Signs and the signs of signs

Marc Riembau
CERN

HEFT 2023, Manchester
19th June 2023

Signs and the signs of signs

the positiveness or negativeness of a quantity

an object, quality, or event whose presence or occurrence indicates the probable presence or occurrence of something else.

Marc Riembau

CERN

HEFT 2023, Manchester

19th June 2023

"Not entirely, dear Adso," my master replied. "True, that kind of print expressed to me, if you like, the idea of 'horse', the verbum mentis, and would have expressed the same to me wherever I might have found it. But the print in that place and at that hour of the day told me that at least one of all possible horses had passed that way. So I found myself halfway between the perception of the concept 'horse' and the knowledge of an individual horse. And in any case, what I knew of the universal horse had been given me by those traces, which were singular. I could say I was caught at that moment between the singularity of the traces and my ignorance, which assumed the quite diaphanous form of a universal idea. If you see something from a distance, and you do not understand what it is, you will be content with defining it as a body of some dimension. When you come closer, you will then define it as an animal, even if you do not yet know whether it is a horse or an ass. And finally, when it is still closer, you will be able to say it is a horse even if you do not yet know whether it is Brunellus or Niger. And only when you are at the proper distance will you see that it is Brunellus (or, rather, that horse and not another, however you decide to call it). And that will be full knowledge, the learning of the singular. So an hour ago I could expect all horses, but not because of the vastness of my intellect, but because of the paucity of my deduction. And my intellect's hunger was sated only when I saw the single horse that the monks were leading by the halter. Only then did I truly know that my previous reasoning, had brought me close to the truth. And so the ideas, which I was using earlier to imagine a horse I had not yet seen, were pure signs, as the hoofprints in the snow were signs of the idea of 'horse'; ***and signs and the signs of signs are used only when we are lacking things."***

William of Baskerville, talking about Effective Field Theory,
in 'The Name of the Rose' by Umberto Eco

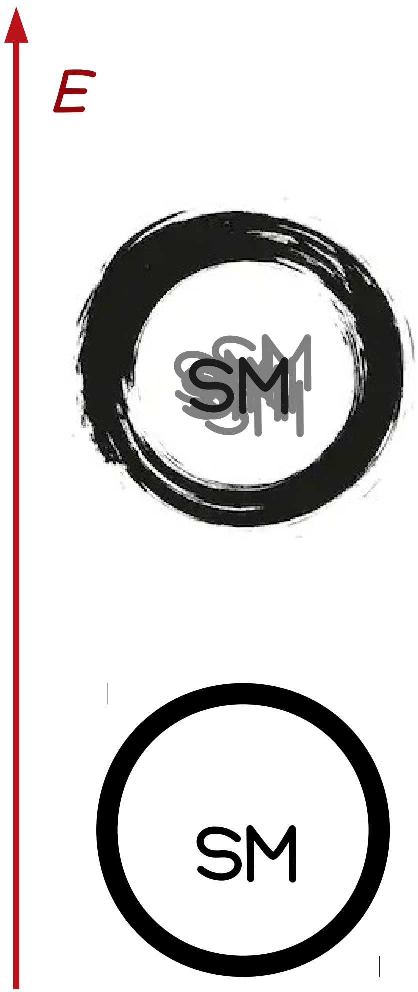
$\mathcal{L}\,?$

E

SM

$$\mathcal{L} = \mathcal{L}_{\text{SM}}$$

$$\mathcal{L}\,?$$



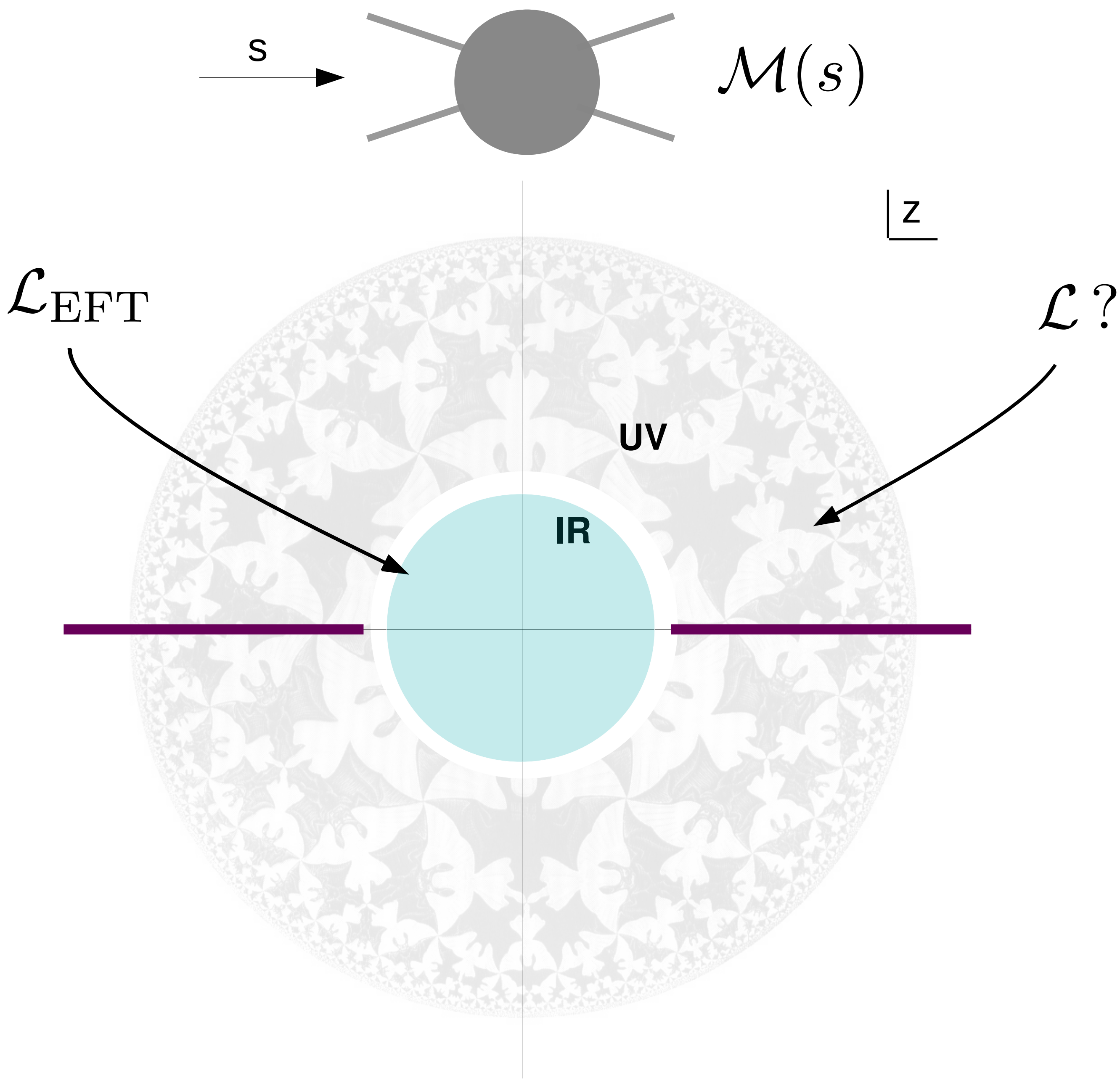
$$\mathcal{L} = \mathcal{L}_{\rm SM} + \sum_i \frac{c_i}{\Lambda}\mathcal{O}_i$$

