Correlations and Fluctuations in Heavy Ion Collisions

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Part I
Very short Introduction

Why correlations and fluctuations?

- Looking only at the averages can deceive you.
  - ∗ (Ex.) Micro-canonical, Canonical, Grand-canonical

    E, $\langle E \rangle$ can be all the same, but *fluctuations* are not.

- To study a blackbox – Shake and listen.
  - ∗ Disturb the system with a *known* force and observe the response
  - ∗ Response function: $\theta(t - t')\left\langle [\rho(x), \rho(x')] \right\rangle$

Why corr. and fluct.? – Cont.

- First order phase transition: Entropy, Volume, Enthalpy becomes discontinuous – Finite latent heat

- Second order phase transition: S, V, H changes smoothly – No latent heat – but their derivative becomes discontinuous

  * In Stat-Mech: Derivative of an average $\Longrightarrow$Susceptibility $\Longrightarrow$Fluctuation

$$\text{(Ex.)}\ \chi_M = \frac{1}{V}\frac{\partial M}{\partial H} \propto \left\langle M^2 \right\rangle - \langle M \rangle^2$$

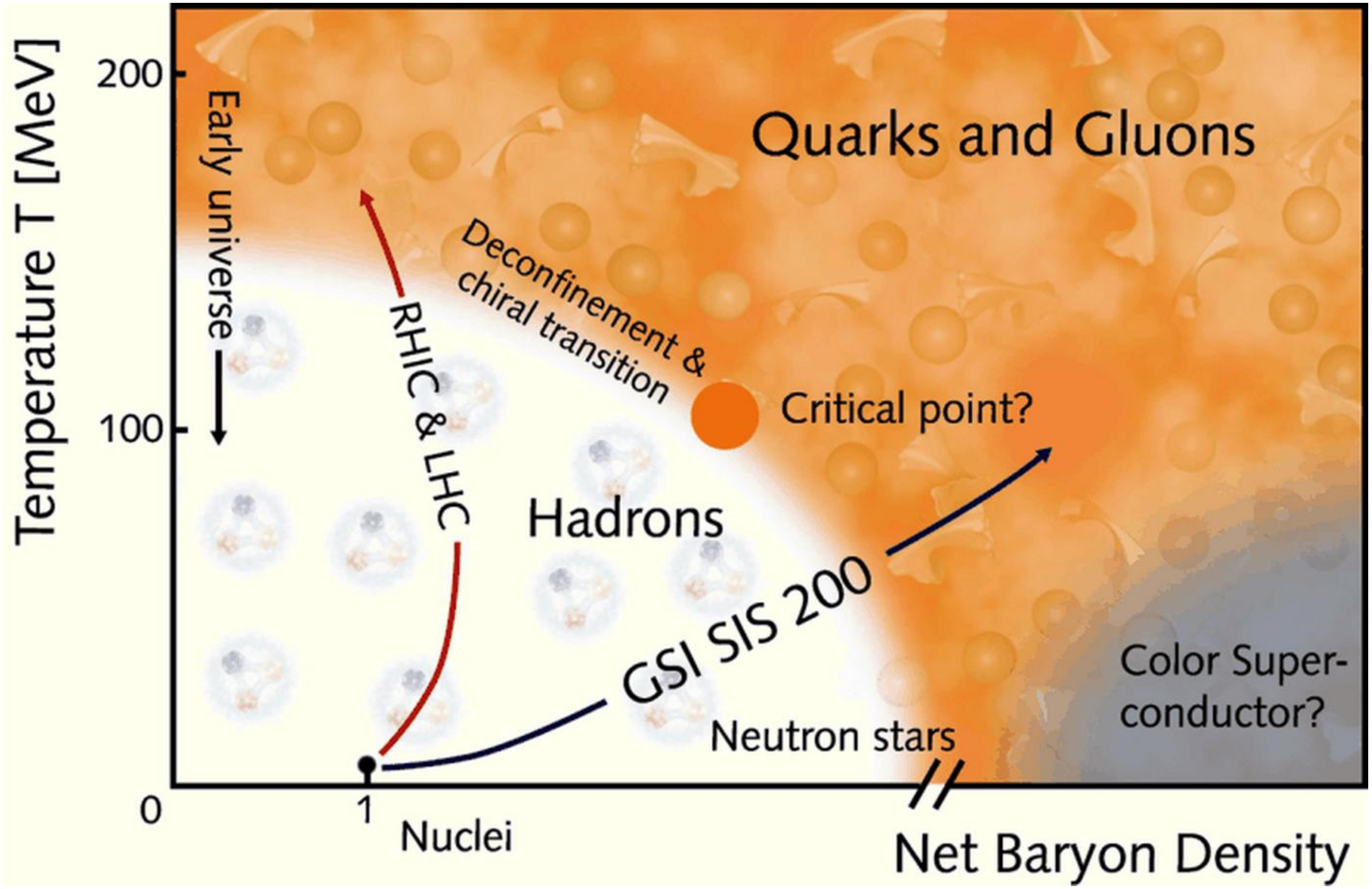
- Second order phase transition: Long range fluctuations. $\Longrightarrow$All wavelengths are excited

- Can also use Quantum correlations – HBT

In heavy ion collisions ...

- A lot of the 'QGP signals' and 'QGP puzzles' involve correlations/fluctuations one way or another
  - HBT : 2-Point momentum correlation function
  - Elliptic flow (v_2) : Conditional probability. (Recall: $P(a|b) = P(a,b)/P(b)$.) Given the reaction plane, what is the excess ...
  - Jet quenching : Conditional probability. Given we see a near side jet ...
  - p_T fluctuations
  - Multiplicity fluctuations
  - Net charge fluctuations
  - Many more ...

Motivation



And this happens with local thermal equilibrium maintained. (That's the assumption anyway.)

Basic Assumptions

- 1 Collision event = 1 member of an statistical ensemble.
- Doesn't have to be an equilibrium ensemble.
- Averages:

$$\langle O \rangle = \mathrm{Tr}\hat{\rho}O \leftrightarrow \frac{1}{N_{\text{events}}} \sum_{i=1}^{N_{\text{events}}} O_i$$

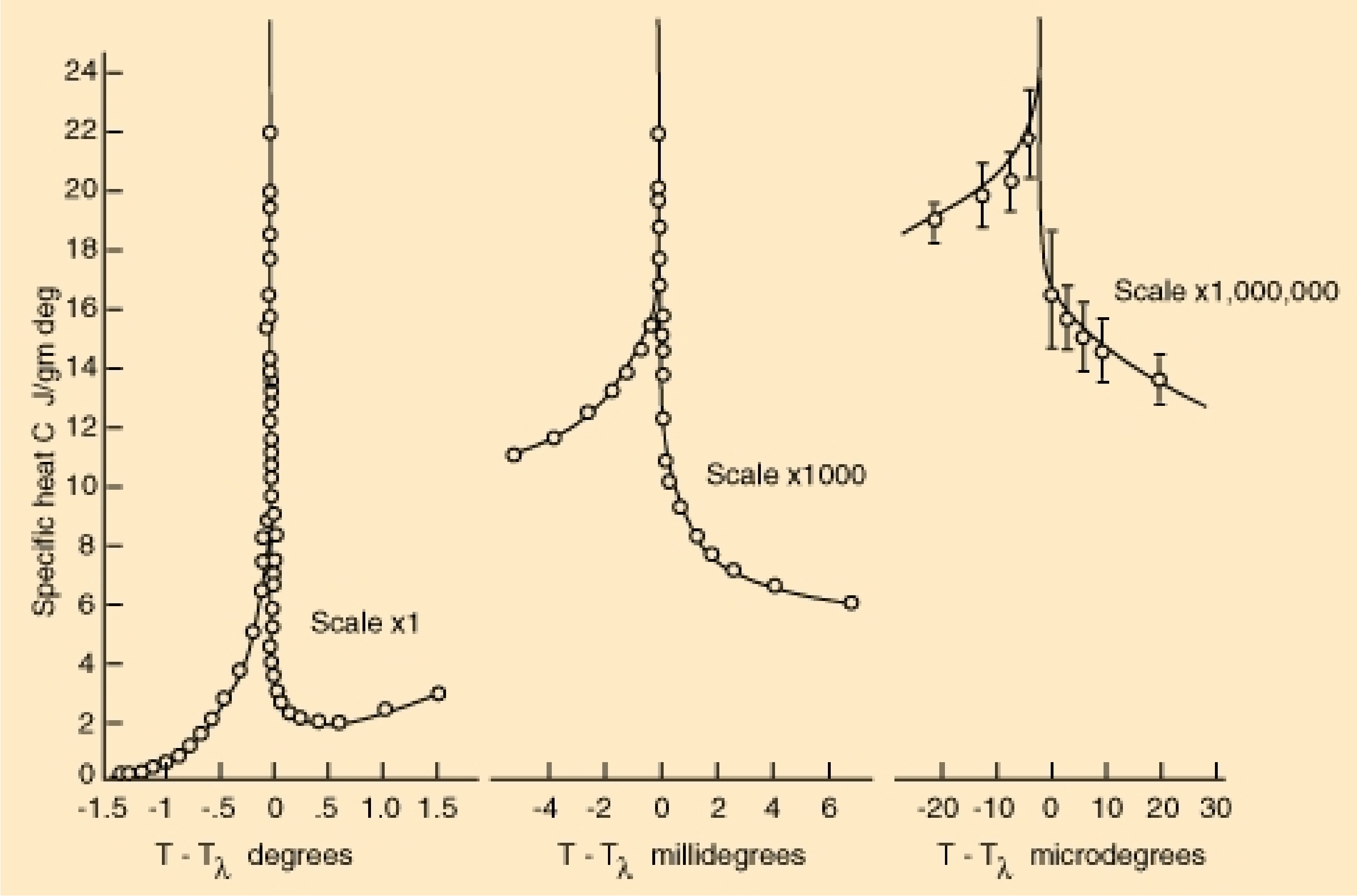
- Correlations:

$$\langle O_x O_y \rangle = \mathrm{Tr}\hat{\rho}O_x O_y \leftrightarrow \frac{1}{N_{\text{events}}} \sum_{i=1}^{N_{\text{events}}} O_x^i O_y^i$$

- Fluctuations:

$$\left\langle \Delta O^2 \right\rangle = \left\langle O^2 \right\rangle - \langle O \rangle^2$$

Hope to see (but we won't)



Part II
Charge Independent Correlations and Fluctuations

Multiplicity Fluctuations

- Simplest test of thermalization

$$\langle N \rangle = gV \int \frac{d^3p}{(2\pi)^3} n(E_p)$$

$$\left\langle \Delta N^2 \right\rangle = gV \int \frac{d^3p}{(2\pi)^3} n(E_p)[1 \pm n(E_p)] \sim \langle N \rangle$$

- In $\bar{p}p$ between $10\,\text{GeV} < \sqrt{s} < 546\,\text{GeV}$, UA1 and UA5 found,

$$E\frac{d\sigma}{d^3p} = \begin{cases} A\exp(-bm_T) & \text{Low } p_T : \text{ Bulk. Looks thermal.} \\ B(1 + p_0/p_T)^{-n} & \text{High } p_T \end{cases}$$

But

$$\left\langle \Delta N^2 \right\rangle = \langle N \rangle (1 + \langle N \rangle / k) \sim \langle N \rangle^2$$

with $k \sim 3 - 4$ and $\langle N \rangle \sim 20 - 30$

No multiple re-scatterings $\Longrightarrow$ No real thermalization

STAR Data from QM01

J.G.Reid, “STAR Event-by-event Fluctuations”.

