Vacuum densities for branes orthogonal to AdS boundary

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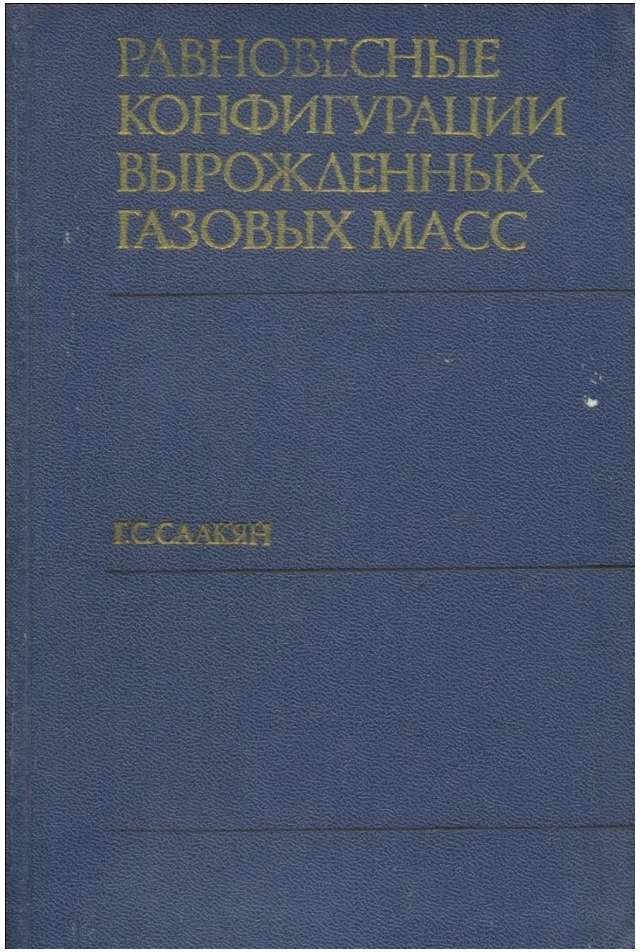
MPCS-23, 12 - 16 September, 2023

Yerevan, Armenia

Outline

- Anti-de Sitter spacetime as a background geometry in QFT
- Geometry, field and boundary conditions
- Mean field squared
- VEV of the energy-momentum tensor and the Casimir forces
- Conclusions

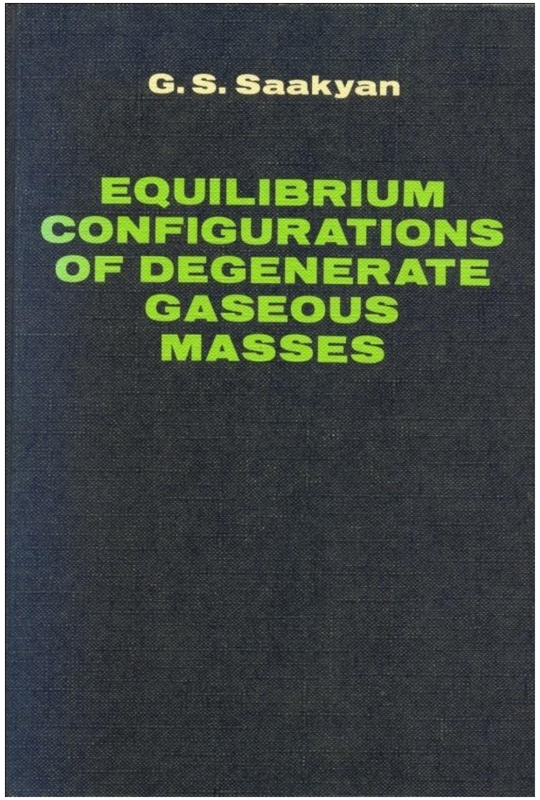


Equilibrium configurations
of degenerate gaseous masses

G.S. Sahakyan

Moscow,1972

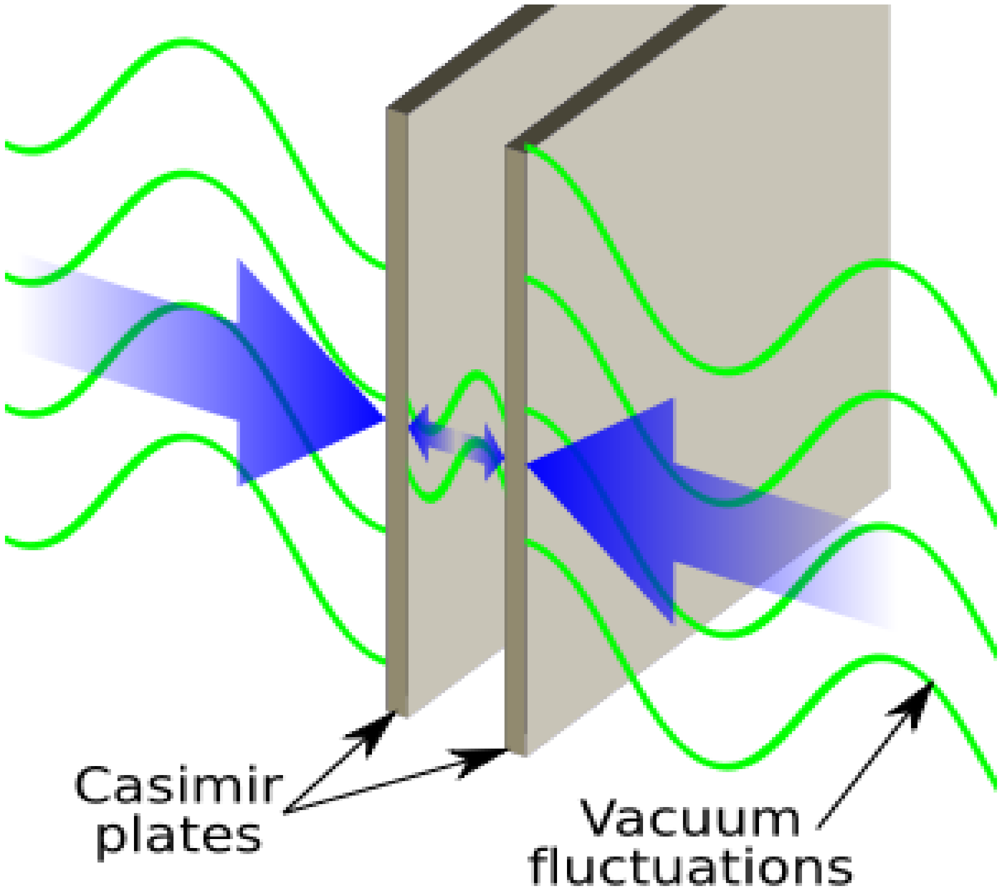
It was translated into English and was issued abroad in 2 years period, that was very short in those days.



The English edition of the monograph by G.S. published by *A Halsted Press Book* in 1974.

Casimir effect

Parallel metallic plates



$$\frac{F_C}{S} = -\frac{\partial}{\partial a}\frac{E_C}{S} = -\frac{\hbar c \pi^2}{240 a^4}$$

Aim

- We aim to consider combined effects of boundaries and gravity on the properties of quantum vacuum
- Gravitational field is considered as a classical curved background
- Back-reaction of quantum effects is described by Einstein equations with the expectation value of the energy-momentum tensor for quantum fields in the right-hand side
- This hybrid but very useful scheme is an important intermediate step to the development of quantum gravity
- Among the most interesting effects in this field are the particle production and the vacuum polarization by strong gravitational fields
- As a background geometry we consider AdS spacetime

