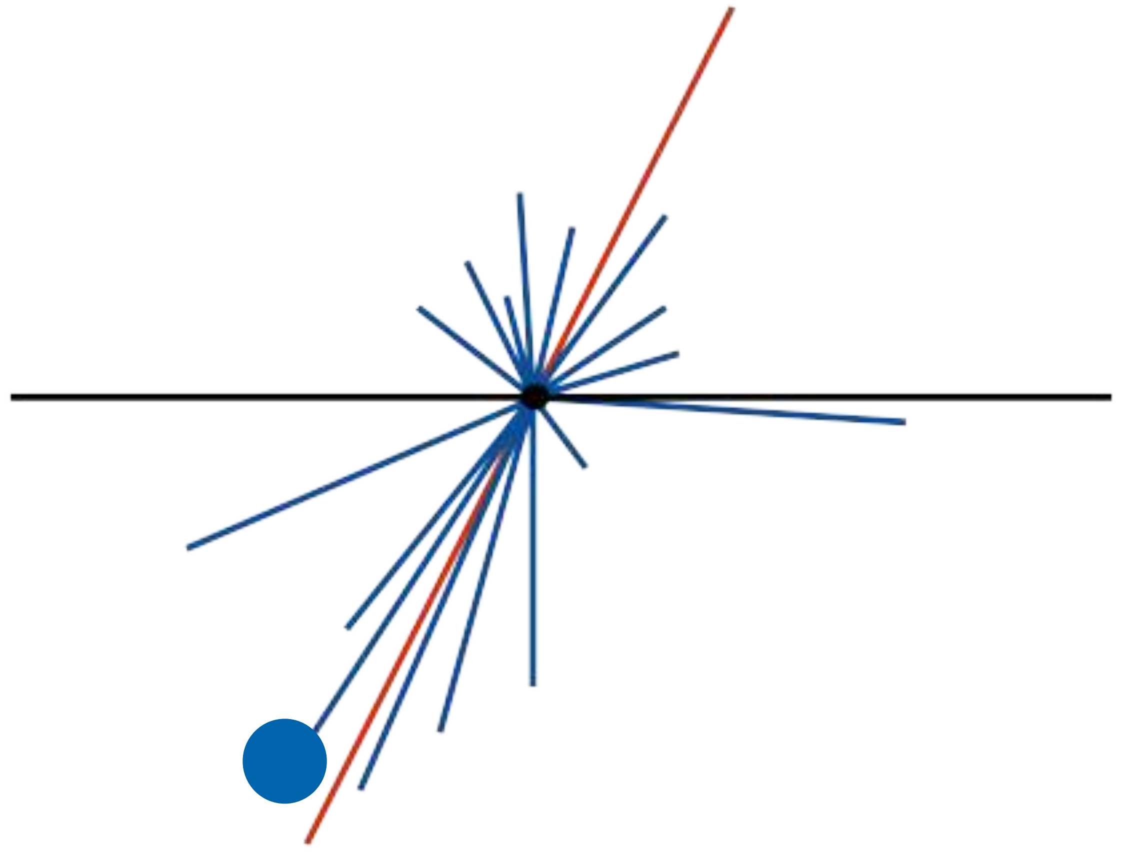

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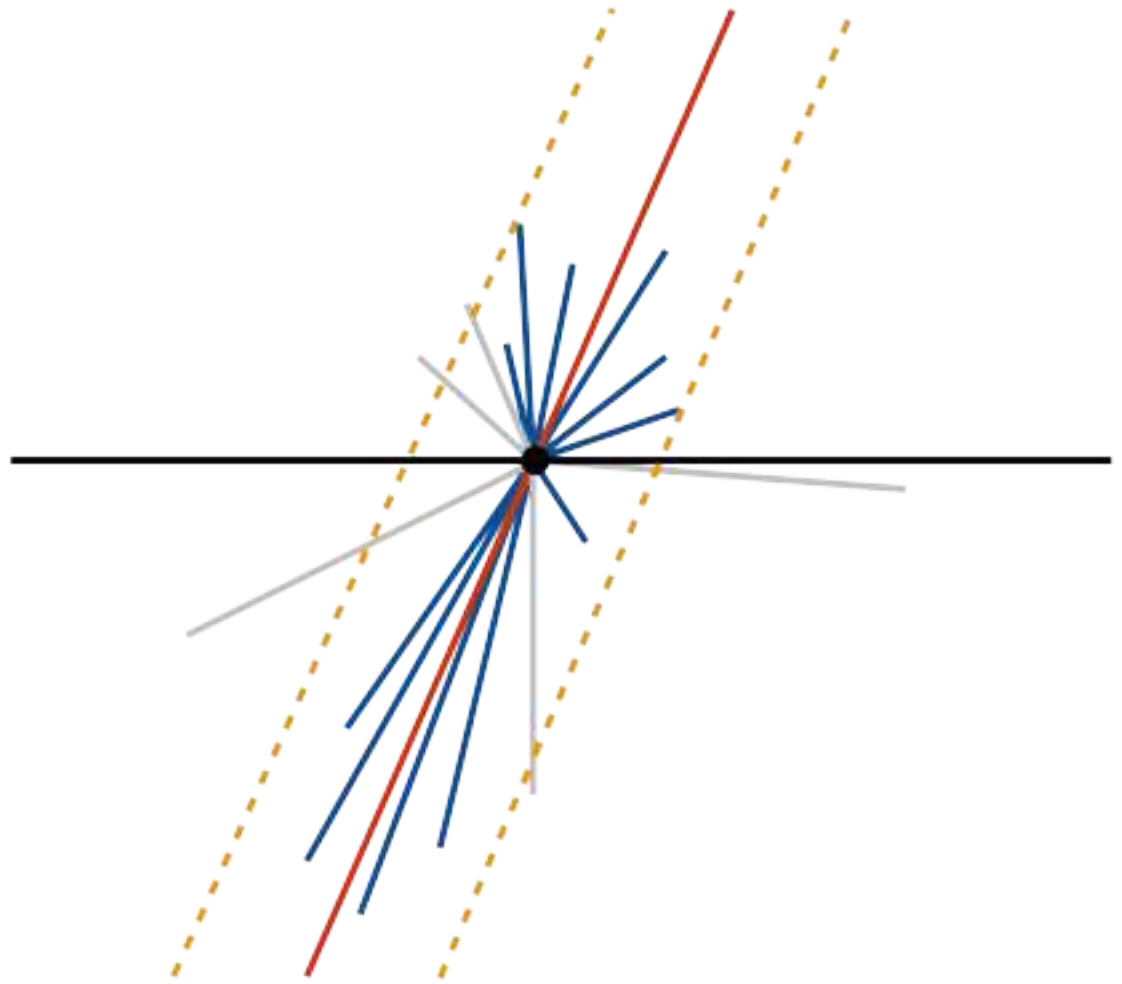

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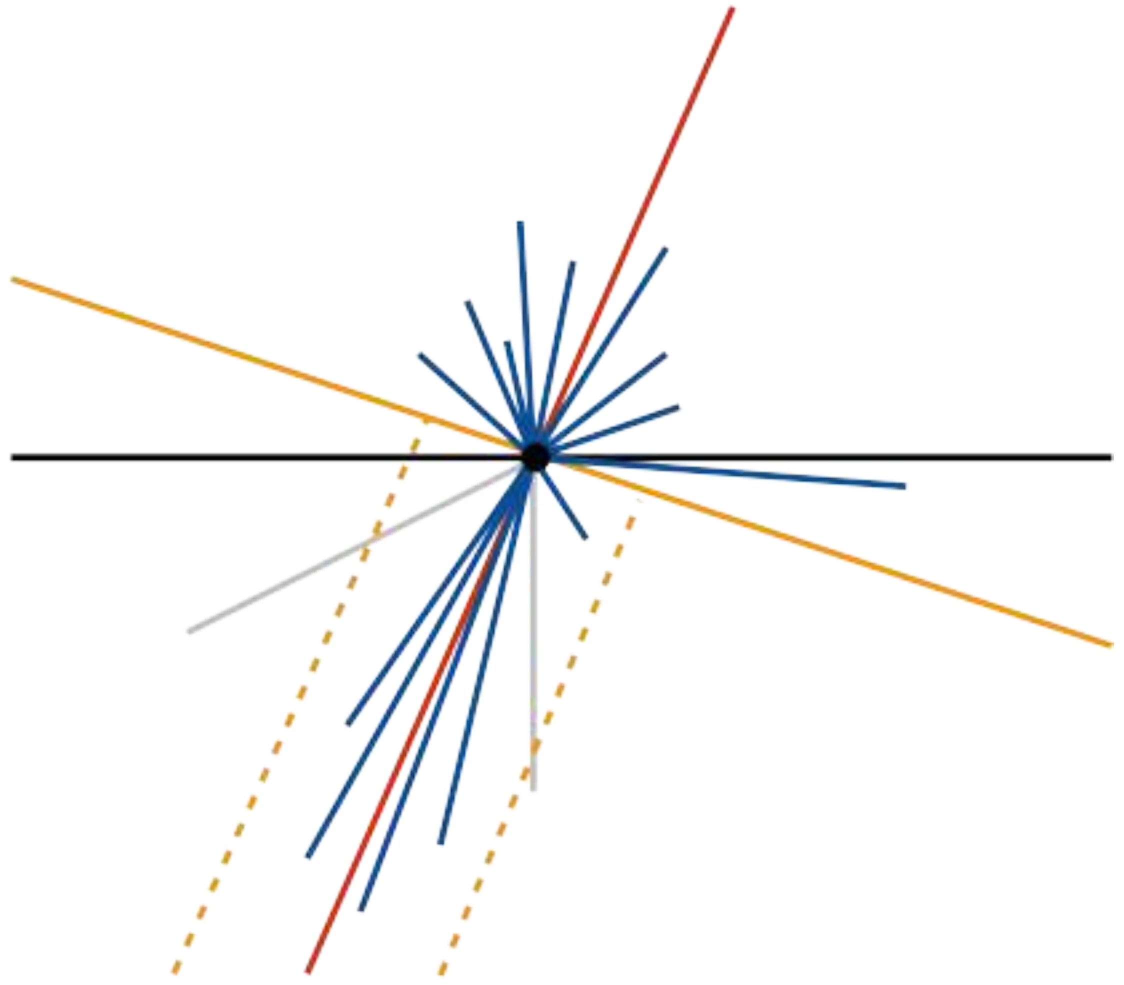
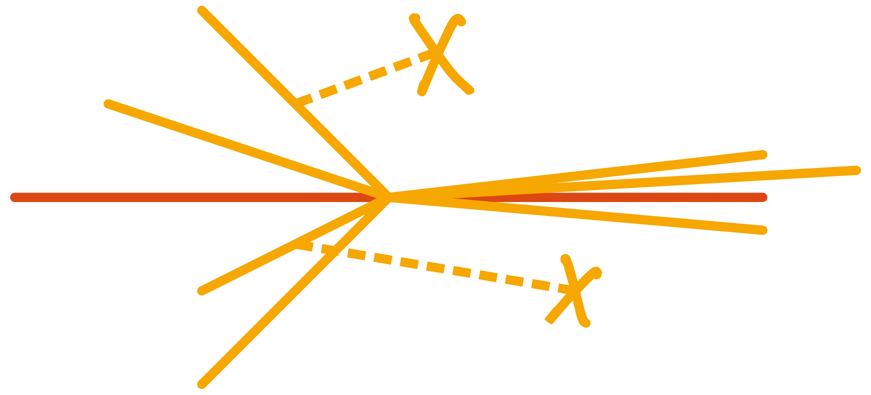
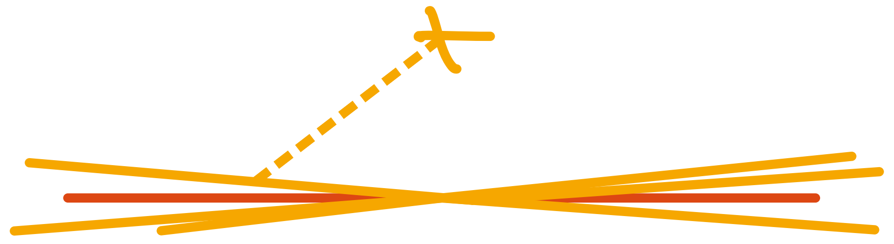


Coherence breaks down for non-global observables.





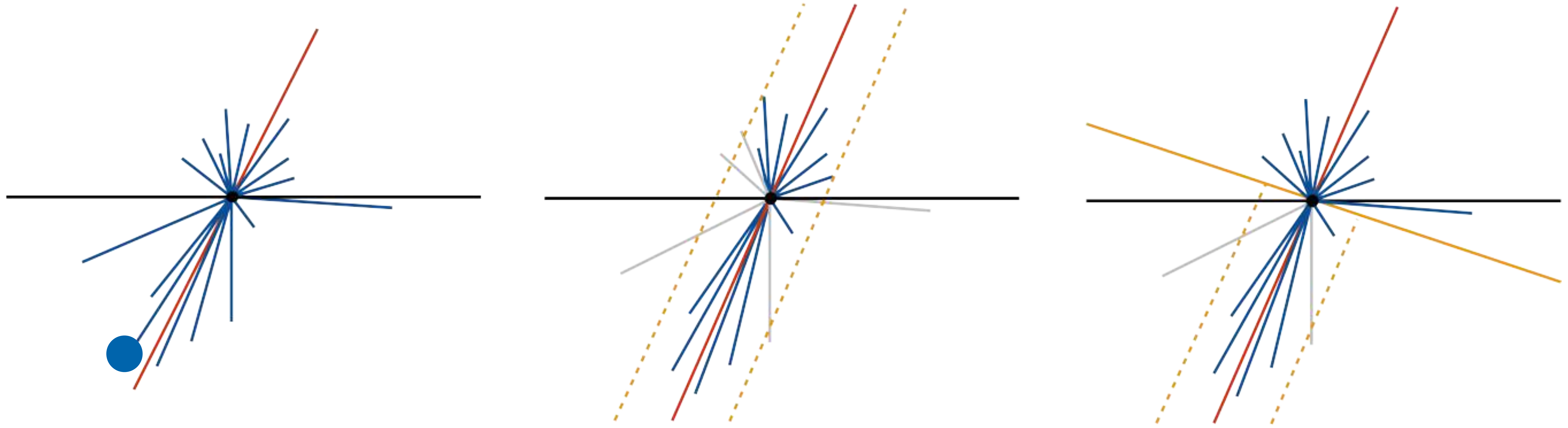
[Banfi, Marchesini, Smye '02]

$T_k T_l T_i \cdot T_j T_m T_n$

large-N limit ↓

$$\frac{\partial G_{ab}(t)}{\partial t} = -\int_{\text{in}} \frac{\mathrm{d}\Omega_k}{4\pi} \omega_{ab}(k) G_{ab}(t) + \int_{\text{out}} \frac{\mathrm{d}\Omega_k}{4\pi} \omega_{ab}(k) \left[G_{ak}(t) G_{kb}(t) - G_{ab}(t) \right]$$

NLL$_{\text{global}}$/LL$_{\text{non-global}}$ showers



(N)NLO with matching

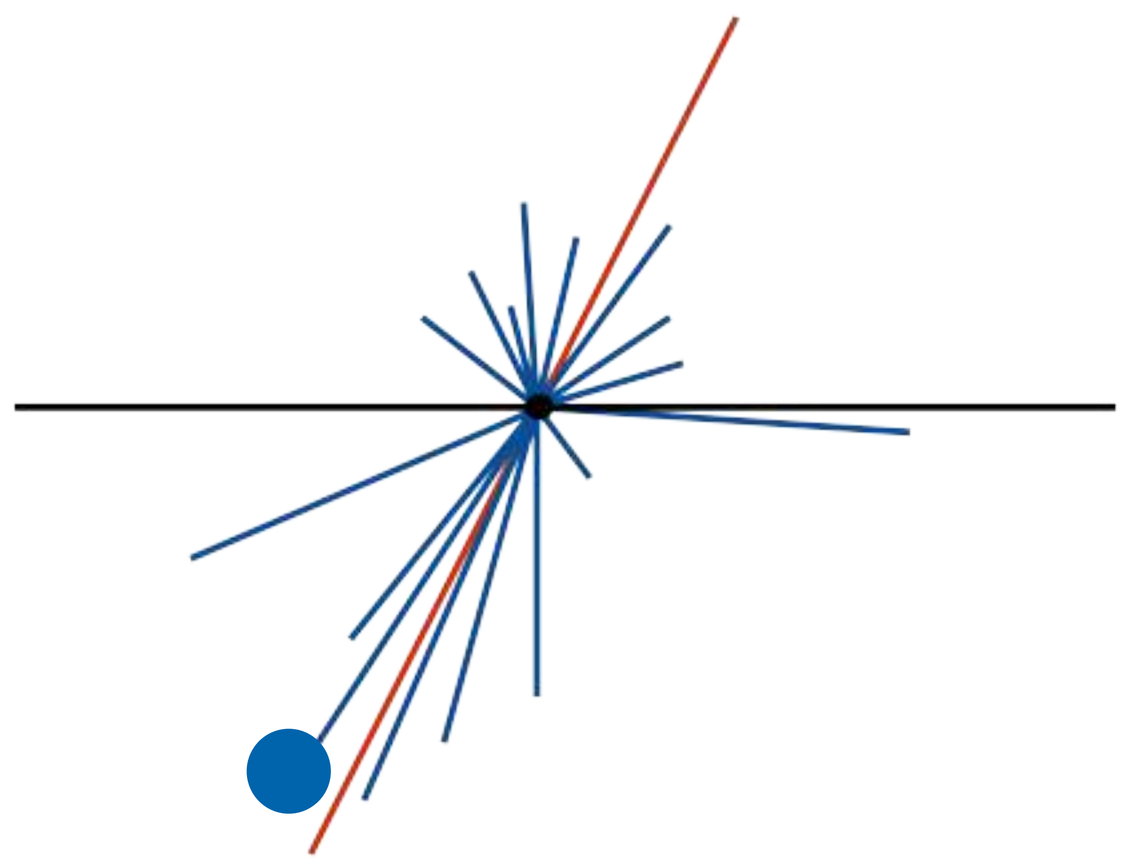
NLL with coherent branching
Issues in dipole showers

Issues in coherent branching
LL with dipole showers

Can we push this to NLL$_{\text{global}}$ / LL$_{\text{non-global}}$ in one (dipole) algorithm?

$\alpha_s L \sim 1 \qquad \alpha_s N^2 \sim 1$

$\mathrm{NLL}_{\mathrm{global}}/\mathrm{LL}_{\mathrm{non\text{-}global}}$ showers



(N)NLO with matching

Demonstrate NLL accurate evolution:

- PanScales — numerical

  [PanScales — Dasgupta, Monni, Salam, Soyez + ….]
- Deductor — numerical/analytical

  [Nagy, Soper]
- Forshaw/Holguin/Plätzer — analytical

  [aim at improving Herwig 7 dipole shower]
- Sherpa — numerical/analytical

  [Herren, Höche, Krauss, Reichelt, Schönherr]

} Based on amplitude evolution.

Can we push this to $\mathrm{NLL}_{\mathrm{global}}$ / $\mathrm{LL}_{\mathrm{non\text{-}global}}$ in one (dipole) algorithm? $\alpha_s L \sim 1 \quad \alpha_s N^2 \sim 1$

$\mathrm{NLL}_{\mathrm{global}}/\mathrm{LL}_{\mathrm{non\text{-}global}}$ showers

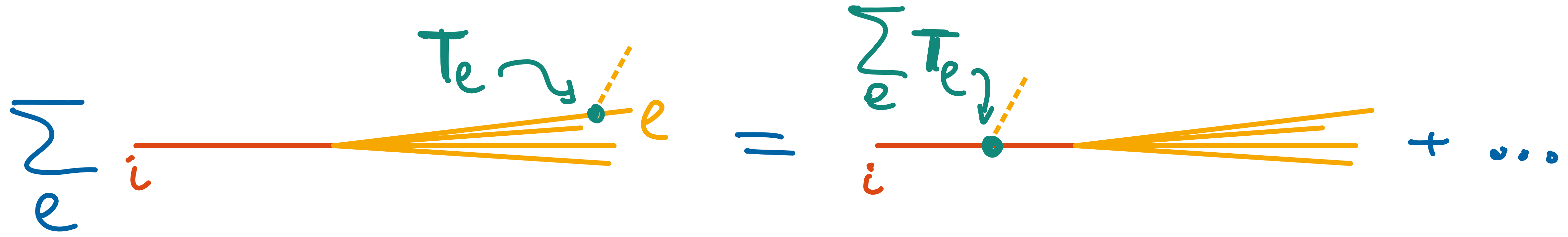
[PanScales — Dasgupta, Monni, Salam, Soyez + … — '18 …]
[Holguin, Forshaw, Plätzer — '19]
[Herren, Höche, Krauss, Reichelt, Schönherr — '22]

Ingredients for NLL accurate, 2-jet global observables:
Compatibility of

- evolution ordering,
- partitioning of soft radiation, and
- initial conditions.