For $\sqrt{s_{NN}} = 130\,\mathrm{GeV}$,

- $\left\langle \Delta N_{\mathrm{ch}}^2 \right\rangle = 2.09 \left\langle N_{\mathrm{ch}} \right\rangle = \left\langle N_{\mathrm{ch}} \right\rangle + \sigma_V^2 + \sigma_R^2$

- With $\sigma_V^2 = 0.83 \left\langle N_{\mathrm{ch}} \right\rangle$ and $\sigma_R^2 = 0.25 \left\langle N_{\mathrm{ch}} \right\rangle$

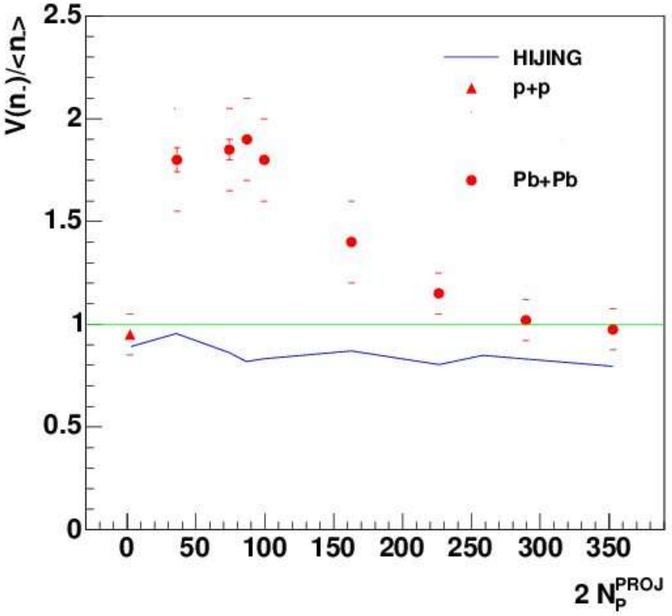
$$\left\langle \Delta N_{\mathrm{ch}}^2 \right\rangle_{\mathrm{corr.}} \approx \left\langle N_{\mathrm{ch}} \right\rangle$$

- Food for thoughts only. This analysis is still ‘preliminary’.

DATA from NA49

Marek Gazdzicki, Correlations and Fluctuations 2005

The scaled variance ...



The scaled variance is corrected for the resolution of ZDC and the final width of the E_{ZDC} interval

Note that the previously shown results for C+C and Si+Si collisions were incorrect, the reanalysis is in progress

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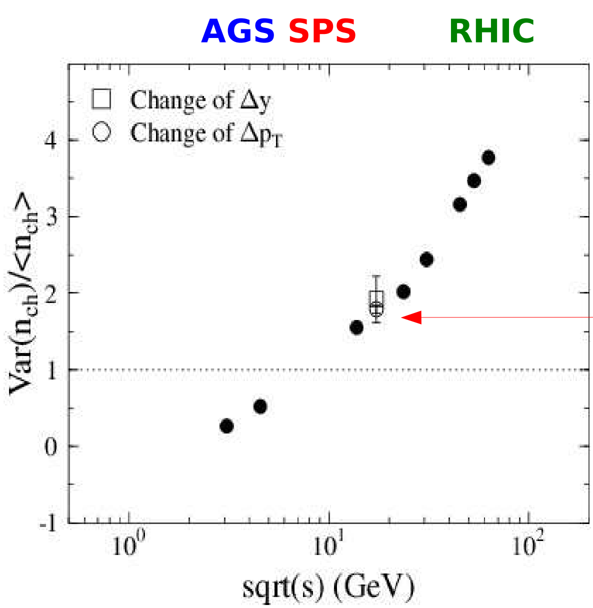
DATA from NA49

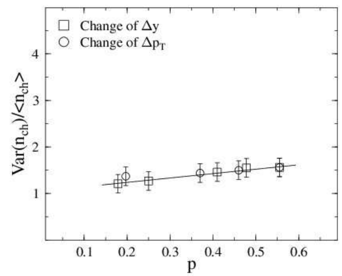
Marek Gazdzicki, Correlations and Fluctuations 2005

The scaled variance in p+p interactions



The scaled variance increases with collision energy

The extrapolated NA49 results agree with the old p+p data



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Again, food for thoughts only. No conclusion can be drawn.

Mean p_T Fluctuations

- Motivation: $\pi - \sigma$

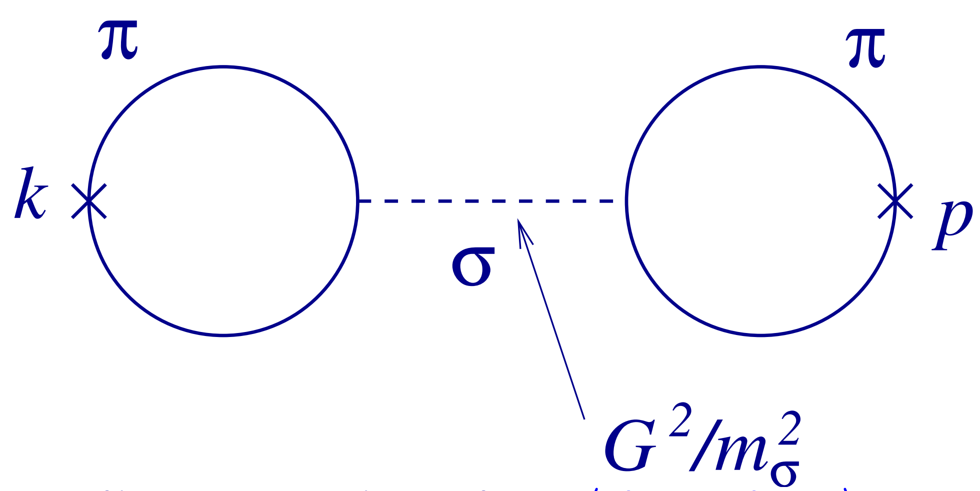
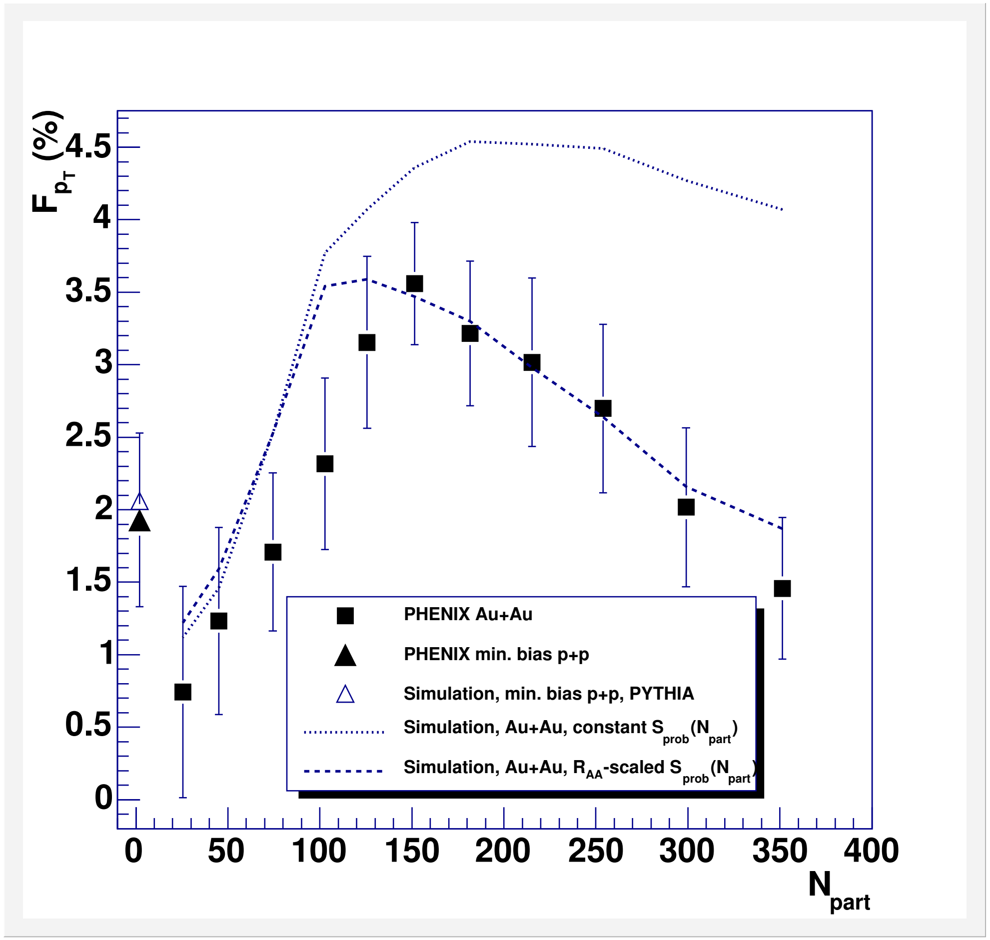


Diagram that contribute to the pion $\langle \Delta n_k \, \Delta n_p \rangle$ correlator.

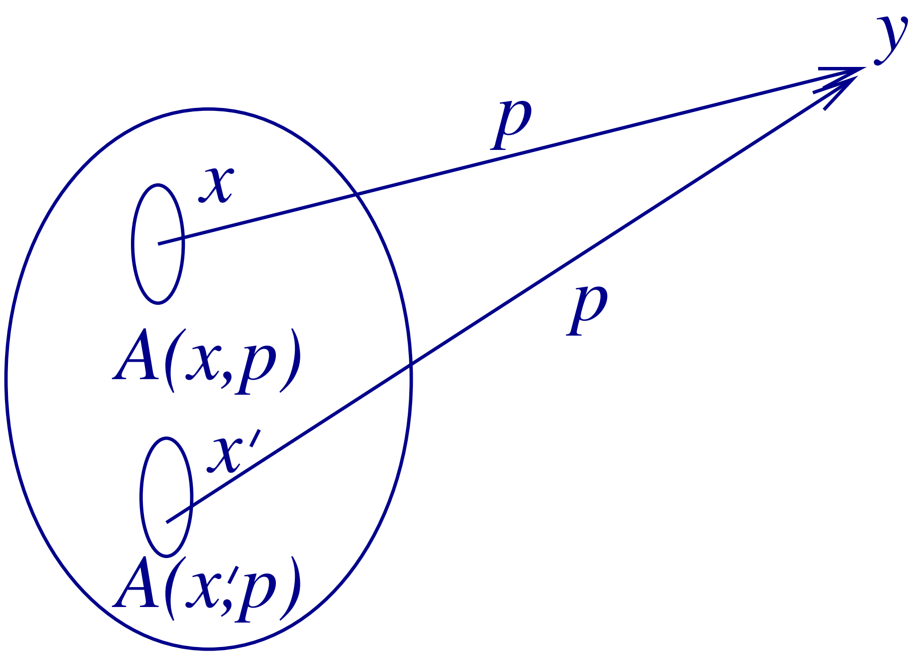
- At the critical point $m_\sigma \to 0$
- If chiral symmetry is not explicitly broken $G \to 0$
- Chiral symmetry is explicitly broken by $m_q \neq 0 \Longrightarrow G$ doesn't have to vanish $\Longrightarrow$Pion p_T fluctuations can be large.

Phenix Data



- Excess seen. But can be explained by correlations among high p_T (jet) particles (with energy loss).

