Optimizing Entanglement and Bell Inequality Violation in $t\bar{t}$ Events

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Introduction: Quantum state and Bell inequalities

The spin density matrix of a spin-1/2 particle is a 2×2 trace-1 hermitian matrix, therefore can be always expanded as $\rho = \frac{1}{2}(I_2 + B_i \sigma_i)$. Likewise, the density matrix in $\mathscr{H}_A \otimes \mathscr{H}_B$ can be parametrized as

$$\rho_{t\bar{t}} = \frac{1}{4}\left(I_2 \otimes I_2 + B_i^+ \sigma_i \otimes I_2 + B_i^- I_2 \otimes \sigma_i + C_{ij} \sigma_i \otimes \sigma_j\right)$$

$B_i^{\pm}$ **parametrize the polarization of each particle;** $\langle \sigma_i^t \rangle = B_i^+, \qquad \langle \sigma_i^{\bar{t}} \rangle = B_i^-$

C_{ij} **parametrize their spin correlation** $\langle \sigma_i^t \sigma_j^{\bar{t}} \rangle = C_{ij}$

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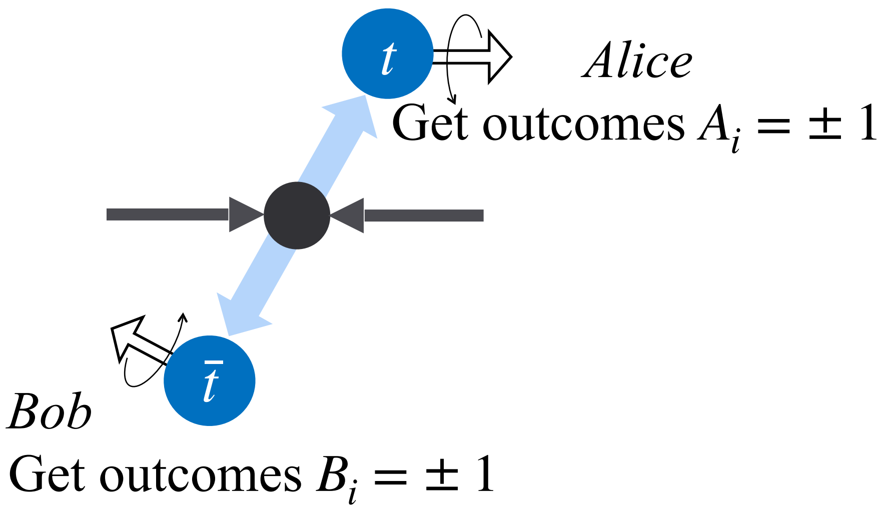
C_{ij} **parametrize their spin correlation** $\boxed{\langle \sigma_i^t \sigma_j^{\bar{t}} \rangle = C_{ij}}$

Bell inequality: For a local theory, the results of two-outcome measurements $\hat{A}_{1,2}$ and $\hat{B}_{1,2}$ satisfy

$$\left|\langle \hat{A}_1\hat{B}_1\rangle + \langle \hat{A}_1\hat{B}_2\rangle + \langle \hat{A}_2\hat{B}_1\rangle - \langle \hat{A}_2\hat{B}_2\rangle\right| \leq 2$$



When choosing $\hat{A}_i$ and $\hat{B}_i$ as the angular momentum measurements along direction $\vec{a}_i$ and $\vec{b}_i$, $\hat{A}_i = \hat{\sigma} \cdot \vec{a}_i$, the Bell inequality is rewritten as:

$$\left|\vec{a}_1 \cdot C \cdot (\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot C \cdot (\vec{b}_1 + \vec{b}_2)\right| \leq 2$$

$$\mathscr{B}(\rho) = \max_{\vec{a}_i, \vec{b}_i}\left|\vec{a}_1 \cdot C \cdot (\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot C \cdot (\vec{b}_1 + \vec{b}_2)\right| = 2\sqrt{\mu_1^2 + \mu_2^2}$$

μ_1^2, μ_2^2 **are the largest two eigenvalue of $C^T C$**
When C_{ij} is symmetric, μ_i is its eigenvalue

Test Bell inequality at collider

The ℓ^+ tends to be along the spin direction of t

A spin-up top quark $t_\uparrow \to \ell^+ \nu b$: $\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta_\ell} \approx \frac{1}{2}(1+\cos\theta_\ell)$

Or generally, $\rho^t = \frac{1}{2}(I_2 + B_i\sigma_i)$, $\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\theta_\ell} \approx \frac{1}{2}(1+\vec{B}\cdot\vec{\ell}) \implies B_i = 3\langle \ell_i \rangle_{\text{av}}$

ℓ_i: cosine of the angle between $\vec{\ell}$ and axis $\hat{e}_i$

For $t\bar{t}$ system, $\rho_{t\bar{t}} = \frac{1}{4}\left(I_2 \otimes I_2 + B_i^+ \sigma_i \otimes I_2 + B_i^- I_2 \otimes \sigma_i + C_{ij}\,\sigma_i\,\sigma_j\right)$. The density matrix constructed from $t \to \ell^+ \nu b$, $\bar{t} \to \ell^- \bar{\nu}\bar{b}$ decay channel is

$$B_i^+ = 3\langle \ell_i^+ \rangle, \quad B_i^- = -3\langle \ell_i^- \rangle, \quad C_{ij} = -9\langle \ell_i^+ \ell_j^- \rangle$$

Measure the momentum $\ell_i^\pm$ → $C_{ij} = -9\langle \ell_i^+ \ell_j^- \rangle_{\text{av}}$ → Obtain a density matrix, and test entanglement…

The ℓ^+ and ℓ^- inside the average are decay from different $t\bar{t}$ system.
We don’t have enough number of events to study $t\bar{t}$ produced at a fixed scattering angle θ and ϕ.

Angular average in $t\bar{t}$ phase space is needed, we obtain an angular averaged states.

The average $\langle \ell_i^+ \ell_j^- \rangle_{\text{av}}$ can be basis dependent.

Fixed basis and event-by-event basis

$$\boxed{\begin{gathered}\rho_{t\bar{t}} = \frac{1}{4}\left(I_2 \otimes I_2 + B_i^+ \sigma_i \otimes I_2 + B_i^- I_2 \otimes \sigma_i + C_{ij}\,\sigma_i\,\sigma_j\right) \\ C_{ij} = -9\langle \ell_i^+ \ell_j^- \rangle_{\text{av}}\end{gathered}}$$

Fixed beam basis:
the spin basis $|\uparrow\rangle$ and $|\downarrow\rangle$ are define as spin eigenstates along $\hat{z}$-direction

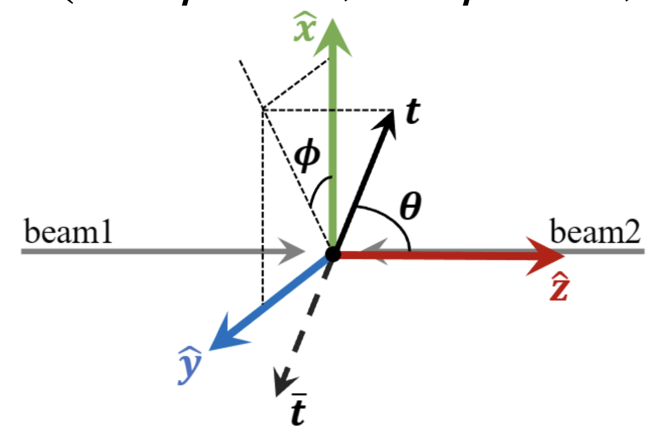
Helicity basis:
the spin basis $|\uparrow\rangle$ and $|\downarrow\rangle$ are define as spin eigenstates along the moving direction of top quark.