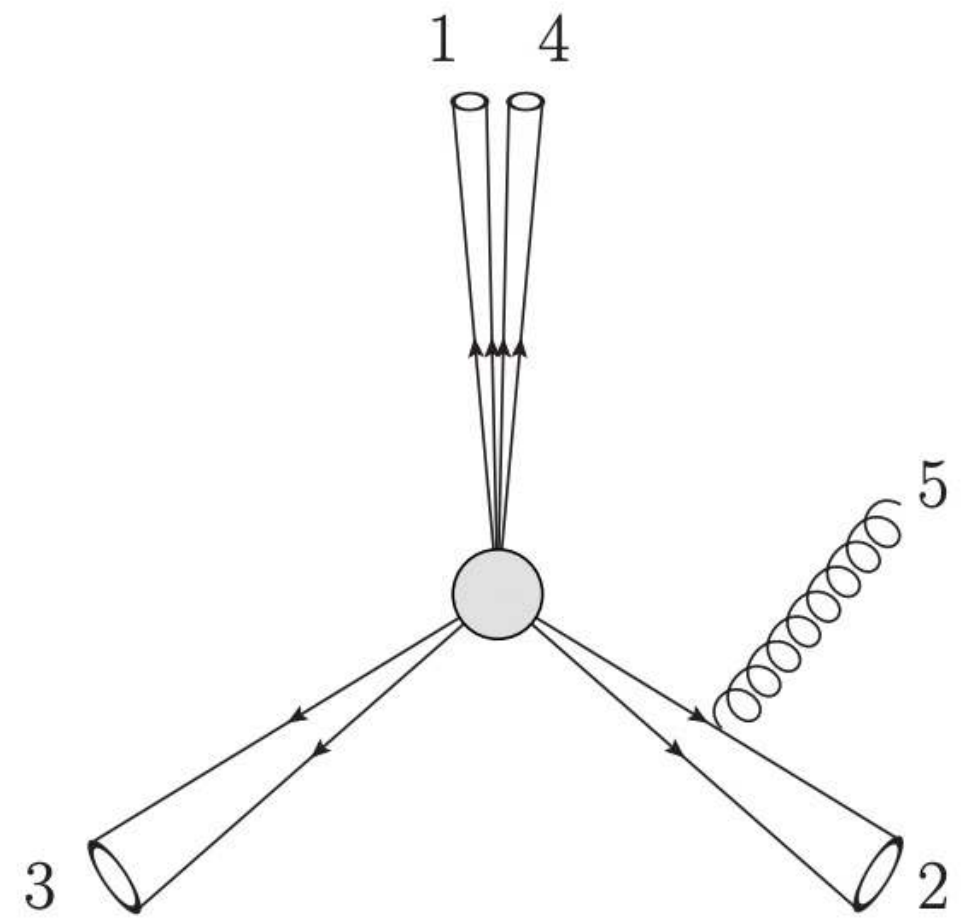
From a colour point of view:

- Simplicity of colour structures at the hard process.
- Colour factors as dictated by coherence.



Non-global observables will stick with the large-N limit.

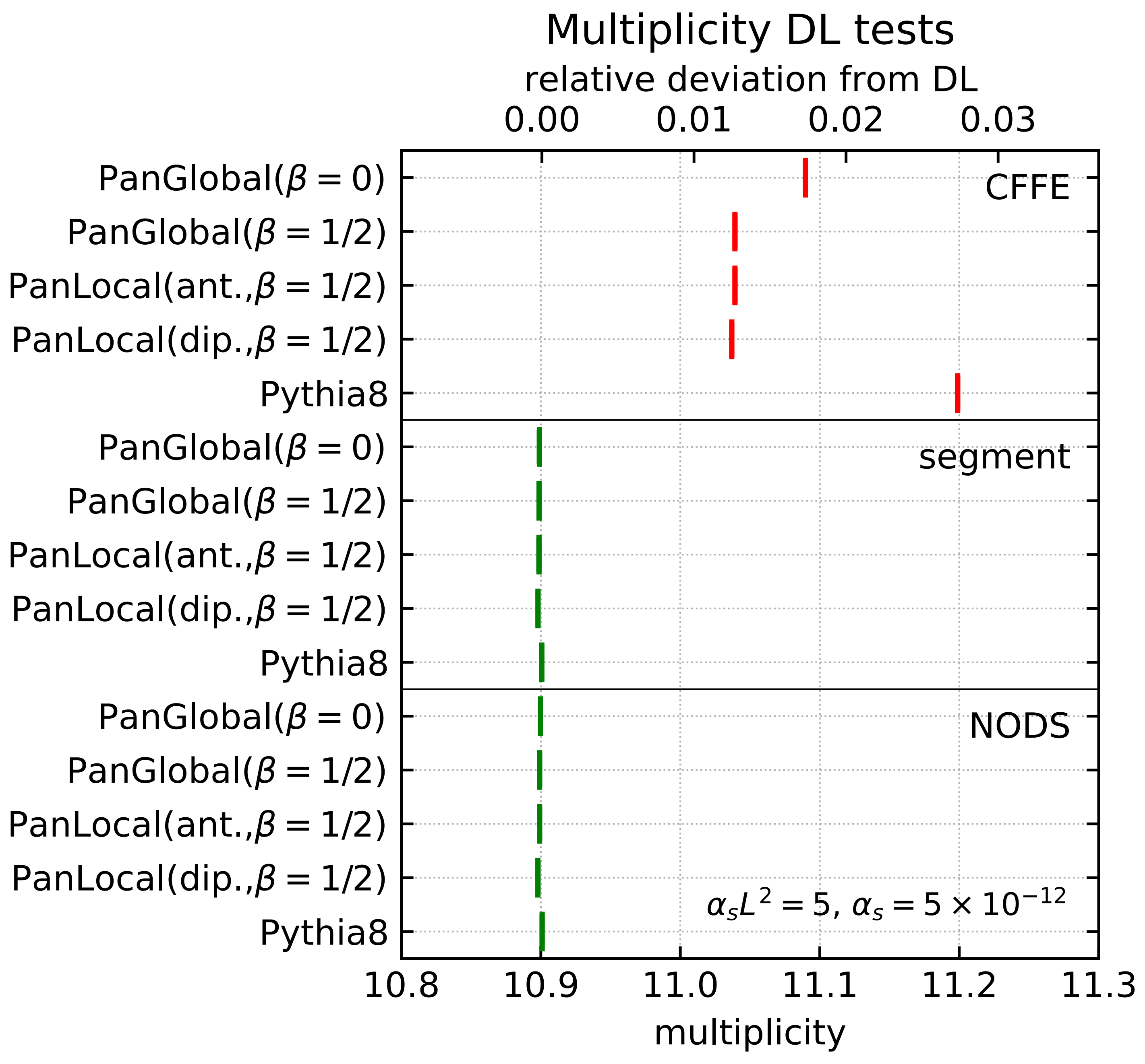
Can extend to three jets where colours are still trivial.



[Holguin, Forshaw, Plätzer — '20]

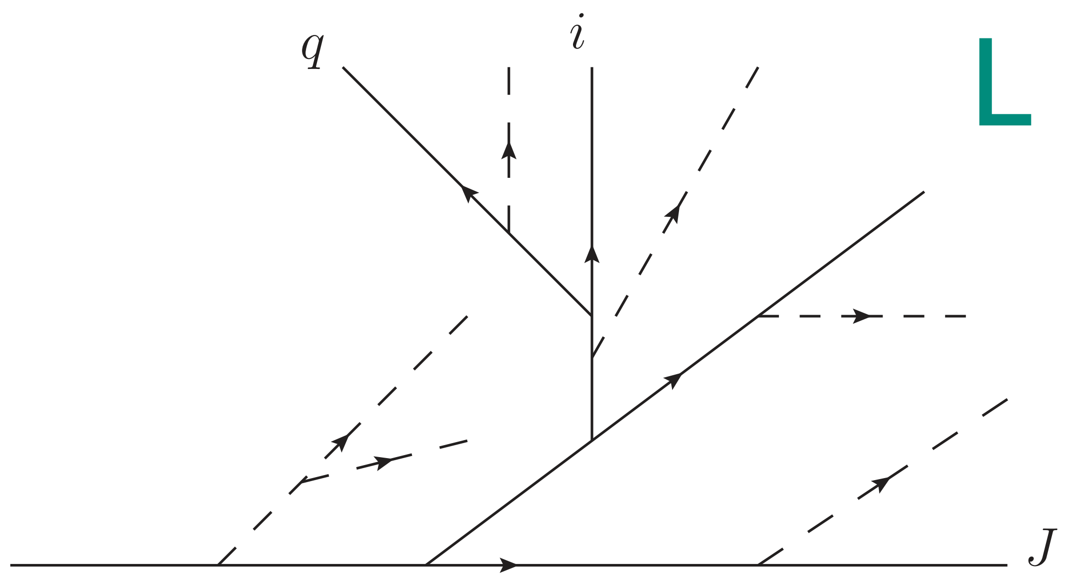
NLL$_{\text{global}}$/LL$_{\text{non-global}}$ showers

[Hamilton, Medves, Salam, Szyboz, Soyez '20]



Errors in dipole showers when patron emitted at an angle larger than the angular extent of the chain which led to the emission: Coherent branching would know through angular ordering.

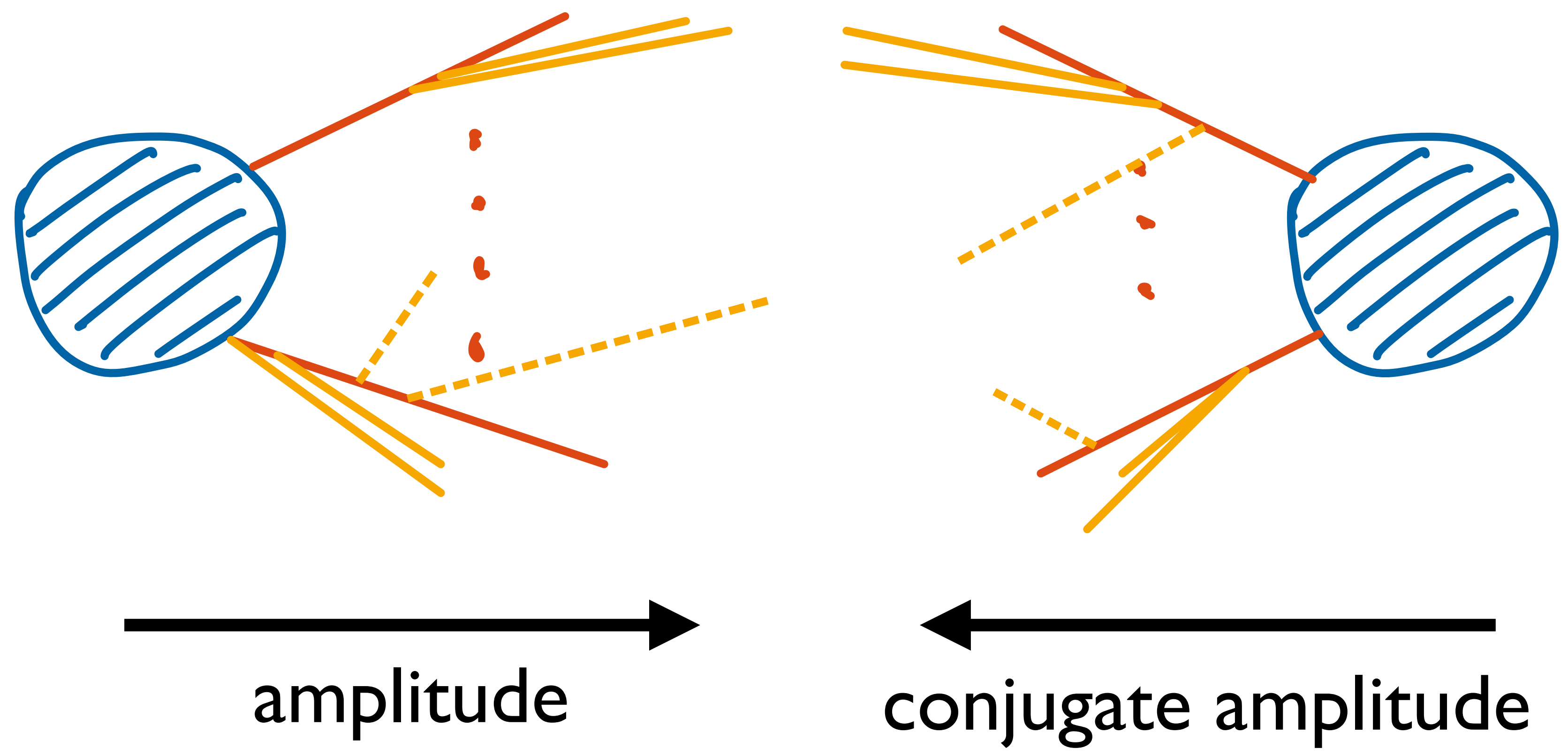
Introduce dynamic colour factors based on branching histories.

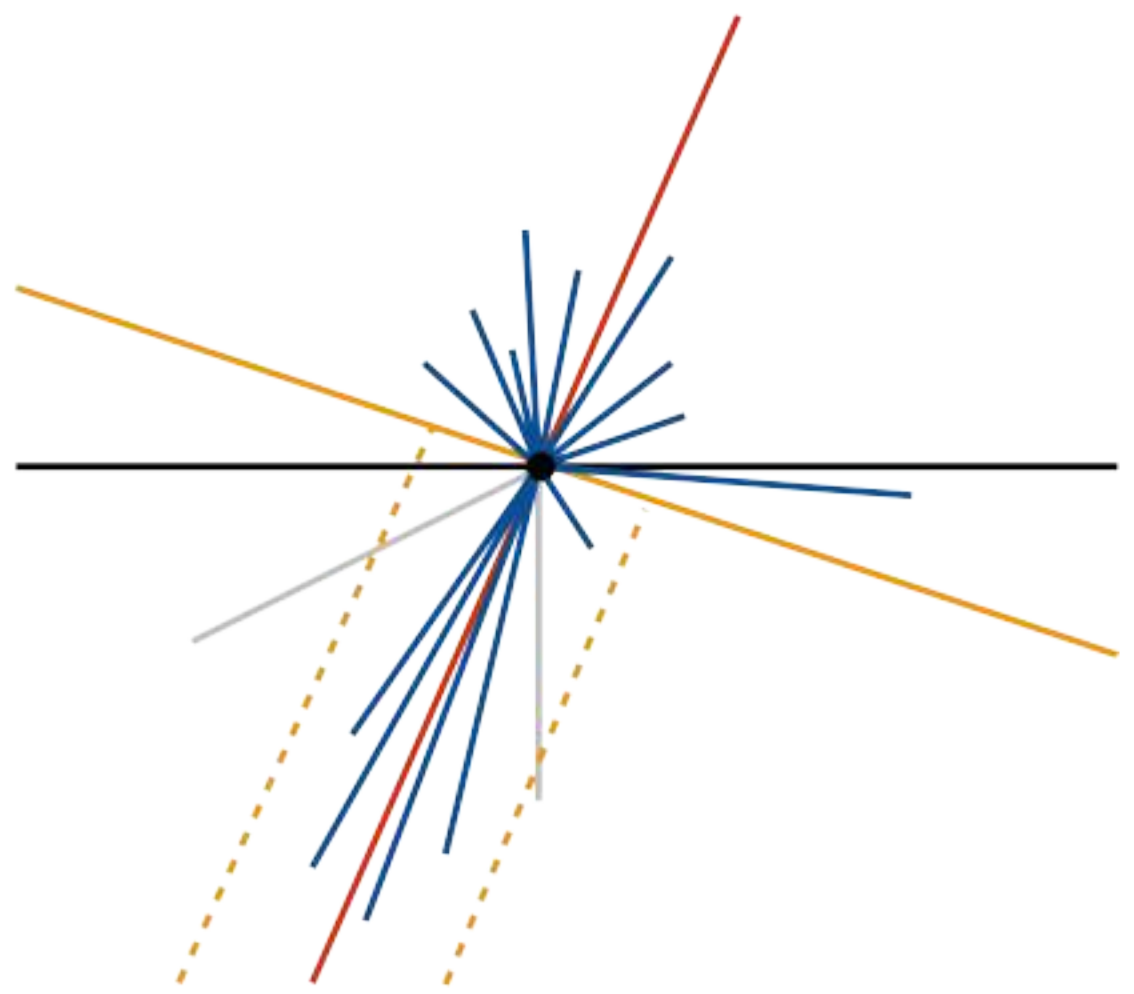


$$\mathcal{C}_{iJ}(\theta_{iq},\theta_{LJ}) = \left(C_F\delta_i^{(q)} + \frac{C_A}{2}\delta_i^{(g)}\right)\theta(\theta_{iq} < \theta_{LJ}) + \left(\frac{C_A}{2}\delta_J^{(g)} + C_F\delta_J^{(q)}\right)\theta(\theta_{iq} > \theta_{LJ})$$

[Holguin, Forshaw, Plätzer — '20]

Tracking (colour) charges





Non-global observables set the level of complexity we need to address.
We cannot tell how subleading finite N is until we have the tools to test.