Hanbury-Brown-Twiss



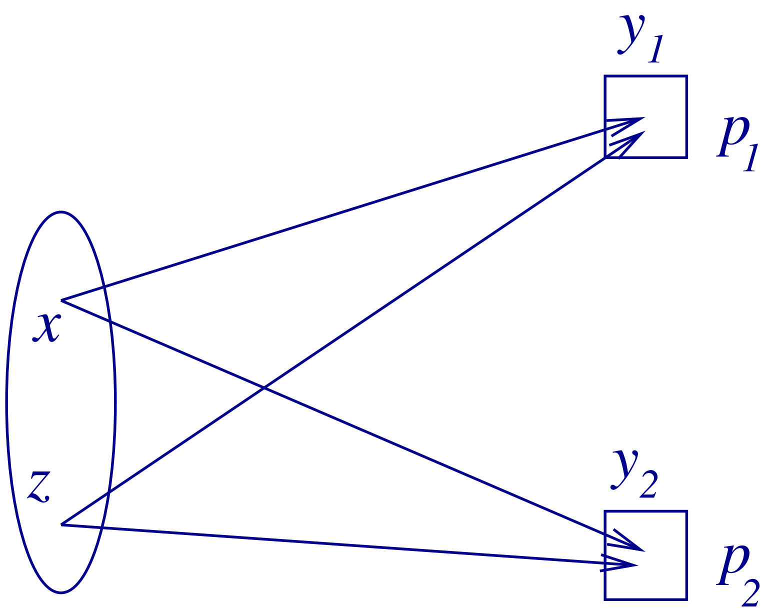
Signal from a distributed source:

$$\Psi_{\mathbf{p}}(\mathbf{y}) = \sum_{\mathbf{x}} A(\mathbf{x},\mathbf{p}) e^{i\phi_{\mathbf{x}}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

Squaring and using $\left\langle e^{i(\phi_{\mathbf{x}}-\phi_{\mathbf{z}})} \right\rangle = \delta_{\mathbf{x},\mathbf{z}}$,

$$P(\mathbf{p}) = \sum_{\mathbf{x}} A(\mathbf{x},\mathbf{p})^2 = \int d\mathbf{x}\, \rho(\mathbf{x})\, A(\mathbf{x},\mathbf{p})^2$$

HBT



Detector 1 got $\mathbf{p}_1$ and detector 2 got $\mathbf{p}_2$.

Total amplitude have $\mathbf{p}_1$ at $\mathbf{y}_1$ and $\mathbf{p}_2$ at $\mathbf{y}_2$:

$$\Psi(\mathbf{y}_1,\mathbf{p}_1;\mathbf{y}_2,\mathbf{p}_2) = \frac{1}{2}\sum_{\mathbf{x},\mathbf{z}}\Big(A(\mathbf{x},\mathbf{p}_1)e^{i\mathbf{p}_1\cdot(\mathbf{x}-\mathbf{y}_1)}e^{\phi_\mathbf{x}}A(\mathbf{z},\mathbf{p}_2)e^{i\mathbf{p}_2\cdot(\mathbf{z}-\mathbf{y}_2)}e^{\phi_\mathbf{z}}$$
$$+ A(\mathbf{z},\mathbf{p}_1)e^{i\mathbf{p}_1\cdot(\mathbf{z}-\mathbf{y}_1)}e^{\phi_\mathbf{z}}A(\mathbf{x},\mathbf{p}_2)e^{i\mathbf{p}_2\cdot(\mathbf{x}-\mathbf{y}_2)}e^{\phi_\mathbf{x}}\Big)$$

HBT

- Squaring and using $\left\langle e^{i(\phi_{\mathbf{x}}-\phi_{\mathbf{z}})} \right\rangle = \delta_{\mathbf{x},\mathbf{z}}$,

$$C(\mathbf{p}_1,\mathbf{p}_2) = \frac{P(\mathbf{p}_1,\mathbf{p}_2)}{P(\mathbf{p}_1)P(\mathbf{p}_2)} = 1 + \left| \int d\mathbf{x}\, \rho_{\text{eff}}(\mathbf{x}) e^{i(\mathbf{p}_1-\mathbf{p}_2)\cdot\mathbf{x}} \right|^2$$

with $\rho_{\text{eff}}(\mathbf{x}) = \rho(\mathbf{x}) \dfrac{A(\mathbf{x},\mathbf{p}_1)A(\mathbf{x},\mathbf{p}_2)}{\sqrt{P(\mathbf{p}_1)P(\mathbf{p}_2)}}$

- Assume plane waves and

$$\rho_{\text{eff}}(\mathbf{x}) = \mathcal{N}\, e^{-\mathbf{x}^2/R^2 - t^2/\Delta\tau^2}$$

With $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$ and $q_0 = E_1 - E_2$, this yields

$$C(\mathbf{p}_1,\mathbf{p}_2) = 1 + e^{-\mathbf{q}^2 R^2/2 - q_0^2 \Delta\tau^2/2}$$

- Now note: $q_0^2 = (E_1 - E_2)^2 = |(\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{p}_1 + \mathbf{p}_2)/(E_1 + E_2)|^2$

  Define $\mathbf{q}_{\text{out}} = (\mathbf{q} \cdot \widehat{\beta})\widehat{\beta}$

  $\widehat{\beta}$: the unit vector in the $\mathbf{p}_1 + \mathbf{p}_2$ direction
  Get

  $$C(\mathbf{p}_1, \mathbf{p}_2) = 1 + \exp\left(-q_{\text{out}}^2 R_{\text{out}}^2/2 - q_{\text{side}}^2 R_{\text{side}}^2/2 - q_{\text{long}}^2 R_{\text{long}}^2/2\right)$$

  with

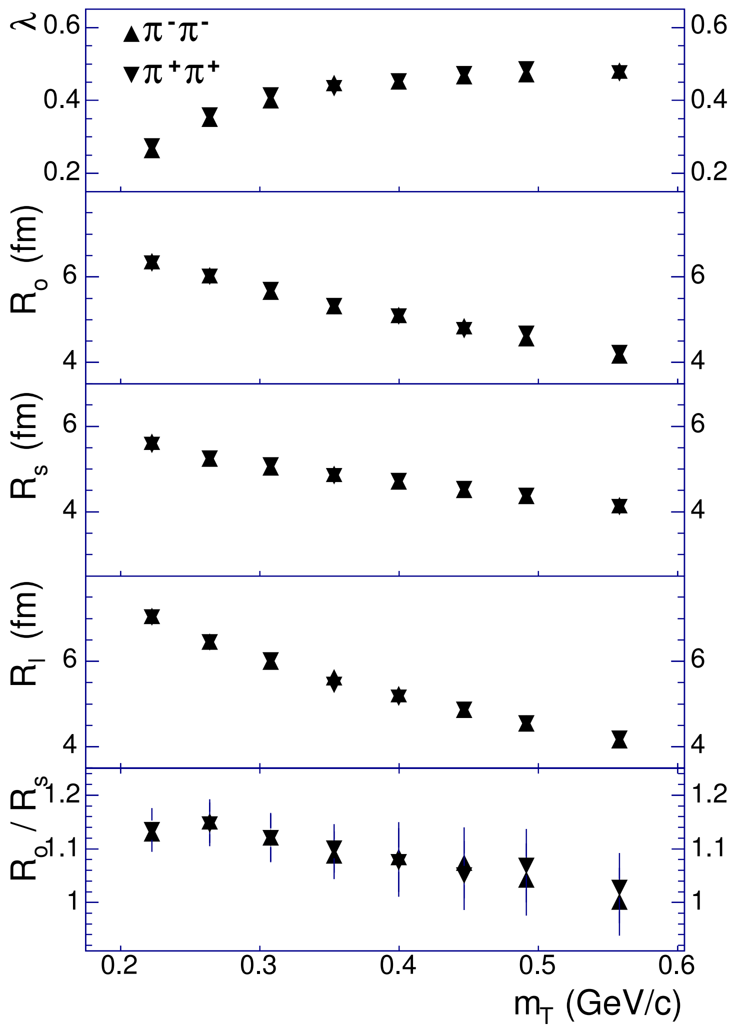
  $$R_{\text{out}}^2 = R^2 + \beta_T^2 \Delta\tau^2$$

  $\mathbf{q}_{\text{long}} || \text{beam}$, $\mathbf{q}_{\text{out}} \perp \mathbf{q}_{\text{side}}$ and $\mathbf{q}_{\text{out}} \perp \mathbf{q}_{\text{long}}$

- Must have

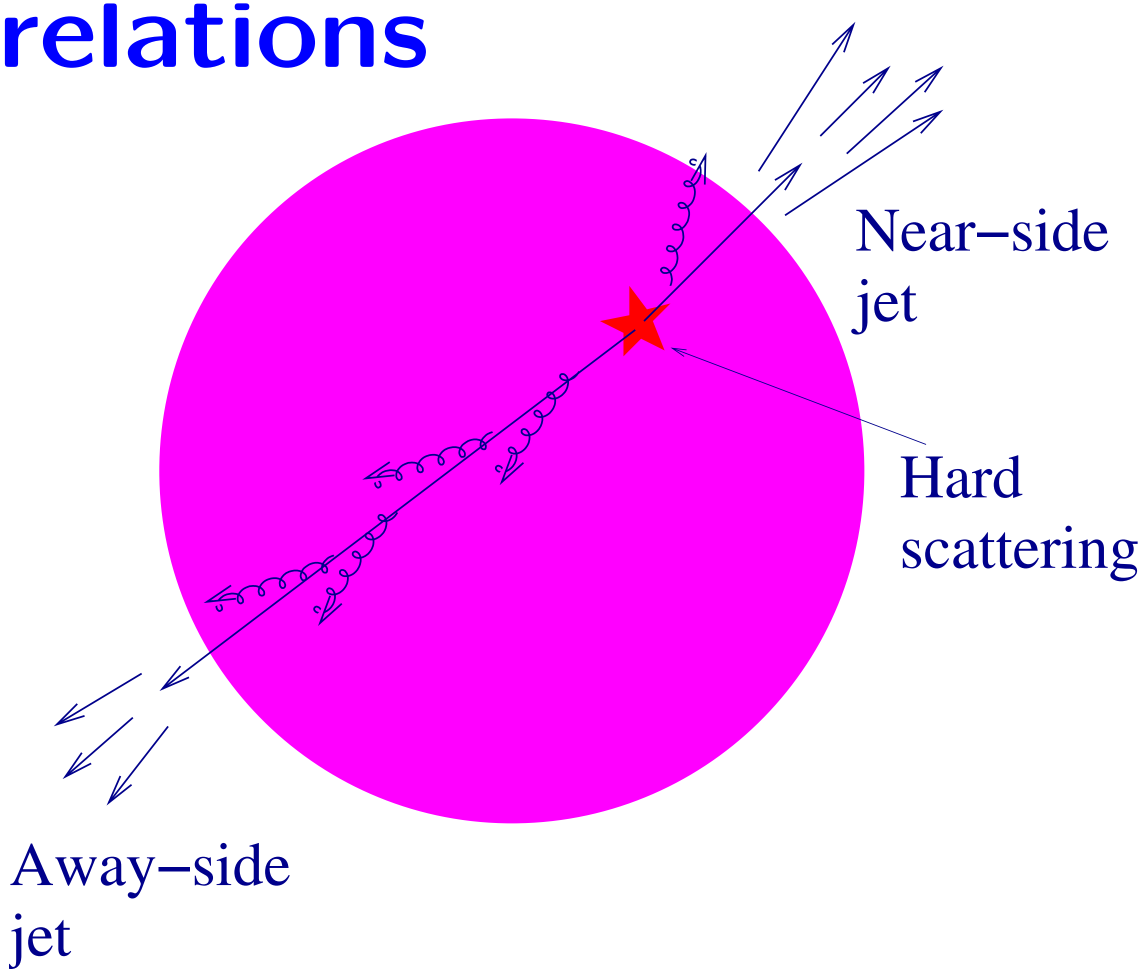
$$R_{\text{side}} < R_{\text{out}}$$

The HBT Puzzle



- $R_{\rm side} \approx R_{\rm out}$!!!
- Conventional interpretation is out.
- What is the source shape?
- Static vs. Evolving system?
- Puzzle remains.