Importance of AdS in QFT on curved backgrounds

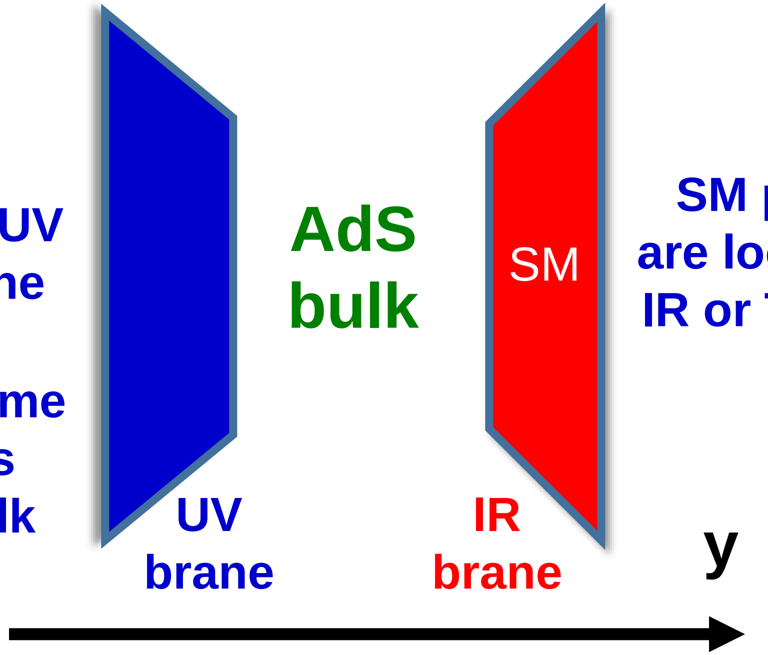
- Because of the high symmetry, numerous problems are exactly solvable on AdS bulk and this may shed light on the influence of a classical gravitational field on the quantum matter in more general geometries
- Questions of principal nature related to the quantization of fields propagating on curved backgrounds
- AdS spacetime generically arises as a ground state in extended supergravity and in string theories
- AdS/Conformal Field Theory correspondence: Relates string theories or supergravity in the bulk of AdS with a conformal field theory living on its boundary
- Braneworld models: Provide a solution to the hierarchy problem between the gravitational and electroweak scales
- Braneworlds naturally appear in string/M-theory context and provide a novel setting for discussing phenomenological and cosmological issues related to extra dimensions

Randall-Sundrum-type braneworlds

- Original Randall-Sundrum model (RS1) offers a solution to the hierarchy problem by postulating 5D AdS spacetime bounded by two (3+1)-dimensional branes

Gravity is localized near UV or Planck brane

Only graviton and some other non SM fields propagate in the bulk



SM particles are localized on IR or TeV brane

Branes are parallel to the AdS boundary

$$ds^2 = e^{-2ky}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$

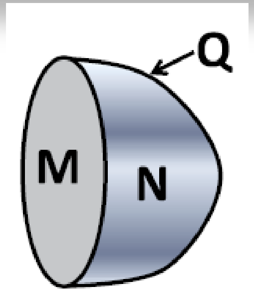
- Hierarchy problem between the gravitational and electroweak scales is solved for

k·distance between branes = 40

- The Casimir effect has been widely investigated in braneworlds setup and provides an attractive mechanism to stabilize the interbrane distance

Surfaces intersecting AdS boundary and Ryu-Takayanagi conjecture

- AdS/BCFT correspondence maps conformal field theory on a manifold with boundary (BCFT) to asymptotically AdS spacetime with additional boundary



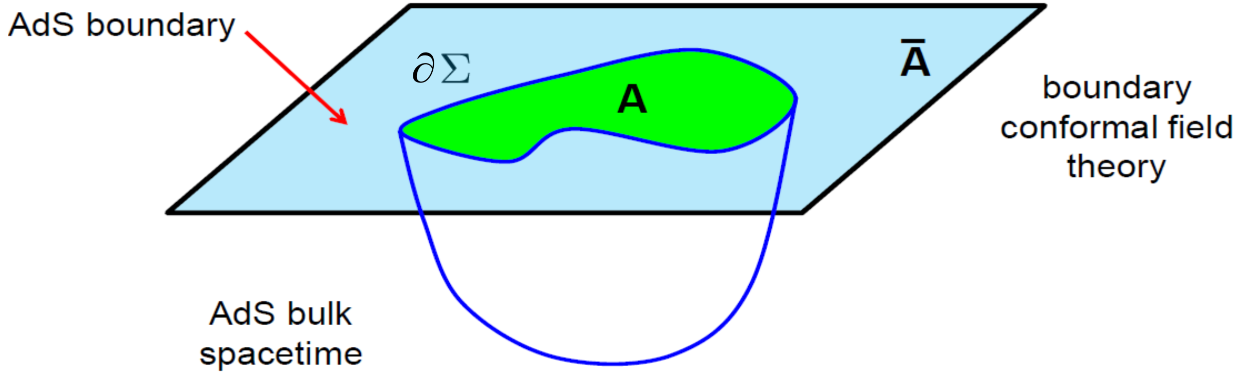
- Consider CFT living on the boundary $\partial\Sigma$ of AdS spacetime
- Conjecture provides a geometric interpretation of the entanglement entropy in CFT
- Entanglement entropy of CFT in spatial sub-region $A \subset \partial\Sigma$ with its complement $\bar{A}$

$$S_A = \frac{\text{Area of } \gamma_A}{4G}$$ Ryu-Takayanagi formula

Compare with Bekenstein-Hawking formula for black hole entropy

$$S_A = \frac{\text{Area of event horizon}}{4G}$$

- Surface γ_A in the bulk satisfies the properties:
  - Has the same boundary as A
  - Is homologous to A
  - Extremizes the area



Geometry

- (D+1)-dimensional AdS spacetime

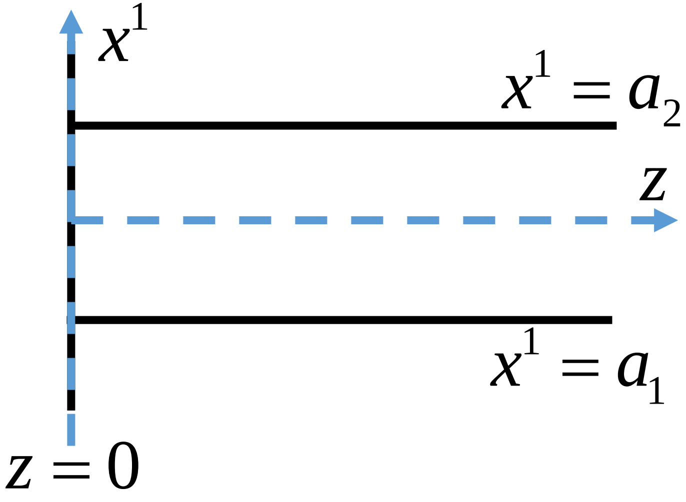
$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{-2y/a}\eta_{ik}dx^i dx^k - dy^2 \qquad \eta_{ik} = \mathrm{diag}(1, -1, \ldots, -1)$$

- New coordinate $z = ae^{y/a},\ 0 \leqslant z < \infty$

$$ds^2 = (a/z)^2(\eta_{ik}dx^i dx^k - dz^2)$$

- AdS boundary ⇨ z=0, horizon ⇨ z=∞

- Branes perpendicular to AdS boundary



Proper distance between the branes $(a/z)(a_2 - a_1)$

Field, boundary conditions and VEVs

- Scalar field with general curvature coupling parameter

$$(g^{ik}\nabla_i\nabla_k + m^2 + \xi R)\varphi(x) = 0,\ \ R = -D(D+1)/\alpha^2$$

- Boundary conditions $(1+\beta_j n_j^i \nabla_i)\varphi(x) = 0,\ n_j^i$ ⇐ Normal to the brane at $x^1 = a_j$

- Boundary condition imposed on the field modify the spectrum of vacuum fluctuations and the VEVs of physical observables are changed

Casimir effect

- We are interested in the VEVs of the field squared and of the energy-momentum tensor

- Scheme of calculation:

Two-point function ⇒ Mean field squared ⇒ VEV of the energy-momentum tensor
⇒ Forces acting on the branes

Decomposition of the Wightman function

- In the region between the branes the eigenvalues of the quantum number corresponding to the direction perpendicular to the branes are expressed in terms of roots of transcendental equation
- In the expression of the Wightman function for summation of the series of those eigenvalues the Generalized Abel-Plana formula is used
- That allows to decompose the Wightman function as:

[Wightman function] = [Wightman function for the geometry without branes] + [Contribution corresponding to the geometry of a single brane]