Colour matrix element corrections

Colour matrix element corrections:
Real emissions only amplitude evolution —
first implementation in a shower algorithm.

$$|\mathcal{M}_n\rangle = \sum_{\alpha=1}^{d_n} c_{n,\alpha}|\alpha_n\rangle$$

$$\mathcal{M}_n = (c_{n,1}, \ldots, c_{n,d_n})^T$$

[Plätzer, Sjödahl '12]
[Plätzer, Sjödahl, Thoren '18]

$$|\mathcal{M}_n|^2 = \mathcal{M}_n^\dagger S_n \mathcal{M}_n = \mathrm{Tr}\left(S_n \times \mathcal{M}_n \mathcal{M}_n^\dagger\right)$$

$$\langle \mathcal{M}_n | \mathbf{T}_{\tilde{ij}} \cdot \mathbf{T}_{\tilde{k}} | \mathcal{M}_n \rangle = \mathrm{Tr}\left(S_{n+1} \times T_{\tilde{k},n} \mathcal{M}_n \mathcal{M}_n^\dagger T_{\tilde{ij},n}^\dagger\right)$$

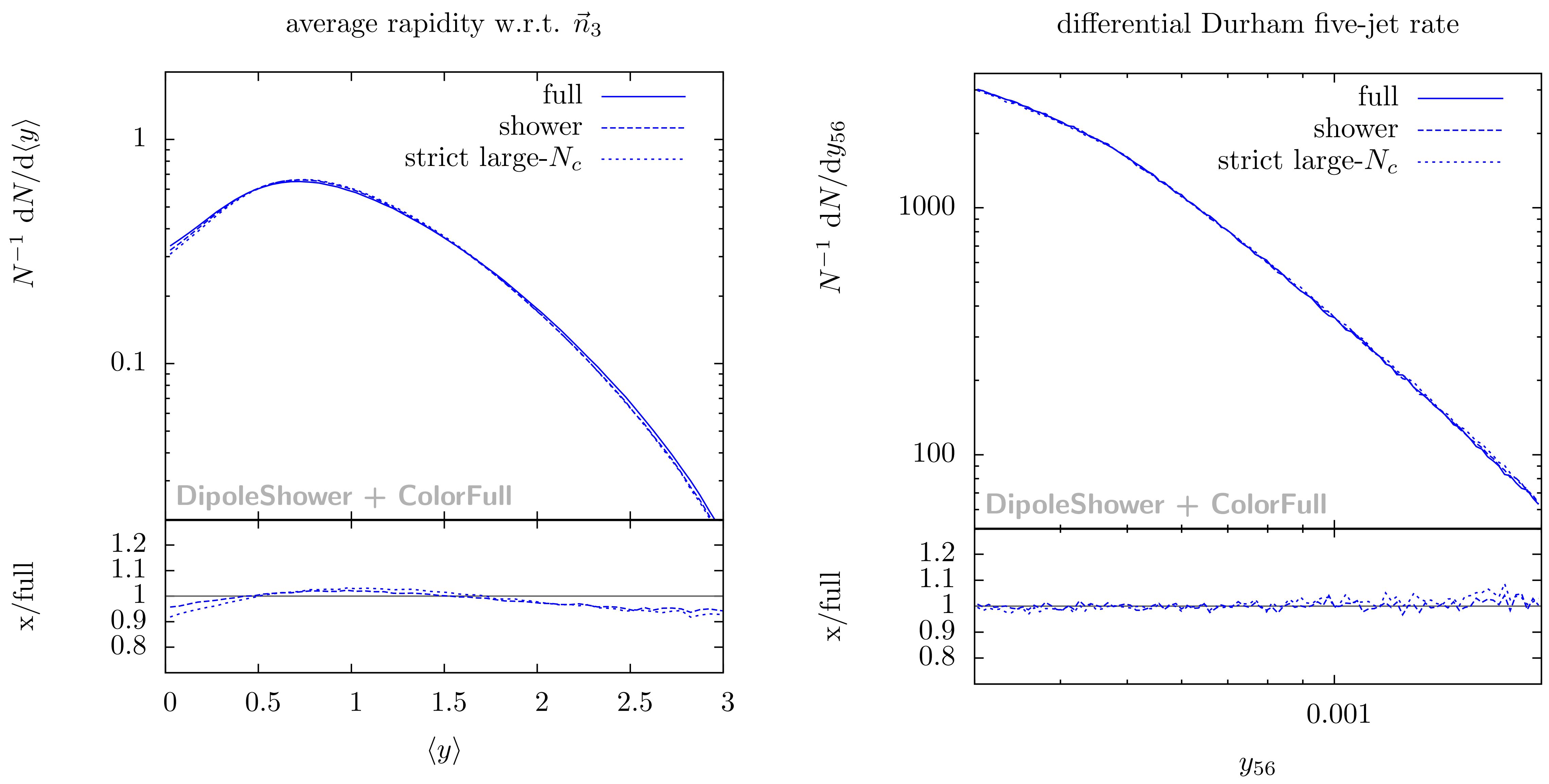
$$V_{ij,k}(p_\perp^2, z; p_{\tilde{ij}}, p_{\tilde{k}}) \times \frac{-1}{\mathbf{T}_{\tilde{ij}}^2} \frac{\langle \mathcal{M}_n | \mathbf{T}_{\tilde{ij}} \cdot \mathbf{T}_k | \mathcal{M}_n \rangle}{|\mathcal{M}_n|^2}$$

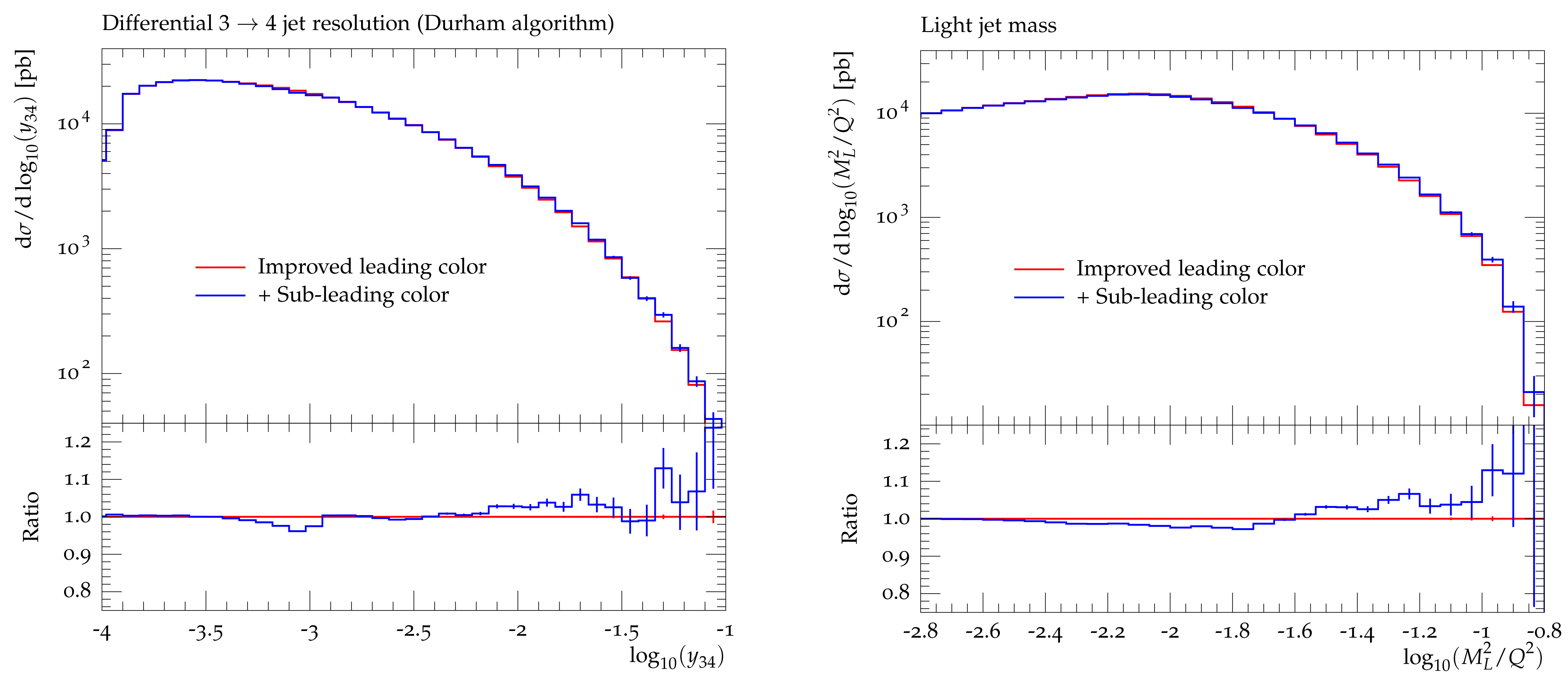
$$M_{n+1} = -\sum_{i \neq j} \sum_{k \neq i,j} \frac{4\pi\alpha_s}{p_i \cdot p_j} \frac{V_{ij,k}(p_i, p_j, p_k)}{\mathbf{T}_{\tilde{ij}}^2} \, T_{\tilde{k},n} M_n T_{\tilde{ij},n}^\dagger$$

More concepts and analytics in [Forshaw, Kyreleis, Seymour '06] [Nagy, Soper '07] [Plätzer '13]

Colour matrix element corrections

Detailed understanding of similarities and differences and comparisons are still needed.





[Plätzer, Sjödahl '12]
[Plätzer, Sjödahl, Thoren '18]

[Höche, Reichelt '20]
"Sampling" also explored [Isaacson, Prestel '19]

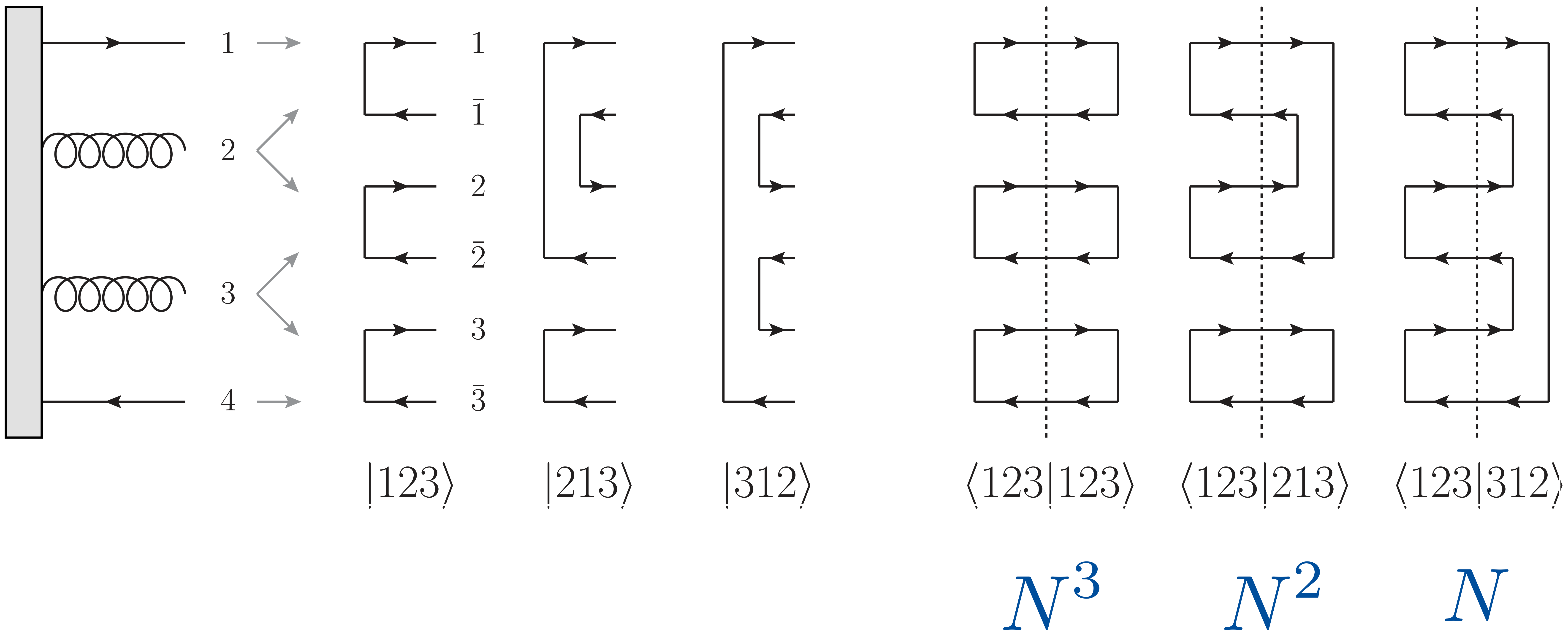
Correct at fixed-order, fixed-multiplicity expansion, not beyond.

[Forshaw, Holguin, Plätzer – '19]

Tracking colour

Decompose amplitudes in flow of colour charge.

$$\mathrm{Tr}\left[\mathbf{A}_n\right] = \sum_{\sigma,\tau} A_{\tau\sigma} \langle \sigma | \tau \rangle$$

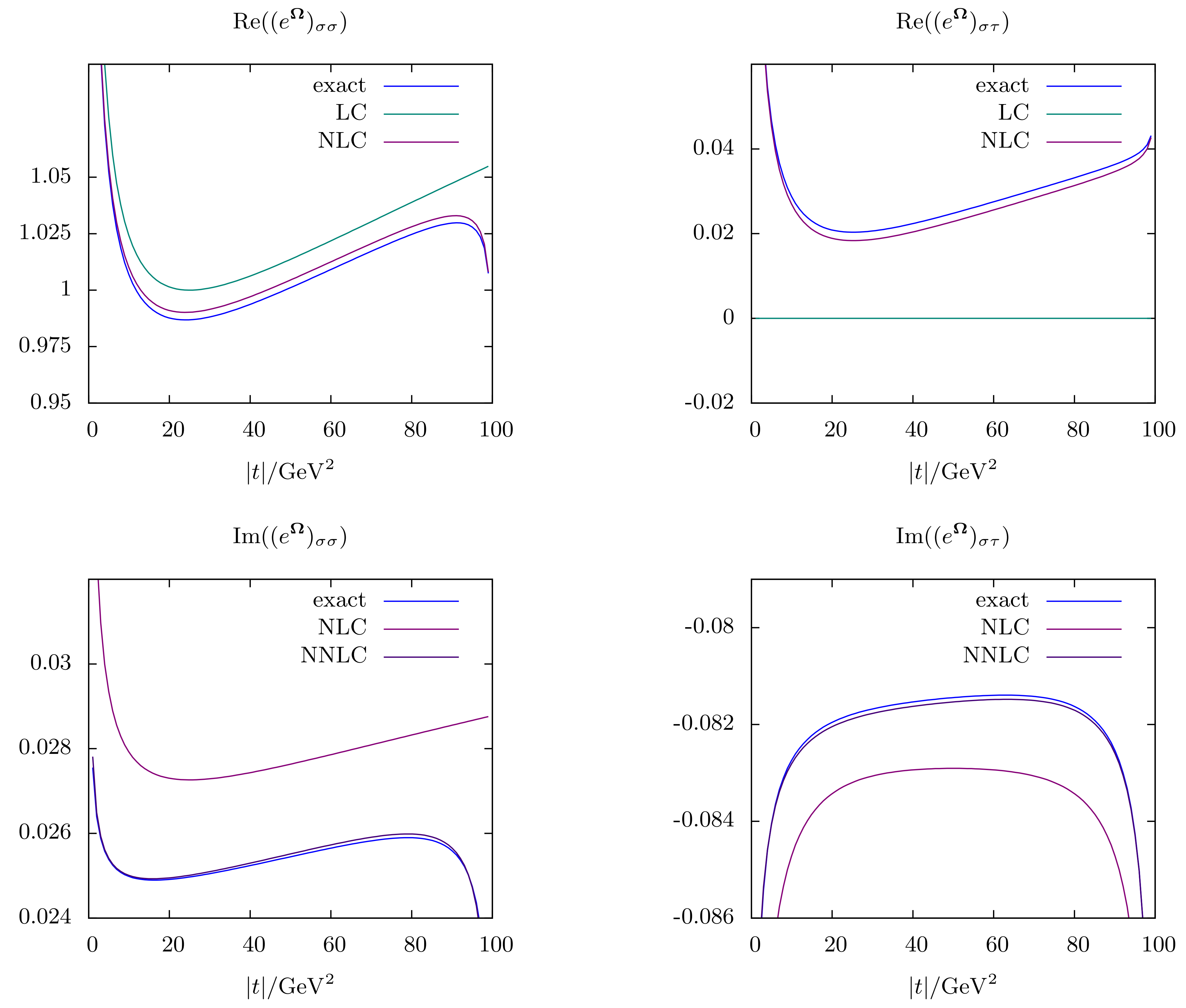


$$(t^a)^i{}_k (t^a)^j{}_l = T_R \left(\delta^i_l \delta^j_k - \frac{1}{N} \delta^i_k \delta^j_l \right)$$

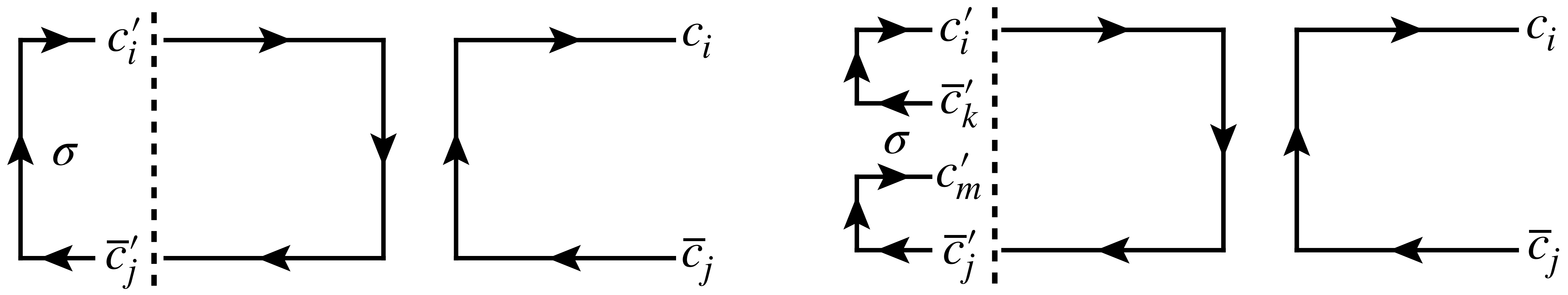
[Plätzer '13]
[Angeles, De Angelis, Forshaw, Plätzer, Seymour '18]

Virtual corrections

Understand structure and kinematic dependence in (soft) anomalous dimensions:



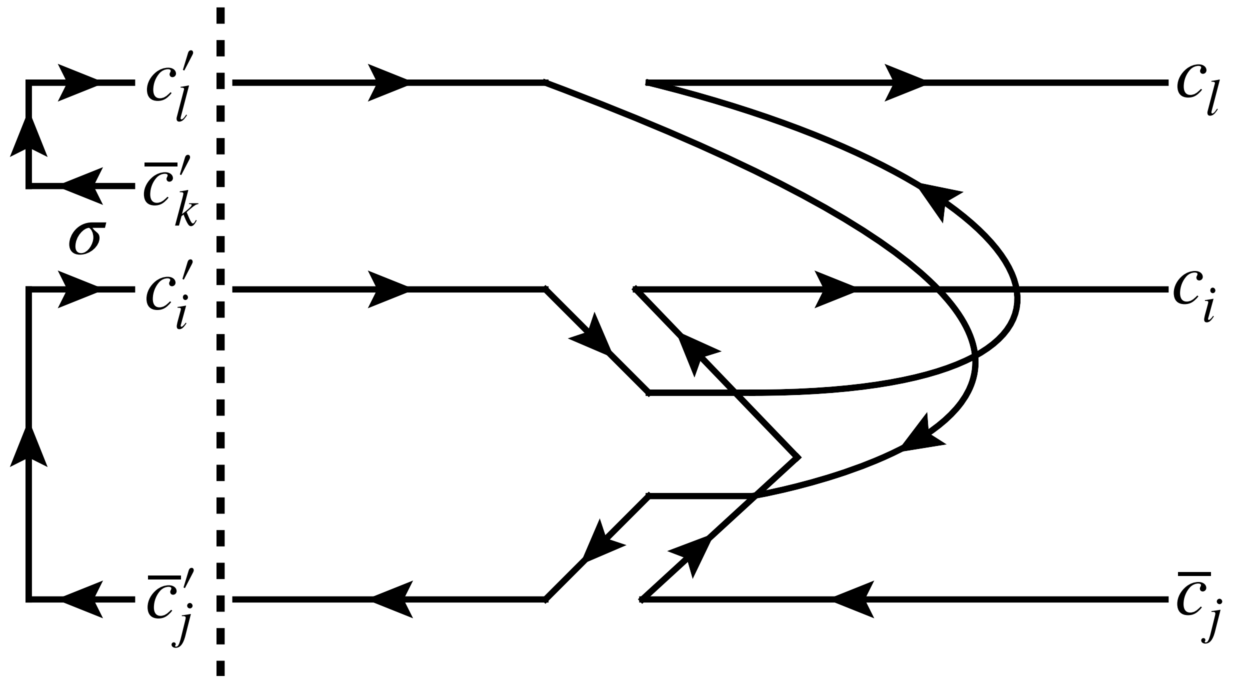
[Plätzer '13]



$$[\tau|\mathbf{\Gamma}^{(1)}|\sigma\rangle = \left(\Gamma^{(1)}_\sigma + \frac{1}{N^2}\rho^{(1)}\right)\delta_{\sigma\tau} + \frac{1}{N}\Sigma^{(1)}_{\sigma\tau}$$