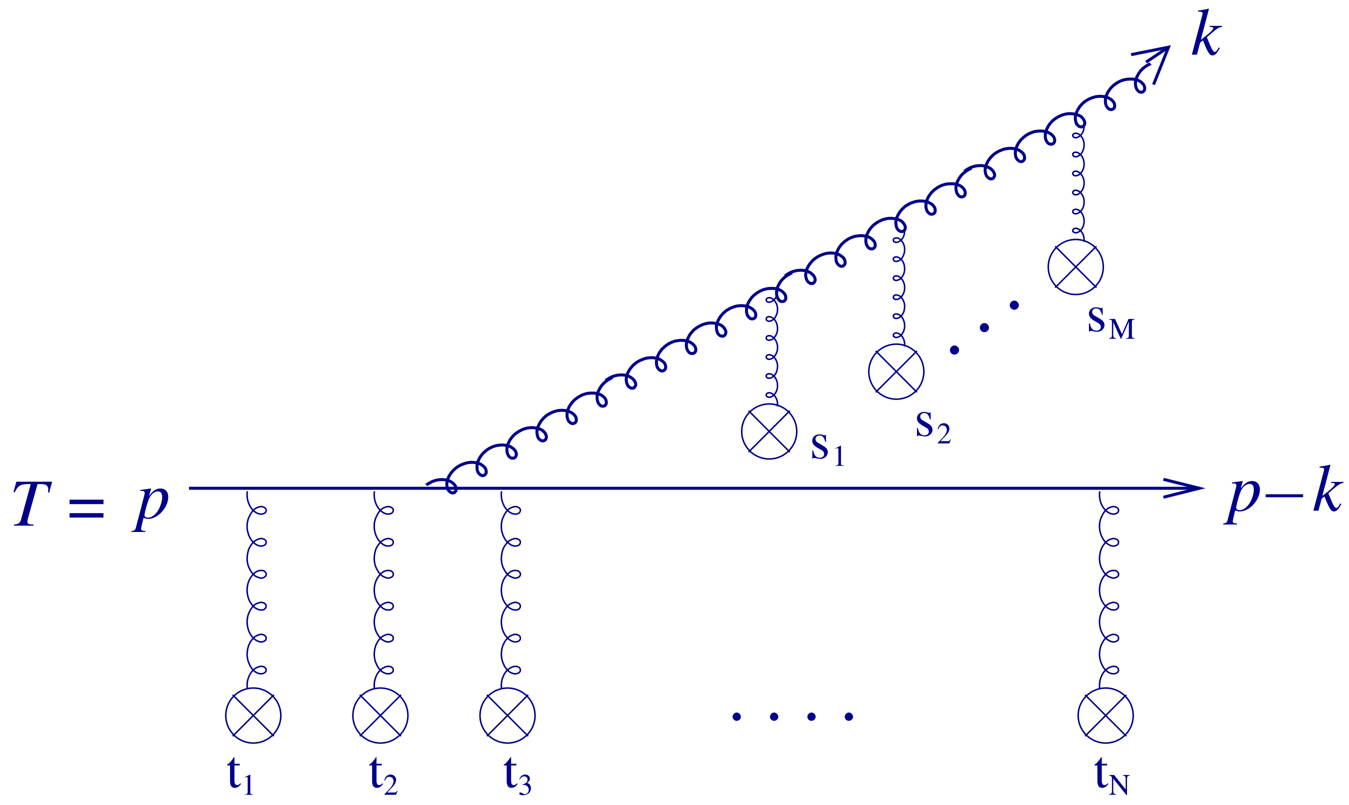
Jet Correlations



Gyulassy, Levai, Vitev (GLV), X.-N. Wang, Baier, Dokshitzer, Mueller, Peigne, Schiff (BDMPS), Zakharov, Wiedemann, Kovner, Turbide, Gale, Jeon, Moore, ...

Jet quenching – Theory

Diagrams to sum:



Need to sum over M and N and then square to get the radiation rate: Landau-Pomeranchuck-Migdal effect

SD Equation for Gluon Radiation

Must take care of:

- Gluon momentum $\mathbf{k}$ can change now.
- Color factors.
- Must keep track of quarks *and* gluons.

$$
\begin{aligned}
2\mathbf{h} &= i\delta E(\mathbf{h},p,k)\mathbf{F}(\mathbf{h}) + g^2 \int \frac{d^2\mathbf{q}_\perp}{(2\pi)^2} C(\mathbf{q}_\perp) \times \\
&\times \Big\{ (C_s - C_{\mathsf{A}}/2)[\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h}{-}k\,\mathbf{q}_\perp)] \\
&\quad +(C_{\mathsf{A}}/2)[\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h}{+}p\,\mathbf{q}_\perp)] \\
&\quad +(C_{\mathsf{A}}/2)[\mathbf{F}(\mathbf{h}) - \mathbf{F}(\mathbf{h}{-}(p{-}k)\,\mathbf{q}_\perp)] \Big\}, \\
\delta E(\mathbf{h},p,k) &= \frac{\mathbf{h}^2}{2pk(p{-}k)} + \frac{m_k^2}{2k} + \frac{m_{p-k}^2}{2(p{-}k)} - \frac{m_p^2}{2p}\,.
\end{aligned}
$$

Gluon Radiation Rate

$$\frac{d\Gamma_g(p,k)}{dkdt} = \frac{C_s g_{\mathrm{s}}^2}{16\pi p^7} \frac{1}{1 \pm e^{-k/T}} \frac{1}{1 \pm e^{-(p-k)/T}} \times$$
$$\times \left\{ \begin{array}{cc} \frac{1+(1-x)^2}{x^3(1-x)^2} & q \to qg \\ N_{\mathrm{f}} \frac{x^2+(1-x)^2}{x^2(1-x)^2} & g \to qq \\ \frac{1+x^4+(1-x)^4}{x^3(1-x)^3} & g \to gg \end{array} \right\}$$
$$\times \int \frac{d^2\mathbf{h}}{(2\pi)^2} 2\mathbf{h} \cdot \mathrm{Re}\, \mathbf{F}(\mathbf{h}, p, k)\,,$$

where $x \equiv k/p$ is the momentum fraction in the gluon (or either quark, for the case $g \to qq$).

$\mathbf{h} \equiv \mathbf{p} \times \mathbf{k}$: 2-D vector. $O(gT^2)$

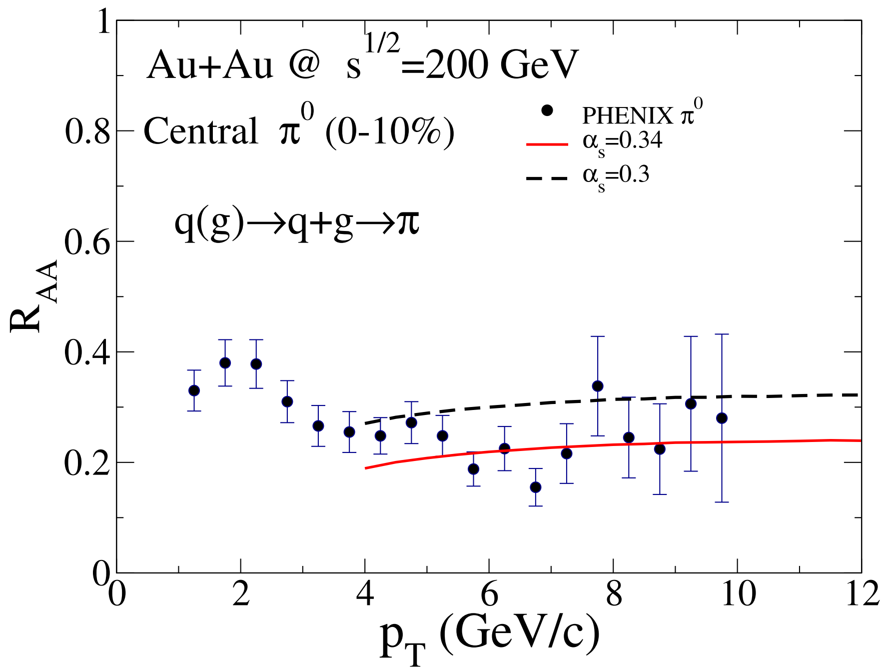
Time evolution equation

$$\frac{dP_{q\bar{q}}(p)}{dt} = \int_k P_{q\bar{q}}(p+k)\frac{d\Gamma^q_{qg}(p+k,k)}{dkdt} - P_{q\bar{q}}(p)\frac{d\Gamma^q_{qg}(p,k)}{dkdt} + 2P_g(p+k)\frac{d\Gamma^g_{q\bar{q}}(p+k,k)}{dkdt},$$

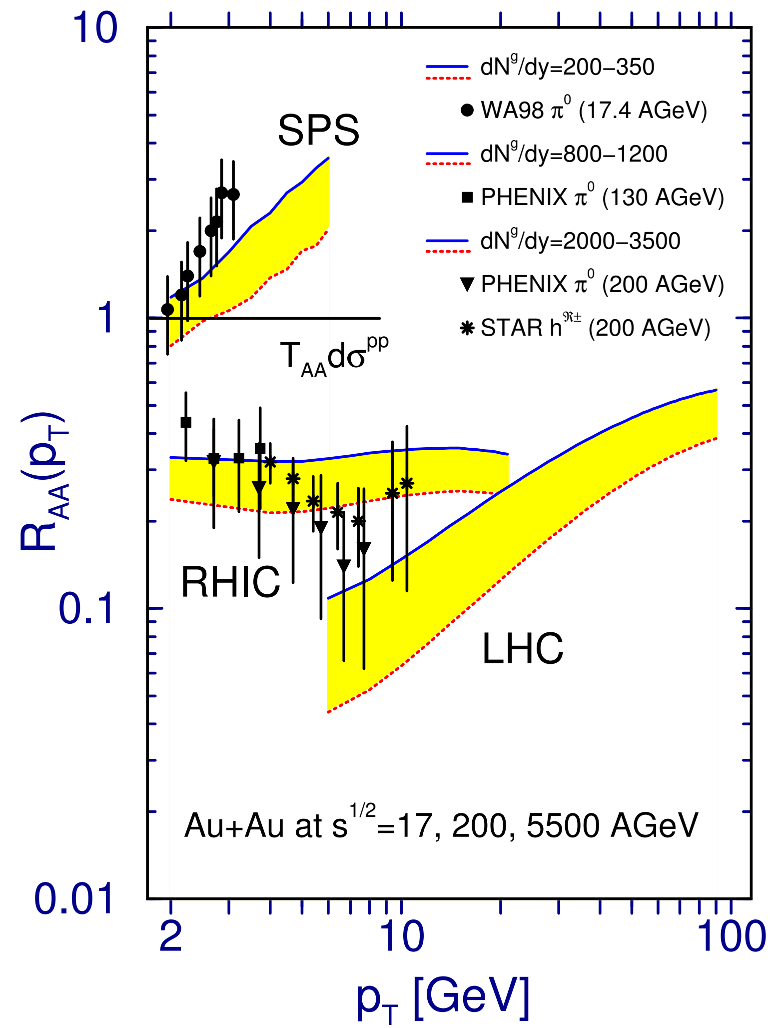
$$\frac{dP_g(p)}{dt} = \int_k P_{q\bar{q}}(p+k)\frac{d\Gamma^q_{qg}(p+k,p)}{dpdt} + P_g(p+k)\frac{d\Gamma^g_{gg}(p+k,k)}{dkdt} - P_g(p)\left(\frac{d\Gamma^g_{q\bar{q}}(p,k)}{dkdt} + \frac{d\Gamma^g_{gg}(p,k)}{dkdt}\Theta(2k-p)\right),$$

- k integrals range: $(-\infty, \infty)$.
- $k < 0$: Absorption of thermal gluons.
- $k > p$: annihilation against and antiquark of energy $(k-p)$.
- $\Theta(2k-p)$: To prevent double counting of final states.