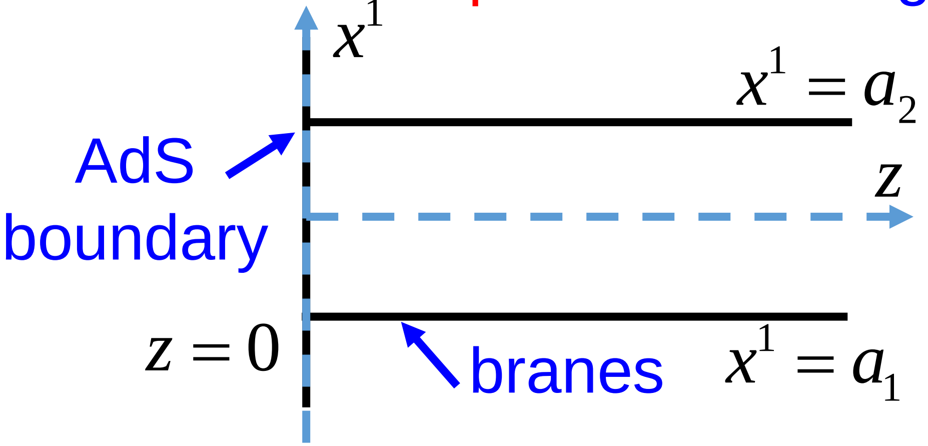
+ [Contribution when we add the second brane to geometry of a single brane]

- For points away from the branes the renormalization of the local VEVs is reduced to the one in the brane-free geometry
- Similar decomposition is obtained for Bulk-to-boundary propagator

among the central objects in the AdS/CFT correspondence

Mean field squared in the single brane geometry

■ Mean field squared in the region between the branes ($a = a_2 - a_1$)



$$\langle \varphi^2 \rangle = \langle \varphi^2 \rangle_0 + \frac{(\sqrt{\pi}\alpha)^{1-D}}{2^{D+2\nu}} \int_0^\infty dx\, x^{D+2\nu-1}$$

$$\times F_\nu^{D/2}(x) \frac{2 + \sum_{j=1,2} e^{2|x^1 - a_j|x/z} c_j(x/z)}{c_1(x/z) c_2(x/z) e^{2ax/z} - 1}$$

Notations: $F_\nu^\mu(u) = \frac{{}_1F_2(\nu + 1/2; \nu + \mu \cdot (x^1,\ a_j,\ \beta_j) - u^2)}{\Gamma(\nu + \mu + 1/2)\Gamma(\nu + 1)}$, $\nu = \sqrt{(D/2)^2 - D(D+1)\xi + m^2\alpha^2}$

$$c_j(\lambda) = \frac{\beta_j \lambda - 1}{\beta_j \lambda + 1}$$

$\langle \varphi^2 \rangle_0$ ⇐ VEV for the geometry without branes

■ VEV depends on $(x^1,\ a_j,\ \beta_j)$ in the form of the dimensionless ratios x^1 / z , a / z and β_j / z

Proper distance from the brane ⇒ $\alpha |x^1 - a_j| / z$ ↺ Maximal symmetry of AdS spacetime

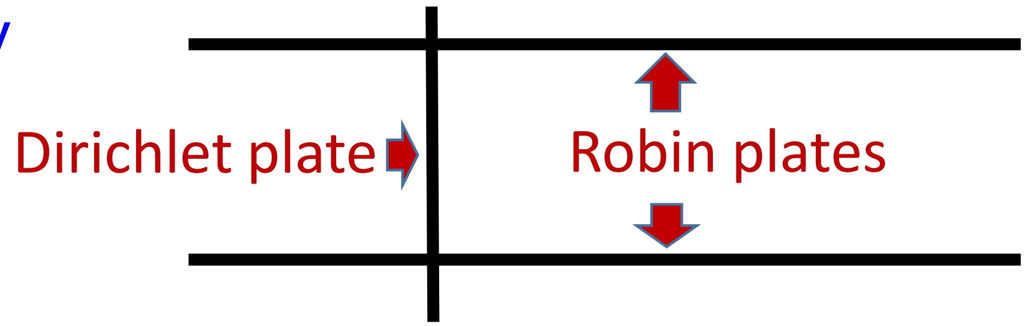
Mean field squared: Special cases and asymptotes

- Conformally coupled massless field: $m=0,\ \xi=\dfrac{D-1}{4D} \Rightarrow \nu=\dfrac{1}{2}$

$$\langle\varphi^2\rangle=\langle\varphi^2\rangle_0+(z/\alpha)^{D-1}\langle\varphi^2\rangle_{(\mathrm{M})},$$

$$\langle\varphi^2\rangle_{(\mathrm{M})}=\frac{1}{2^D\pi^{\frac{D}{2}}}\int_0^\infty d\lambda\,\lambda^{D-2}\left[\frac{1}{\Gamma(D/2)}-\frac{J_{D/2-1}(2z\lambda)}{(z\lambda)^{D/2-1}}\right]\frac{2+\sum_{j=1,2}e^{2|x^1-a_j|\lambda}c_j(\lambda)}{c_1(\lambda)c_2(\lambda)e^{2a\lambda}-1}$$

VEV in Minkowski spacetime for geometry



- Small distances from the brane ($|x^1-a_j|\ll z, |\beta_j|$) for non-Dirichlet BC

$$\langle\varphi^2\rangle\approx\frac{\Gamma(\frac{D-1}{2})}{(4\pi)^{\frac{D+1}{2}}}\left(\frac{z}{\alpha|x^1-a_j|}\right)^{D-1}$$

(opposite sign for Dirichlet BC)

Mean field squared: Special cases and asymptotes

- Points near the AdS boundary: $z \ll |x^1 - a_j|,\ j = 1, 2,$

$$\langle\varphi^2\rangle \approx \langle\varphi^2\rangle_0 + \frac{F_\nu^{\frac{D}{2}}(0) z^{D+2\nu}}{2^{D+2\nu}(\sqrt{\pi}\alpha)^{D-1}} \int_0^\infty d\lambda\, \lambda^{D+2\nu-1} \frac{2 + \sum_{j=1,2} e^{2|x^1 - a_j|\lambda} c_j(\lambda)}{c_1(\lambda) c_2(\lambda) e^{2a\lambda} - 1}$$

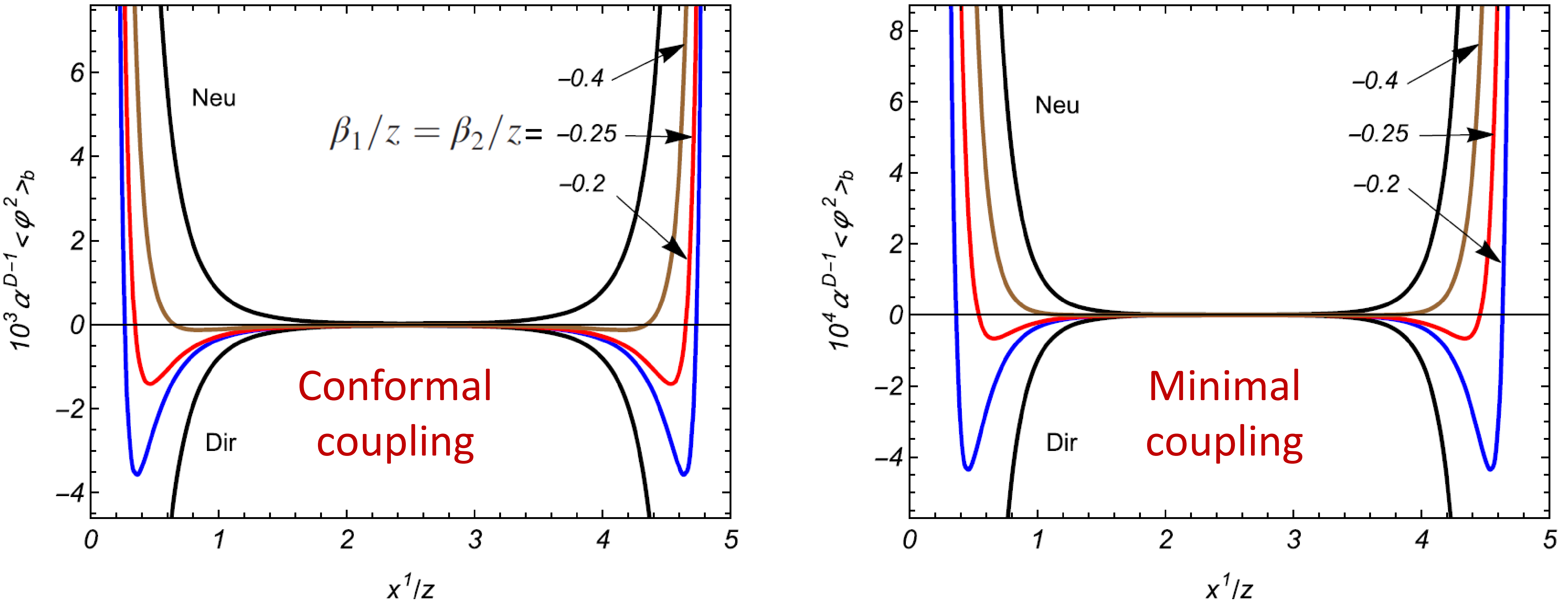
For points near the AdS boundary and not too close to the branes, the brane-induced part in the mean field squared tends to zero like $z^{D+2\nu}$