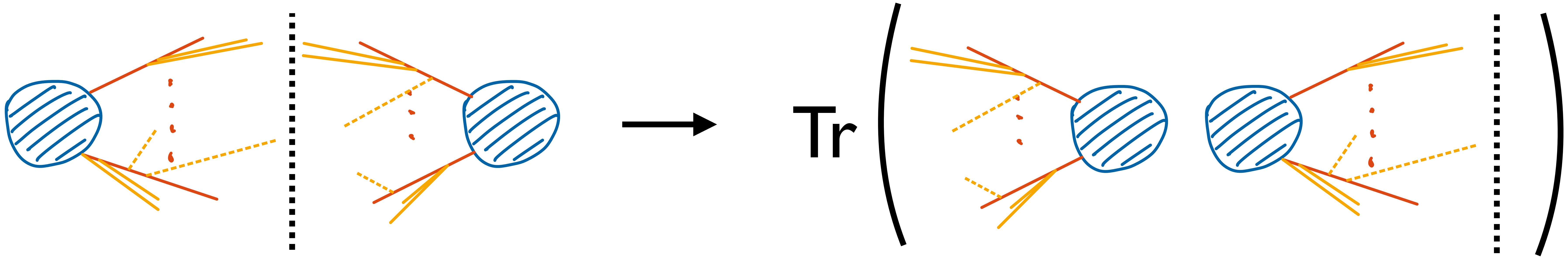
[Plätzer '13]

$$[\tau|\mathbf{\Gamma}|\sigma\rangle = (\alpha_s N)[\tau|\mathbf{\Gamma}^{(1)}|\sigma\rangle + (\alpha_s N)^2[\tau|\mathbf{\Gamma}^{(2)}|\sigma\rangle + \ldots$$



[Plätzer, Ruffa '21]

Amplitude evolution: CVolver

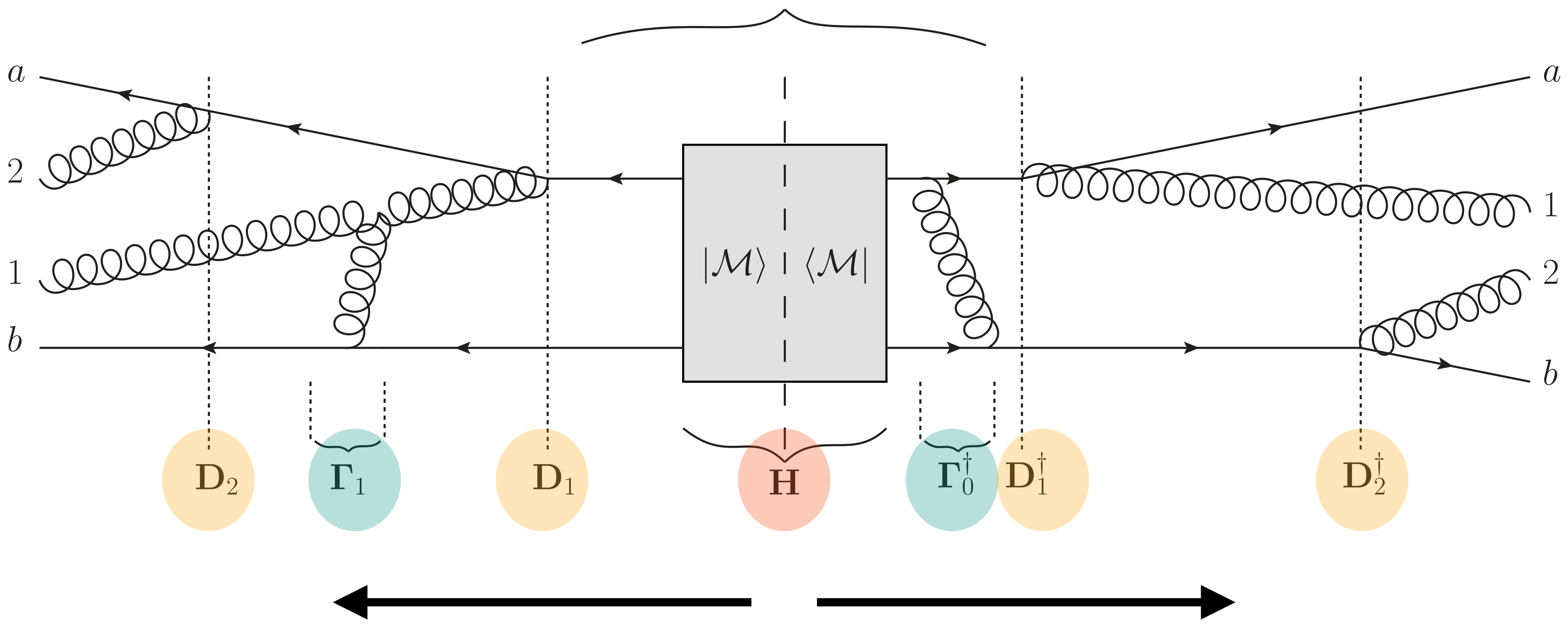


$$\mathbf{A}_n(q) = \int_q^Q \frac{\mathrm{d}k}{k}\, \mathrm{P}e^{-\int_q^k \frac{\mathrm{d}k'}{k'}\mathbf{\Gamma}(k')}\, \mathbf{D}_n(k)\, \mathbf{A}_{n-1}(k)\, \mathbf{D}_n^\dagger(k)\, \overline{\mathrm{P}}e^{-\int_q^k \frac{\mathrm{d}k'}{k'}\mathbf{\Gamma}^\dagger(k')}$$

Markovian algorithm at the amplitude level:
Iterate gluon exchanges and emission.

Different histories in amplitude and conjugate amplitude needed to include interference.

[Angeles, De Angelis, Forshaw, Plätzer, Seymour – '18]
[Forshaw, Holguin, Plätzer – '19]



amplitude   conjugate amplitude

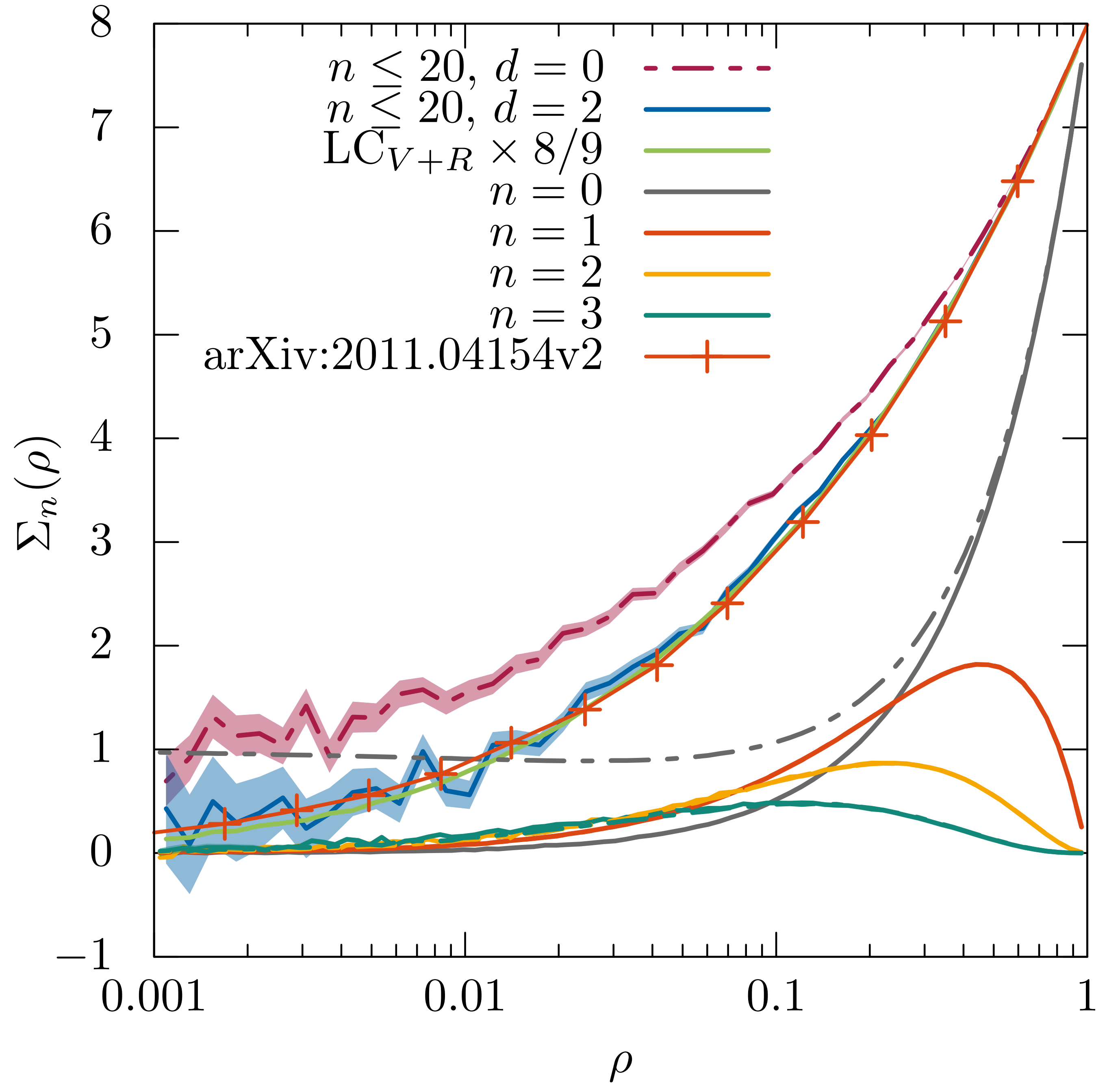
Amplitude evolution: CVolver

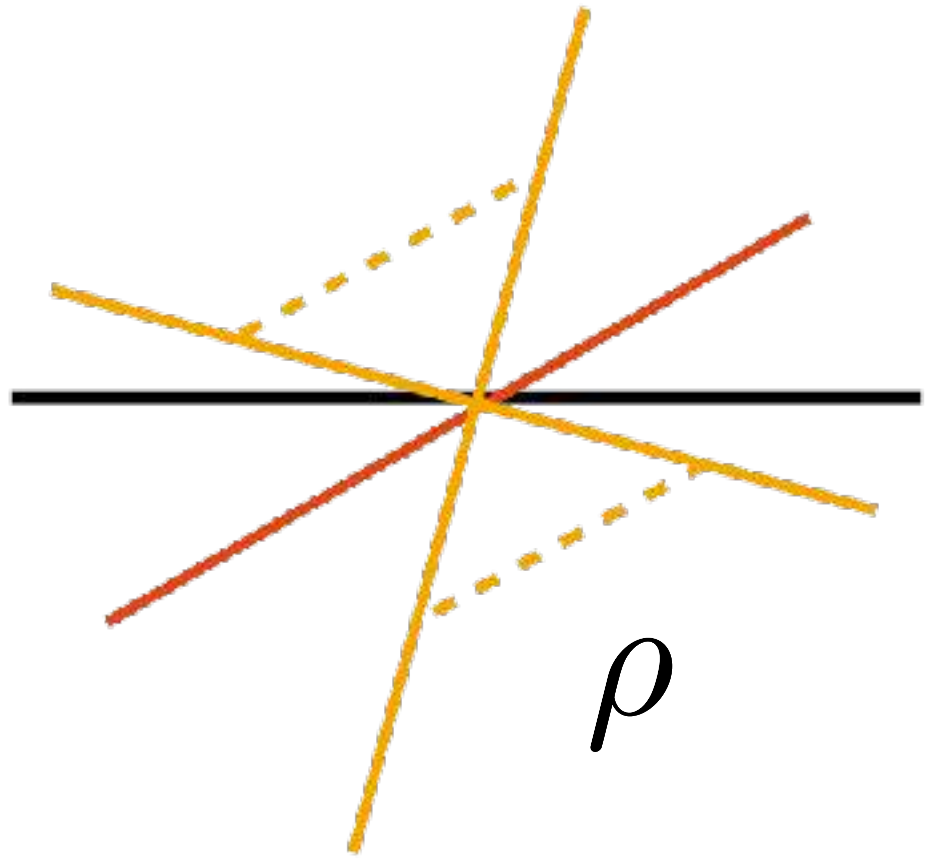
CVolver solves evolution equations in colour flow space

[De Angelis, Forshaw, Plätzer '21]
[Plätzer '13]

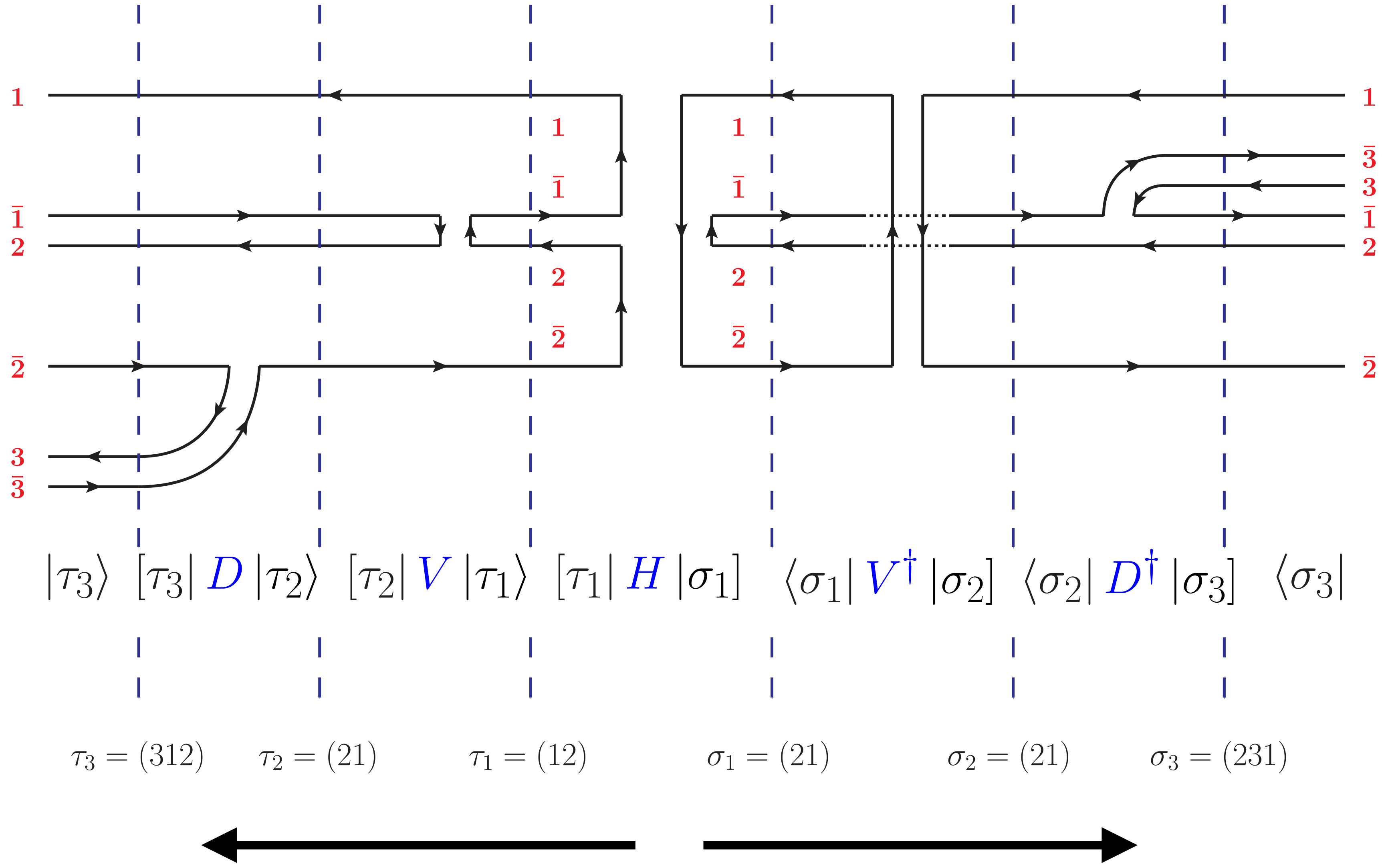
$$\mathbf{A}_n(q) = \int_q^Q \frac{\mathrm{d}k}{k}\, \mathrm{P}e^{-\int_q^k \frac{\mathrm{d}k'}{k'}\boldsymbol{\Gamma}(k')}\, \mathbf{D}_n(k)\, \mathbf{A}_{n-1}(k)\, \mathbf{D}_n^\dagger(k)\, \overline{\mathrm{P}}e^{-\int_q^k \frac{\mathrm{d}k'}{k'}\boldsymbol{\Gamma}^\dagger(k')}$$

singlet $\to gg$ spectrum





$$\Sigma(\rho) = \sum_n \int \mathrm{d}\sigma(\{p_i\}) \prod_i \theta_{\rm in}(\rho - E_i)$$