Data & Theory

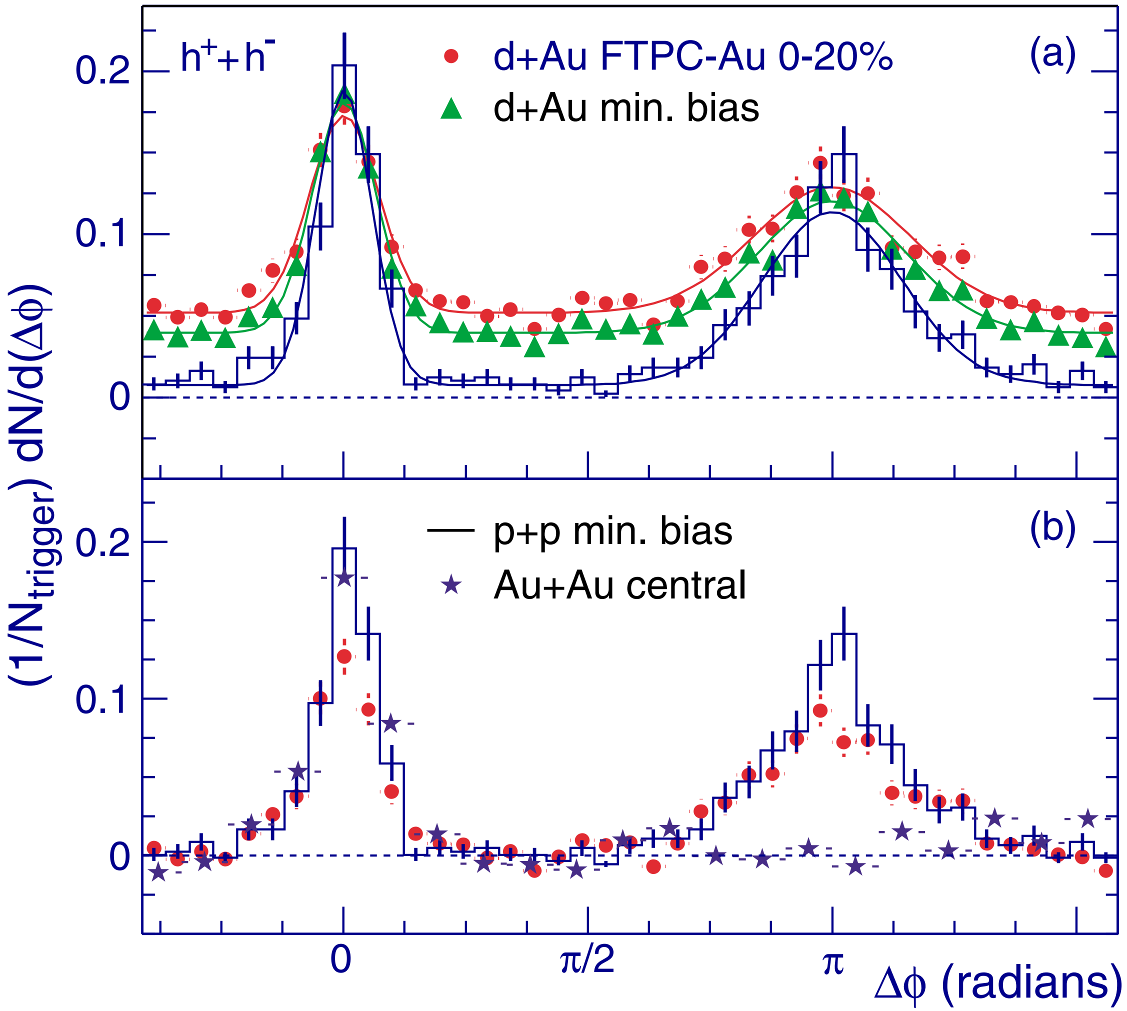


Turbide, Gale, Jeon and Moore, Phys.Rev.C72:014906,2005



Gyulassy and Vitev Phys.Rev.Lett. 89 (2002) 252301

Data – Disappearance of Awayside



STAR: Phys. Rev. Lett. 91 (2003) 072304

Part III
Charge Dependent Correlations and Fluctuations

Net Charge Fluctuations

- Motivations [Jeon & Koch + Asakawa, Heinz & Muller]
  - ∗ Quarks carry fractional charges
  - ∗ Gluons are abundant
  - ∗ In QGP (with appropriate degeneracy factors (12+12+16))

$$\left\langle \Delta Q^2 \right\rangle = (9/4) \left\langle \Delta N_u^2 \right\rangle + (9/1) \left\langle \Delta N_d^2 \right\rangle$$

and invoking 'parton-hadron duality'

$$\langle N_{\text{ch}} \rangle = (2/3)(\langle N_u \rangle + \langle N_d \rangle + \langle N_g \rangle)$$

we get (and Lattice confirms it)

$$\frac{\left\langle \Delta Q^2 \right\rangle}{\langle N_{\text{ch}} \rangle} \approx 1/4 - 1/3$$

STAR Data

- Usually given in terms of the 'ν-Dynamic'

  Define

$$\nu_{+-} = \left\langle \left(\frac{N_+}{\langle N_+ \rangle} - \frac{N_-}{\langle N_- \rangle} \right)^2 \right\rangle = \left\langle \left(\frac{\Delta N_+}{\langle N_+ \rangle} - \frac{\Delta N_-}{\langle N_- \rangle} \right)^2 \right\rangle$$

  and

$$\nu_{+-,\text{dyn}} = \nu_{+-} - \frac{1}{\langle N_+ \rangle} - \frac{1}{\langle N_- \rangle} = \frac{4}{\langle N_{\text{ch}} \rangle} \left(\frac{\langle \Delta Q^2 \rangle}{\langle N_{\text{ch}} \rangle} - 1 \right)$$

- Data

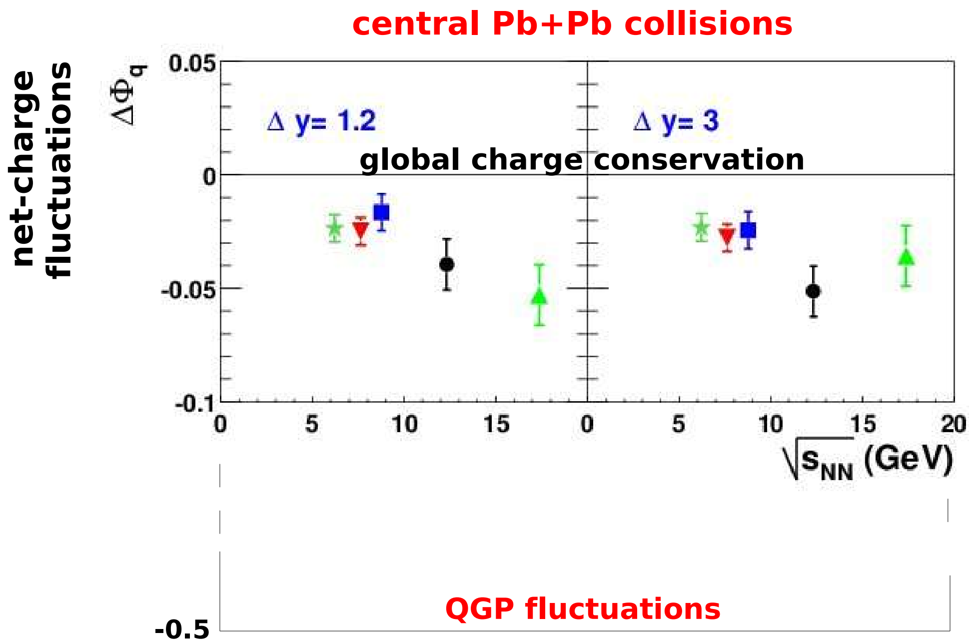
  * $\langle N_{\text{ch}} \rangle \, \nu_{+-,\text{dyn}} \approx -1 - -1.4$

    or $\dfrac{\langle \Delta Q^2 \rangle}{\langle N_{\text{ch}} \rangle} \approx 0.75 - 0.65 \Longrightarrow$ Consistent with hadronic gas

DATA from NA49

Marek Gazdzicki, Correlations and Fluctuations 2005

... and the experimental data



A predicted large suppression of the net-charge fluctuations is not observed!

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NA49, Phys.Rev.C70:064903,2004

Why not?

- Rescatterings in the hadronic phase can be fatal

- QGP content may be small and local.
  Averaging over can hide it.
  Need a more local measure.

Balance Functions

[Pratt, Bass, Danielewicz]

- Definition: Within the given phase space,

$$B(\Delta y) = \frac{1}{2}\left[\frac{N_{+-}(\Delta y) - N_{++}(\Delta y)}{\langle N_+ \rangle} + \frac{N_{-+}(\Delta y) - N_{--}(\Delta y)}{\langle N_- \rangle}\right]$$

where $N_{ij}(\Delta y) =$ Number of ij-pairs with Δy difference (in rapidity, q_{inv}, ...)

- In terms of 2 particle correlation functions

$$N_{ij}(\Delta y) = \int_Y dy_1 \int_Y dy_2\, \rho_{ij}(y_1, y_2)\delta(\Delta y - |y_1 - y_2|)$$

where the subscript Y implies $|y_i| < Y$.

Normalization:

$$\int_Y dy_1 \int_Y dy_2\, \rho_{ij}(y_1, y_2) = \left\langle N_i N_j \right\rangle_Y - \langle N_i \rangle_Y \delta_{ij}$$

Balance Func. – Cont.

- (My) Interpretation using a neutral cluster model

  Assume all charged particles come from neutral clusters.

  $$P_{N_C}(\{y_i^+, y_i^-\}) = \prod_{i=1}^{N_C} f(y_i^+, y_i^-)$$

  with $f(x,y) = \mathcal{N}\ \mathrm{exp}(-|x-y|/\gamma)\, F((x+y)/2)$
  [Thomas, Chao, Quigg]

  If only a single species of clusters,

  $$B(\Delta y) \propto e^{-|\Delta y|/\gamma_1}$$

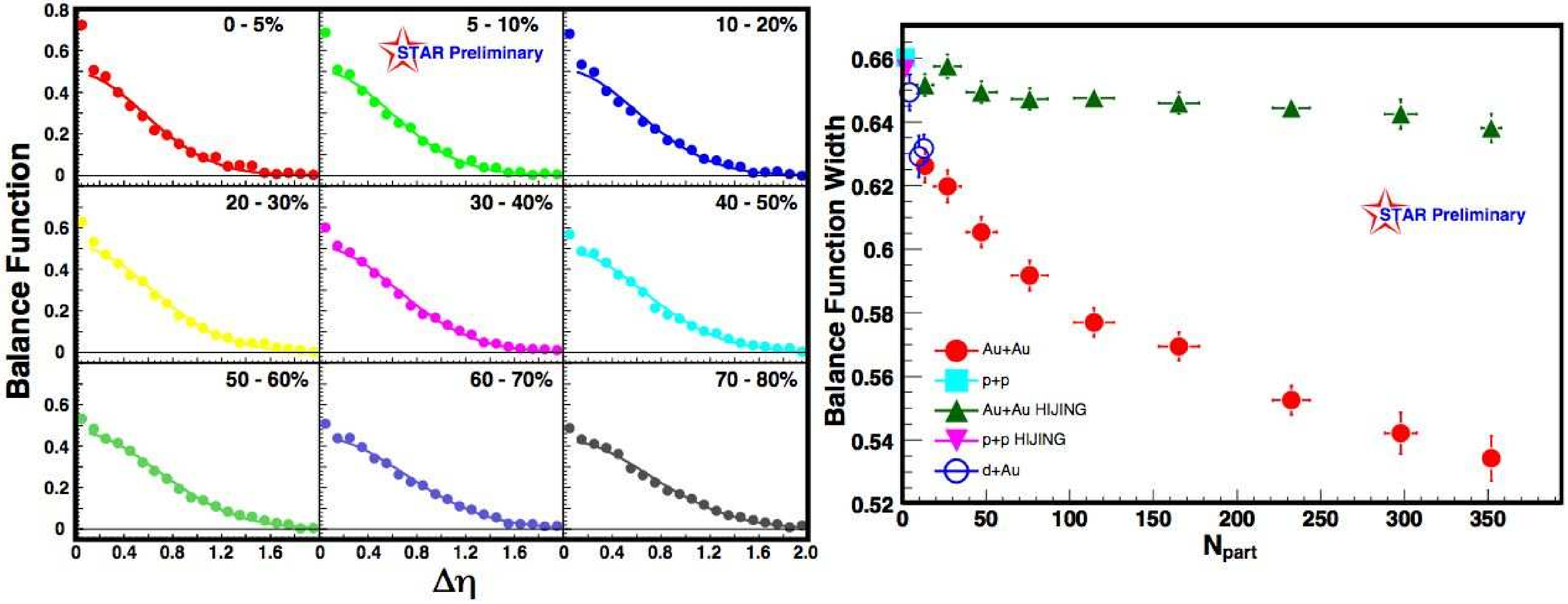
- If a second QGP component develops,

  $$B(\Delta y) \propto p_{\mathrm{full}} p_{\Delta} e^{-|\Delta y|/\gamma_{\mathrm{HG}}} + (1 - p_{\mathrm{full}})(1 - p_{\Delta}) e^{-|\Delta y|/\gamma_{\mathrm{QGP}}}$$

  with $\gamma_{\mathrm{HG}} > \gamma_{\mathrm{QGP}}$.