- Points near the horizon: $\langle\varphi^2\rangle \approx \langle\varphi^2\rangle_0 + (z/\alpha)^{D-1} \langle\varphi^2\rangle_{(\mathrm{M})}^{(0)}|_{m=0}$

Near the horizon the effects of the curvature on the brane induced VEV are weak

Brane induced mean field squared

$$D = 4,\ m\alpha = 0.5,\ a_1 = 0,\ a_2 / z = 5$$



Vacuum energy-momentum tensor

VEV of the energy-momentum tensor

$$\langle T_{ik}\rangle = \frac{1}{2}\lim_{x'\to x}(\partial_i\partial'_k + \partial_k\partial'_i)W(x,x') + \hat{B}_{ik}\langle\varphi^2\rangle,\ \ \hat{B}_{ik} = \left(\xi - \frac{1}{4}\right)g_{ik}g^{lm}\nabla_l\nabla_m - \xi(\nabla_i\nabla_k + R_{ik})$$

Diagonal components

$$\langle T_i^i\rangle = \langle T_i^i\rangle_0 - \frac{\alpha^{-1-D}}{2^{D+2\nu}\pi^{\frac{D-1}{2}}}\int_0^\infty dx\, x\left\{\frac{E_i x^{D+2\nu}F_\nu^{\frac{D}{2}}(x)}{c_1(x/z)c_2(x/z)e^{2ax/z}-1}\right.$$
$$\left. + \frac{2+\sum_{j=1,2}e^{2|x^1-a_j|x/z}c_j(x/z)}{c_1(x/z)c_2(x/z)e^{2ax/z}-1}\left[A_i x^{D+2\nu}F_\nu^{\frac{D}{2}+1}(x) + \hat{B}_i x^{D+2\nu}F_\nu^{\frac{D}{2}}(x)\right]\right\}$$

Off-diagonal component

$$\langle T_D^1\rangle = -\frac{2\alpha^{-1-D}}{2^{D+2\nu}\pi^{\frac{D-1}{2}}}\int_0^\infty dx\frac{\sum_{j=1,2}(-1)^j e^{2|x^1-a_j|x/z}c_j(x/z)}{c_1(x/z)c_2(x/z)e^{2ax/z}-1}\left[\left(\xi-\frac{1}{4}\right)x\partial_x + \xi\right]x^{D+2\nu}F_\nu^{\frac{D}{2}}(x)$$

Covariant conservation and trace relation

- Covariant conservation equation $\nabla_k \langle T_i^k \rangle_b = 0$

$$\partial_1 \langle T_1^1 \rangle_b + \partial_D \langle T_1^D \rangle_b - \frac{D+1}{z} \langle T_1^D \rangle_b = 0,$$

$$\partial_1 \langle T_D^1 \rangle_b + \partial_D \langle T_D^D \rangle_b - \frac{D}{z} \langle T_D^D \rangle_b + \frac{1}{z} \sum_{k=0}^{D-1} \langle T_k^k \rangle_b = 0.$$

- Trace relation

$$\langle T_i^i \rangle_{\mathrm{b}} = D(\xi - \xi_D) \nabla_l \nabla^l \langle \varphi^2 \rangle_{\mathrm{b}} + m^2 \langle \varphi^2 \rangle_{\mathrm{b}}$$

Vacuum energy-momentum tensor: Asymptotes

- VEV of the energy-momentum tensor near branes

$$\langle T_0^0\rangle_{\rm b} \approx \frac{(1-D)\langle T_1^1\rangle_{\rm b}}{(|x^1-a_j|/z)^2} \approx \frac{z\langle T_D^1\rangle_{\rm b}}{x^1-a_j} \approx \frac{2D(\xi_D-\xi)\Gamma(\frac{D+1}{2})}{\pi^{\frac{D+1}{2}}(2\alpha|x^1-a_j|/z)^{D+1}}$$

- Minkowski limit $\alpha \to \infty$

$$\langle T_i^i\rangle_{(\mathrm{M})}^{(0)} = -\frac{(4\pi)^{-\frac{D}{2}}}{D\Gamma(D/2)}\int_m^\infty d\lambda \frac{(\lambda^2-m^2)^{D/2}}{c_1(\lambda)c_2(\lambda)e^{2a\lambda}-1} \qquad i \neq 1$$

$$\times\left[2+\frac{4D(\xi-\xi_D)w^2-m^2}{\lambda^2-m^2}\sum_{j=1,2} e^{2|x^1-a_j|\lambda}c_j(\lambda)\right],$$

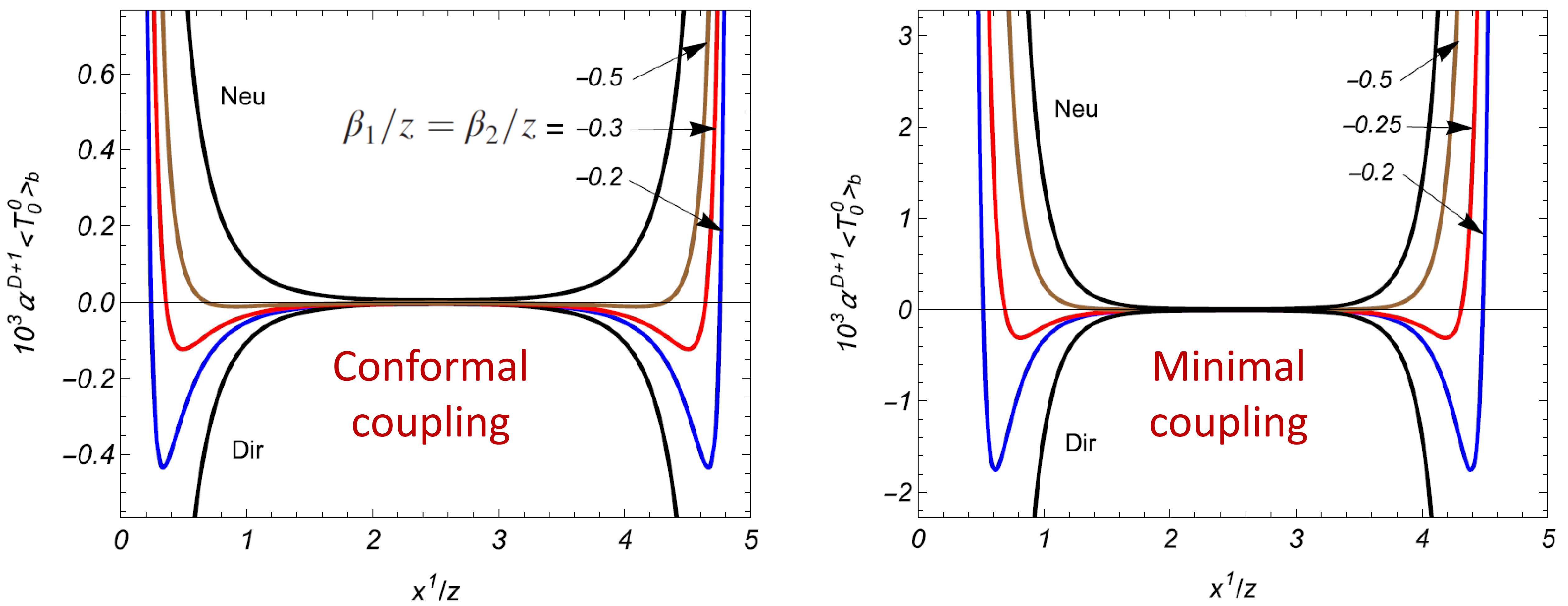
$$\langle T_1^1\rangle_{(\mathrm{M})}^{(0)} = \frac{2(4\pi)^{-\frac{D}{2}}}{\Gamma(D/2)}\int_m^\infty d\lambda \frac{\lambda^2(\lambda^2-m^2)^{D/2-1}}{c_1(\lambda)c_2(\lambda)e^{2a\lambda}-1},$$

$$\langle T_D^1\rangle \approx -\frac{2(4\pi)^{-\frac{D}{2}}}{\Gamma(D/2)\alpha}\int_m^\infty d\lambda \frac{\sum_{j=1,2}(-1)^j e^{2|x^1-a_j|\lambda}c_j(\lambda)}{c_1(\lambda)c_2(\lambda)e^{2a\lambda}-1}$$

$$\times\lambda(\lambda^2-m^2)^{D/2-2}\left[D(\xi-\xi_D)\lambda^2-\left(2\xi-\frac{1}{4}\right)m^2\right].$$