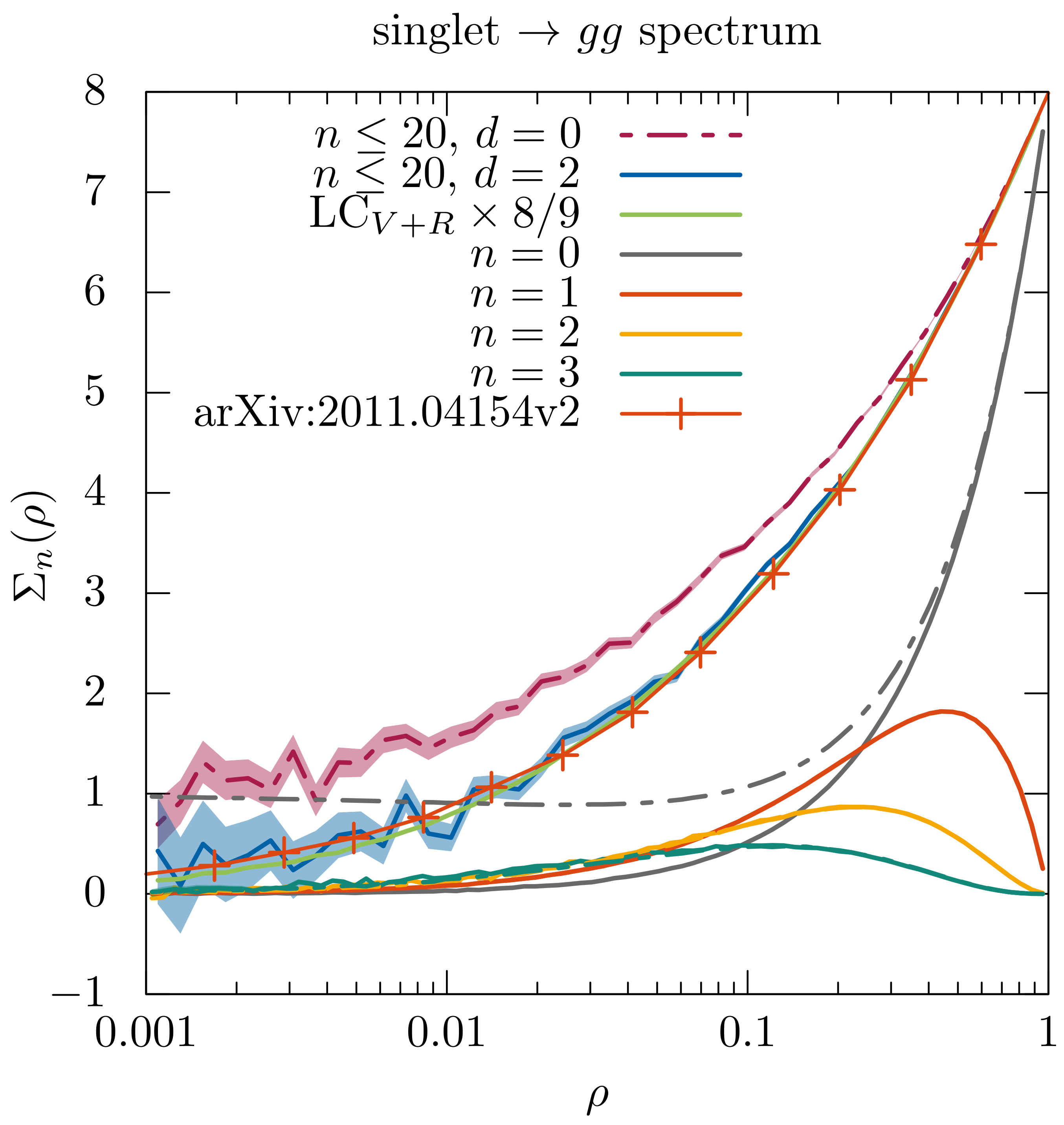


amplitude

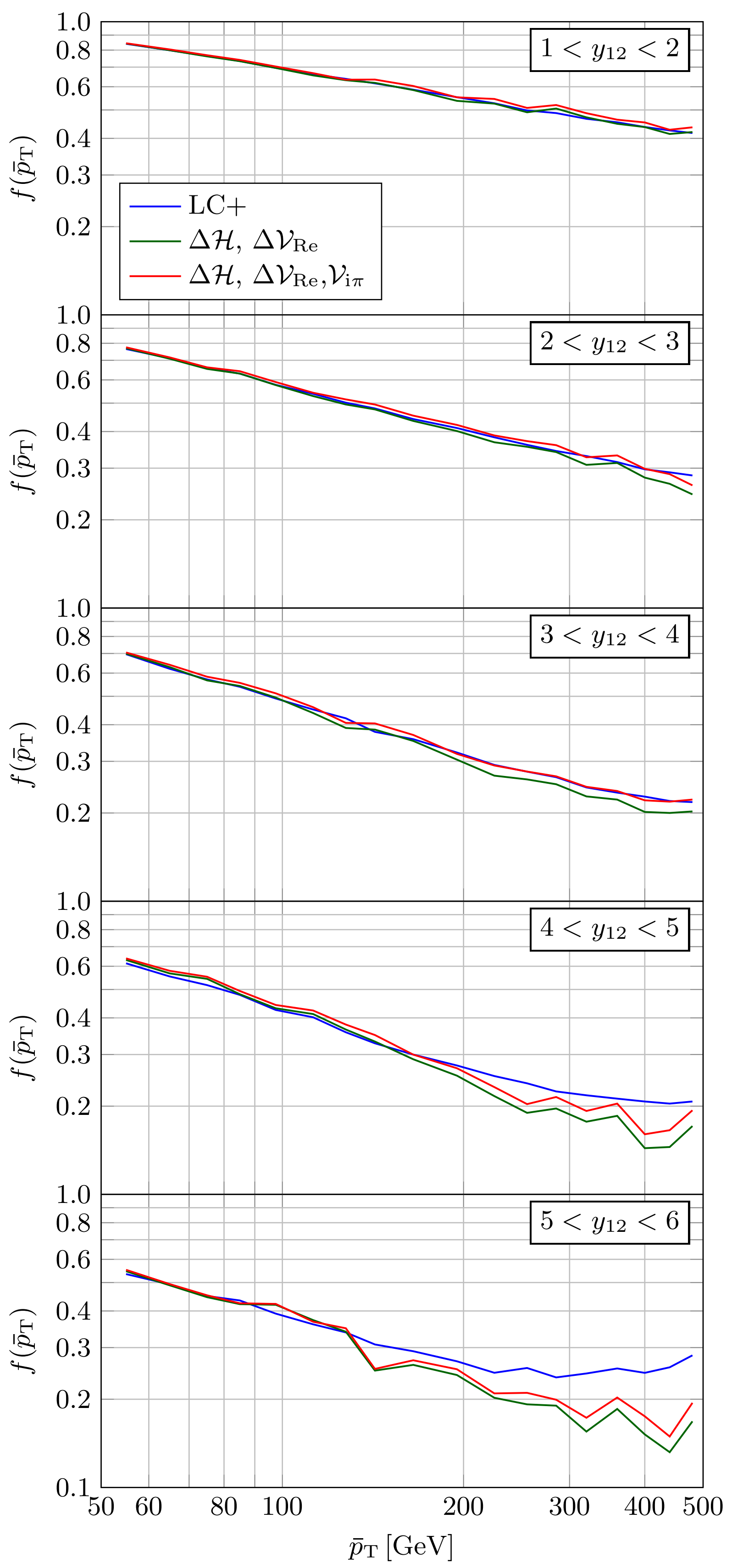
conjugate amplitude

Comparing approximations & results

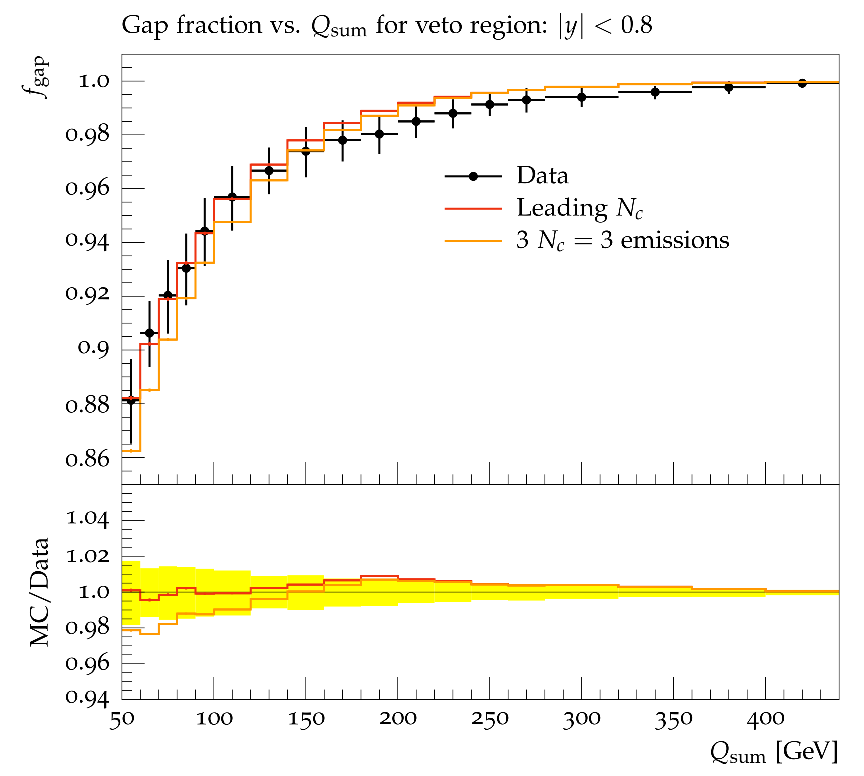
Level of approximation and definition of observables crucial.
Start simple, don't claim everything …



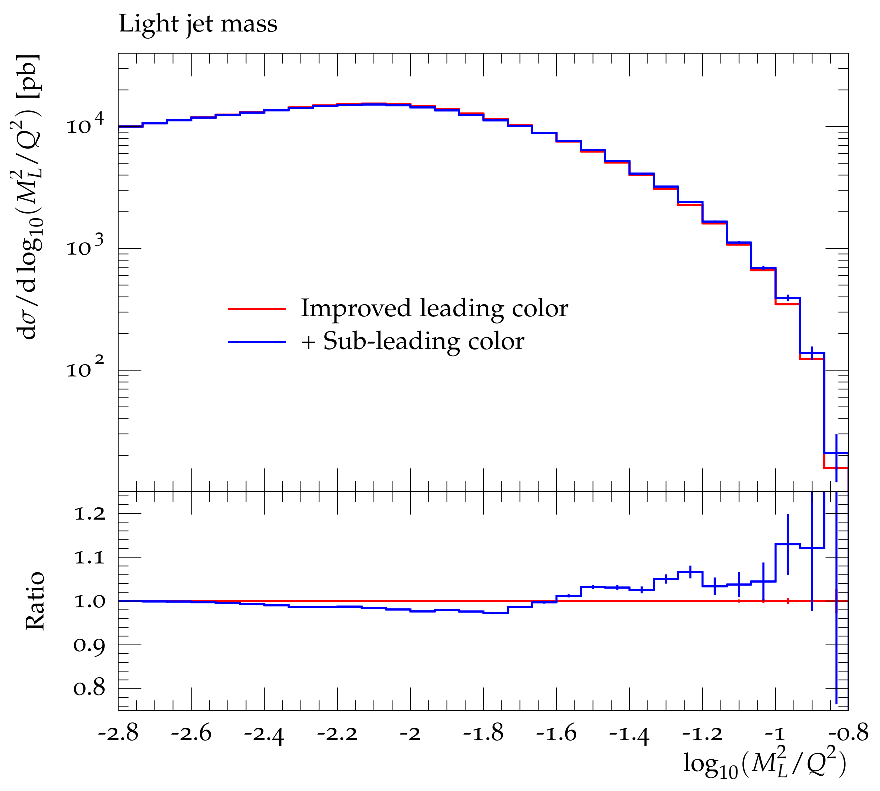
[De Angelis, Forshaw, Plätzer '21] = [Hatta et al. '21]



[Nagy, Soper '19]



[Plätzer, Sjödahl '12]
[Plätzer, Sjödahl, Thoren '18]



[Höche, Reichelt '20]

Comparing approximations & results

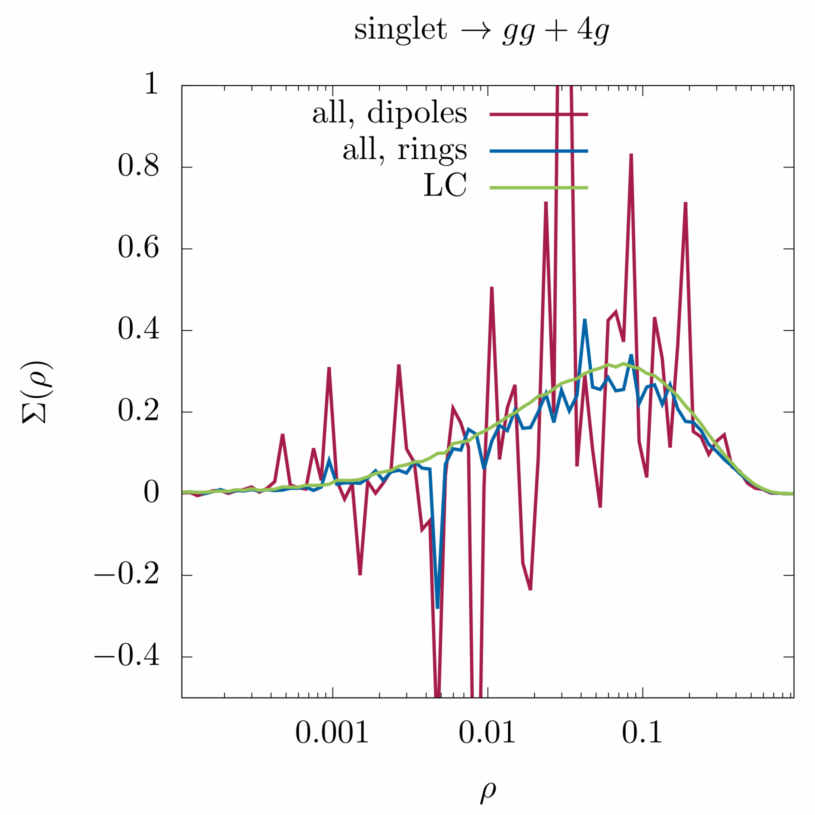
Level of approximation and definition of observables crucial.
Start simple, don't claim everything …

Promising basis of functions to express sub-leading colour.

ω_{ij} dipole

$\omega_{ij} + \omega_{ik} - \omega_{jk}$

string



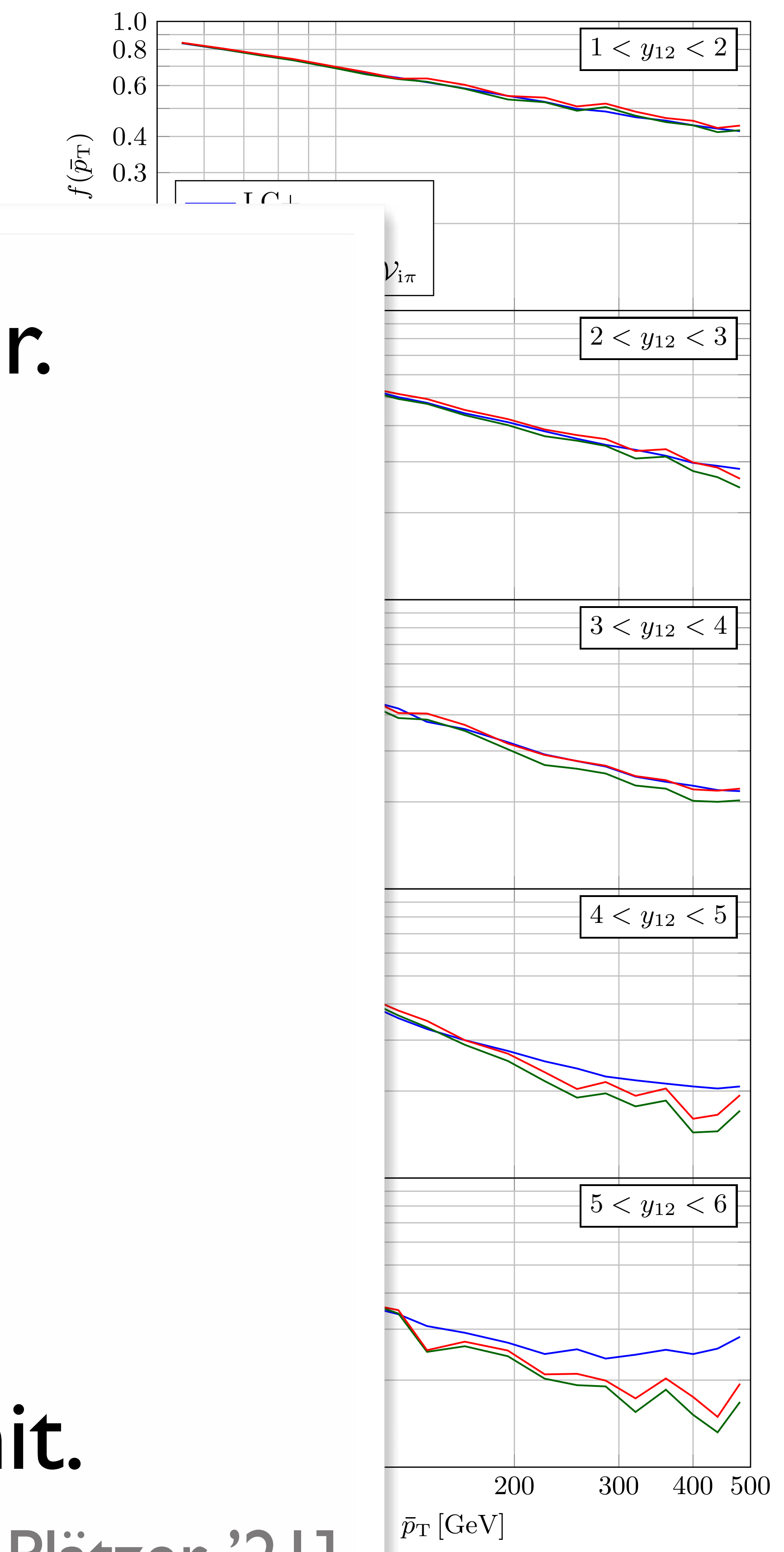
$\omega_{il} + \omega_{kj} - \omega_{kl} - \omega_{ij}$

ring — collinear finite

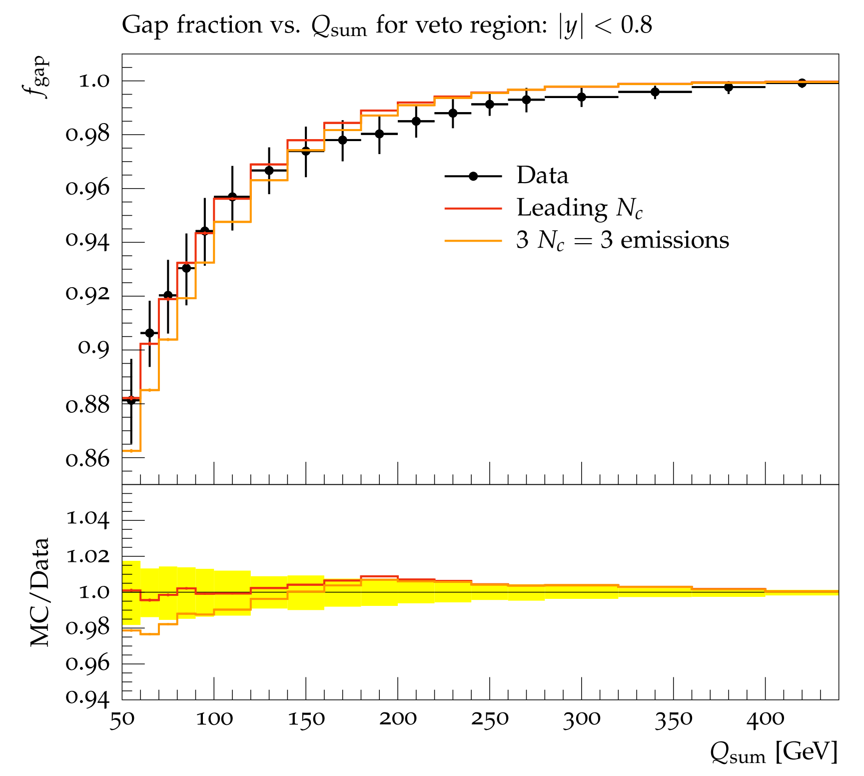
Gives us a notion of coherent branching beyond 2-jet limit.