  $\Longrightarrow$ More central collisions should have smaller width.

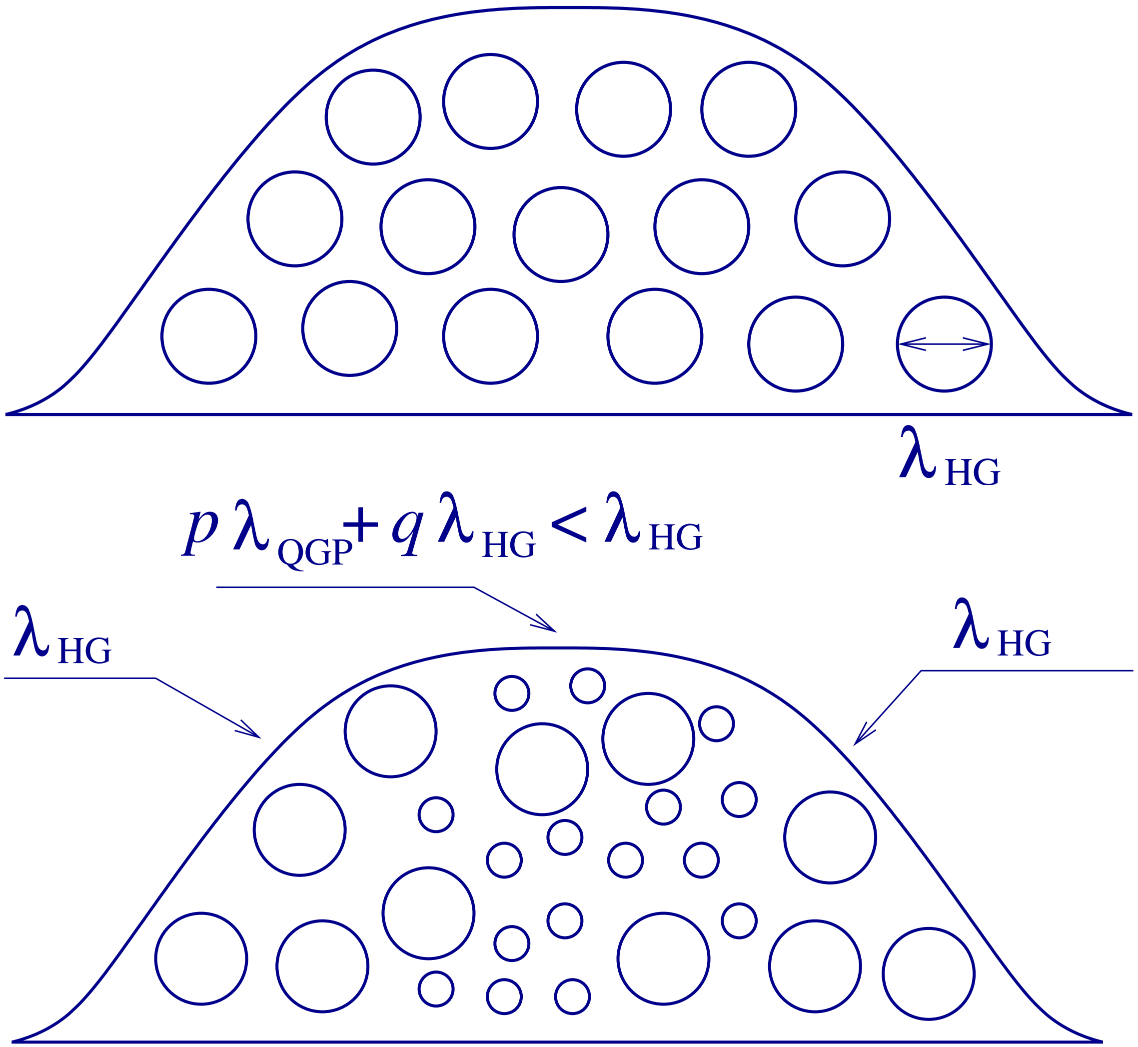
Data

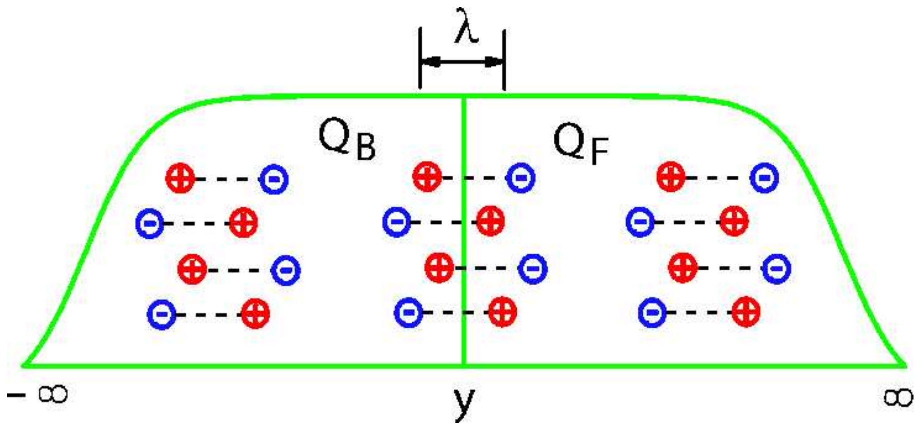


- Reductions seen.
- Still averaging over too much.
- How to estimate fractions and γ_{HG}/γ_{QGP}?

Charge Transfer Fluctuations

Idea:





- Observable:
  - ∗ Define: $u(y) = [Q_F(y) - Q_B(y)]/2$

- Suppose a neutral cluster R decays near y.
  - ∗ $R \longrightarrow h^+ + h^-$ with a typical $\Delta y = \lambda$
  - ∗ For each R decay, $u(y)$ changes by $\pm 1 \Longrightarrow$ Random walk
  - ∗ $D_u(y) = \left\langle \Delta u(y)^2 \right\rangle = N_{\text{steps}}(y) \approx \lambda \dfrac{dN_{\text{cluster}}}{dy}$
  - ∗ Since $dN_{\text{cluster}}/dy \propto dN_{\text{ch}}/dy$, $\boxed{\kappa(y) \equiv \dfrac{D_u(y)}{dN_{\text{ch}}/dy} \propto \lambda(y)}$

Interpretations

- Neutral cluster models

  Assume

$$P_{N_C}(\{y_i^+, y_i^-\}) = \prod_{i=1}^{N_C} f(y_i^+, y_i^-)$$

Then,

$$D_u(y) = \frac{\left\langle \Delta Q^2 \right\rangle}{4} + 2\left\langle N_C \right\rangle \int_{y_o}^{y} dy_- \int_{y}^{y_0} dy_+ \, f(y_+, y_-)$$

with $y_r \equiv y - y', Y = (y + y')/2$

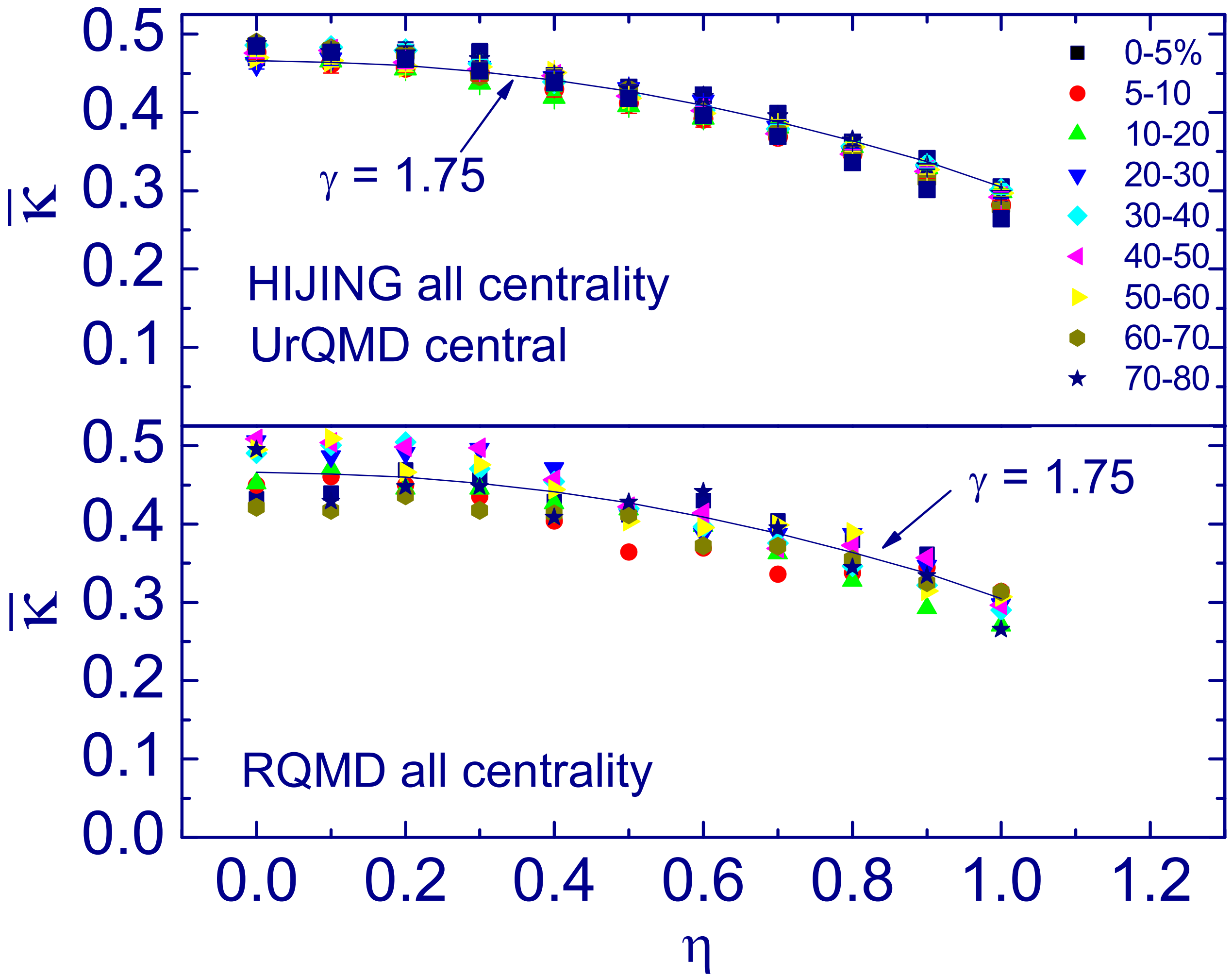
$$\begin{aligned} f(y, y') &= (1-p)\,\mathrm{exp}(-|y_r|/\gamma_{\mathrm{HG}})F_{\mathrm{HG}}(Y) \\ &\quad + p\,\mathrm{exp}(-|y_r|/\gamma_{\mathrm{QGP}})F_{\mathrm{QGP}}(Y) \end{aligned}$$