Example:
Near threshold, the $q_R\bar{q}_L / e_R^+ e_L^- \to t\bar{t}$ processes produces a pure state $|\uparrow_z\uparrow_z\rangle$

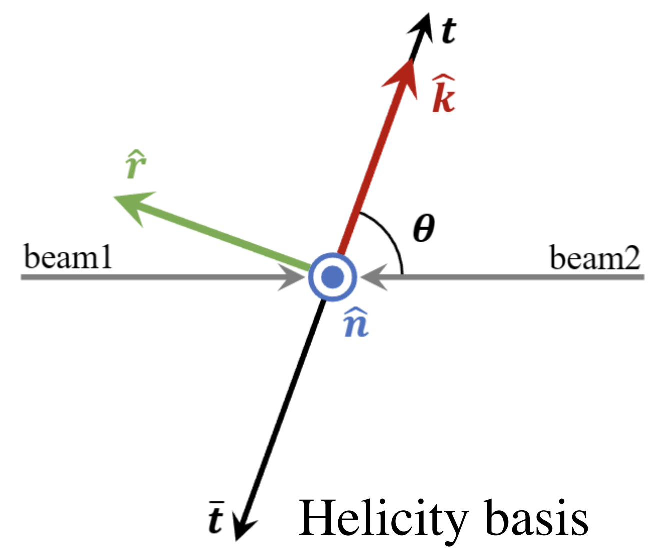
$$\bar{\rho}^{\text{fixed}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \bar{\rho}^{\text{helicity}} = \begin{pmatrix} \frac{8}{3} & -\frac{\pi}{2} & -\frac{\pi}{2} & \frac{4}{3} \\ -\frac{\pi}{2} & \frac{4}{3} & \frac{4}{3} & -\frac{\pi}{2} \\ -\frac{\pi}{2} & \frac{4}{3} & \frac{4}{3} & -\frac{\pi}{2} \\ \frac{4}{3} & -\frac{\pi}{2} & -\frac{\pi}{2} & \frac{8}{3} \end{pmatrix}$$

$$\mathrm{Tr}[(\bar{\rho}^{\text{fixed}})^2] = 1 \qquad \mathrm{Tr}[(\bar{\rho}^{\text{helicity}})^2] \approx 0.7 < 1$$

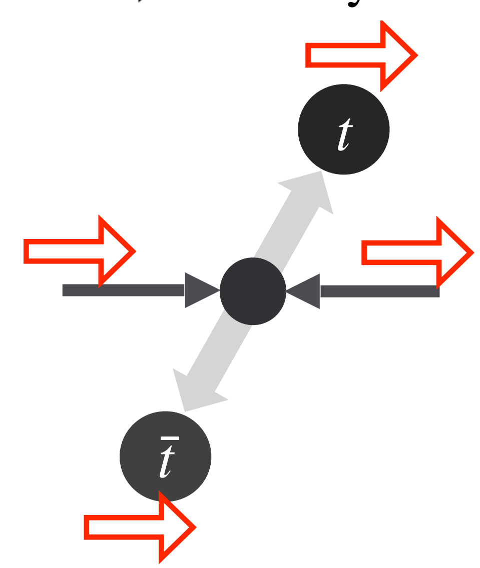
In the c.m. frame of $t\bar{t}$
$\mathbf{k} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$



Fixed beam basis



Helicity basis



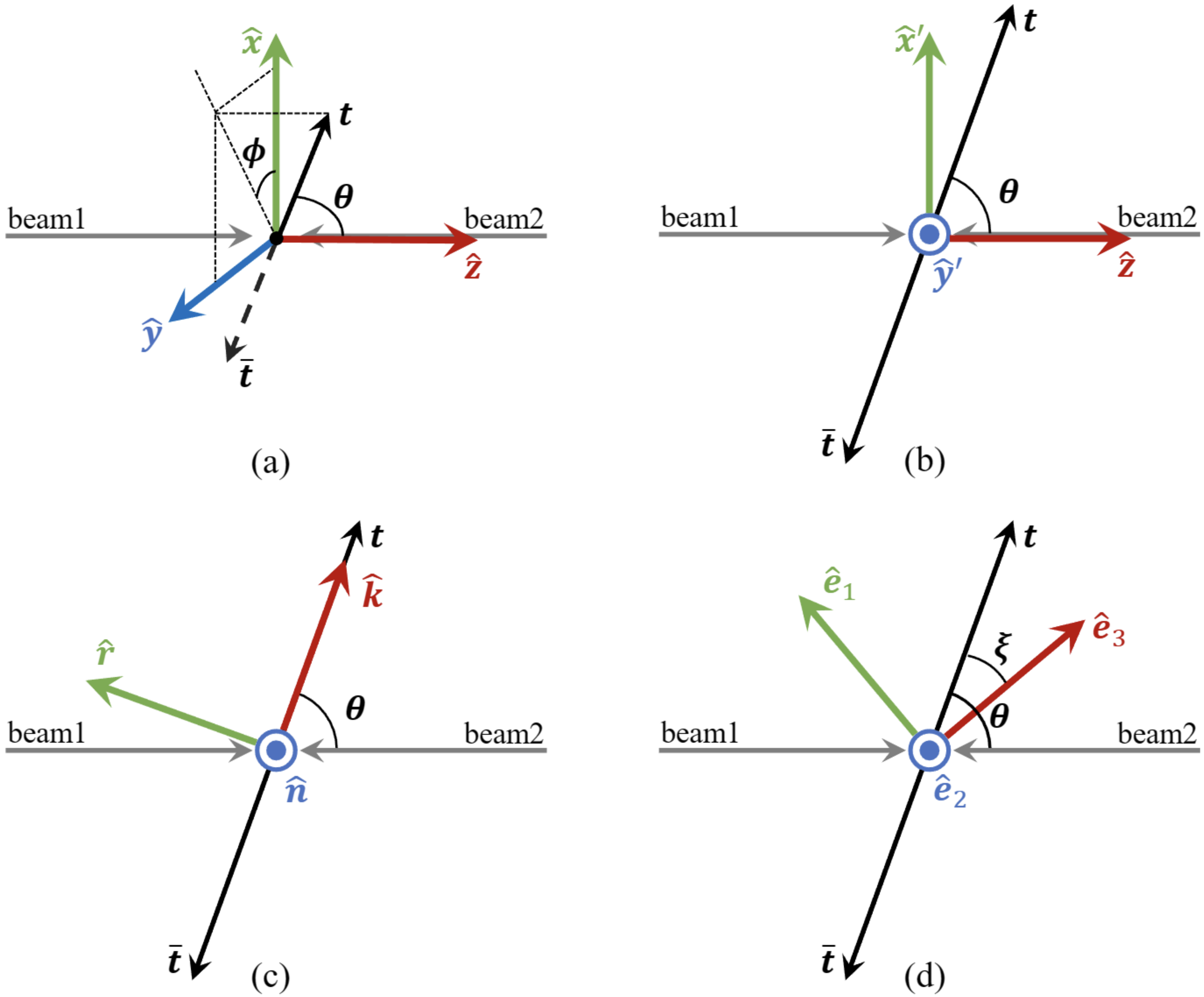
Choose a basis to maximize the entangle of angular-averaged state