[Forshaw, Holguin, Plätzer '21]

[De Angelis, Forshaw, Plätzer '21] = [Hatta et al. '21]



[Nagy, Soper '19]



[Plätzer, Sjödahl '12]
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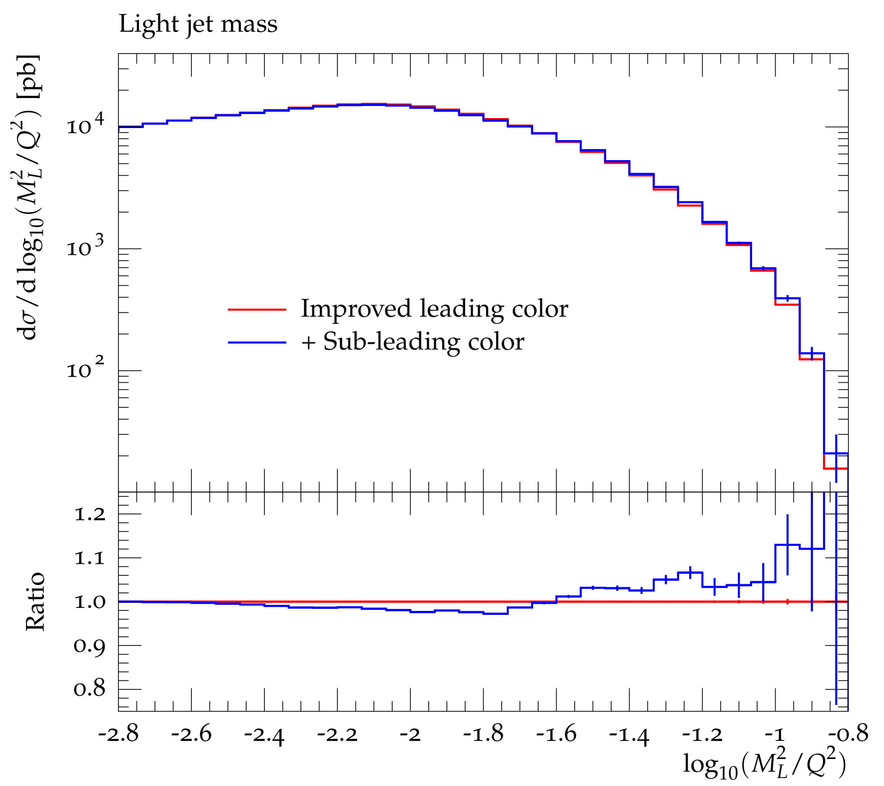
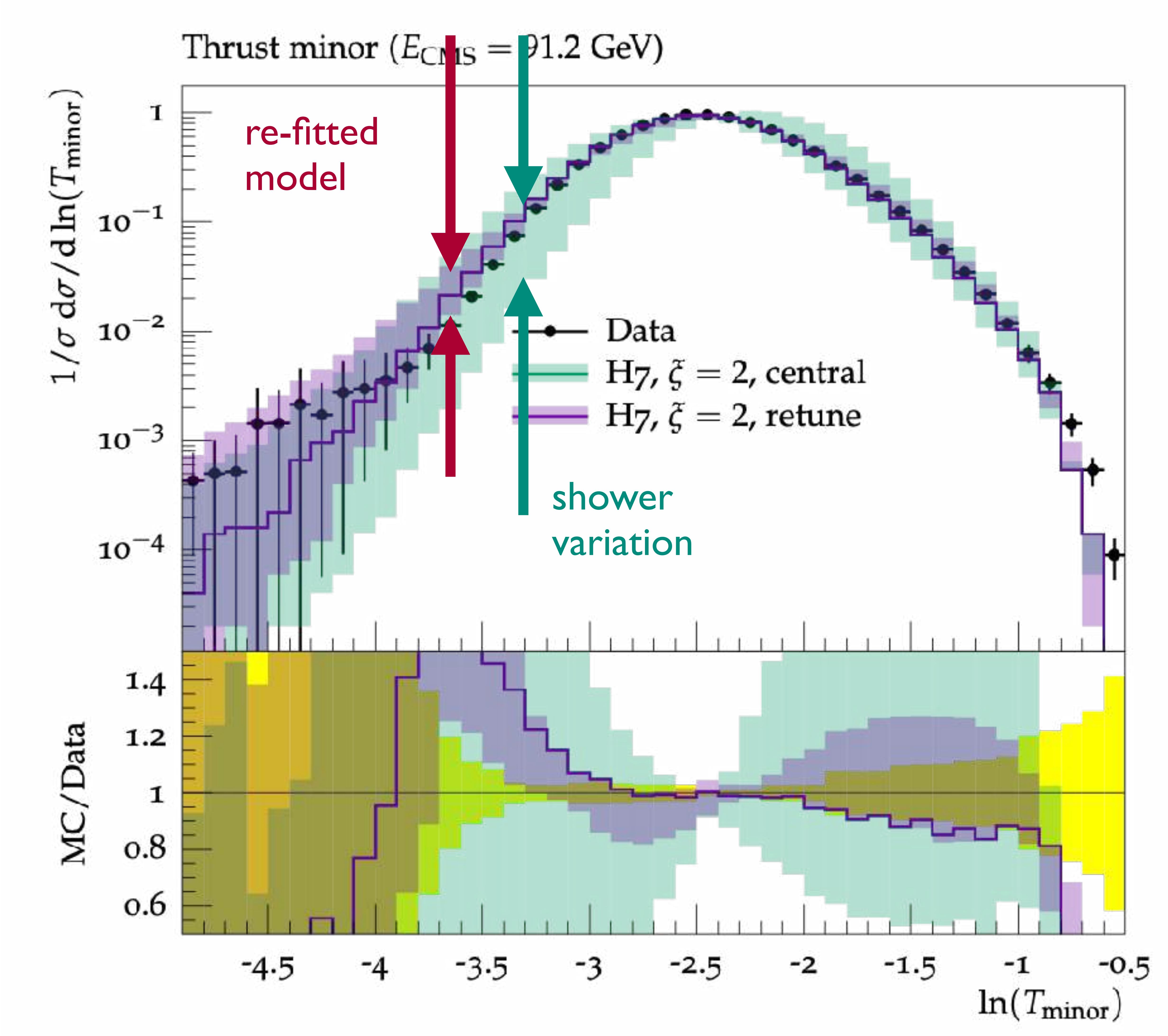


[Höche, Reichelt '20]

The interface to hadronization

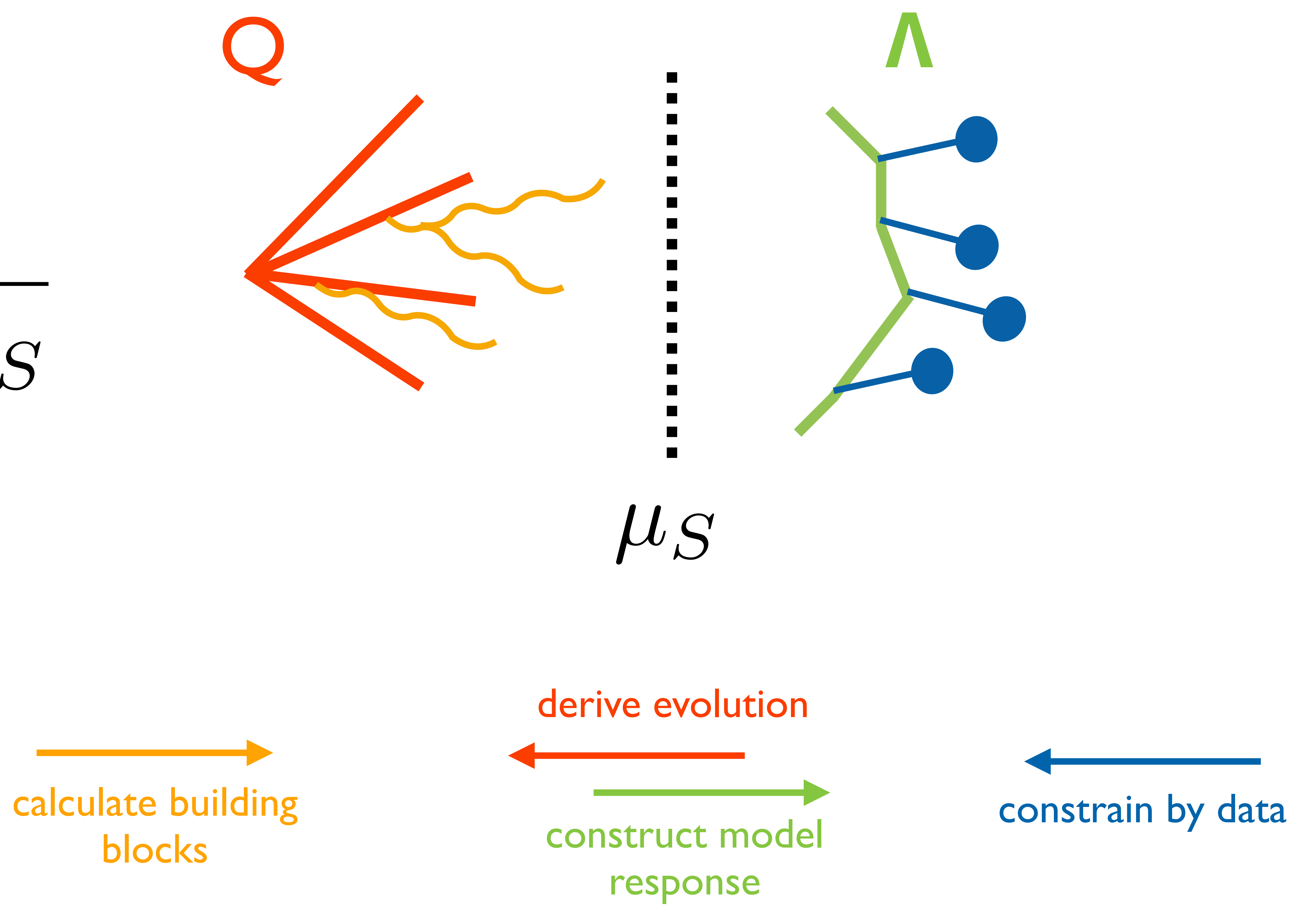
[Bellm, Lönnblad, Prestel, Plätzer, Samitz, Siodmok, Hoang — for Les Houches 2017]



Constructing evolution algorithms

IR cutoff of shower is UV cutoff of hadronization. Cross section is invariant under varying unphysical scales.

$$0 = \frac{\mathrm{d}}{\mathrm{d}\mu_S}$$



[Plätzer – '22]

Constructing evolution algorithms

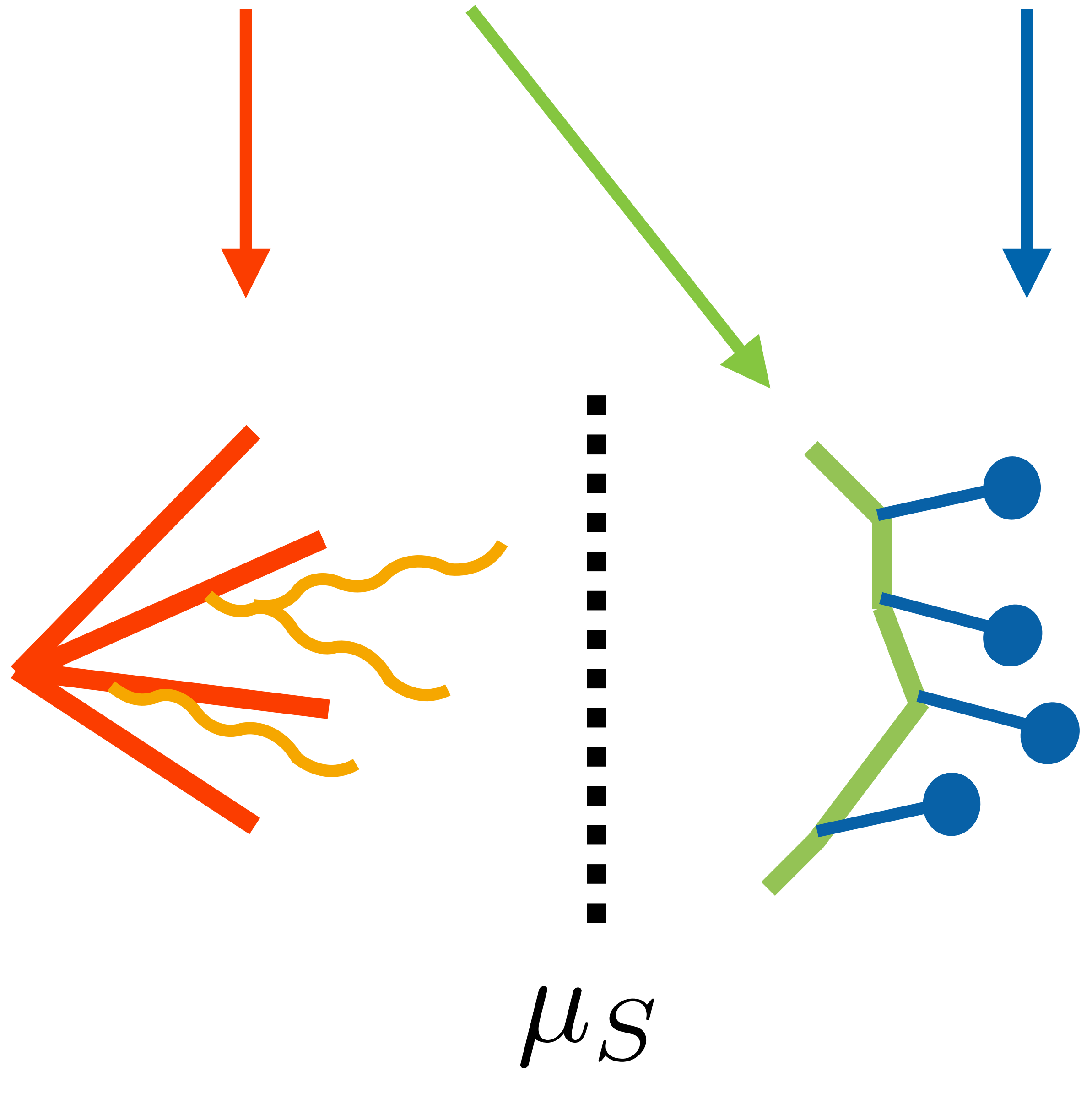
How do we consistently hadronize in light of (improved) shower algorithms?

How to do this at subleading N and higher order shower evolution?

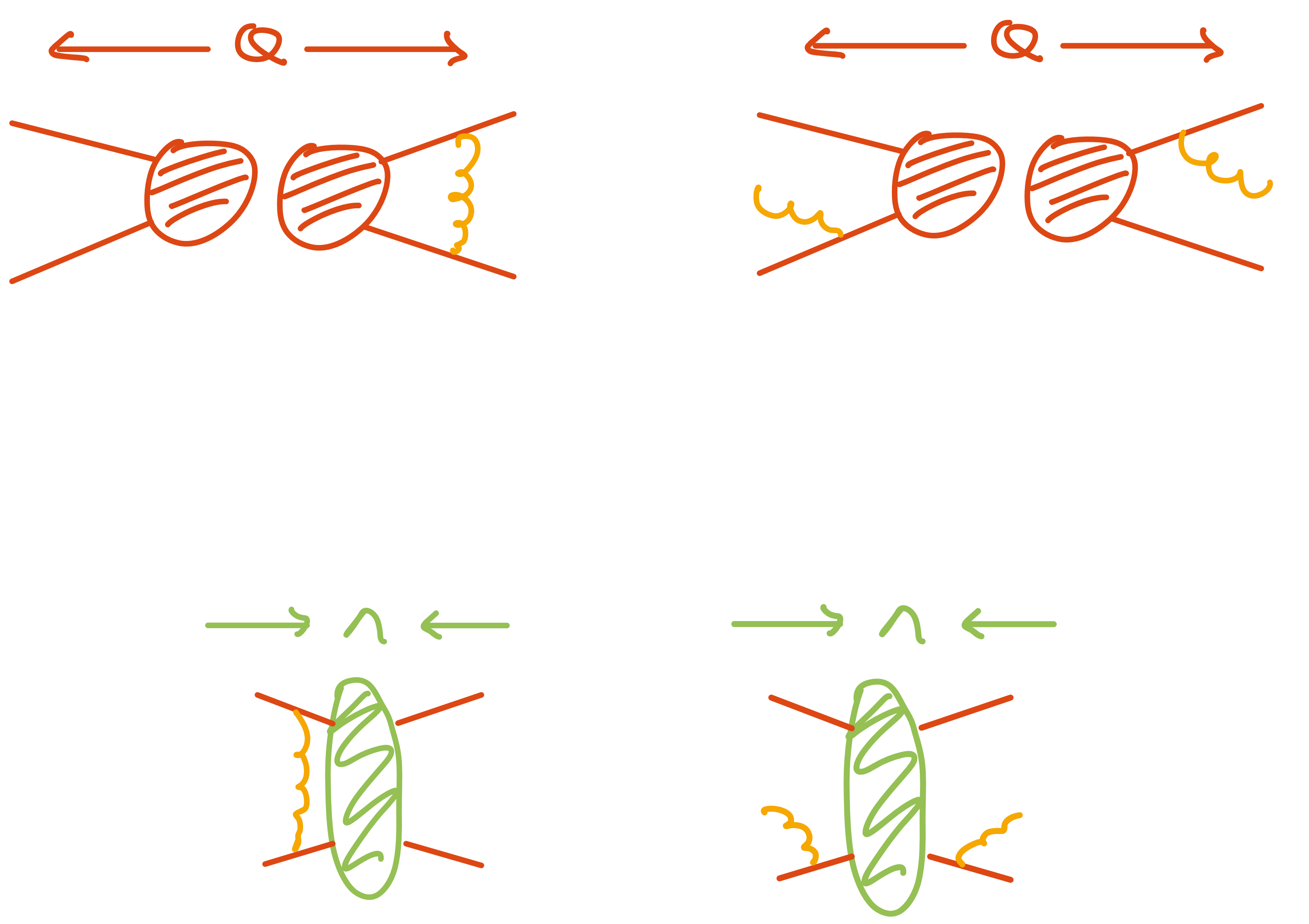
[Plätzer – '22]

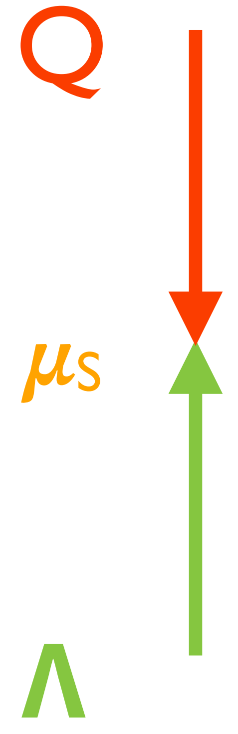
$$\sigma = \sum_{n,m} \int \int \mathrm{Tr}_n \left[\mathbf{M}_n \mathbf{U}_{nm}\right] \mathrm{d}\phi_m u(\phi_m)$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}\mu_S}$$



Implies evolution equations,
cross section invariant after redefinition.





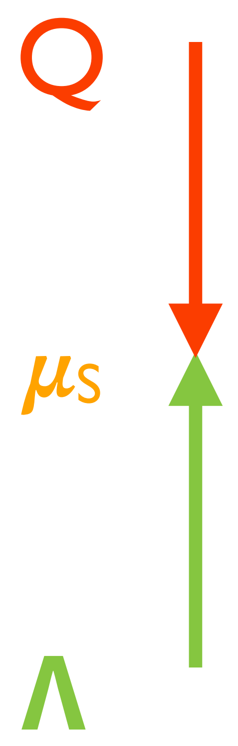
Constructing evolution algorithms

How do we consistently hadronize in light of (improved) shower algorithms?

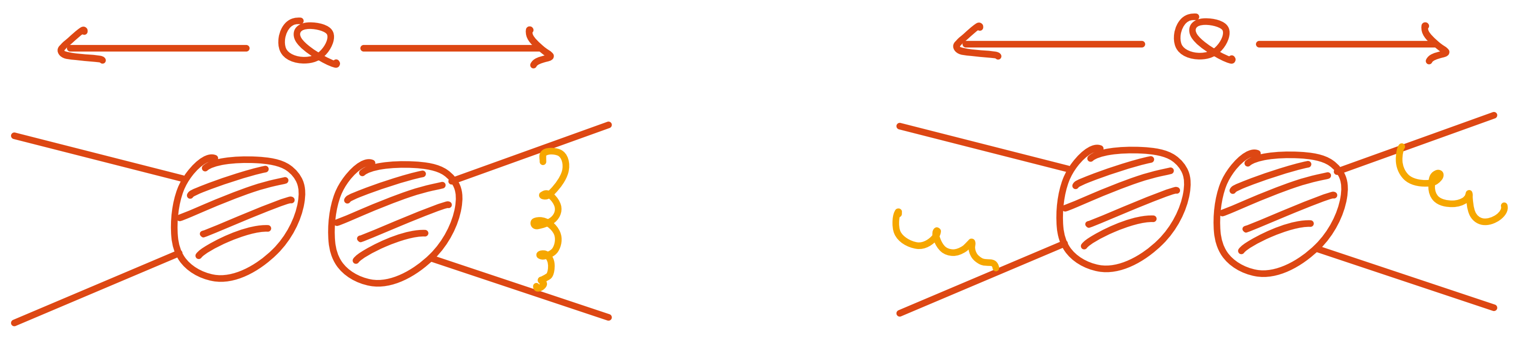
How to do this at subleading N and higher order shower evolution?

[Plätzer – ’22]

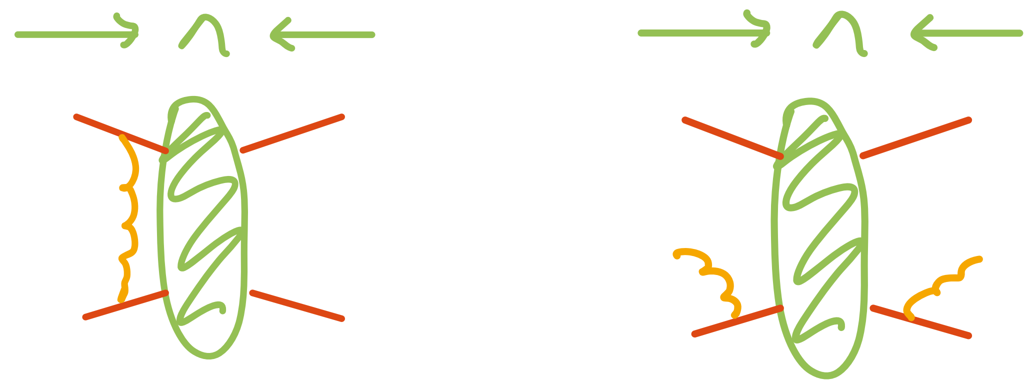
Hard factor reproduces CVolver algorithm and predicts key features of second order evolution.



$$\partial_S \mathbf{A}_n = \mathbf{\Gamma}_{n,S}\mathbf{A}_n + \mathbf{A}_n \mathbf{\Gamma}^\dagger_{n,S} - \sum_{s\geq 1} \alpha_S^s \mathbf{R}^{(s)}_{S,n} \mathbf{A}_{n-s} \mathbf{R}^{(s)\dagger}_{S,n}$$



$$\partial_S \mathbf{S}_n = -\tilde{\mathbf{\Gamma}}^\dagger_{S,n}\mathbf{S}_n - \mathbf{S}_n \tilde{\mathbf{\Gamma}}_{S,n} + \sum_{s\geq 1} \alpha_S^s \int \tilde{\mathbf{R}}^{(s)\dagger}_{S,n+s} \mathbf{S}_{n+s} \tilde{\mathbf{R}}^{(s)}_{S,n+s} \prod_{i=n+1}^{n+s} [\mathrm{d}p_i]\tilde{\delta}(p_i)$$



Soft factor governed by evolution in the inverse direction.