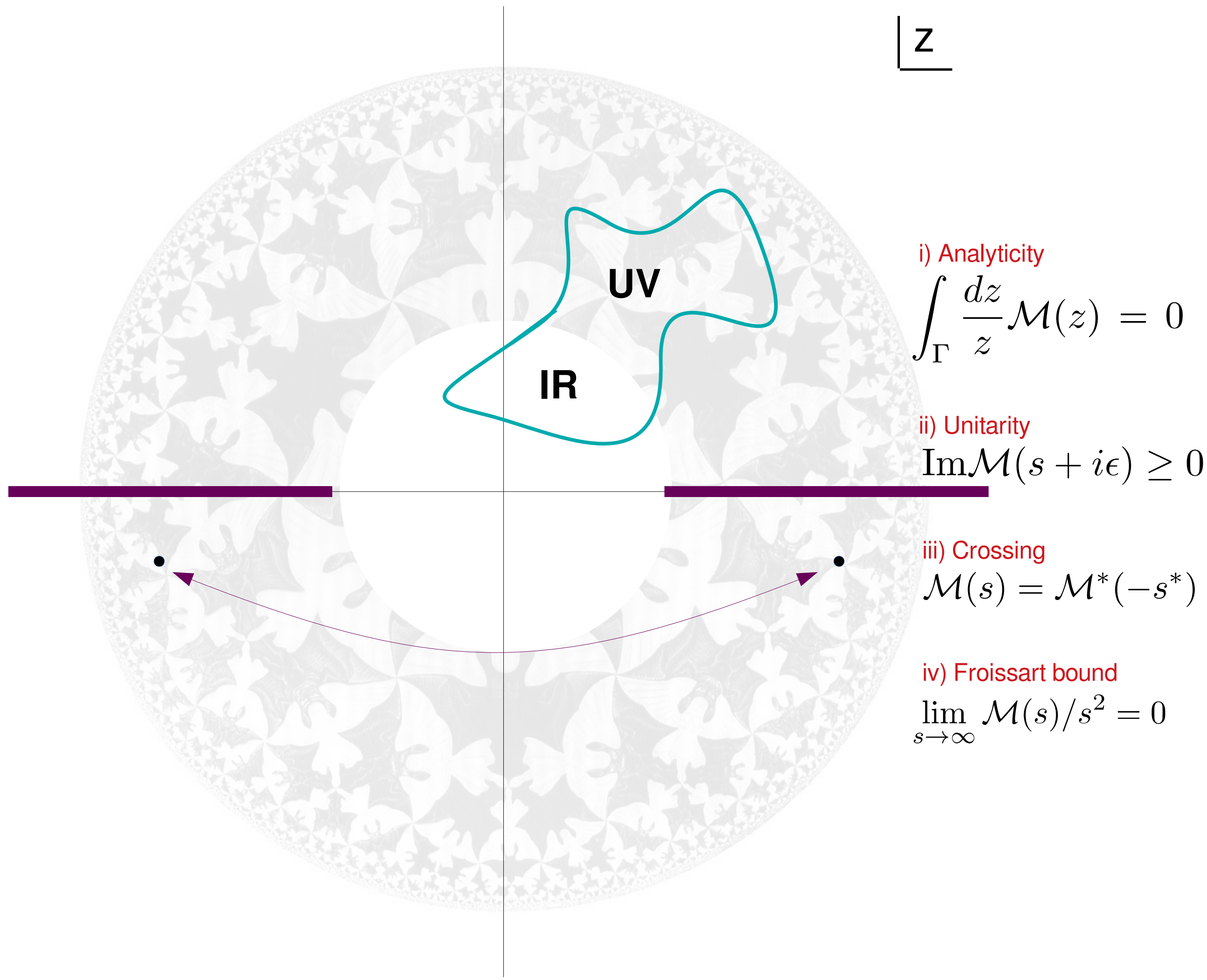
EFT operators encode information about the heavy dynamics, and tells us in which way the SM is deformed.

What is the space of allowed deformations?

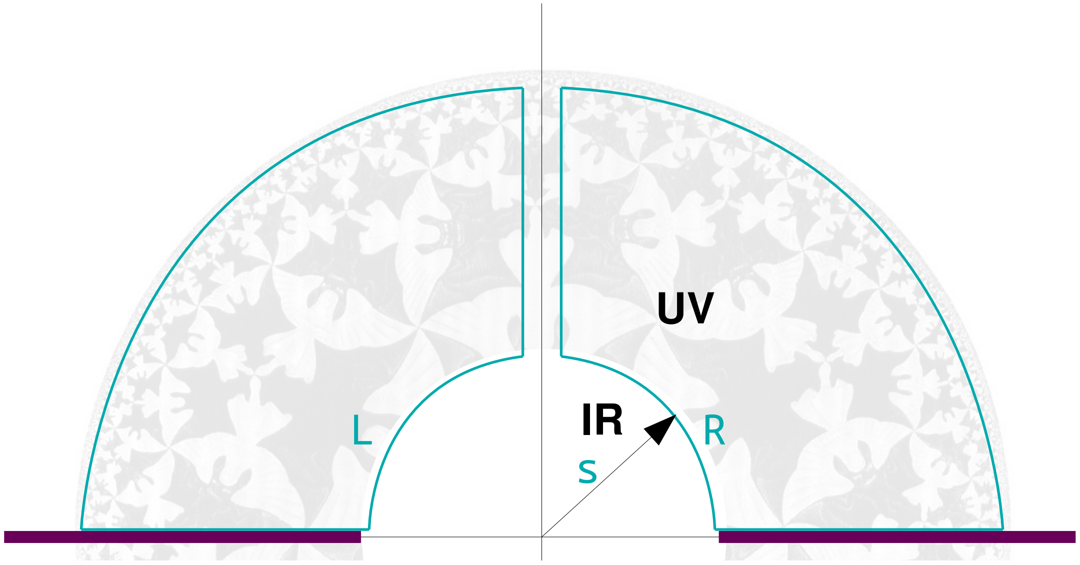
How to chart this space and map it to microscopic dynamics?

$$\mathcal{L} = \mathcal{L}_{\rm SM}$$


s
$\mathcal{M}(s)$
z
$\mathcal{L}_{\rm EFT}$
$\mathcal{L}\,?$
UV
IR



z
UV
IR
i) Analyticity
$\int_\Gamma \frac{dz}{z}\mathcal{M}(z) \; = \; 0$
ii) Unitarity
$\mathrm{Im}\mathcal{M}(s+i\epsilon) \geq 0$
iii) Crossing
$\mathcal{M}(s) = \mathcal{M}^*(-s^*)$
iv) Froissart bound
$\lim_{s\to\infty} \mathcal{M}(s)/s^2 = 0$

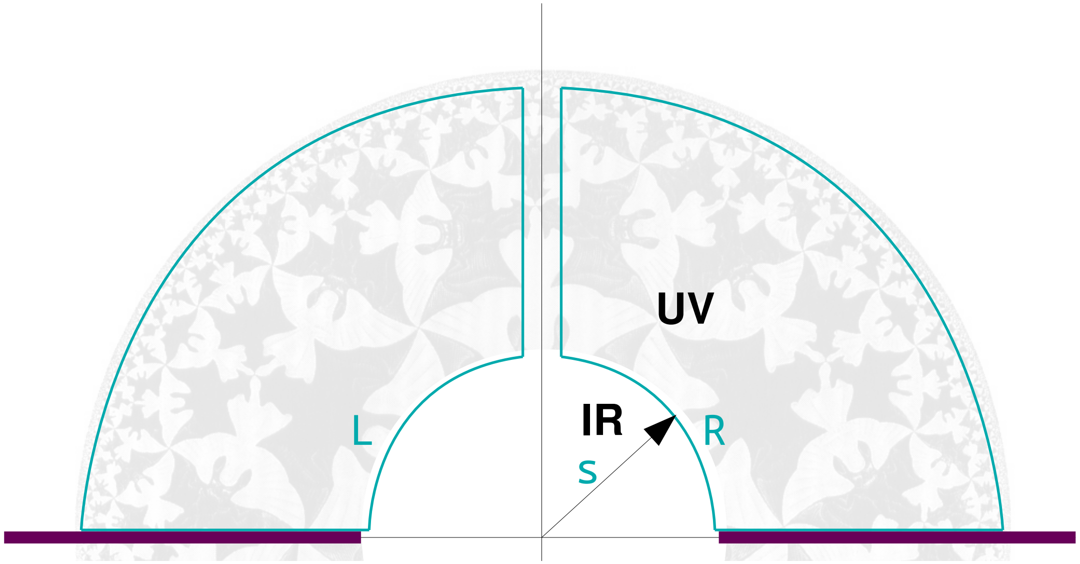


$$a_n^R = \int_R \frac{dz}{i\pi}\frac{\mathcal{M}(z)}{z^{3+n}} = \int_s^\infty \frac{dz}{i\pi}\frac{\mathcal{M}(z)}{z^{3+n}} + \int_{i\infty}^{is} \frac{dz}{i\pi}\frac{\mathcal{M}(z)}{z^{3+n}}$$