(a) Fixed beam basis

(b) Rotated beam basis

(c) Helicity basis

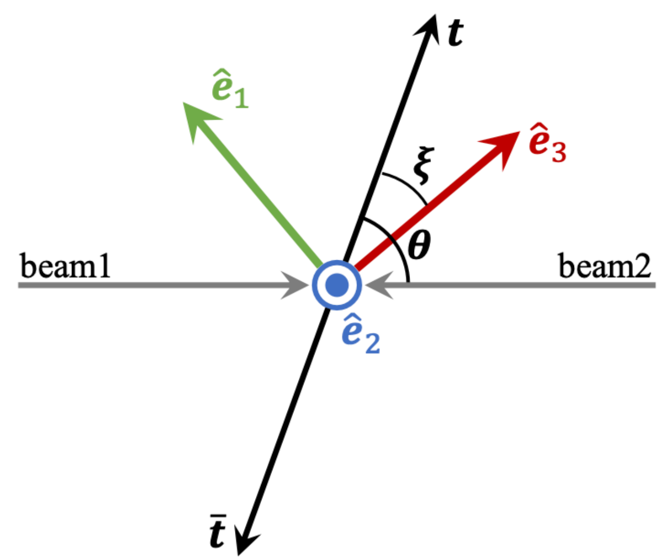
(d) Optimal basis??

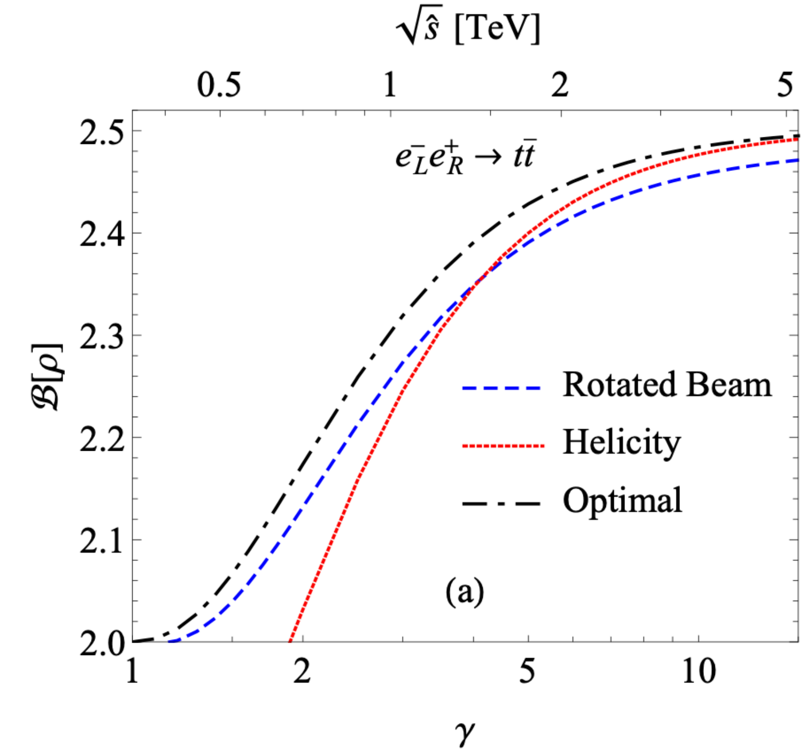


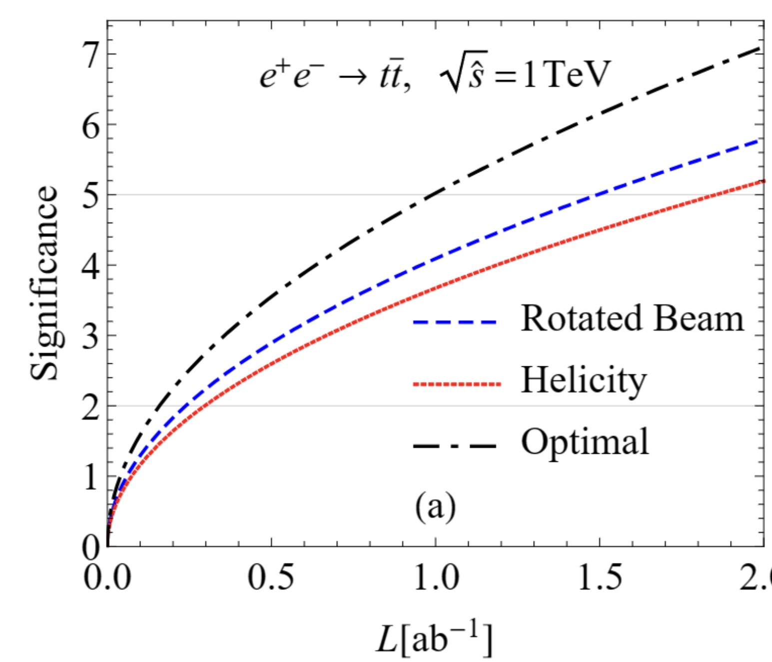
Parton-level processes: $q\bar{q} \to t\bar{t}$, $gg \to t\bar{t}$, $e^+e^- \to t\bar{t}$

initial state	$\sum\lvert\mathcal{M}\rvert^2$	ξ
$q\bar{q}$	$\kappa_q\left(2-\beta^2 s_\theta^2\right)$	$\tan\xi = \frac{1}{\gamma}\tan\theta$
$g_L g_R$	$\kappa_g \beta^2 s_\theta^2 (2-\beta^2 s_\theta^2)$	$\tan\xi = \frac{1}{\gamma}\tan\theta$
$g_L g_L / g_R g_R$	$\kappa_g(1-\beta^4)$	$\xi = 0$

$$C^{\mathrm{diag}}(\mathbf{k}) = \begin{pmatrix} \mu_1(\mathbf{k}) & 0 & 0 \\ 0 & \mu_2(\mathbf{k}) & 0 \\ 0 & 0 & \mu_3(\mathbf{k}) \end{pmatrix}$$