Vacuum energy density

$$D = 4,\ m\alpha = 0.5,\ a_1 = 0,\ a_2 / z = 5$$



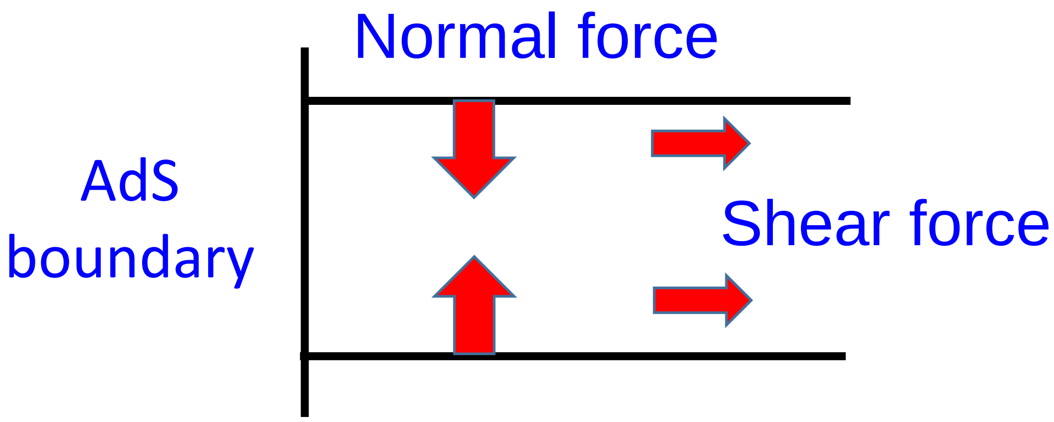
The Casimir forces

- The i-th component of the force acting on the surface element dS of the brane at $x^1 = a_j$ is given by

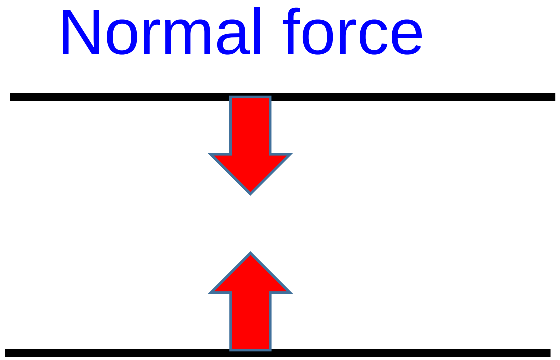
$$dF^i_{(j)} = \langle T^i_1 \rangle \Big|^{x^1 = a_j - 0}_{x^1 = a_j + 0} dS$$

Due to the nonzero off-diagonal stress, in addition to the normal component this force has nonzero component parallel to the brane (shear force)

AdS bulk



Minkowski bulk



Directions of the forces depend on the coefficients in the boundary conditions on the branes

Normal force

- Normal force acting on the brane at $x^1 = a_j$: $dF^1_{(j)} = \langle T^1_1 \rangle |^{z=a_j-0}_{z=a_j+0} dS$
- Vacuum effective pressure

$$P_j = \frac{\alpha^{-1-D}}{2^{D+2\nu}\pi^{\frac{D-1}{2}}} \int_0^\infty dx\, x \frac{-2 + [2 + c_j(x/z) + 1/c_j(x/z)]\hat{B}_1}{c_1(x/z)c_2(x/z)e^{2ax/z} - 1} x^{D+2\nu} F_\nu^{\frac{D}{2}}(x)$$

$$\hat{B}_1 = \left(\xi - \frac{1}{4}\right)\partial_x^2 + \left[\frac{D-1}{4} - (D-2)\xi\right]\frac{\partial_x}{x} - \frac{D\xi}{x^2}$$

- Small separations between the branes

$$P_j \approx -\frac{D\zeta(D+1)}{(2\sqrt{\pi}\alpha a/z)^{D+1}}\Gamma\left(\frac{D+1}{2}\right)$$ Attractive force

- For non-Dirichlet BC one brane and Dirichlet BC on other:

$$P_j \approx \frac{D\zeta(D+1)}{(2\sqrt{\pi}\alpha a/z)^{D+1}}\left(1 - \frac{1}{2^D}\right)\Gamma\left(\frac{D+1}{2}\right)$$ Repulsive force

At small separations the effect of gravity is weak

Normal force

- The effect of gravity is essential at large separations between the branes

$$P_j \approx -\frac{2(D+2\nu+1)(4\nu B_\nu \beta_j^2/z^2+1)}{\pi^{\frac{D}{2}}\Gamma(1+\nu)\alpha^{D+1}(2a/z)^{D+2\nu+2}}\zeta(D+2\nu+2)\Gamma\left(\frac{D}{2}+\nu+1\right)$$

For non-Neumann boundary conditions on both branes

$$B_\nu = (D+2\nu+1)\xi - \frac{D+2\nu}{4}$$

- For non-Neumann BC on the brane and Neumann BC on the second brane

$$P_j \approx \frac{2(D+2\nu+1)(4\nu B_\nu \beta_j^2/z^2+1)}{\pi^{\frac{D}{2}}\Gamma(1+\nu)\alpha^{D+1}(2a/z)^{D+2\nu+2}}\left(1-\frac{1}{2^{D+2\nu+1}}\right)\zeta(D+2\nu+2)\Gamma\left(\frac{D}{2}+\nu+1\right)$$

- For Neumann BC on the brane and non-Neumann BC on the second brane

$$P_j \approx -\frac{4\nu B_\nu(1-2^{1-D-2\nu})\zeta(D+2\nu)}{\pi^{\frac{D}{2}}\Gamma(1+\nu)\alpha^{D+1}(2a/z)^{D+2\nu}}\Gamma\left(\frac{D}{2}+\nu\right)$$