Constructing evolution algorithms

[Plätzer – '22]

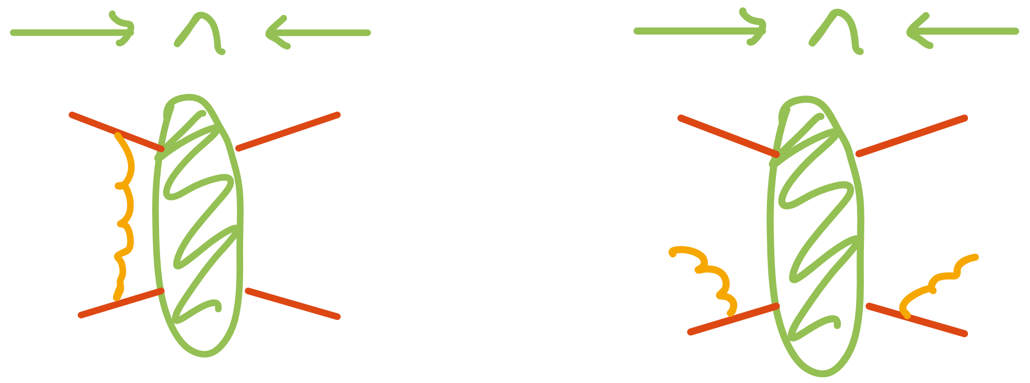
How do we consistently hadronize in light of (improved) shower algorithms?

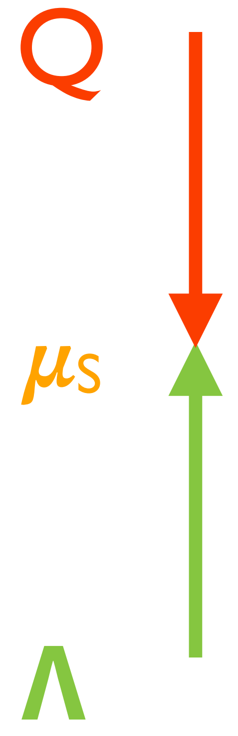
How to do this at subleading N and higher order shower evolution?

Hard factor reproduces CVolver algorithm and predicts key features of second order evolution.

Subtract iterated contribution in ordered phase space.

$$\mathbf{R}_n^{(2,0)} \circ \mathbf{R}_n^{(2,0)\dagger} = \left(\hat{\mathbf{D}}_n^{(0,2)} \circ \hat{\mathbf{D}}_n^{(0,2)\dagger}\hat{\Theta}_{n,2} - \hat{\mathbf{D}}_n^{(0,1)}\hat{\mathbf{D}}_{n-1}^{(0,1)} \circ \hat{\mathbf{D}}_{n-1}^{(0,1)\dagger}\hat{\mathbf{D}}_n^{(0,1)\dagger}\hat{\Theta}_{n-1,1}\hat{\Theta}_{n,1}\right) \times\theta(E_{n-1}-\mu_S)\delta(E_n-\mu_s) + \hat{\mathbf{D}}_n^{(0,2)} \circ \hat{\mathbf{D}}_n^{(0,2)\dagger}\hat{\Theta}_{n,2}\theta(E_n-\mu_S)\delta(E_{n-1}-\mu_S)$$





Use full double gluon matrix element outside.

Similar consequences for virtual corrections.

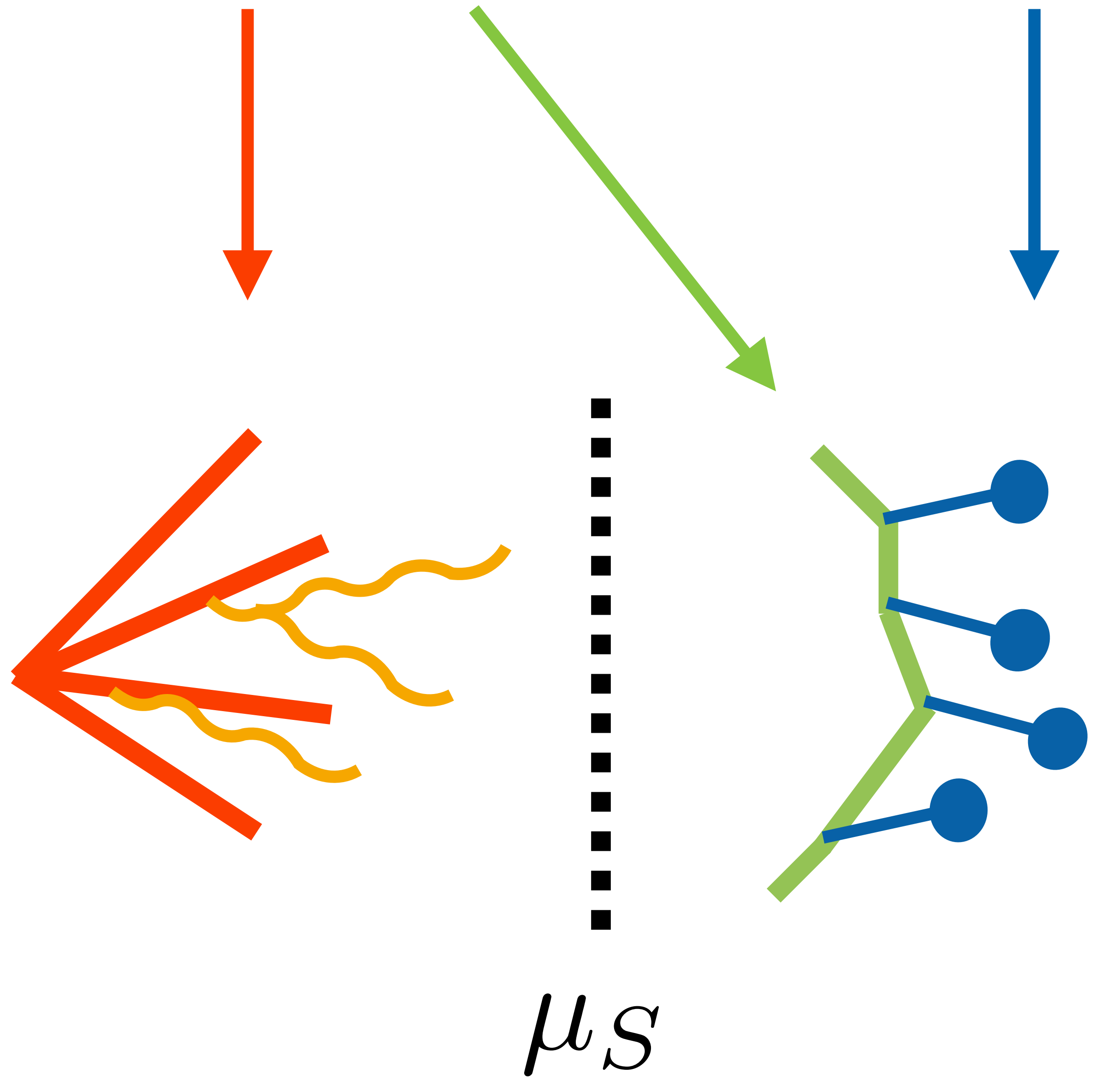
Constructing evolution algorithms

[Plätzer – '22]

How do we consistently hadronize in light of (improved) shower algorithms?

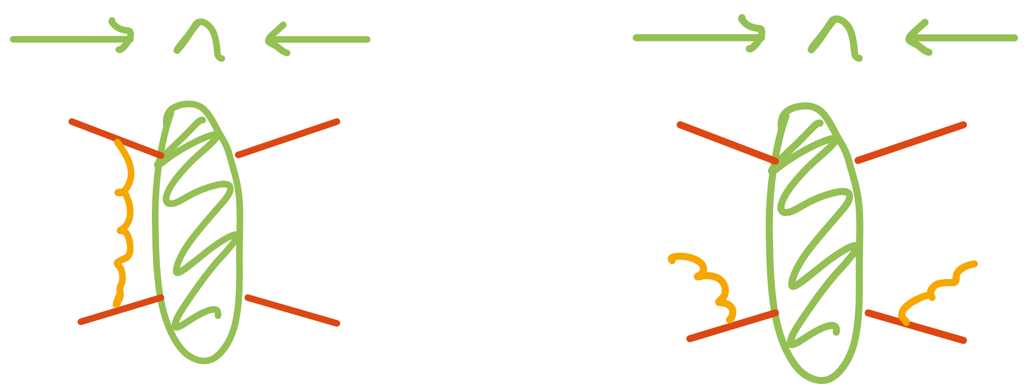
How to do this at subleading N and higher order shower evolution?

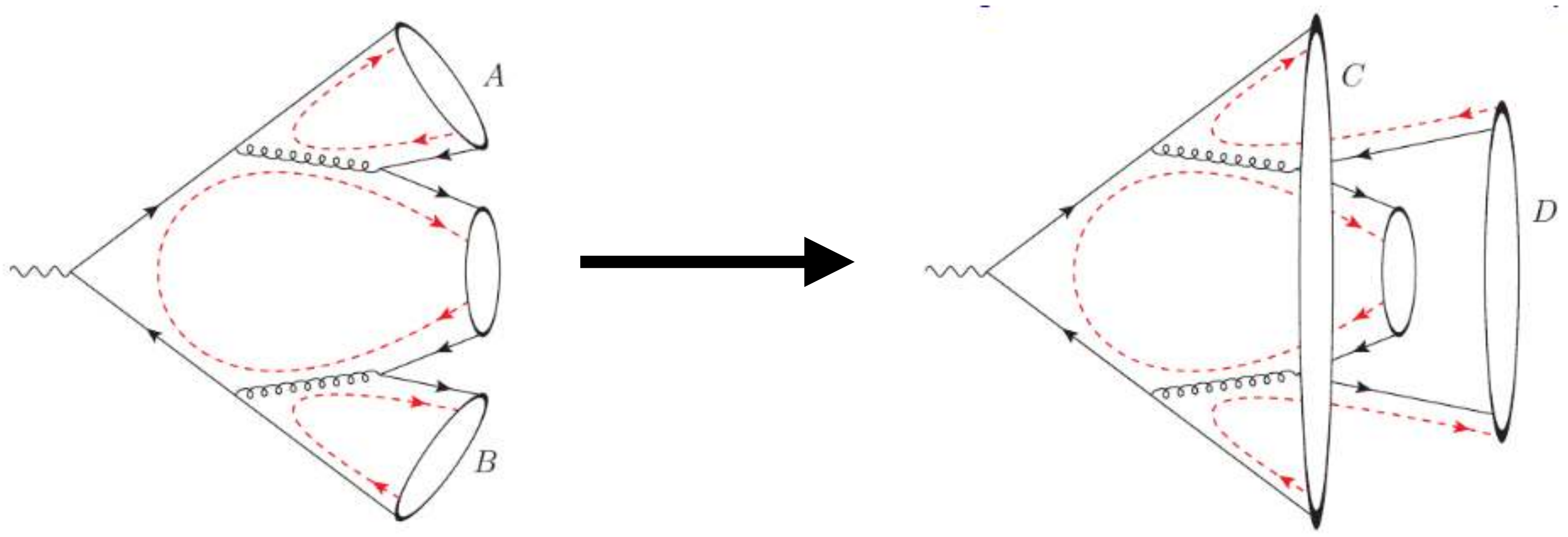
$$\sigma = \sum_{n,m} \int \int \mathrm{Tr}_n \left[\mathbf{M}_n \mathbf{U}_{nm}\right] \mathrm{d}\phi_m u(\phi_m)$$



$$0 = \frac{\mathrm{d}}{\mathrm{d}\mu_S}$$

Construct perturbative end of hadronization.





e.g. colour reconnection *implied* just as observed in [Gieseke, Kirchgaesser, Plätzer – '18 …]

Algorithms & Efficiency

UNI GRAZ

Sudakov-type densities central to Showers

[Olsson, Plätzer, Sjödahl — '20]
[Plätzer, Sjödahl — '12]

no emission

$$\frac{\mathrm{d}S_P(q|Q,z,x)}{\mathrm{d}q\ \mathrm{d}z} = \Delta_P(Q_0|Q,x)\delta(q-Q_0)\delta(z-z_0)$$
$$+\Delta_P(q|Q,x)P(q,z,x)\theta(Q-q)\theta(q-Q_0)$$

emission

$Q' \leftarrow Q, w \leftarrow w_0$
loop
  A trial splitting scale and variables, q, z, are generated according to $S_R(q|Q',z,x)$, for example using Alg. 1.
  if $q = Q_0$ **then**
    There is no emission and the cut-off scale Q_0 is returned while the event weight is kept at w.
  else
    if rnd$\leq \epsilon$ **then**
      The trial splitting variables q, z are accepted, and

$$w \leftarrow w \times \frac{1}{\epsilon} \times \frac{P(Q',z,x)}{R(Q',z,x)}. \tag{3}$$

    else
      The emission is rejected, and the algorithm continues with

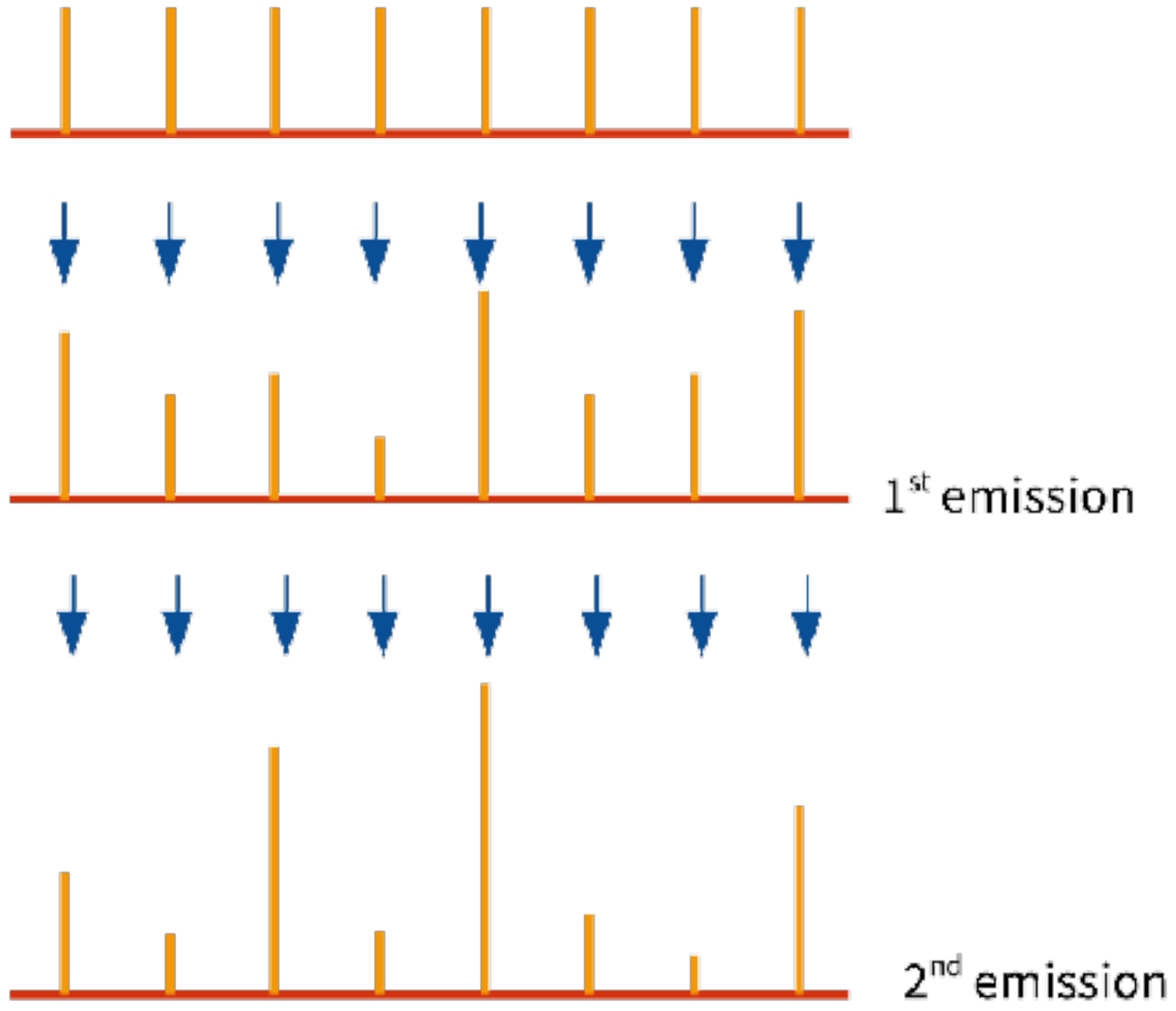
$$w \leftarrow w \times \frac{1}{1-\epsilon} \times \left(1 - \frac{P(q,z,x)}{R(q,z,x)}\right)$$
$$Q' \leftarrow q. \tag{4}$$

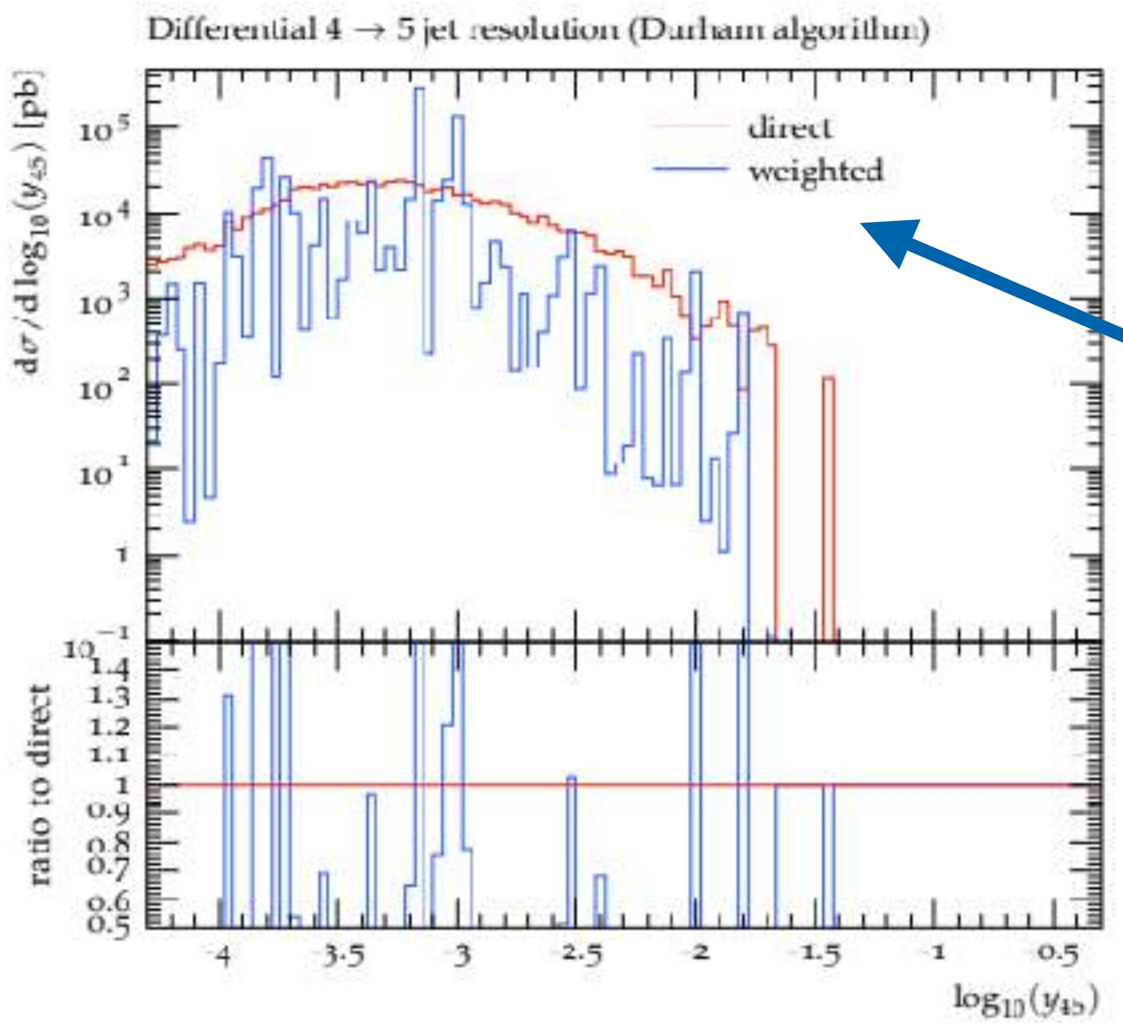
    end if
  end if
end loop

Negative P or unknown overestimate requires weighted veto algorithm, with in principle arbitrary proposal kernel and veto probability.