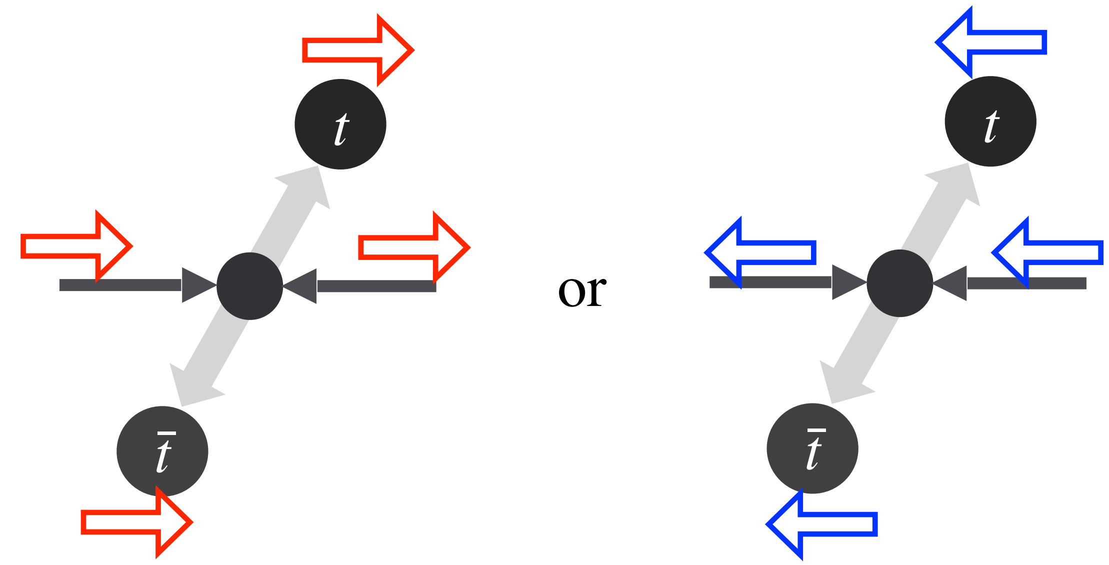
The third direction (with the largest eigenvalue of correlation matrix) is exactly the optimal basis of spin correlation found by Parke, Shadmi and Mahlon.

At sufficient high energy, the helicity basis approximates the optimal basis very well for all these processes.

LHC: $\rho^{t\bar{t}} = \omega^{q\bar{q}}\rho^{q\bar{q}\to t\bar{t}} + \omega^{gg}\rho^{gg\to t\bar{t}}$

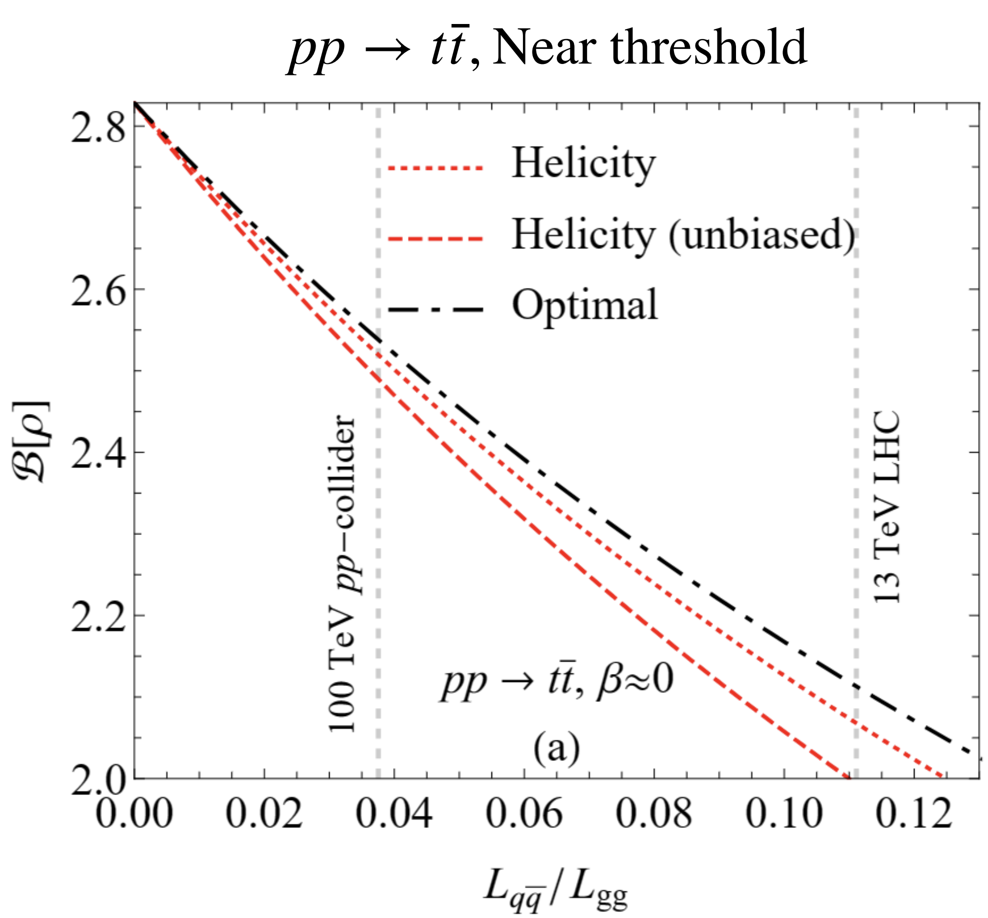
Boosted region: unlike-helicity gluon dominates, $gg \to t\bar{t}$ and $q\bar{q} \to t\bar{t}$ produce the same spin correlation.

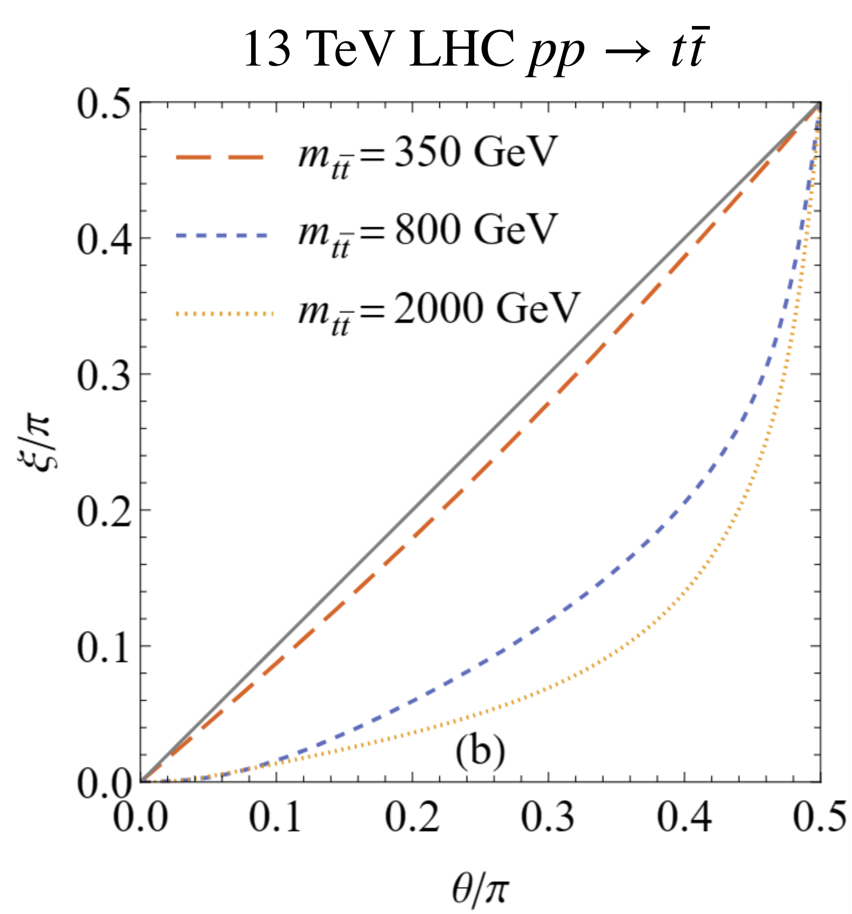
Near threshold: like-helicity gluon dominates, $gg \to t\bar{t}$ and $q\bar{q} \to t\bar{t}$ produce different spin correlation. The spin correlation from different initial state cancel with each other.