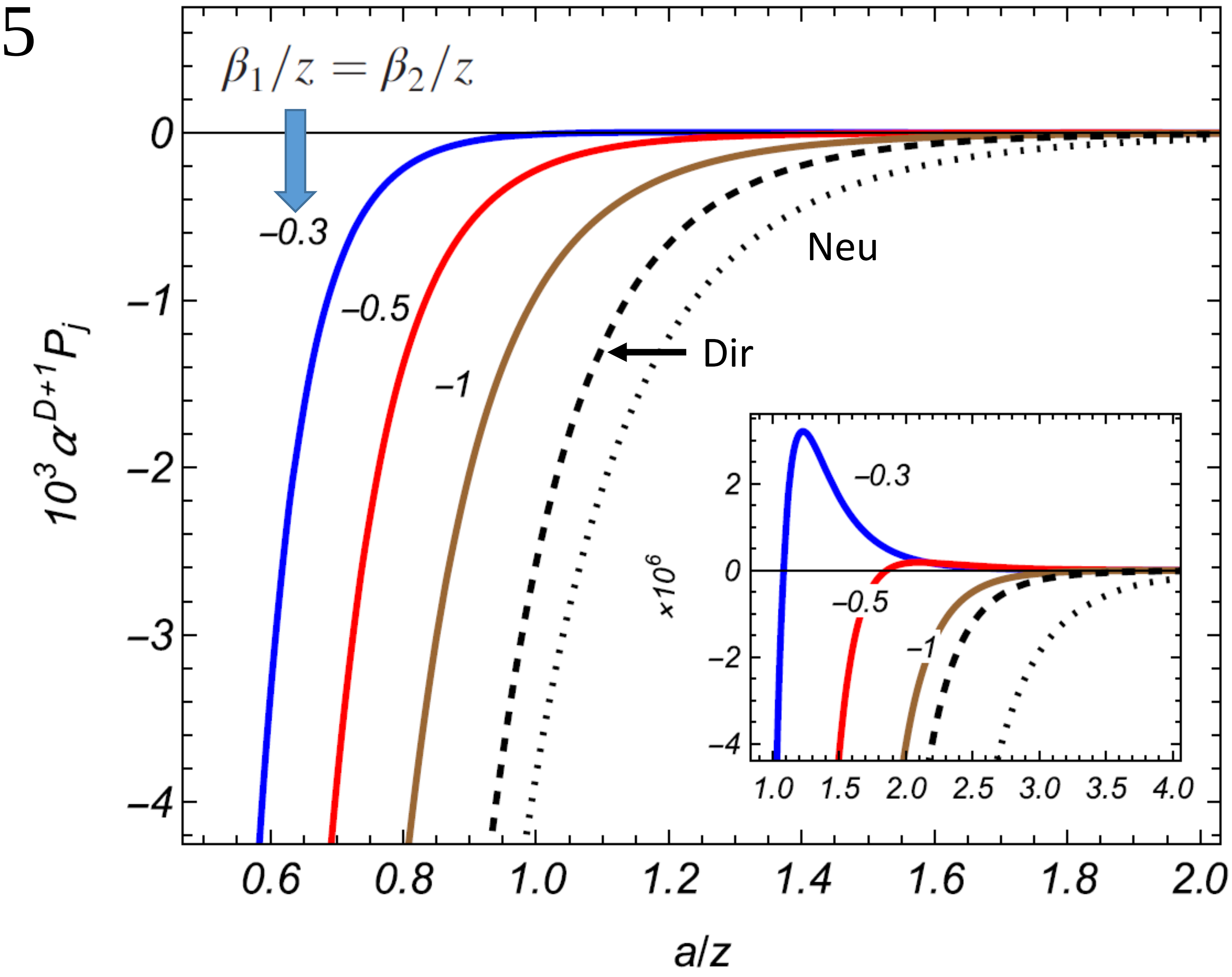
- For Neumann BC on both branes

$$P_j \approx \frac{4\nu B_\nu \zeta(D+2\nu)\Gamma(D/2+\nu)}{\pi^{\frac{D}{2}}\Gamma(1+\nu)\alpha^{D+1}(2a/z)^{D+2\nu}}$$