n even:

$$a_n^R + a_n^L = \frac{2}{\pi}\int_s^\infty \frac{dz}{z}\frac{\mathrm{Im}\mathcal{M}(z)}{z^{2+n}}$$

$$a_n^R - a_n^L = \frac{2}{i\pi}\int_s^\infty \frac{dz}{z}\frac{\mathrm{Re}\mathcal{M}(z)}{z^{2+n}} + 2I_n$$



$$a_n^R = \int_R \frac{dz}{i\pi}\frac{\mathcal{M}(z)}{z^{3+n}} = \int_s^\infty \frac{dz}{i\pi}\frac{\mathcal{M}(z)}{z^{3+n}} + \int_{i\infty}^{is} \frac{dz}{i\pi}\frac{\mathcal{M}(z)}{z^{3+n}}$$

n even:

$$a_n^R + a_n^L = \frac{2}{\pi}\int_s^\infty \frac{dz}{z}\frac{\mathrm{Im}\mathcal{M}(z)}{z^{2+n}}$$

$$a_n^R - a_n^L = \frac{2}{i\pi}\int_s^\infty \frac{dz}{z}\frac{\mathrm{Re}\mathcal{M}(z)}{z^{2+n}} + 2I_n$$

Bellazzini, Elias-Miro, Rattazzi, MR, Riva '20

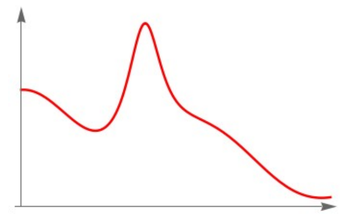
$$a_n^R + a_n^L = \frac{2}{\pi}\int_s^\infty \frac{dz}{z}\frac{\mathrm{Im}\mathcal{M}(z)}{z^{2+n}}$$

$\{a_n\}$

IR–calculable

$$\int_s^\infty \frac{dz}{z}\frac{1}{z^{2+n}}\bullet$$

$\mathrm{Im}\mathcal{M}(z) \geq 0$



is this mapping complete? Yes.

$$m_n = \int_0^1 d\mu(x)x^n\,, \quad d\mu(x)\geq 0$$

Knowledge of the moments = knowledge of the measure

Knowledge of IR arcs = knowledge of UV spectrum

When is an EFT UV-completable? → **When a bunch of numbers can be identified with moments of a positive distribution?**

When can a bunch of numbers be identified with moments of a positive distribution?

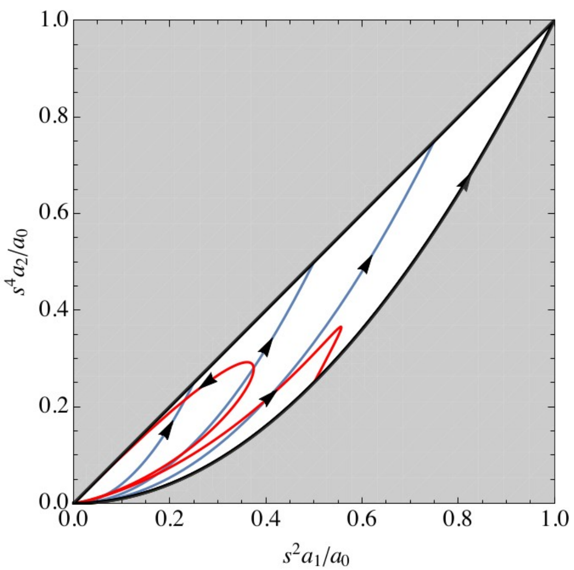
$\{a_0,\, a_1\,, \dots\}$ moments of a positive distribution in [0,1] **iff**

$$\begin{pmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & & \\ \dots & & \ddots & \vdots \\ a_n & & \dots & a_{2n} \end{pmatrix} \succ 0$$

$$\begin{pmatrix} a_0 - a_1 & a_1 - a_2 & \dots & a_n - a_{n+1} \\ a_1 - a_2 & a_2 - a_3 & & \\ \dots & & \ddots & \vdots \\ a_n - a_{n+1} & & \dots & a_{2n} - a_{2n+1} \end{pmatrix} \succ 0$$

2d

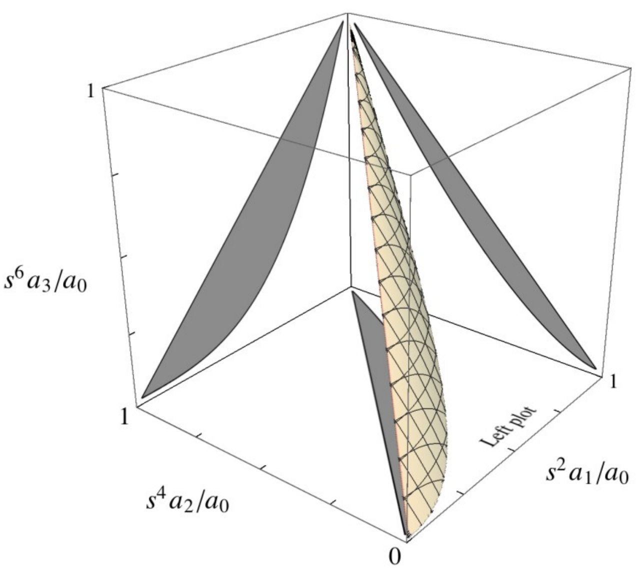
$$\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} \succ 0\,, \quad a_1 > 0\,, \quad a_0 > \hat{s}^2 a_1\,, \quad a_1 > \hat{s}^2 a_2$$



3d

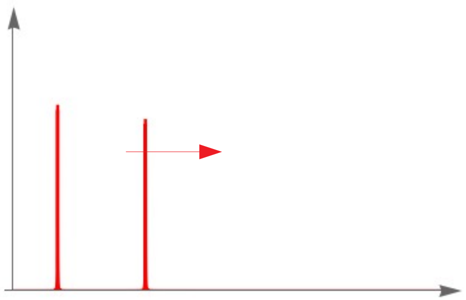
$$\begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix} \succ 0\,, \quad \begin{pmatrix} a_0 - a_1\hat{s}^2 & a_1 - a_2\hat{s}^2 \\ a_1 - a_2\hat{s}^2 & a_2 - a_3\hat{s}^2 \end{pmatrix} \succ 0\,,$$

$$\begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \succ 0\,, \quad a_1 > \hat{s}^2 a_2\,,$$

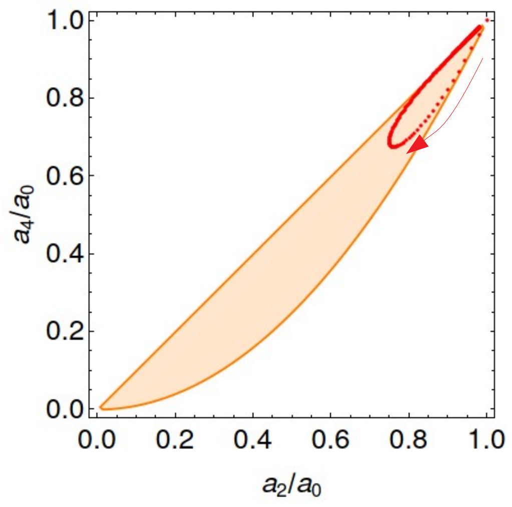


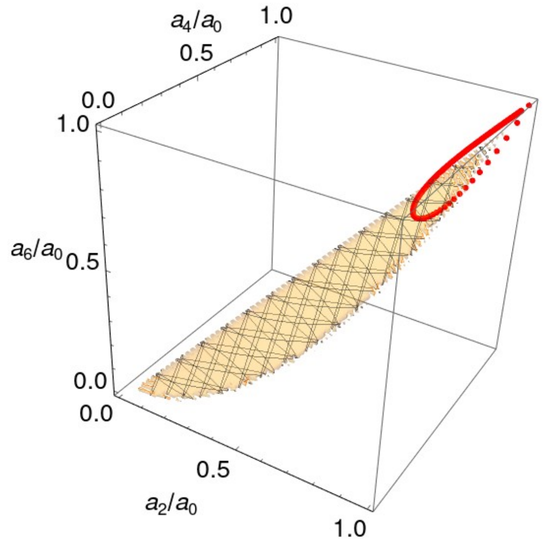
Convexity → Understanding of the boundary = Understanding of the entire space