$q\bar{q} \to t\bar{t}$: positive spin correlation

$g_L g_L / g_R g_R \to t\bar{t}$: spin singlet $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$





Summary

By looking at the distribution of $t\bar{t}$ decay products, which quantum state we are studying?

- **Ideally, the best way to test entanglement of $t\bar{t}$ is to divide $t\bar{t}$ scattering angle (θ, ϕ) into infinitesimal bins and measure the entanglement of $t\bar{t}$ produced at different phase space point separately.**

- **Using angular-averaged state in event-by-event basis $\big(\hat{e}_1(\mathbf{k}), \hat{e}_2(\mathbf{k}), \hat{e}_3(\mathbf{k})\big)$:**
  - **Basis dependent**
  - **Optimal basis exists**

- **Current studies of $t\bar{t}$ at the LHC:**
  - **Entanglement (concurrence) is easier to test than Bell violation.**
  - **Helicity basis is mostly used.**
  - **At boosted region, the helicity basis is usually good enough (e.g. $m_{t\bar{t}} > 1\,\mathrm{TeV}$, in 20% agreement with optimal basis)**
  - **An improvement on testing the Bell inequality violation can be very useful.**

Backup

Backup: basis transformation

The spin density matrix of a spin-1/2 particle is a 2×2 trace-1 hermitian matrix, therefore can be always expanded as $\rho = \frac{1}{2}(I_2 + B_i\sigma_i)$. Likewise, the density matrix in $\mathscr{H}_A \otimes \mathscr{H}_B$ can be parametrized as

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It is convenient to discuss different basis choices using this parametrization.

$$\sigma_1 = |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|, \qquad \sigma_2 = -i\,|\uparrow\rangle\langle\downarrow| + i\,|\downarrow\rangle\langle\uparrow|, \qquad \sigma_3 = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$$

The basis transformation $U \otimes U$ on $\rho_{t\bar{t}}$ is now a simple rotation on C_{ij}……

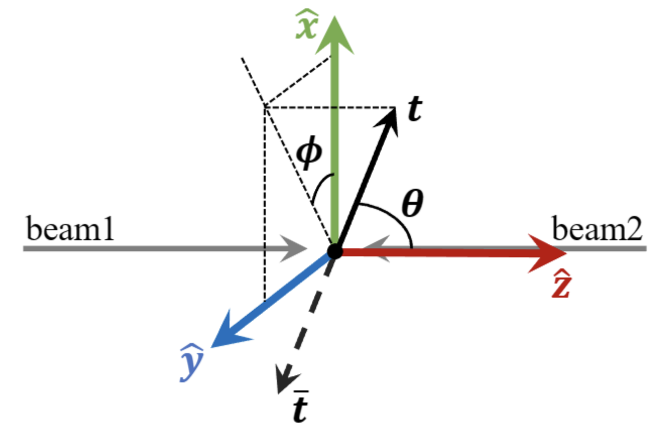
Treating $t\bar{t}$ produce at colliders as quantum states

The quantum state produced at collider is defined in $\mathscr{H}_k \otimes \mathscr{H}_{\rm spin} \otimes \mathscr{H}_{\rm color}$, we can expand it in terms of $|\mathbf{k}, \alpha\bar{\alpha}\rangle$

$$|t\bar{t}\rangle \propto \int \mathrm{d}\mathbf{k} \sum_{\alpha\bar{\alpha}} |\mathbf{k},\alpha\bar{\alpha}\rangle \langle \mathbf{k},\alpha\bar{\alpha}| T |I,\lambda\rangle = \int \mathrm{d}\mathbf{k} \sum_{\alpha\bar{\alpha}} \mathcal{M}^{\lambda}_{\alpha\bar{\alpha}}(\mathbf{k}) |\mathbf{k},\alpha\bar{\alpha}\rangle$$

In the c.m. frame of $t\bar{t}$

$\mathbf{k} = (\cos\phi \sin\theta, \sin\phi\sin\theta, \cos\theta)$



$|\uparrow\rangle$: *defined along $\hat{e}_3$-direction*

To obtain a physical density matrix in the spin space:

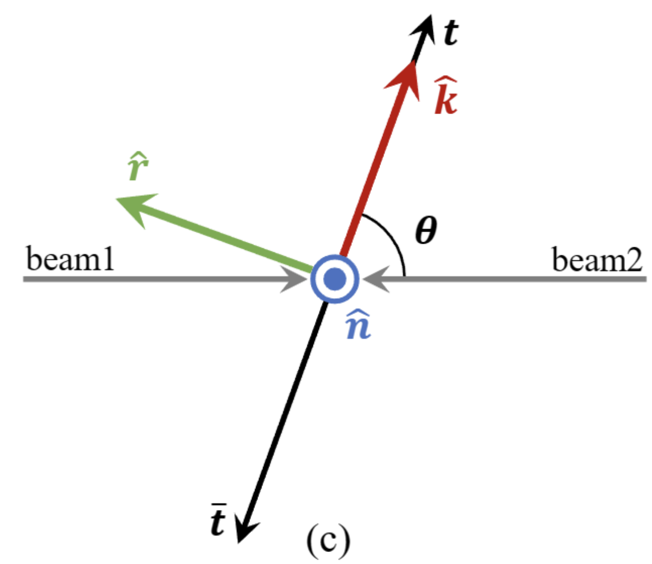
1) Project the states to a momentum eigenstate

$$\rho(\mathbf{k}) = \langle \mathbf{k}|\rho|\mathbf{k}\rangle = \rho(\mathbf{k})_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'} |\alpha\bar{\alpha}\rangle\langle\alpha'\bar{\alpha}'|$$

Need infinitesimal bins
"Quantum sub-states"

2) Trace in the momentum space.

$$\rho_{\Pi} = \mathrm{Tr}_{\mathbf{k}\in\Pi}\Big(\rho|\mathbf{k}\rangle\langle\mathbf{k}|\Big) = \frac{1}{\sigma_{\Pi}} \int_{\Omega\in\Pi} \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \rho(\mathbf{k})_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'} |\alpha\bar{\alpha}\rangle\langle\alpha'\bar{\alpha}'|$$