Casimir normal force

$$D = 4,\ m\alpha = 0.5$$

Minimal coupling

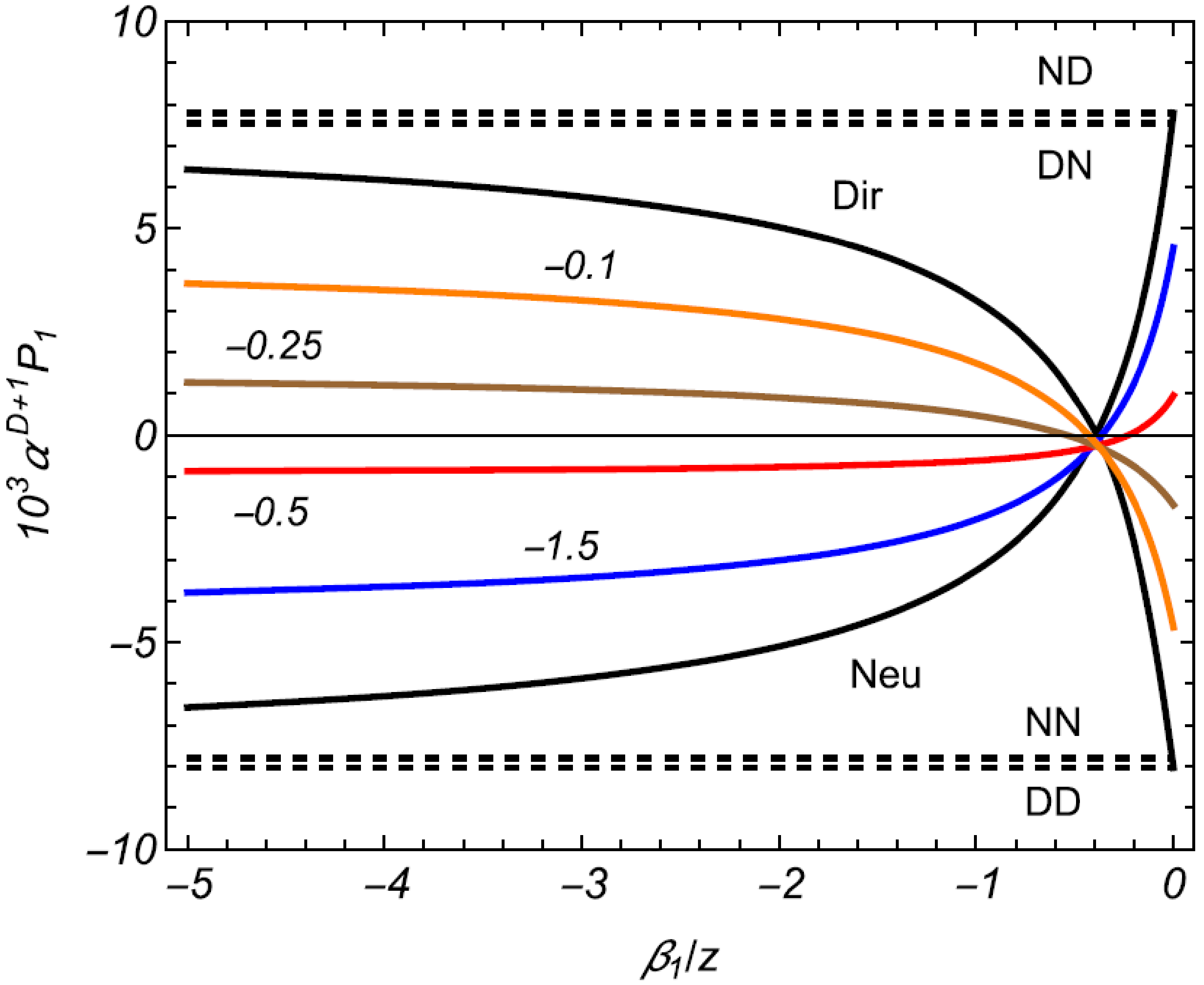


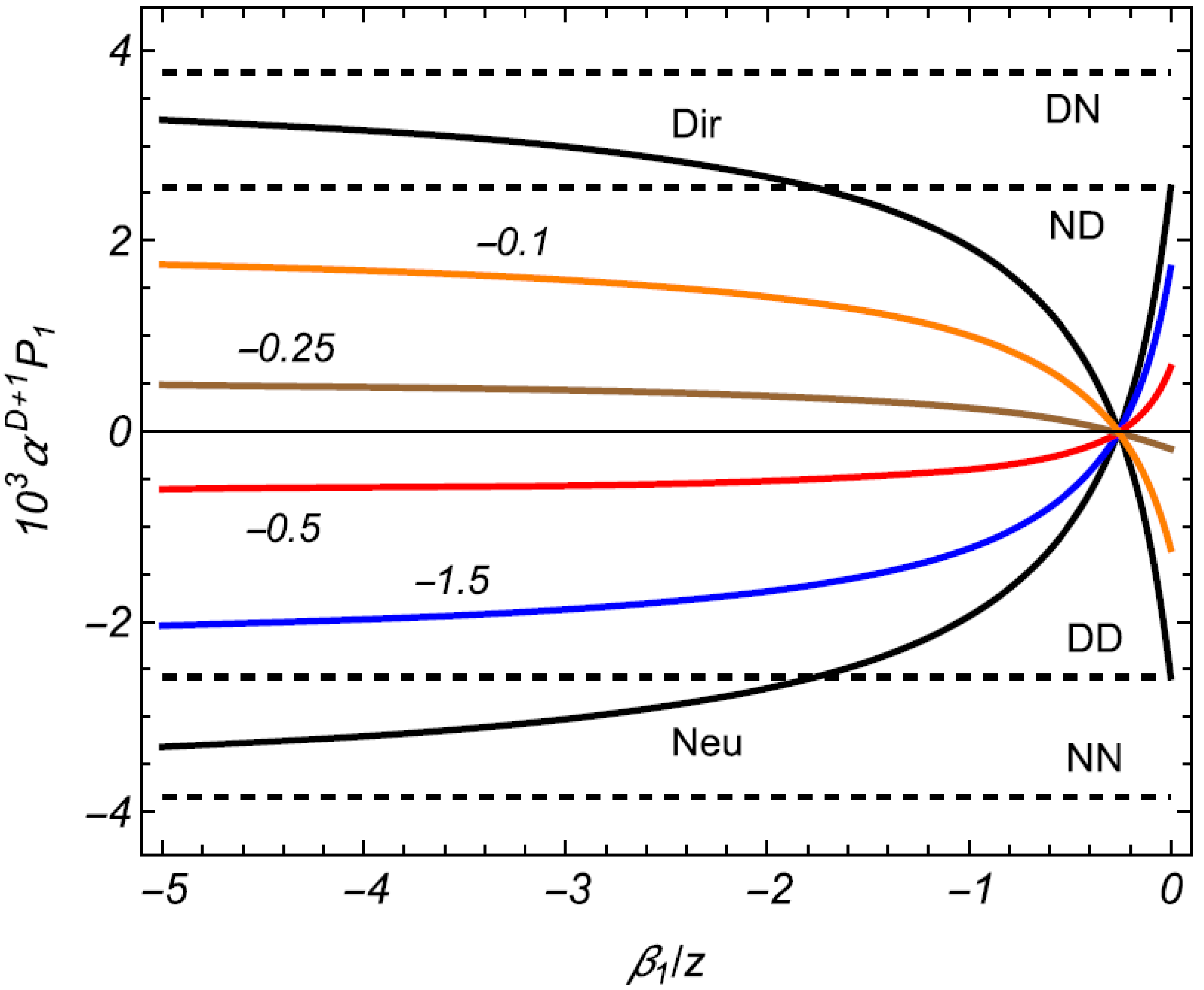
Casimir normal force

$$D=4,\ m\alpha=0.5,\ a/z=1$$

Conformal coupling

Minimal coupling





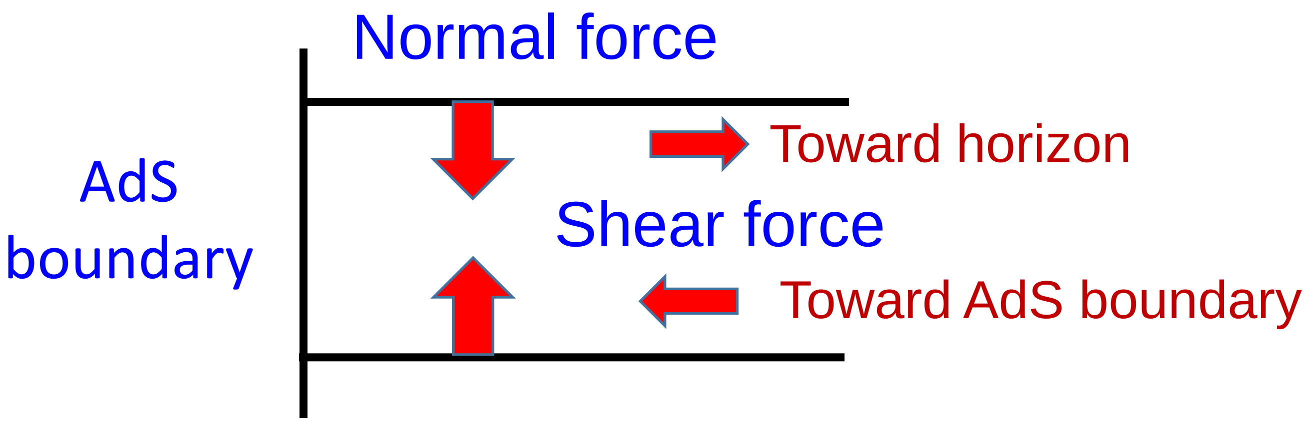
Shear force

- The part in the shear force induced by the second brane

$$f_j^{(\mathrm{int})} = -\frac{2\alpha^{-1-D}}{2^{D+2\nu}\pi^{\frac{D-1}{2}}}\int_0^\infty dx \frac{c_j(x/z) - 1/c_j(x/z)}{c_1(x/z)c_2(x/z)e^{2ax/z} - 1}\left[\left(\xi - \frac{1}{4}\right)x\partial_x + \xi\right]x^{D+2\nu}F_\nu^{\frac{D}{2}}(x)$$

- Shear force is zero for Dirichlet and Neumann boundary conditions on the brane regardless of boundary conditions on the second brane

- Shear force is directed toward the horizon for $f_j^{(\mathrm{int})} > 0$ and toward the AdS boundary for $f_j^{(\mathrm{int})} < 0$



Shear force: Asymptotes

- Shear force at small separations between the branes

$$f_j^{(\mathrm{int})} \approx \frac{4D(\xi-\xi_D)\zeta(D-1)}{\pi^{\frac{D+1}{2}}\alpha^{D+1}(2a/z)^D b_j}\Gamma\left(\frac{D-1}{2}\right)(2^{2-D}-1)^{\delta_{0b_{j'}}}, \quad b_j = \beta_j / a$$

At small separations, the shear component of the force has opposite signs for Dirichlet and non-Dirichlet boundary conditions on the second brane

- Shear force at large separations between the branes

$$f_j^{(\mathrm{int})} \approx -\frac{4b_j B_\nu(D+2\nu+1)\zeta(D+2\nu+2)}{\pi^{\frac{D}{2}}\Gamma(\nu+1)\alpha^{D+1}(2a/z)^{D+2\nu+1}}\Gamma\left(\frac{D}{2}+\nu+1\right)\left(\frac{1}{2^{D+2\nu+1}}-1\right)^{\delta_{\infty b_{j'}}}$$

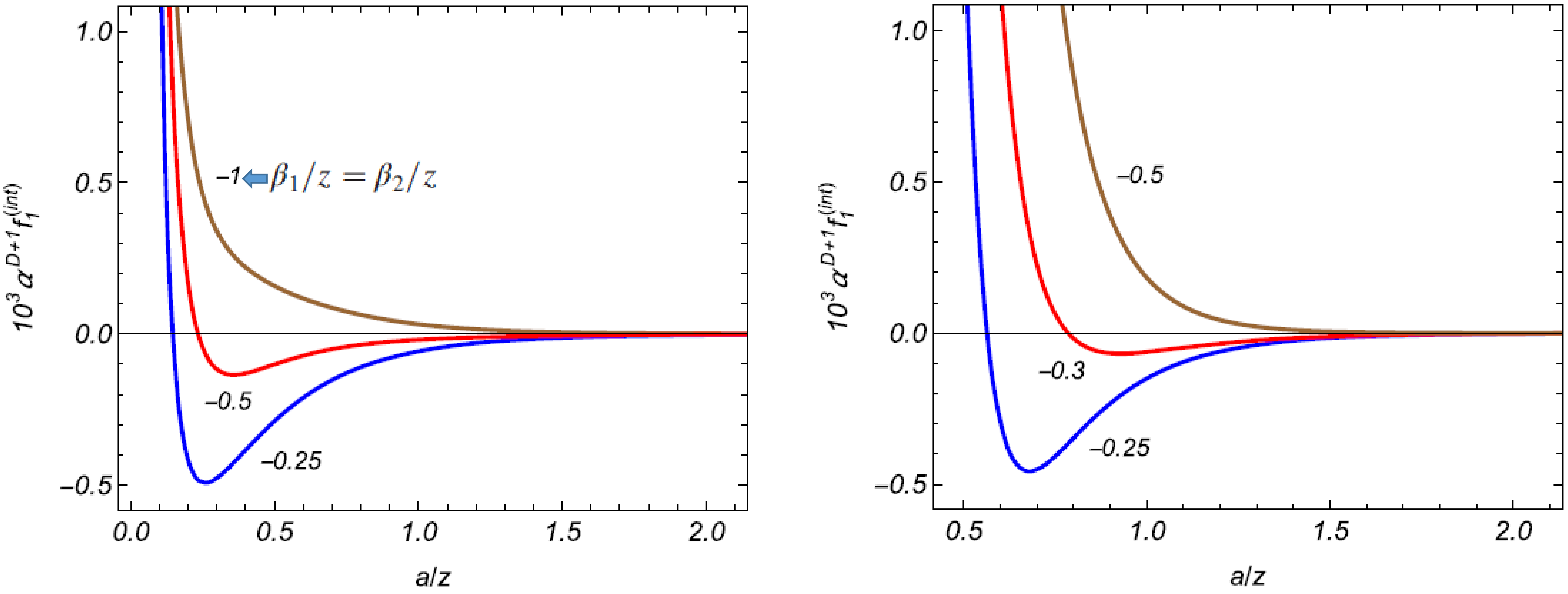
Force has opposite signs for Neumann and non-Neumann boundary conditions on the brane