Weighted Veto Algorithms & Resampling

[Olsson, Plätzer, Sjödahl —'20]

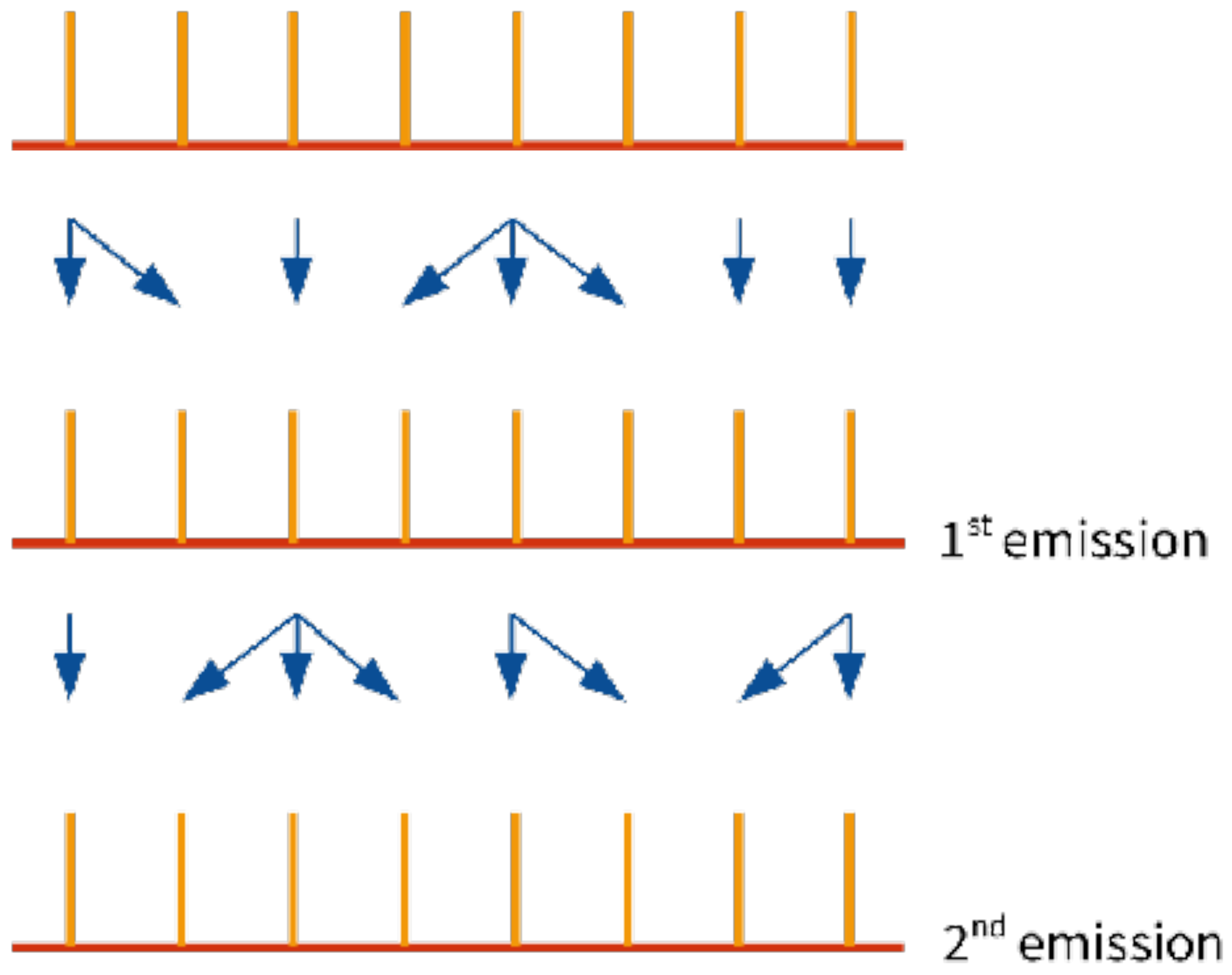


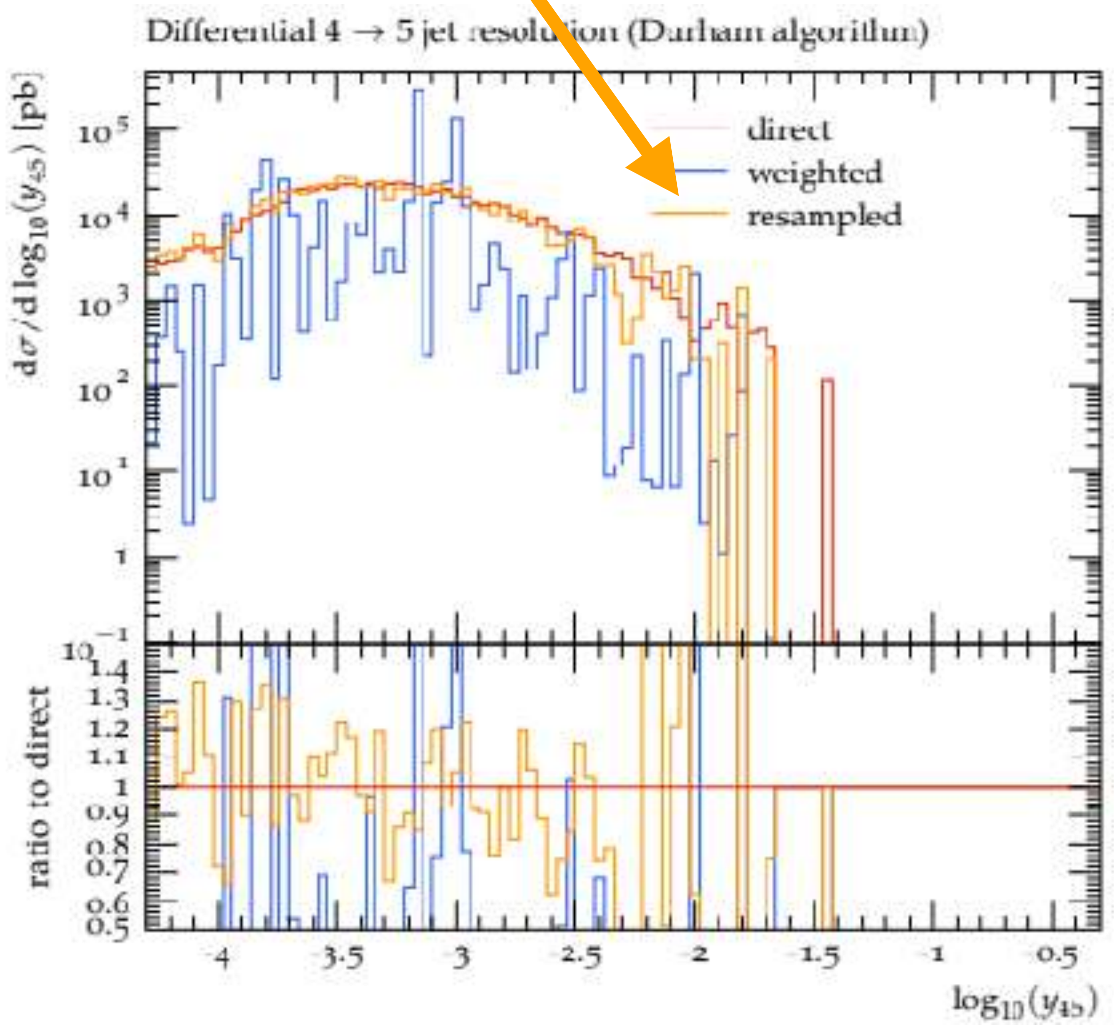


Weighted branching algorithms exhibit prohibitive weight distributions & convergence issues.

Result without resampling

Result with resampling





Resampling algorithms can compress weight distributions at intermediate steps.

Different resampling method developed as event generator after-burner.

[Andersen, Gütschow, Maier, Prestel — '20]

PSR 23 & 24

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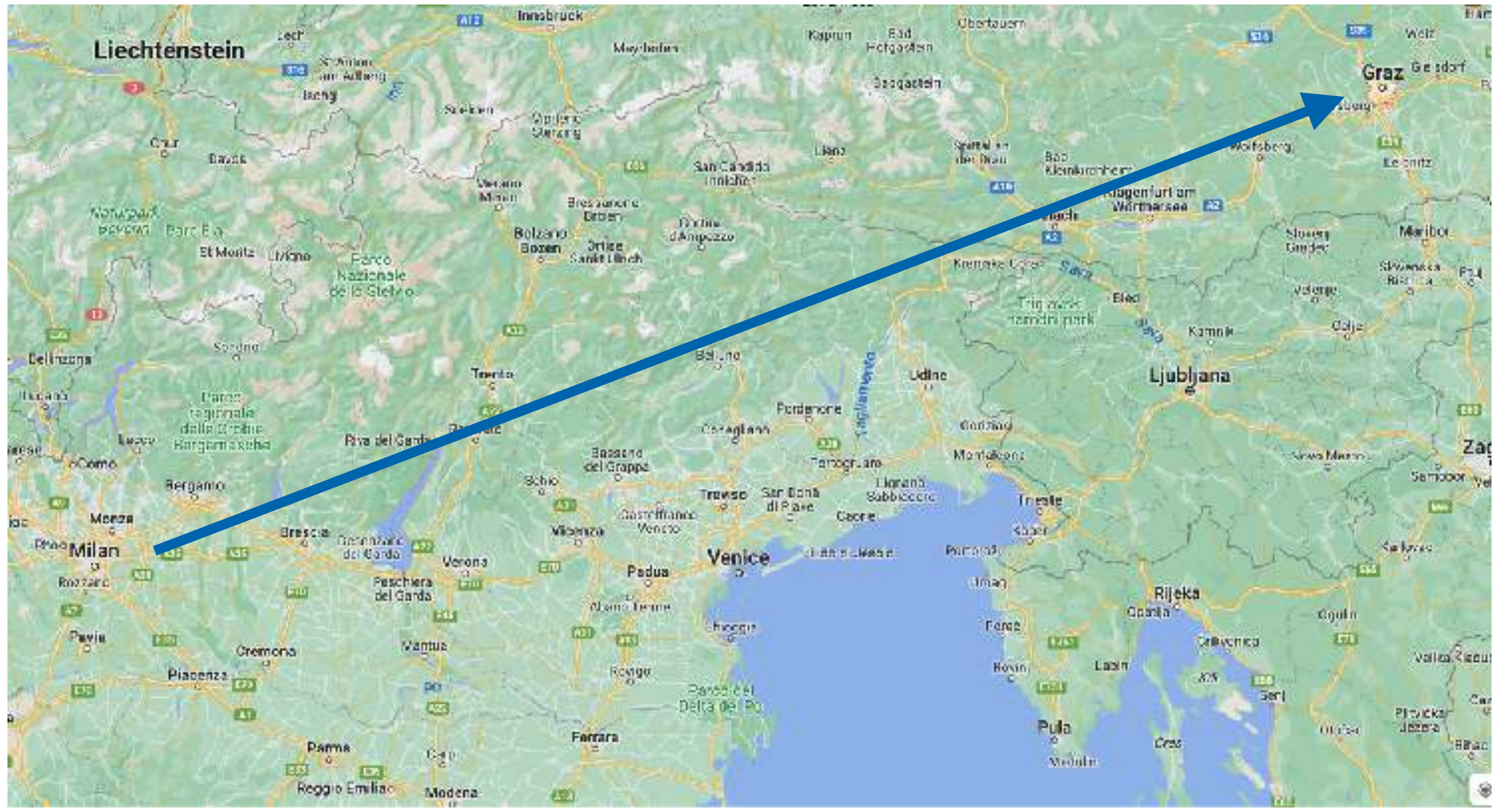


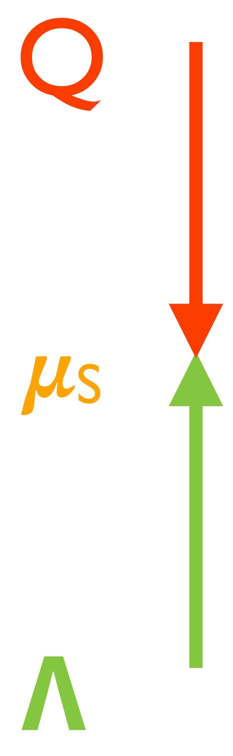
Thank you.

Redefinitions of "bare" operators

$$\sigma[\mathbf{U}] = \sum_n \int \alpha_0^n \, \mathrm{Tr}\left[\mathbf{M}_n(Q;p_1,...,p_n)\mathbf{U}_n(Q;p_1,...,p_n)\right] \mathrm{d}\phi(Q) \prod_{i=1}^{n} (4\pi\mu^2)^\epsilon [\mathrm{d}p_i]\tilde{\delta}(p_i)$$

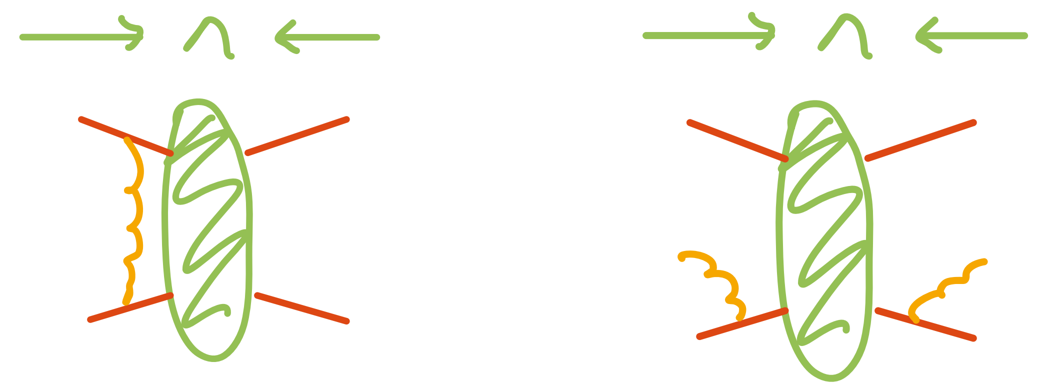
Remove UV divergencies

$$\alpha_0 \left(4\pi\mu^2\right)^\epsilon = \alpha_S(\mu_R)\mu_R^{2\epsilon} Z_g$$



Subtract IR divergencies in unresolved regions

$$\mathbf{U}_n = \mathbf{X}_n^\dagger \mathbf{S}_n \mathbf{X}_n - \sum_{s=1}^{\infty} \alpha_S^s \int \mathbf{F}_{n+s}^{(s)\dagger} \mathbf{S}_{n+s} \mathbf{F}_{n+s}^{(s)} \prod_{i=n+1}^{n+s} \mu_R^{2\epsilon} [\mathrm{d}p_i]\tilde{\delta}(p_i)$$



Re-arrange to resum IR enhancements

$$\mathbf{M}_n Z_g^n = \mathbf{Z}_n \mathbf{A}_n \mathbf{Z}_n^\dagger + \sum_{s=1}^{n} \alpha_S^s \mathbf{E}_n^{(s)} \mathbf{A}_{n-s} \mathbf{E}_n^{(s)\dagger}$$

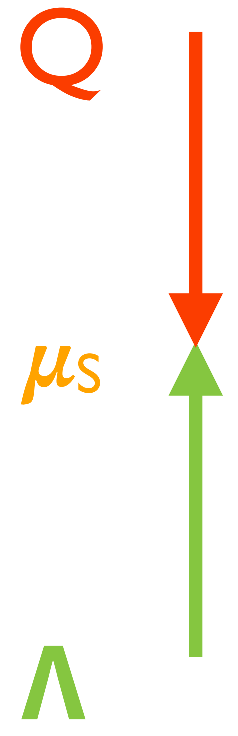
[Plätzer – '22]

Redefinitions of “bare” operators

$$\sigma[\mathbf{U}] = \sum_n \int \alpha_0^n \, \mathrm{Tr}\left[\mathbf{M}_n(Q;p_1,...,p_n)\mathbf{U}_n(Q;p_1,...,p_n)\right] \mathrm{d}\phi(Q) \prod_{i=1}^n (4\pi\mu^2)^\epsilon [\mathrm{d}p_i]\tilde{\delta}(p_i)$$

Redefinitions of hard and soft factor **inverse** to each other:

$$\mathbf{Z}_n = \mathbf{X}_n^{-1} \qquad \mathbf{X}_n \mathbf{E}_n^{(s)} \circ \mathbf{E}_n^{(s)\dagger} \mathbf{X}_n^\dagger - \mathbf{F}_n^{(s)} \mathbf{Z}_{n-s} \circ \mathbf{Z}_{n-s}^\dagger \mathbf{F}_n^{(s)\dagger} - \sum_{t=1}^{s-1} \mathbf{F}_n^{(t)} \mathbf{E}_{n-t}^{(s-t)} \circ \mathbf{E}_{n-t}^{(s-t)\dagger} \mathbf{F}_n^{(t)\dagger} = 0$$

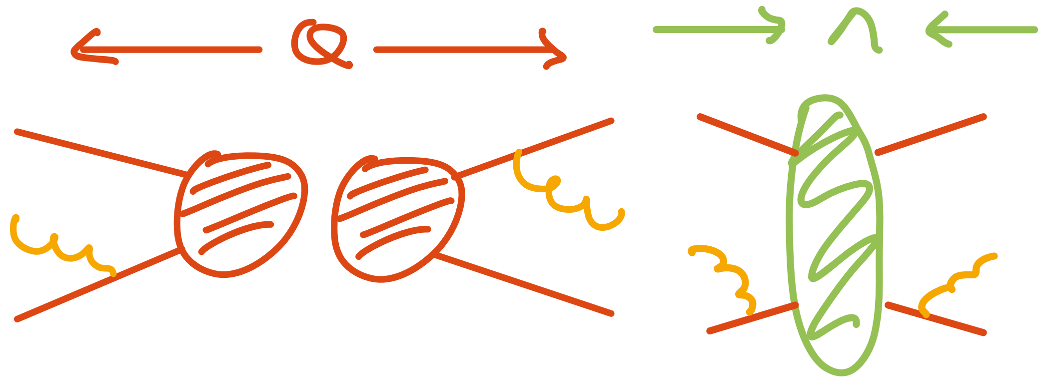


dressing of hard process ~ parton shower

soft evolution ~ hadronization model

$$\sum_n \int \alpha_S^n \, \mathrm{Tr}\left[(\mathbf{A}_n + \mathbf{\Delta}_n)\mathbf{S}_n\right] \mathrm{d}\phi(Q) \prod_{i=1}^n \mu_R^{2\epsilon} [\mathrm{d}p_i]\tilde{\delta}(p_i)$$

α_s corrections to tower of logarithms in A —
truncation error of relation of Z factors



[Plätzer – ’22]