The basis $|\alpha\bar{\alpha}\rangle$ can be take out of the integral if it is defined in a fixed reference axis independent of $\mathbf{k}$

$$\rho^{\Pi}_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'} = \frac{1}{\sigma} \int_{\Omega\in\Pi} \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \rho(\mathbf{k})_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'}$$

However, it is a usual practice to average the density matrix in an event-by-event basis such as the helicity basis

$$\bar{\rho}^{\rm helicity}_{\alpha\bar{\alpha},\alpha'\bar{\alpha}'} = \frac{1}{\sigma} \int \mathrm{d}\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} U^{\dagger}_{\alpha\bar{\alpha},\beta\bar{\beta}} \rho(\mathbf{k})_{\beta\bar{\beta},\beta'\bar{\beta}'} U_{\beta'\bar{\beta}',\alpha'\bar{\alpha}'}$$

Fictitious state

It is still fine to use angular-averaged state (fictitious state)

Assume $C(k)_{ij}$ is the correlation matrix written in a event-by-event basis, then the angular averaged state is

$$\bar{C}^{\text{fic}}_{ij} = \frac{1}{\sigma_\Pi} \int_{\Omega \in \Pi} \mathrm{d}\Omega \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} C(\mathbf{k})_{ij}.$$

If the Bell inequality is not violated for any quantum sub-states, then for any directions $(\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2)$

$$\vec{a}_1 \cdot C(\mathbf{k})(\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot C(\mathbf{k})(\vec{b}_1 + \vec{b}_2) \in [-2, 2]$$

Then the Bell inequality is also conserved for the angular averaged state.

$$\begin{aligned} &\frac{1}{\sigma_\Pi} \int_{\Omega \in \Pi} \mathrm{d}\Omega \, \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \left(\vec{a}_1 \cdot C(\mathbf{k})(\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot C(\mathbf{k})(\vec{b}_1 + \vec{b}_2) \right) \\ &= \vec{a}_1 \cdot \bar{C} \cdot (\vec{b}_1 - \vec{b}_2) + \vec{a}_2 \cdot \bar{C} \cdot (\vec{b}_1 + \vec{b}_2) \\ &\in [-2, 2], \end{aligned}$$

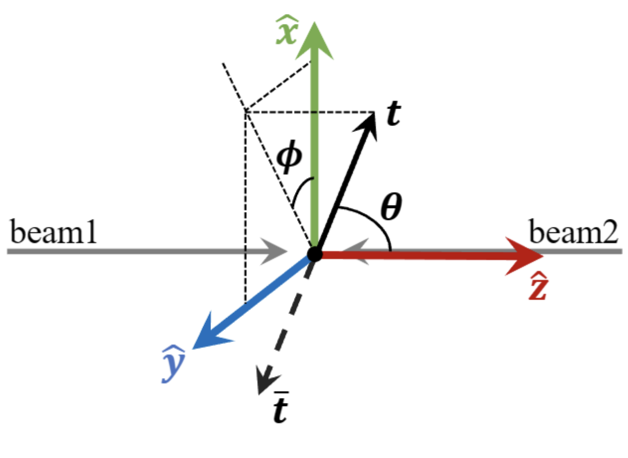
The Bell inequality violation of the angular-averaged state implies the Bell inequality violation in some quantum sub-states

From angular momentum conservation, the $t\bar{t}$ quantum state can be written as

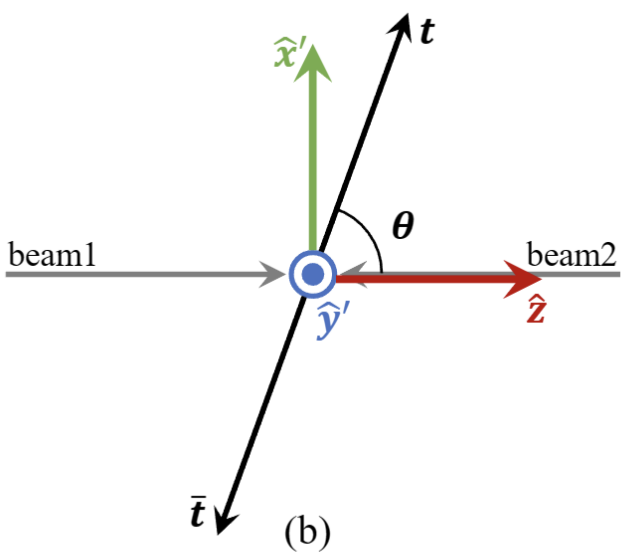
$$e^{iS_z^I\phi}\left(e^{-i\phi}\mathcal{M}_{\uparrow\uparrow}\left|\uparrow\uparrow\right\rangle + \mathcal{M}_{\uparrow\downarrow}\left|\uparrow\downarrow\right\rangle + \mathcal{M}_{\downarrow\uparrow}\left|\downarrow\uparrow\right\rangle + e^{i\phi}\mathcal{M}_{\downarrow\downarrow}\left|\downarrow\downarrow\right\rangle\right)$$

At leading order, the helicity amplitudes are real. After rotating the azimuthal angle ϕ to zero, the correlation matrix C_{ij} is diagonal in the second direction.

The normal direction of the scattering plane is a eigenvector of C_{ij} at LO



(a)



(b)

Maximizing spin correlation vs. Maximizing entanglement of angular averaged state

The coordinate $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ that diagonalizes the correlation matrix maximizes the Bell inequality violation of angular averaged states.

$$C^{\mathrm{diag}}(\mathbf{k}) = \begin{pmatrix} \mu_1(\mathbf{k}) & 0 & 0 \\ 0 & \mu_2(\mathbf{k}) & 0 \\ 0 & 0 & \mu_3(\mathbf{k}) \end{pmatrix}$$

Choosing a basis to maximize the entanglement of angular-averaged state is different from choosing a basis to maximize spin correlation

The spin correlation $\langle S_3^t \otimes S_3^{\bar{t}} \rangle$ is simply a function of reference axis $\hat{e}_3$, while the basis dependence of entanglement is introduced from angular averaging.
When using event-by-event basis $\hat{e}_i(\mathbf{k})$, the entanglement of angular averaged state is basis dependent is because the angular averaged state is a functional of $\hat{e}_i(\mathbf{k})$

$$\bar{C}_{ij} \propto \int \mathrm{d}\mathbf{k}\, |\mathcal{M}|^2 \; \hat{e}_i(\mathbf{k}) \cdot C(\mathbf{k}) \cdot \hat{e}_j(\mathbf{k})$$