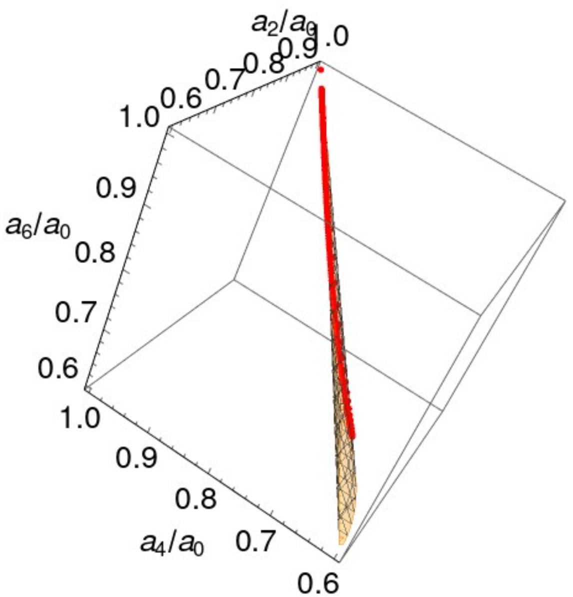
Boundary of n-dimensional space given by (n-1)-particles



Arcs from two narrow resonances

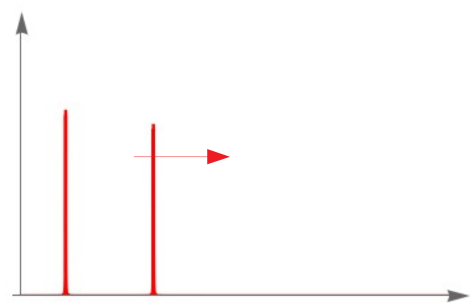






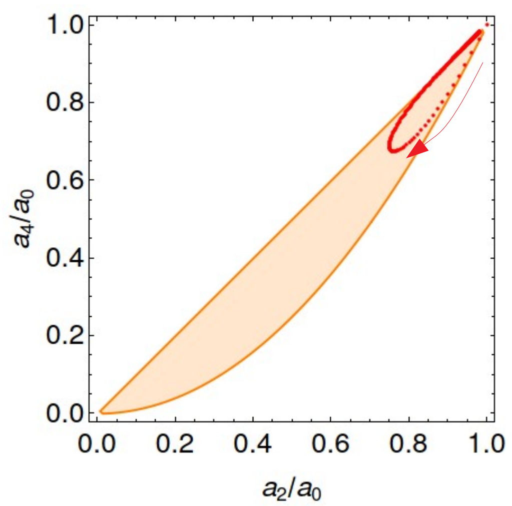
Convexity ➜ **Understanding of the boundary = Understanding of the entire space**

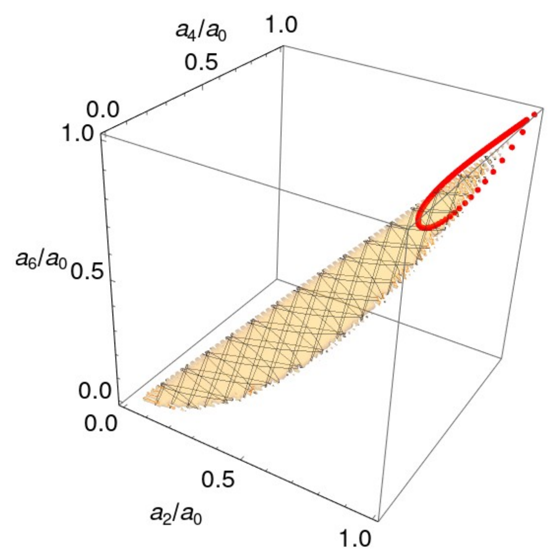
Boundary of n-dimensional space given by (n-1)-particles

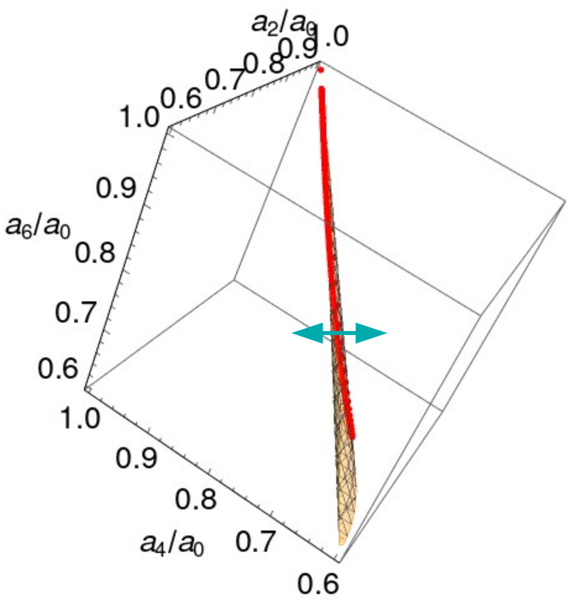


Arcs from two narrow resonances

Space of arcs shrinks
faster than exponentially with n!

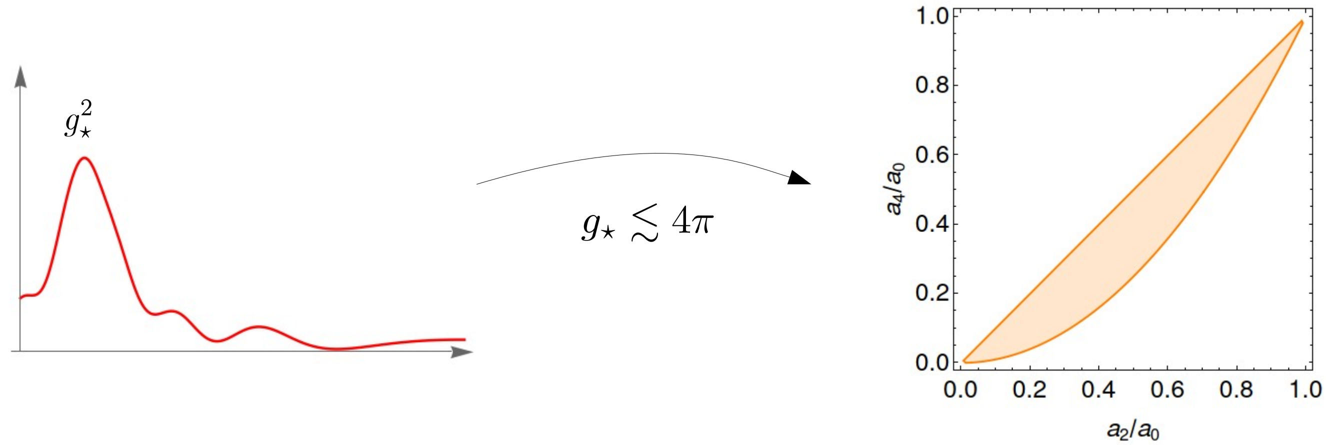






Full unitarity? $2\text{Im} f_\ell(s) \geq |f_\ell(s)|^2$

“Should impose an upper bound on the first arc, which gives the overall coupling”
After all, *any* spectrum leads to moment structure.