where $\gamma_{\mathrm{HG}} > \gamma_{\mathrm{QGP}}$

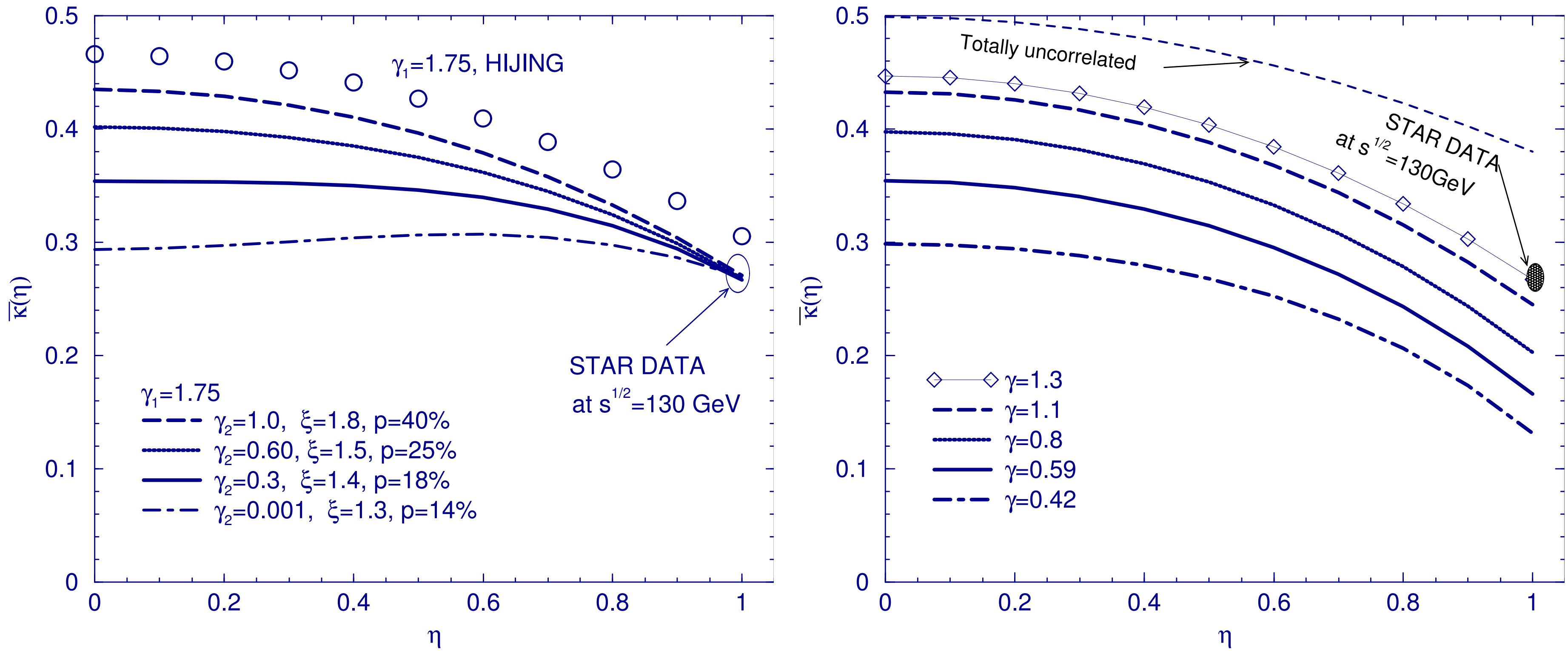
p: Overall QGP fraction.

HG – STAR acceptance

Hadronic models with the single component results



Predictions [Jeon, Shi]



- Hadronic models have *no* centrality dependence
- Predict
  - ∗ QGP width ≈ 1.4
  - ∗ QGP content $\approx 20\,\%$
  - ∗ QGP correlation length $= 0.3 - 0.6$

A lot more data available

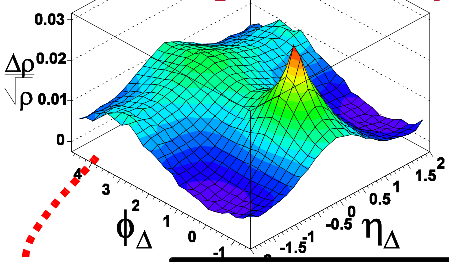
Followings copied from presentation by
L.Ray, and C.Roland in Correlations and fluctuations workshop, 2005, MIT.

η, ϕ correlations for 62 GeV Au-Au:

L. Ray

p- p 200 GeV

STAR preliminary



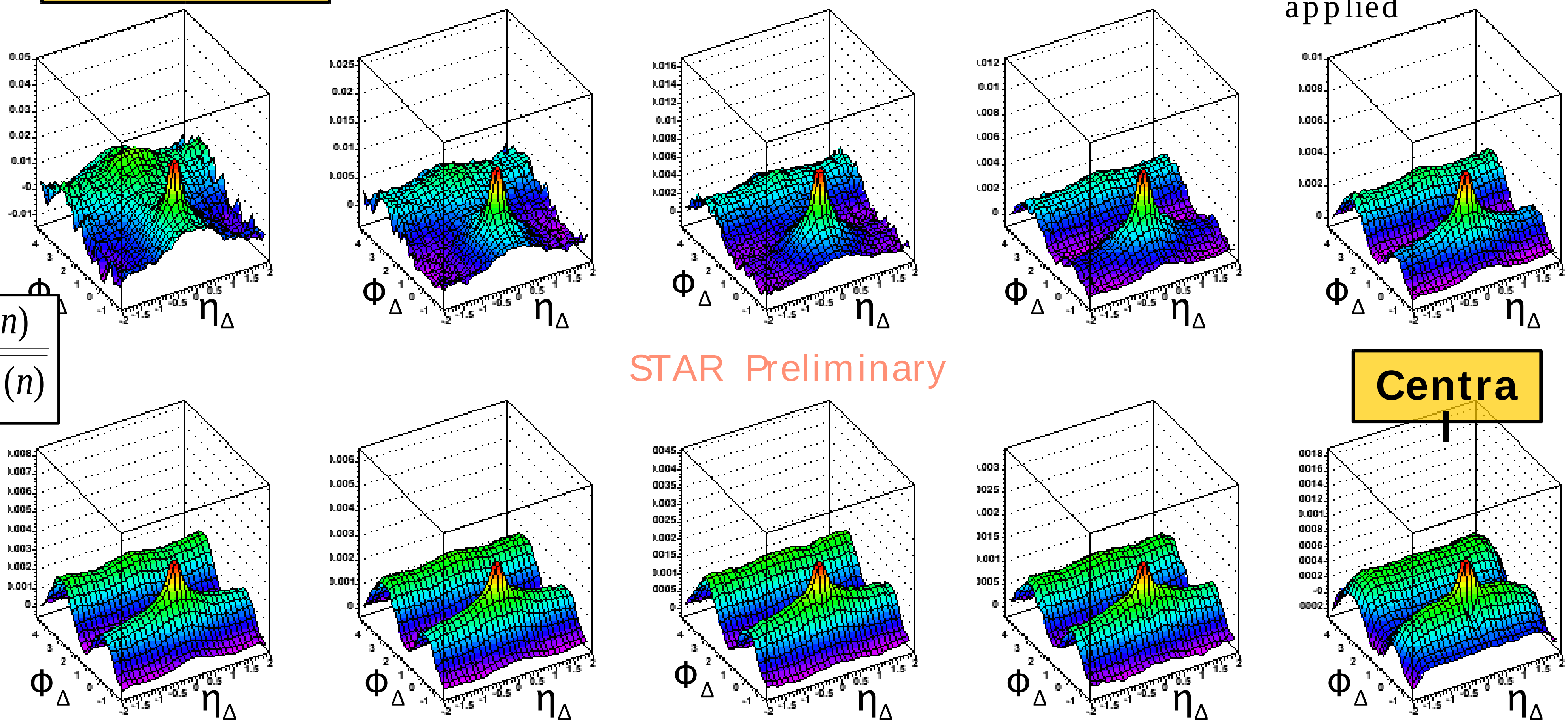
Correlation structure evolves smoothly

from p- p to central Au- Au

$0.15 < p_t < 2$ GeV/c
$|\eta| < 1.0$, full $\phi = 2\pi$
merging & HBT cuts applied