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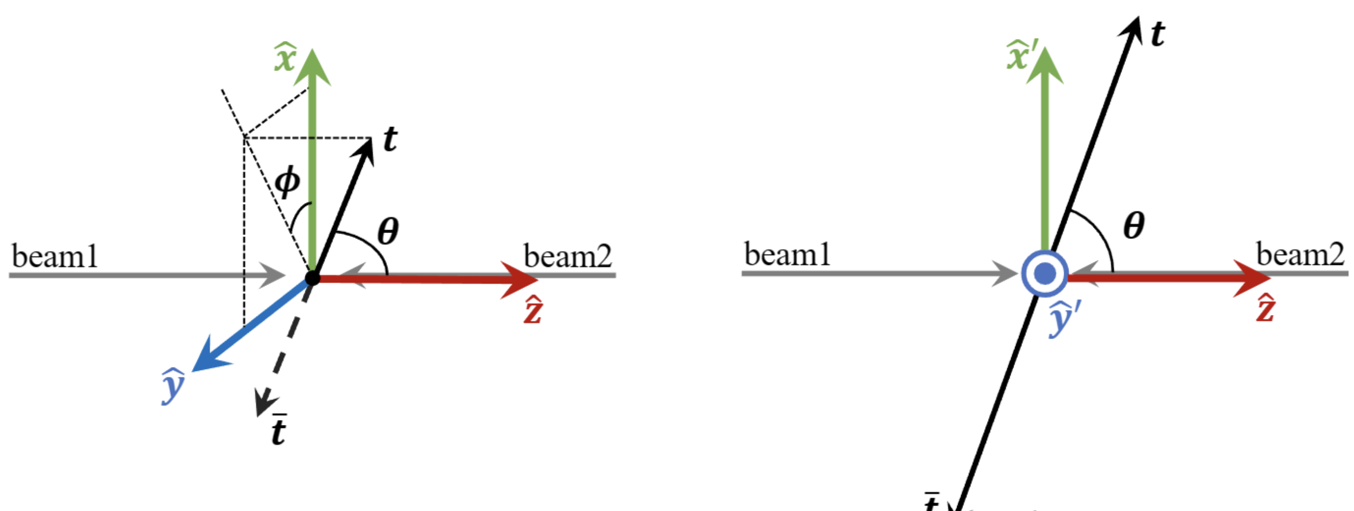


(a) (b)

Maximizing the spin correlation only need to find a proper z direction to define $|\uparrow\rangle$ and $|\downarrow\rangle$, and the phase of $|\uparrow\rangle$ and $|\downarrow\rangle$ (the direction of $\hat{x}$ and $\hat{y}$) is irrelevant.

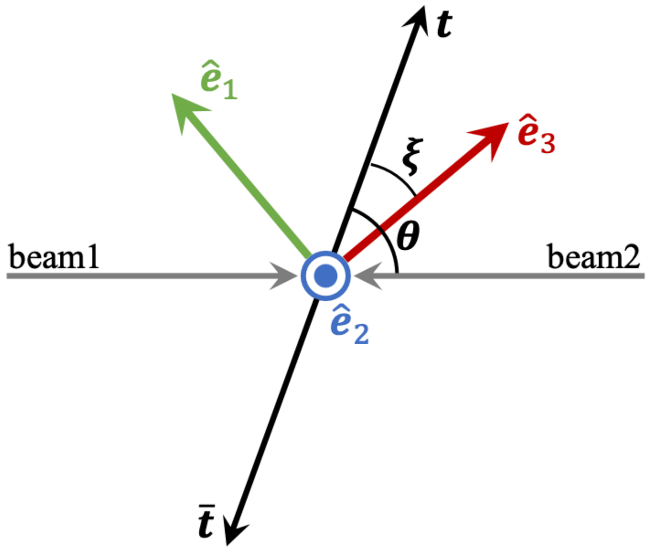
At high-p_T region, $q\bar{q}/gg$ scattering produce a triplet Bell state, $|\Psi_\phi\rangle = i\dfrac{e^{-i\phi}|\uparrow\uparrow\rangle + e^{i\phi}|\downarrow\downarrow\rangle}{\sqrt{2}}$

Rotated beam basis: $$\bar{\rho}^{\text{rotated}}_{triplet} = \frac{1}{2\pi}\int \mathrm{d}\phi\, |\Psi_0\rangle\langle\Psi_0| = |\Psi_0\rangle\langle\Psi_0|$$

Fixed beam basis $$\bar{\rho}^{\text{fixed}}_{triplet} = \frac{1}{2\pi}\int \mathrm{d}\phi\, |\Psi_\phi\rangle\langle\Psi_\phi| = \frac{|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|}{2}$$

Maximum spin correlated
Separable, entangle…….

The optimal basis for angular averaged state



The correlation matrix C_{ij} is symmetric for unpolarized final states,

$$\text{eigenvalues}\left(C_{ij}(\mathbf{k})\right) = (\ \mu_1(\mathbf{k}), \mu_2(\mathbf{k}), \mu_3(\mathbf{k})\ )$$

Diagonal basis:

$$C^{\text{diag}}(\mathbf{k}) = \begin{pmatrix} \mu_1(\mathbf{k}) & 0 & 0 \\ 0 & \mu_2(\mathbf{k}) & 0 \\ 0 & 0 & \mu_3(\mathbf{k}) \end{pmatrix}$$

The diagonal basis maximizes the signal of entanglement of angular averaged states.

- **The angular averaged state in the diagonal basis:**

$$\bar{C}^{\text{diag}} = \left\langle C^{\text{diag}}(\mathbf{k})\right\rangle_{\mathbf{k}\in\Pi} = \begin{pmatrix} \bar{\mu}_1 & 0 & 0 \\ 0 & \bar{\mu}_2 & 0 \\ 0 & 0 & \bar{\mu}_3 \end{pmatrix} \qquad \bar{\mu}_i = \langle \mu_i(\mathbf{k})\rangle_{\mathbf{k}\in\Pi}$$

- **The angular averaged state in any other basis: (denote the eigenvalues of $\bar{C}^{\text{basis}}$ as $\bar{c}_i$)**

$$\bar{C}^{\text{basis}} = \left\langle C^{\text{basis}}(\mathbf{k})\right\rangle_{\mathbf{k}\in\Pi} \qquad \bar{c}_1 + \bar{c}_2 + \bar{c}_3 = \bar{\mu}_1 + \bar{\mu}_2 + \bar{\mu}_3 = \mathrm{Tr}(\bar{C}),$$

  The diagonal terms of a matrix are always bounded by its eigenvalues

$$\bar{\mu}_1 \geq \bar{c}_i \geq \bar{\mu}_3.$$

To show that the diagonal basis maximize the violation of Bell inequalities, we need to prove that for any $i \neq j$,

$$\bar{c}_i^2 + \bar{c}_j^2 \ \leq\ \max_{k\neq\ell}\left[\bar{\mu}_k^2 + \bar{\mu}_\ell^2\right]$$

$$\bar c_1+\bar c_2+\bar c_3=\bar\mu_1+\bar\mu_2+\bar\mu_3=\mathrm{Tr}(\bar C),\qquad \bar\mu_1\ge\bar c_i\ge\bar\mu_3. \quad \xRightarrow{i\neq j} \quad \bar c_i^2+\bar c_j^2 \le \max_{k\neq\ell}\left[\bar\mu_k^2+\bar\mu_\ell^2\right]$$

Case (a) $\bar\mu_1\ge\bar\mu_2\ge\bar\mu_3\ge 0$

- **If:** $0\le\bar c_i\le\bar\mu_2 \quad \bar c_j\le\bar\mu_1 \implies \bar c_i^2+\bar c_j^2\le\bar\mu_1^2+\bar\mu_2^2$
- **Else:** $\bar\mu_2\le\bar c_i\le\bar\mu_1$