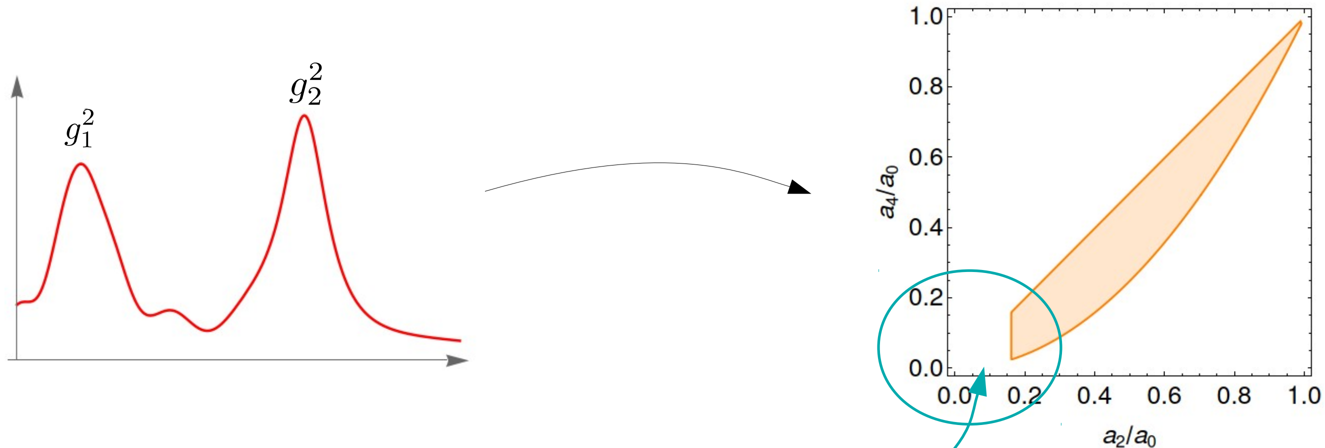
~ bounds unchanged, only overall normalization affected.

Effects only on strongly coupled EFTs

Full unitarity? $2\mathrm{Im} f_\ell(s) \geq |f_\ell(s)|^2$

"Should impose an upper bound on the first arc, which gives the overall coupling"
After all, *any* spectrum leads to moment structure.

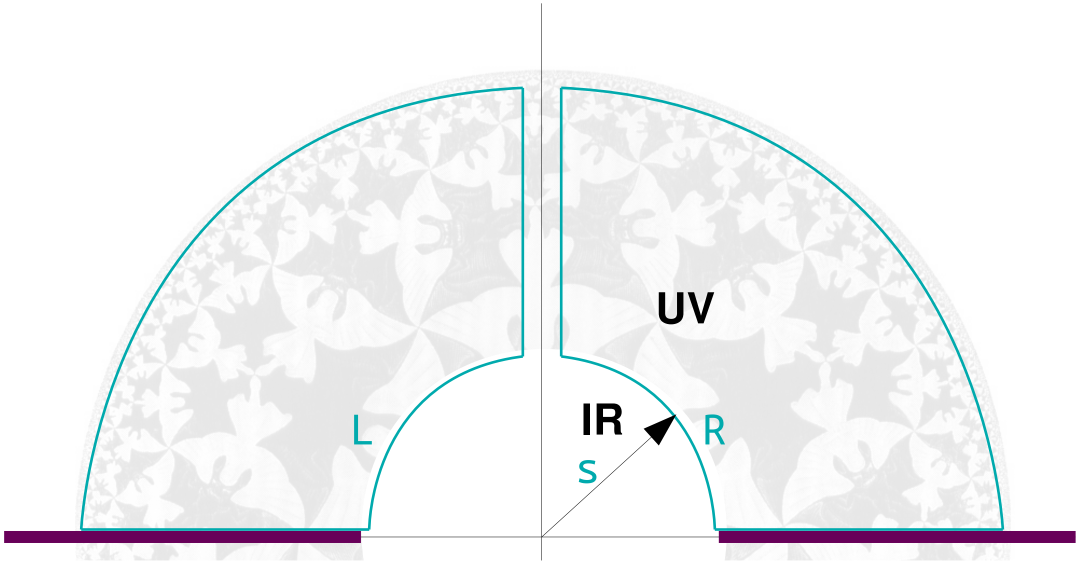
But not *any* spectrum is unitary.



$$\frac{a_2}{a_0} \simeq 1 - \frac{g_2^2}{g_1^2}\frac{m_1^2}{m_2^2} + \mathcal{O}(m_1^2/m_2^2)$$

Keeping first arc fixed, making second one small for m1<<m2 requires arbitrarly large coupling g2/g1~m1/m2

Full unitarity has impact on EFTs under perturbative control.



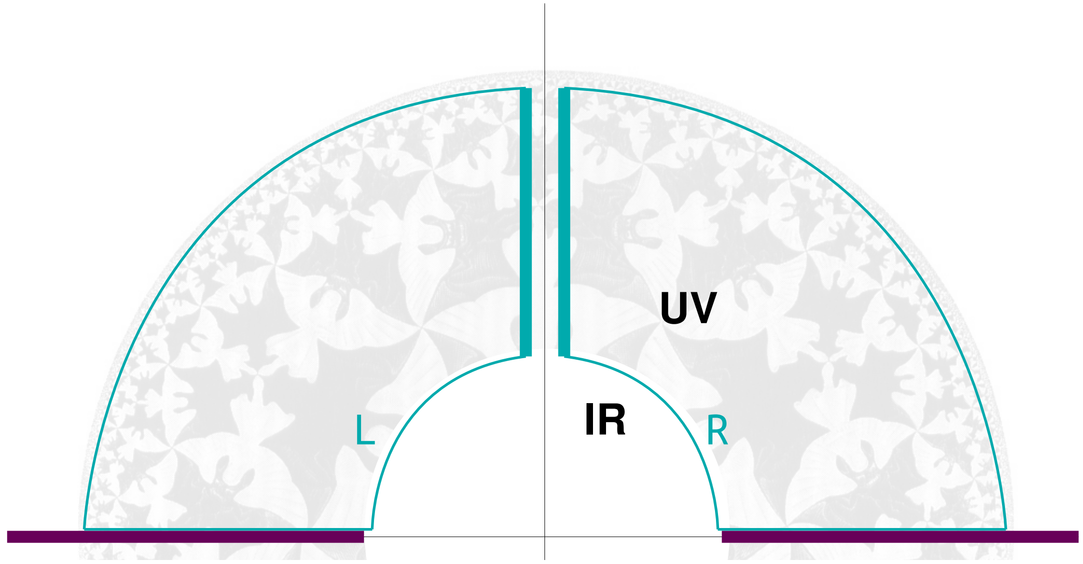
$$a_n^R = \int_R \frac{dz}{i\pi}\frac{\mathcal{M}(z)}{z^{3+n}} = \int_s^\infty \frac{dz}{i\pi}\frac{\mathcal{M}(z)}{z^{3+n}} + \int_{i\infty}^{is} \frac{dz}{i\pi}\frac{\mathcal{M}(z)}{z^{3+n}}$$

n even:

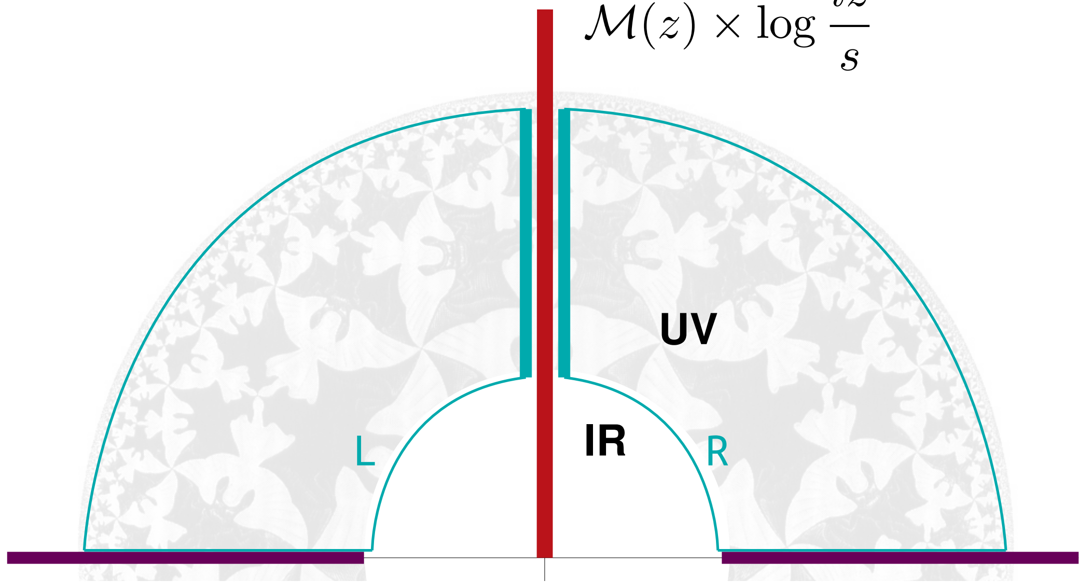
$$a_n^R + a_n^L = \frac{2}{\pi}\int_s^\infty \frac{dz}{z}\frac{\mathrm{Im}\mathcal{M}(z)}{z^{2+n}}$$

$$a_n^R - a_n^L = \frac{2}{i\pi}\int_s^\infty \frac{dz}{z}\frac{\mathrm{Re}\mathcal{M}(z)}{z^{2+n}} + 2I_n$$