Shear force: Numerics

$$D = 4,\ m\alpha = 0.5$$

Conformal coupling

Minimal coupling



Shear force: Numerics

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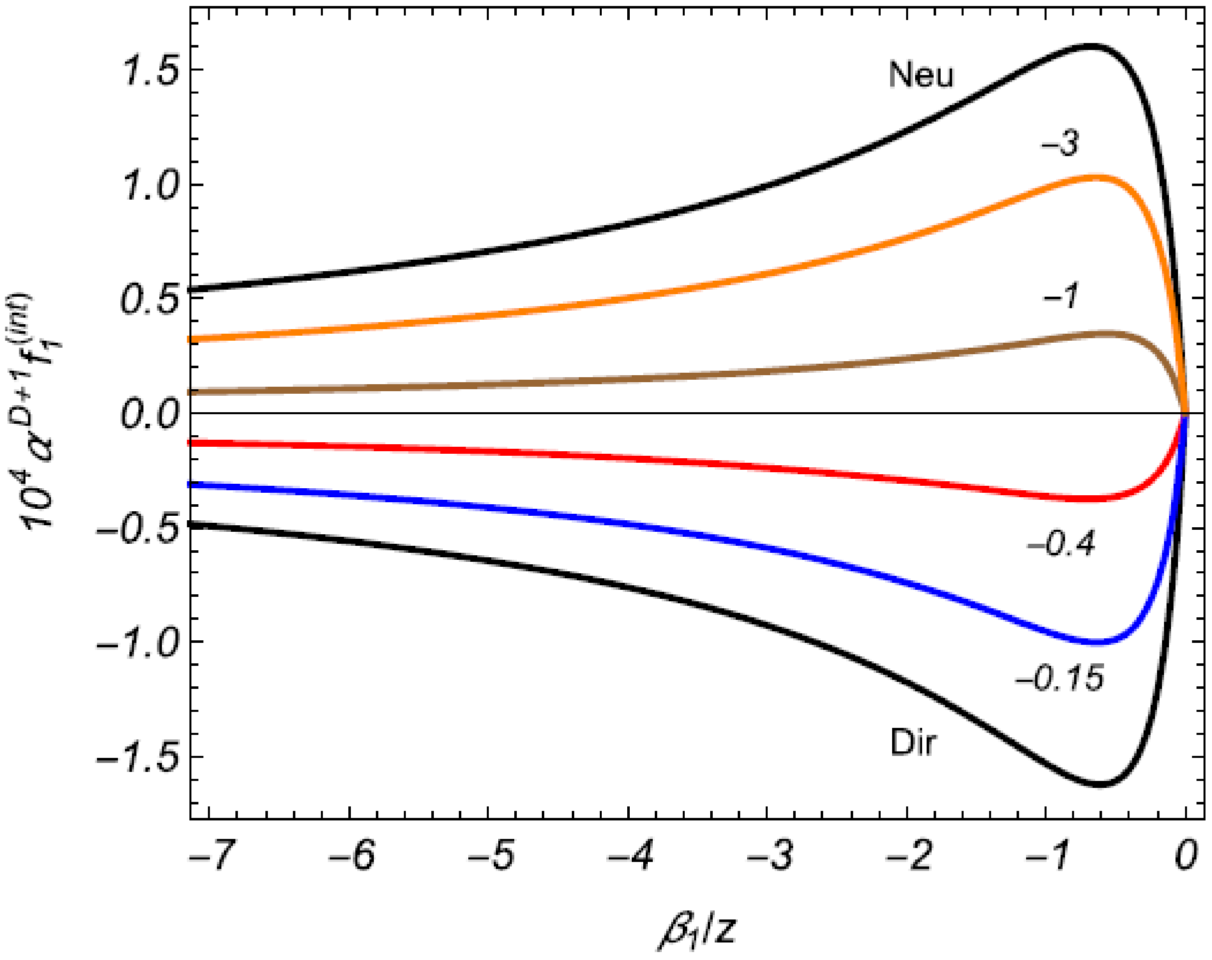
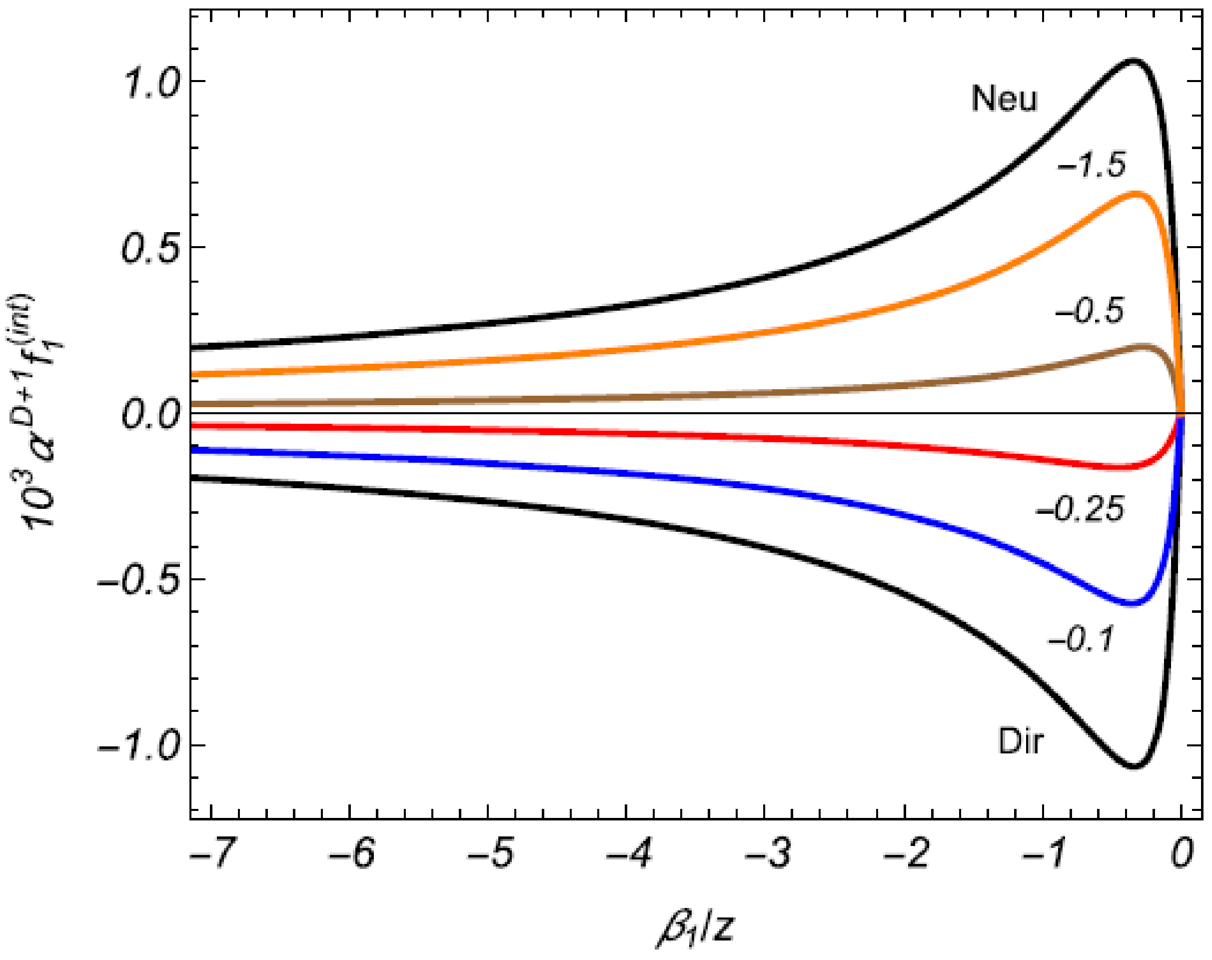


Minimal coupling



Conclusions

- We have investigated combined effects of gravity and boundaries on local characteristics of the scalar vacuum
- Near the boundaries the effects of gravity on the mean field squared, on the energy density and parallel stresses are weak
- The effect of gravity is essential at distances from the brane larger than the curvature radius
- Off-diagonal component of the vacuum energy-momentum tensor is induced that gives arise to shear force acting on branes
- VEVs vanish on the AdS boundary and diverge on the horizon
- Signs for both the normal and shear Casimir forces depend on the Robin coefficients

Thank you