  $\bar c_i+\bar c_j\le\bar\mu_1+\bar\mu_2 \implies \bar c_i^2+\bar c_j^2\le\bar c_i^2+(\bar\mu_1+\bar\mu_2-\bar c_i)^2$

We need to prove:

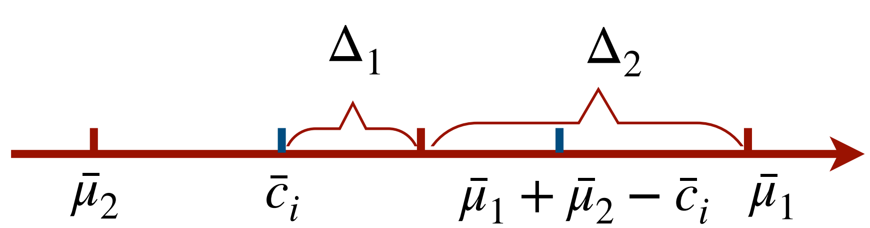
$$\bar c_i^2+\bar c_j^2\le\underbrace{\bar c_i^2+(\bar\mu_1+\bar\mu_2-\bar c_i)^2}_{f(\Delta_1)}\le\underbrace{\bar\mu_1^2+\bar\mu_2^2}_{f(\Delta_2)}$$

Define $f(\Delta)=\dfrac{(\bar\mu_1+\bar\mu_2)^2}{2}+2\Delta^2$

$f(\Delta_1)<f(\Delta_2)$ **when** $|\Delta_1|<|\Delta_2|$

$$\Delta_1=\frac{\bar\mu_1+\bar\mu_2}{2}-\bar c_i$$

$$\Delta_2=\frac{\bar\mu_1-\bar\mu_2}{2}$$



Case (b) $0\ge\bar\mu_1\ge\bar\mu_2\ge\bar\mu_3$

Case (c) $\bar\mu_1\ge 0\ge\bar\mu_3$

Spin correlation matrix of different processes

$$\rho^{t\bar{t}} = \omega^{q\bar{q}}\rho^{q\bar{q}\to t\bar{t}} + \omega^{gg}\rho^{gg\to t\bar{t}},$$
$$\omega^I = \frac{L_I|\mathcal{M}_{I\to t\bar{t}}|^2}{L_{q\bar{q}}|\mathcal{M}_{q\bar{q}\to t\bar{t}}|^2 + L_{gg}|\mathcal{M}_{gg\to t\bar{t}}|^2}, \quad I = q\bar{q}, gg,$$

TABLE I. Here, the correlation matrix is expressed in the helicity basis (r, n, k). The QCD color factor $\kappa_q = g_s^2\frac{N^2-1}{N^2}$, $\kappa_g = \frac{2g_s^2}{(\beta^2 c_\theta^2-1)^2}\frac{N^2(\beta^2 c_\theta^2+1)-2}{N(N^2-1)}$, and $N=3$ is the number of colors.

initial state	$\overline{\sum}\|\mathcal{M}\|^2$	Correlation matrix	ξ
$q\bar{q}$	$\kappa_q\left(2-\beta^2 s_\theta^2\right)$	$\begin{pmatrix} \frac{(2-\beta^2)s_\theta^2}{2-\beta^2 s_\theta^2} & 0 & -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2 s_\theta^2} \\ 0 & \frac{-\beta^2 s_\theta^2}{2-\beta^2 s_\theta^2} & 0 \\ -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2 s_\theta^2} & 0 & \frac{2c_\theta^2+\beta^2 s_\theta^2}{2-\beta^2 s_\theta^2} \end{pmatrix}$	$\tan\xi = \frac{1}{\gamma}\tan\theta$
$g_L g_R$	$\kappa_g\beta^2 s_\theta^2(2-\beta^2 s_\theta^2)$	$\begin{pmatrix} \frac{(2-\beta^2)s_\theta^2}{2-\beta^2 s_\theta^2} & 0 & -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2 s_\theta^2} \\ 0 & \frac{-\beta^2 s_\theta^2}{2-\beta^2 s_\theta^2} & 0 \\ -\frac{2c_\theta s_\theta\sqrt{1-\beta^2}}{2-\beta^2 s_\theta^2} & 0 & \frac{2c_\theta^2+\beta^2 s_\theta^2}{2-\beta^2 s_\theta^2} \end{pmatrix}$	$\tan\xi = \frac{1}{\gamma}\tan\theta$
$g_L g_L/g_R g_R$	$\kappa_g(1-\beta^4)$	$\begin{pmatrix} \frac{\beta^2-1}{\beta^2+1} & 0 & 0 \\ 0 & \frac{\beta^2-1}{\beta^2+1} & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\xi = 0$

$$\tan 2\xi = \frac{(L_{qq}\kappa_q + L_{gg}\kappa_g\beta^2 s_\theta^2)s_{2\theta}\sqrt{1-\beta^2}}{(L_{qq}\kappa_q + L_{gg}\kappa_g\beta^2 s_\theta^2)(c_{2\theta}+\beta^2 s_\theta^2) + L_{gg}\kappa_g\beta^2(1-\beta^2)}$$

Spin correlation matrix of different processes

$$\mathcal{M}_{t\bar{t}} = \frac{e^2 g_{\mu\nu}}{s} J^{\mu}_{\mathrm{in},\pm 1} \left(f_V J^{\nu}_{\mathrm{out},V} + f_A J^{\nu}_{\mathrm{out},A} \right)$$

$$\tan\xi = \frac{1}{\gamma} \frac{f_V s_\theta}{f_V c_\theta \pm f_A \beta}.$$

$$\mathcal{M}_{e^-_L e^+_R \to t\bar{t}} = \frac{e^2 g_{\mu\nu}}{s} J^{\mu}_{\mathrm{in},-1} \left(f^L_V J^{\nu}_{\mathrm{out},V} + f^L_A J^{\nu}_{\mathrm{out},A} \right),$$

$$\mathcal{M}_{e^-_R e^+_L \to t\bar{t}} = \frac{e^2 g_{\mu\nu}}{s} J^{\mu}_{\mathrm{in},1} \left(f^R_V J^{\nu}_{\mathrm{out},V} + f^R_A J^{\nu}_{\mathrm{out},A} \right),$$

$$f_{LV} = Q_e Q_t + \frac{(I^3_e - Q_e s^2_W)(I^3_t - 2Q_t s^2_W)}{2c^2_W s^2_W} \frac{\hat{s}}{\hat{s} - m^2_Z},$$

$$f_{LA} = -\frac{(I^3_e - Q_e s^2_W) I^3_t}{2c^2_W s^2_W} \frac{\hat{s}}{\hat{s} - m^2_Z},$$

$$f_{RV} = Q_e Q_t - \frac{Q_e (I^3_t - 2Q_t s^2_W)}{2c^2_W} \frac{\hat{s}}{\hat{s} - m^2_Z},$$

$$f_{RA} = \frac{Q_e I^3_t}{2c^2_W} \frac{\hat{s}}{\hat{s} - m^2_Z},$$