MR '22



$$I_n = \int_{is}^{i\infty} \frac{dz}{i\pi} \frac{\mathcal{M}(z)}{z^{2+n}} = \;??$$



MR '22

$$I_n = \int_{is}^{i\infty} \frac{dz}{i\pi} \frac{\mathcal{M}(z)}{z^{2+n}}$$

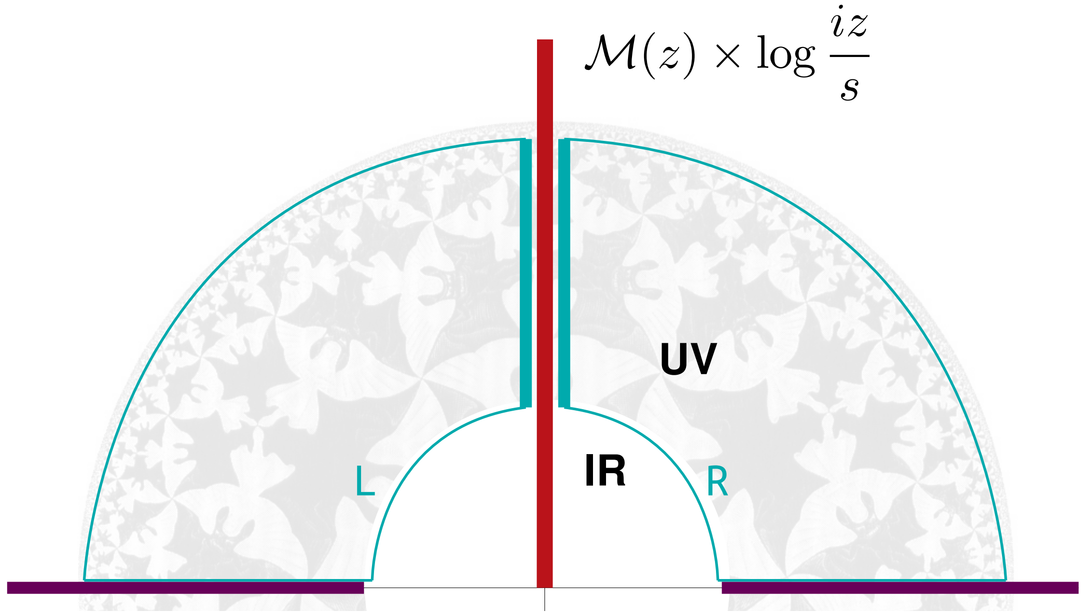
IR - calculable

$$= -\frac{1}{i\pi}\left(\int_s^\infty \frac{dz}{i\pi}\frac{\mathrm{Im}\mathcal{M}(z)}{z^{2+n}}\log\frac{z}{s} + \int_{\mathrm{arc}}\frac{dz}{i\pi}\frac{\mathcal{M}(z)}{z^{2+n}}\log\frac{z}{s}\right)$$

$$x\log\frac{1}{x} = \sum_{ij} c_{ij} x^i (1-x)^j$$

Also IR calculable,
but unconvenient for analytical understanding

MR '22



$$I_n = \int_{is}^{i\infty} \frac{dz}{i\pi} \frac{\mathcal{M}(z)}{z^{2+n}}$$

IR - calculable

$$= -\frac{1}{i\pi}\left(\int_s^\infty \frac{dz}{i\pi} \frac{\mathrm{Im}\mathcal{M}(z)}{z^{2+n}} \log \frac{z}{s} + \int_{\mathrm{arc}} \frac{dz}{i\pi} \frac{\mathcal{M}(z)}{z^{2+n}} \log \frac{z}{s}\right)$$

$$\frac{1}{2} \log \frac{a_n}{s^2 a_{n+2}} \leq \frac{A_n^{\mathrm{Im},\log}}{a_n} \leq \frac{1}{2} \log \frac{a_{n-2}}{s^2 a_n}$$

$$\frac{2}{\pi}\int_s^\infty \frac{dz}{z}\frac{\mathrm{Im}\mathcal{M}(z)}{z^{2+n}} \rightarrow A_n^{\mathrm{Im}} = a_n^R + a_n^L$$

$$\frac{2}{\pi}\int_s^\infty \frac{dz}{z}\frac{\mathrm{Re}\mathcal{M}(z)}{z^{2+n}} \rightarrow A_n^{\mathrm{Re}} = a_n^R - a_n^L - \frac{4}{\pi}\mathrm{Re}(a_n^{R,\log}) + \frac{2}{\pi}A_n^{\mathrm{Im},\log}$$

IR – calculable quantities

Having control on the real part allows to use the full unitarity constraints:

$$\mathcal{M}(z) = \sum_\ell 16\pi(2\ell+1)\, f_\ell(z)$$

$$2\mathrm{Im} f_\ell(s) \geq |f_\ell(s)|^2$$

$$\Rightarrow \quad s^n A_n^{\mathrm{Im}} \geq \frac{s^{n+k+1}}{64}\frac{1}{1+2\ell_{\mathrm{eff}}}\left[\left(A^{\mathrm{Im}}_{\frac{n+k-1}{2}}\right)^2 + \left(A^{\mathrm{Re}}_{\frac{n+k-1}{2}}\right)^2\right]$$

Both sides IR calculable… but what is l_eff?

ℓ_{eff} is $\sim$ the partial wave that dominates the integral

Let us show it cannot be infinite!

Arcs at finite t

$$\mathcal{M}(s,t) = \sum_{ij} g_{i,j}(s^2+t^2+u^2)^i(stu)^j$$

Each monomial is mapped to a dispersive integral:

$$s^{n+2}t^k \to \partial_t^{(k)} a_n = \int_s^\infty \frac{dz}{z} \frac{\sum_\ell \mathrm{Im} f_\ell(z) \partial_t^{(k)} P_\ell(1+2t/z)}{z^{2+n}}$$

But there are more integrals than independent monomials!

$$(s^2+t^2+u^2)^2 \supset 4s^4 + 12s^2t^2 + \dots \quad \Rightarrow \quad 0 = \int_s^\infty \frac{dz}{z} \frac{\sum_\ell \mathrm{Im} f_\ell(z)}{z^4} \left(8\ell(\ell+1) - (\ell(\ell+1))^2\right)$$

$$\Downarrow$$

$$A_2^{\mathrm{Im},\ell=2} = \sum_{\ell \geq 4} \frac{\ell(\ell+1)(\ell(\ell+1)-8)}{12} A_2^{\mathrm{Im},\ell}$$

$$\Downarrow$$

$$\ell_{\mathrm{eff}} \lesssim 2.75$$

The UV ansatz makes unitarity manifest, at the expenses of crossing.
Imposing dispersion relations with an IR crossing symmetric ansatz induces sum rules on the integrals
This procedure is perturbative since it relies on IR explicit arcs.