Peripheral

$$\frac{\Delta\rho(n)}{\sqrt{\rho_{ref}(n)}}$$



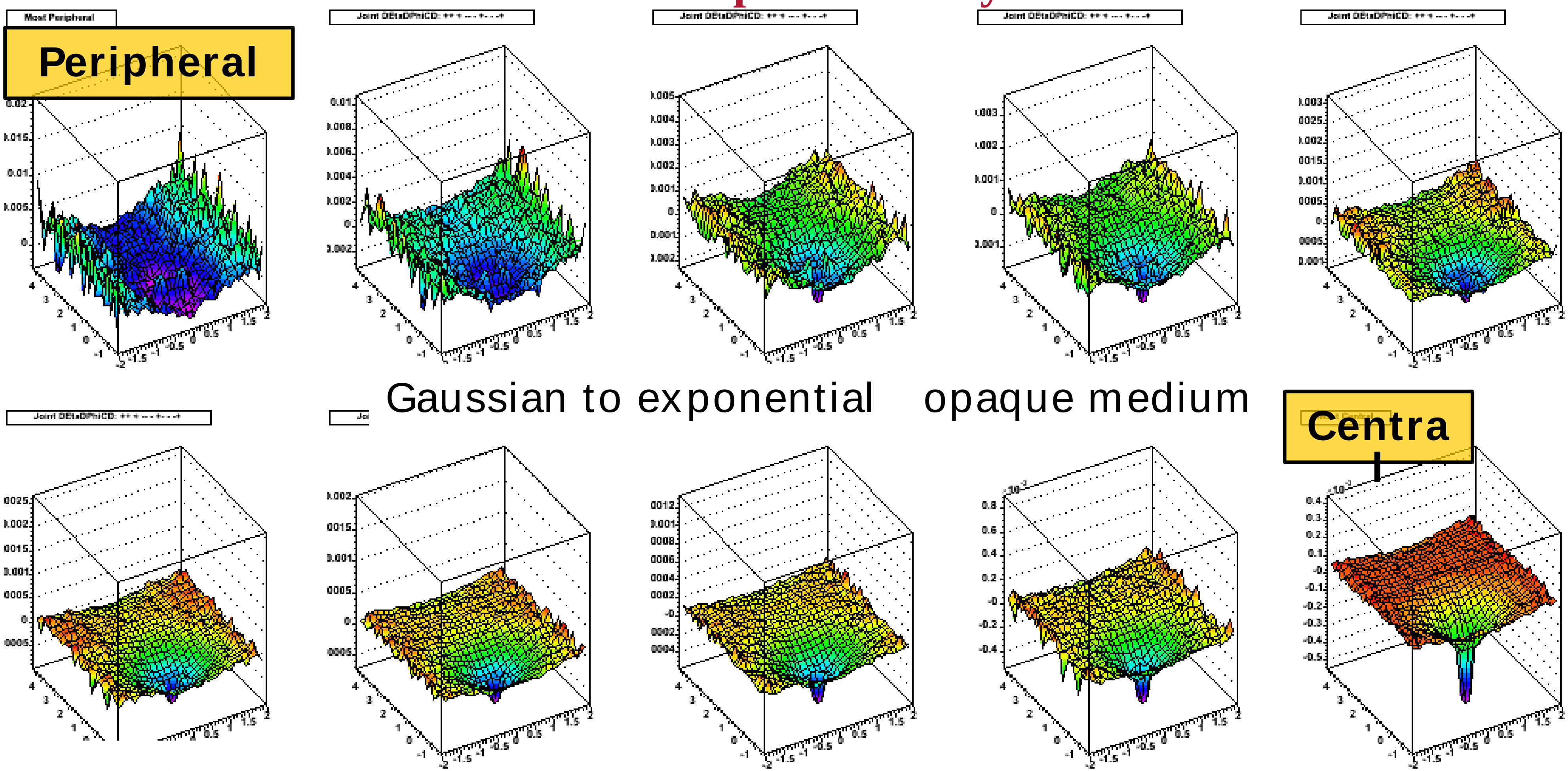
STAR Preliminary

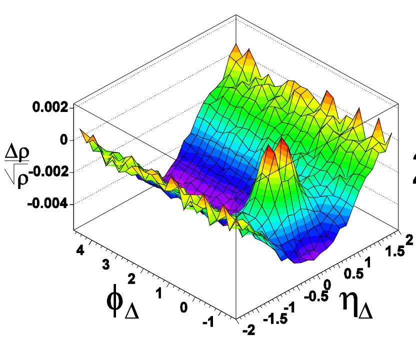
Central

η, ϕ charge difference correlations for 62 GeV Au-Au

L. Ray

STAR preliminary



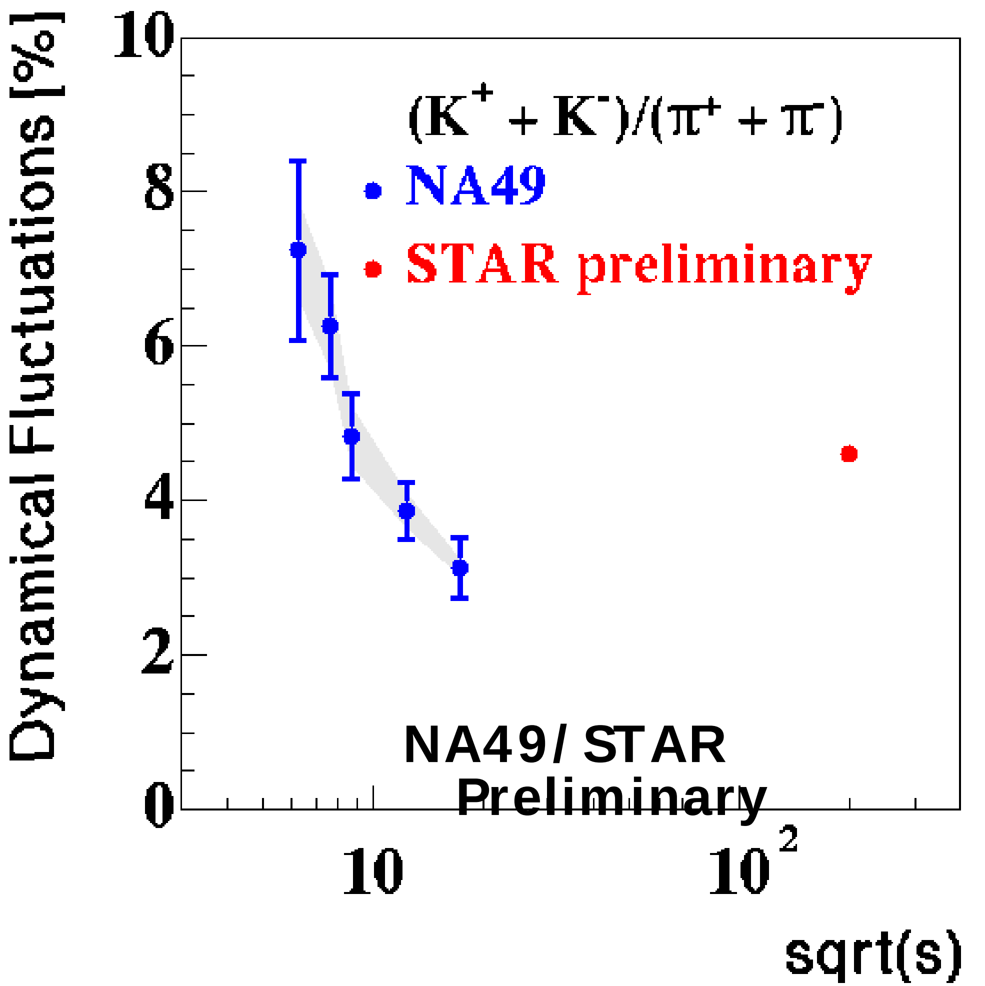


p-p
200 GeV

Evolution from 1D string fragmentation to at least 2D hadronization

$0.15 < p_t < 2$ GeV/c
$|\eta| < 1.0$, full $\phi = 2\pi$
merging & HBT cuts applied

What about RHIC



STAR: Supriya Das, ICPAQGP2005, Kolkata INDIA

Perspectives instead of conclusions

There is no doubt that we have an elephant.

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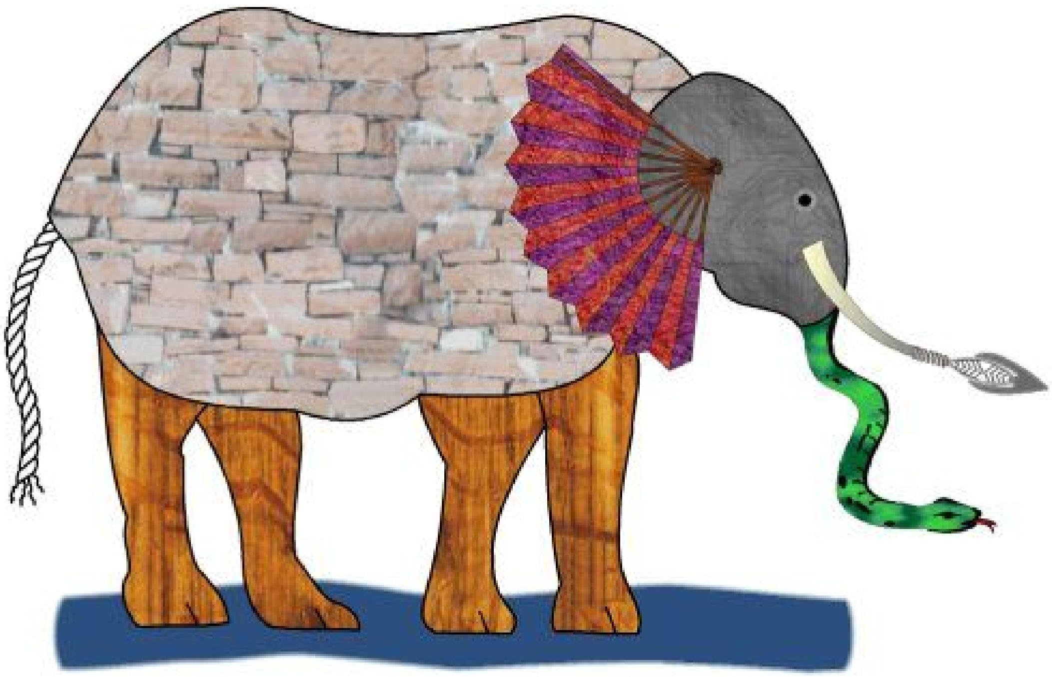

But is it a Wall, Spear, Snake, Tree, Fan or Rope?

Perspectives instead of conclusions

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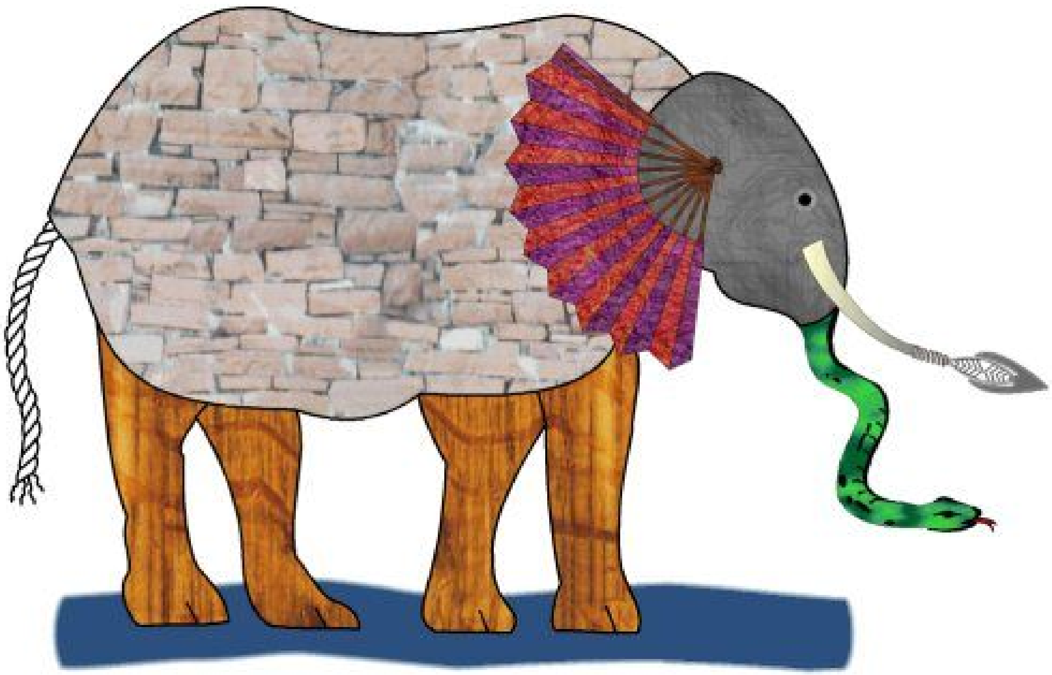

But is it a Wall, Spear, Snake, Tree, Fan or Rope?

We need a wide, comprehensive perspective! $\Longrightarrow$Need correlations!

What needs to be done

- A lot.
- Get as much as we can from *perturbative* QCD – Thermal QCD, CGC, ...
- Coordinated and concerted Lattice Effort – Viscosities, Spectral functions, Susceptibilities that are *relevant* to the experiment [This is HARD.]
- Need a good formulation of non-equilibrium QFTs and ways to solve them.
- Failing that, need to build a physically motivated *consistent* (with QCD, ChPT, ...) model that can explain *majority* of SPS and RHIC phenomena and predict LHC
- New era for high energy QCD/Hadronic physics –

  Good: Lots of exciting phenomena

  Bad?: No systematic ways to get them (yet)

Backup Slides

Φ

Takes the baseline from Single particle inclusive distribution.

$$\Phi_x = \sqrt{\frac{\left\langle \Delta X^2 \right\rangle}{\langle N_e \rangle}} - \sqrt{\left\langle \Delta x^2 \right\rangle_{\text{incl}}}$$

$$\Delta X = \sum_{i=1}^{N_e} (x_i - \langle x \rangle_{\text{incl}})$$

- Inclusive momentum distribution:

$$P_{\text{incl}}(p) = \frac{\langle n_p \rangle}{\sum_p \langle n_p \rangle} = \frac{\langle n_p \rangle}{\langle N_e \rangle}$$

  * $\langle p \rangle_{\text{incl}} = \sum_p p P(p) = \frac{1}{\langle N_e \rangle} \sum_p p \langle n_p \rangle$
  * $\left\langle \Delta p^2 \right\rangle_{\text{incl}} = \left\langle p^2 \right\rangle_{\text{incl}} - \langle p \rangle^2_{\text{incl}}$

- Event-by-event averages

$$\begin{aligned} \langle p \rangle_{\text{ebe}} &= \frac{1}{N_{\text{event}}} \sum_{i_e=1}^{N_{\text{event}}} \frac{1}{N_e} \sum_{i_e=1}^{N_e} p_{i_e} \\ &= \langle p \rangle_{\text{incl}} - \frac{\langle \Delta N_e \Delta M_e(p) \rangle}{\langle N_e \rangle} \end{aligned}$$

with $M_e(x) = (1/N_e) \sum_{i_e} x_{i_e}$

But

$$P(N) = \left(\begin{array}{c} N+k-1 \\ k-1 \end{array} \right) \left(\frac{\langle N \rangle /k}{1+\langle N \rangle /k} \right)^N \frac{1}{(1+\langle N \rangle /k)^k}$$

Variance:

$$\left\langle \Delta N^2 \right\rangle = \langle N \rangle (1 + \langle N \rangle /k)$$

with $k \sim 3-4$ and $\langle N \rangle \sim 20-30$