Implications for a goldstone scalar

$$\mathcal{M}(s) = c_2 s^2 + c_4 s^4 + c_6 s^6 + \ldots$$

$$A_n^{\text{Im}} = c_n$$

Moment structure

$$A_n^{\text{Re}} \sim s^{n-2} c_2$$

$$s^n c_{n+2} \gtrsim \frac{1}{1+2\ell_{\text{eff}}} \frac{c_2^2}{16\pi^2}$$

Always violated deep enough in the IR!
Need to include loop corrections

Implications for a goldstone scalar

$$\mathcal{M}(s) = c_2 s^2 + s^4(c_4 + \beta_4 \log(-is)) + s^6(c_6 + \beta_6 \log(-is)) + \dots$$

$$A_2^{\rm Im} = c_2 + s^2\beta_4 + \dots$$

$$A_4^{\rm Im} = c_4 + \beta_4 + \dots$$

$$A_6^{\rm Im} = -\frac{\beta_4}{s^2} + c_6 + \beta_6 + \dots$$

Higher arcs receive much more relevant contributions at one loop

Beta function affects tree level relations

$$A_n^{\rm Re} \sim s^{n-2} c_2 + \dots$$

$$-\frac{\beta_4}{2} + c_6 s^2 \geq \frac{1}{1+2\ell_{\rm eff}} \frac{1}{4} \frac{c_2^2}{16\pi^2}$$

Lower bound on the first running coefficient

Explicit calculation: $\beta_4 = -\frac{7}{10}\frac{c_2^2}{16\pi^2}$, so bound is satisfied even for $\ell_{\rm eff} = 0$ as it had to be!

Implications for a goldstone scalar

$$\mathcal{M}(s) = c_2 s^2 + s^4(c_4 + \beta_4 \log(-is)) + s^6(c_6 + \beta_6 \log(-is)) + \dots$$

$$A_2^{\rm Im} = c_2 + s^2\beta_4 + \dots$$

$$A_4^{\rm Im} = c_4 + \beta_4 + \dots$$

$$A_6^{\rm Im} = -\frac{\beta_4}{s^2} + c_6 + \beta_6 + \dots$$

$$A_n^{\rm Re} \sim s^{n-2} c_2 + \dots$$

Convexity: $A_2^{\rm Im} A_6^{\rm Im} > (A_4^{\rm Im})^2$

Unitarity:

$$-\frac{\beta_4}{2} + c_6 s^2 \geq \frac{c_4^2(s)s^2}{c_2}$$

$$-\frac{\beta_4}{2} + c_6 s^2 \geq \frac{1}{1+2\ell_{\rm eff}}\frac{1}{4}\frac{c_2^2}{16\pi^2}$$

$$\frac{\text{relevance of Unitarity bounds}}{\text{relevance of Convexity bounds}} \sim \frac{\beta_4}{c_4}\frac{c_2}{c_4 s^2} \sim \frac{\text{loop expansion}}{\text{derivative expansion}}$$

$$c_2 \sim \frac{g^2}{M^4} \rightarrow \frac{g^2}{16\pi^2}\frac{M^4}{s^2}$$

Bounds from Full Unit. Become more relevant in theories and regimes where the loop expansion is more relevant than the derivative expansion,
i.e. for strongly coupled UV completions well below threshold

Multichannel EFTs

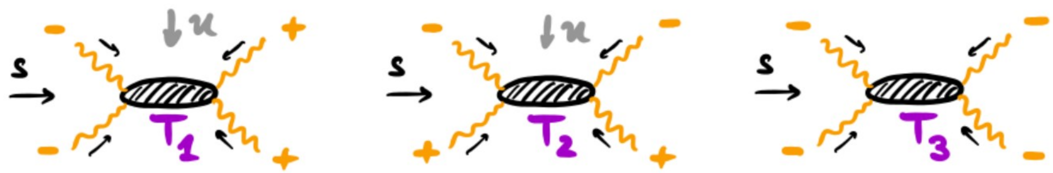
Multichannel EFTs

Remmen, Rodd '19
Li, Xu, Yang, Zhang, Zhou '21
Haring, Hebbar, Karateev, Meineri, Penedones '22
Durieux, Remmen, MR, Rodd 'WIP

Example, a U(1) vector:

$$\mathcal{L} = -(FF) + c_1(FF)^2 + c_2(F\widetilde{F})^2 + c_3(FF)(F\widetilde{F}) + \ldots \qquad (FF) \equiv \frac{1}{4}F_{\mu\nu}F_{\mu\nu} \qquad (F\widetilde{F}) = \frac{1}{4}F_{\mu\nu}\widetilde{F}_{\mu\nu}$$

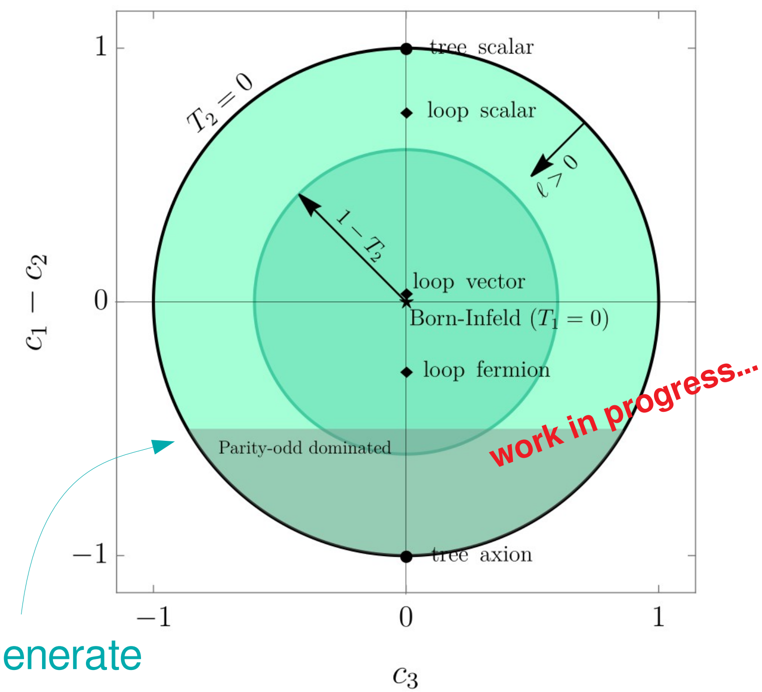
Three distinct scattering channels in the forward limit, and Wilson coeff. written as integrals of them:



$$c_1 = T^1 + T^2 + \mathrm{Re}T^3$$
$$c_2 = T^1 + T^2 - \mathrm{Re}T^3$$
$$c_3 = \mathrm{Im}T^3$$

$$\begin{pmatrix} T^1 & 0 & 0 & T^3_r - iT^3_i \\ 0 & T^2 & 0 & 0 \\ 0 & 0 & T^2 & 0 \\ T^3_r + iT^3_i & 0 & 0 & T^1 \end{pmatrix} \succ 0$$

Now the measure is a positive matrix, relating arcs with same subtractions but different channels.



Boundary made of degenerate scattering channels

Conclusions

Conclusions

Generic UV properties

Moment structure
in the forward limit

Bounds on arc
hierarchies from
full unitarity

Bounds on finite
angle scattering

Relations between
different scattering
channels

Dispersion relations for scattering amplitudes allow to constrain the type of EFTs

and

map regions of parameter space to generic features of the UV completion

"Not entirely, dear Adso," my master replied. "True, that kind of print expressed to me, if you like, the idea of 'horse', the verbum mentis, and would have expressed the same to me wherever I might have found it. But the print in that place and at that hour of the day told me that at least one of all possible horses had passed that way. So I found myself halfway between the perception of the concept 'horse' and the knowledge of an individual horse. And in any case, what I knew of the universal horse had been given me by those traces, which were singular. I could say I was caught at that moment between the singularity of the traces and my ignorance, which assumed the quite diaphanous form of a universal idea. If you see something from a distance, and you do not understand what it is, you will be content with defining it as a body of some dimension. When you come closer, you will then define it as an animal, even if you do not yet know whether it is a horse or an ass. And finally, when it is still closer, you will be able to say it is a horse even if you do not yet know whether it is Brunellus or Niger. And only when you are at the proper distance will you see that it is Brunellus (or, rather, that horse and not another, however you decide to call it). And that will be full knowledge, the learning of the singular. So an hour ago I could expect all horses, but not because of the vastness of my intellect, but because of the paucity of my deduction. And my intellect's hunger was sated only when I saw the single horse that the monks were leading by the halter. Only then did I truly know that my previous reasoning, had brought me close to the truth. And so the ideas, which I was using earlier to imagine a horse I had not yet seen, were pure signs, as the hoofprints in the snow were signs of the idea of 'horse'; and signs and the signs of signs are used only when we are lacking things."

William of Baskerville, talking about Effective Field Theory,
in 'The Name of the Rose' by Umberto Eco

Thank you

Backup

Finite t

Finite t

At finite t, one can define a 2d moment problem, in energy and spin:

$$a_n(s,t) = \sum_k \partial_t^k a_n(s,t)|_{t=0} = \sum_{k,j} c_{nkj}\mu_k^j \qquad \mu_n^q = \int_0^1 d\mu(x,\ell)x^n\ell^{2q}$$

$$H_{(0,0)} = \left(\begin{array}{c|cc|ccc|c} \mu_0^0 & \mu_1^0 & \mu_1^1 & \mu_2^0 & \mu_2^1 & \mu_2^2 & \cdots \\ \hline \mu_1^0 & \mu_2^0 & \mu_2^1 & \mu_3^0 & \mu_3^1 & \mu_3^2 & \cdots \\ \mu_1^1 & \mu_2^1 & \mu_2^2 & \mu_3^1 & \mu_3^2 & \mu_3^3 & \cdots \\ \hline \mu_2^0 & \mu_3^0 & \mu_3^1 & \mu_4^0 & \mu_4^1 & \mu_4^2 & \cdots \\ \mu_2^1 & \mu_3^1 & \mu_3^2 & \mu_4^1 & \mu_4^2 & \mu_4^3 & \cdots \\ \mu_2^2 & \mu_3^2 & \mu_3^3 & \mu_4^2 & \mu_4^3 & \mu_4^4 & \cdots \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right) \succ 0$$

Contrary to the 1d problem, there is no optimal bounds on a finite set of moments: need to study the convergence.

Example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_{1,0} \\ c_{0,1} \\ c_{2,0} \\ c_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/2 & -2 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5/4 & 31/12 & -43/72 & 1/36 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5/2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_0^0 \\ \mu_1^0 \\ \mu_1^1 \\ \mu_2^0 \\ \mu_2^1 \\ \mu_2^2 \\ \mu_3^0 \\ \mu_3^1 \\ \mu_3^2 \\ \mu_3^3 \end{pmatrix}$$

Wilson coeff. 2d moments

Finite t

At finite t, one can define a 2d moment problem, in energy and spin:

$$a_n(s,t) = \sum_k \partial_t^k a_n(s,t)|_{t=0} = \sum_{k,j} c_{nkj}\mu_k^j \qquad \mu_n^q = \int_0^1 d\mu(x,\ell)x^n\ell^{2q}$$

$$H_{(0,0)} = \left(\begin{array}{c|cc|ccc|c}
\mu_0^0 & \mu_1^0 & \mu_1^1 & \mu_2^0 & \mu_2^1 & \mu_2^2 & \cdots \\
\hline
\mu_1^0 & \mu_2^0 & \mu_2^1 & \mu_3^0 & \mu_3^1 & \mu_3^2 & \cdots \\
\mu_1^1 & \mu_2^1 & \mu_2^2 & \mu_3^1 & \mu_3^2 & \mu_3^3 & \cdots \\
\hline
\mu_2^0 & \mu_3^0 & \mu_3^1 & \mu_4^0 & \mu_4^1 & \mu_4^2 & \cdots \\
\mu_2^1 & \mu_3^1 & \mu_3^2 & \mu_4^1 & \mu_4^2 & \mu_4^3 & \cdots \\
\mu_2^2 & \mu_3^2 & \mu_3^3 & \mu_4^2 & \mu_4^3 & \mu_4^4 & \cdots \\
\hline
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right) \succ 0$$

Contrary to the 1d problem, there is no optimal bounds on a finite set of moments: need to study the convergence.

Example:

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 3/2 & 0 \\
0 & 0 & 0 & -1/2 \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} c_{1,0} \\ c_{0,1} \\ c_{2,0} \\ c_{1,1} \end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3/2 & -2 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -5/4 & 31/12 & -43/72 & 1/36 \\
\hline
0 & 0 & 0 & 0 & -2 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 25/12 & -43/72 & 1/36
\end{pmatrix}
\begin{pmatrix} \mu_0^0 \\ \mu_1^0 \\ \mu_1^1 \\ \mu_2^0 \\ \mu_2^1 \\ \mu_2^2 \\ \mu_3^0 \\ \mu_3^1 \\ \mu_3^2 \\ \mu_3^3 \end{pmatrix}$$

Some moments are not independent: "null constraints"

Finite t

Bellazzini, MR, Riva '21

At one loop, some moments diverge:

$$a_0(s,t) \supset t^2 \log t + \ldots \quad \rightarrow \quad \partial_t^2 a_0 = \int_s^\infty \frac{dz}{z} \sum_\ell \frac{\ell^4 \mathrm{Im} f_\ell(z)}{z^4} \rightarrow \infty \;\; \text{as} \;\; t \rightarrow 0$$

This divergence tells us about the large-l, large impact parameter behaviour of the masure

Only a subset of moments can be used to set bounds around t=0.

a_0	$\partial_t a_0$	$\partial_t^2 a_0$	$\partial_t^3 a_0$	$\partial_t^4 a_0$	$\partial_t^5 a_0$	$\cdots$
a_1	$\partial_t a_1$	$\partial_t^2 a_1$	$\partial_t^3 a_1$	$\partial_t^4 a_1$	$\partial_t^5 a_1$	$\cdots$
a_2	$\partial_t a_2$	$\partial_t^2 a_2$	$\partial_t^3 a_2$	$\partial_t^4 a_2$	$\partial_t^5 a_2$	$\cdots$
a_3	$\partial_t a_3$	$\partial_t^2 a_3$	$\partial_t^3 a_3$	$\partial_t^4 a_3$	$\partial_t^5 a_3$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

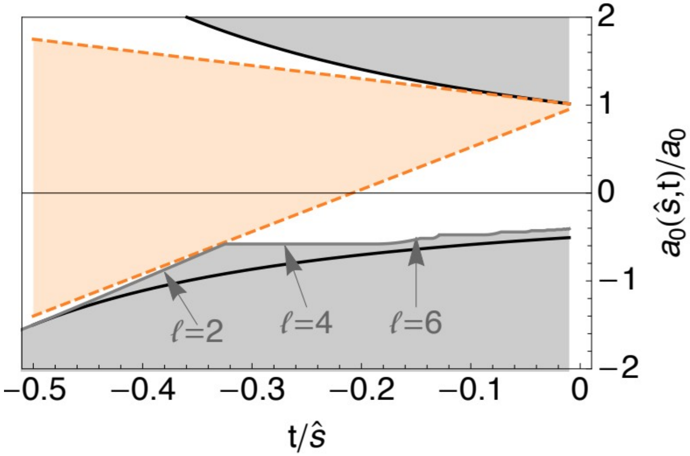
Finite t

Bellazzini, MR, Riva ‘21

Arcs at finite t get a t- and l- dependent kernel

$$a_n(s,t) \;=\; \frac{2}{\pi}\int_s^\infty \frac{dz}{z}\sum_\ell \frac{\mathrm{Im} f_\ell(z)}{z^{2+n}} I_{n,\ell}(t,z)$$

$$\min_{\ell,z} I_{n,\ell}(t,z) \le \frac{a_n(s,t)}{a_n(s,0)} \le \frac{s+t/2}{(s(s+t))^{n+2}}$$



Upper bound on the arc is t-dependent but lower bound is t- and l- dependent.

$$a_n(s,t) \;=\; c_2 - t c_3 + \ldots$$

Upper bound comes from t=0,

but lower bound from finite t,
at the intersection of l=2 and l=4 partial waves