Search of QCD phase transitions at finite density with the canonical approach

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Collaborations

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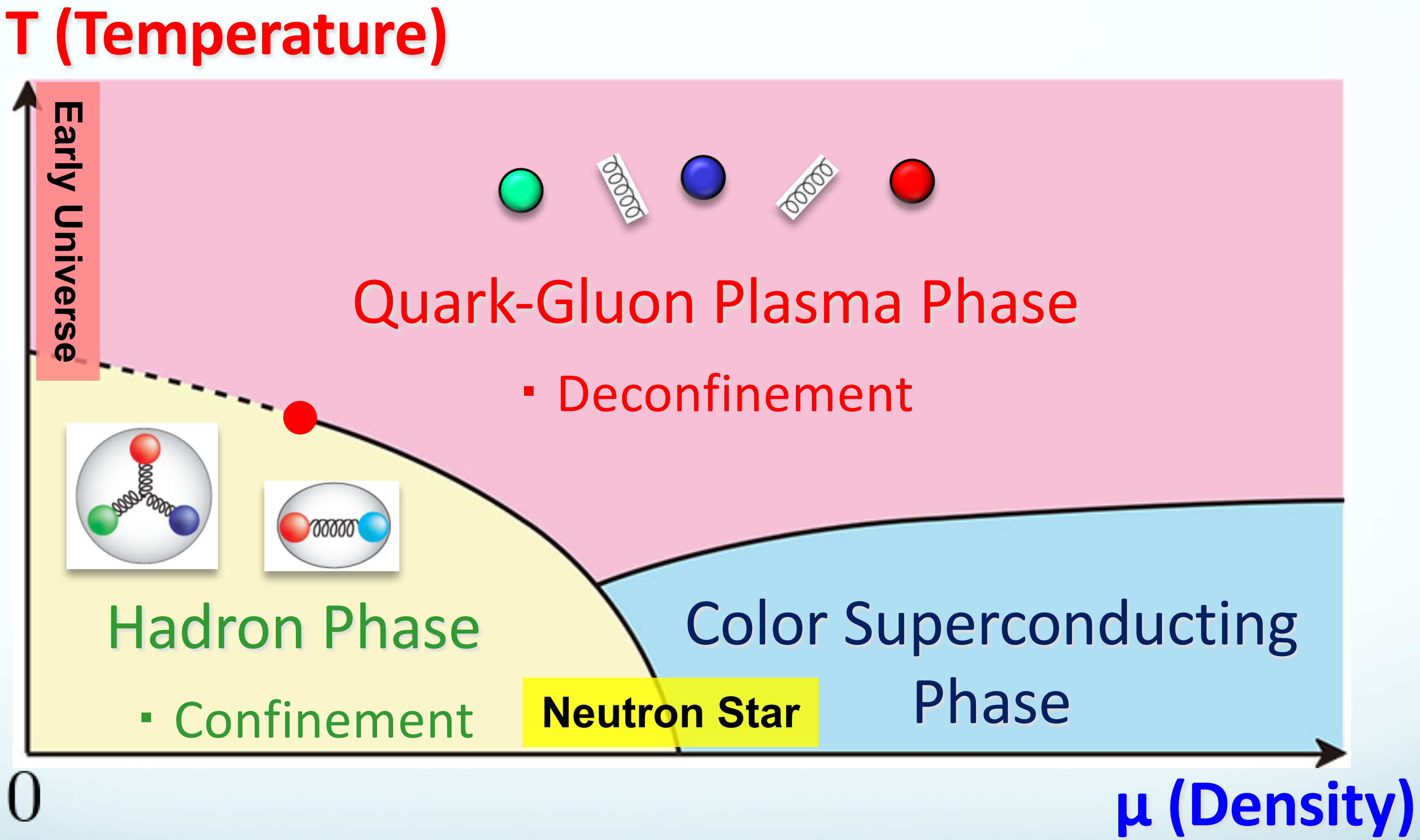
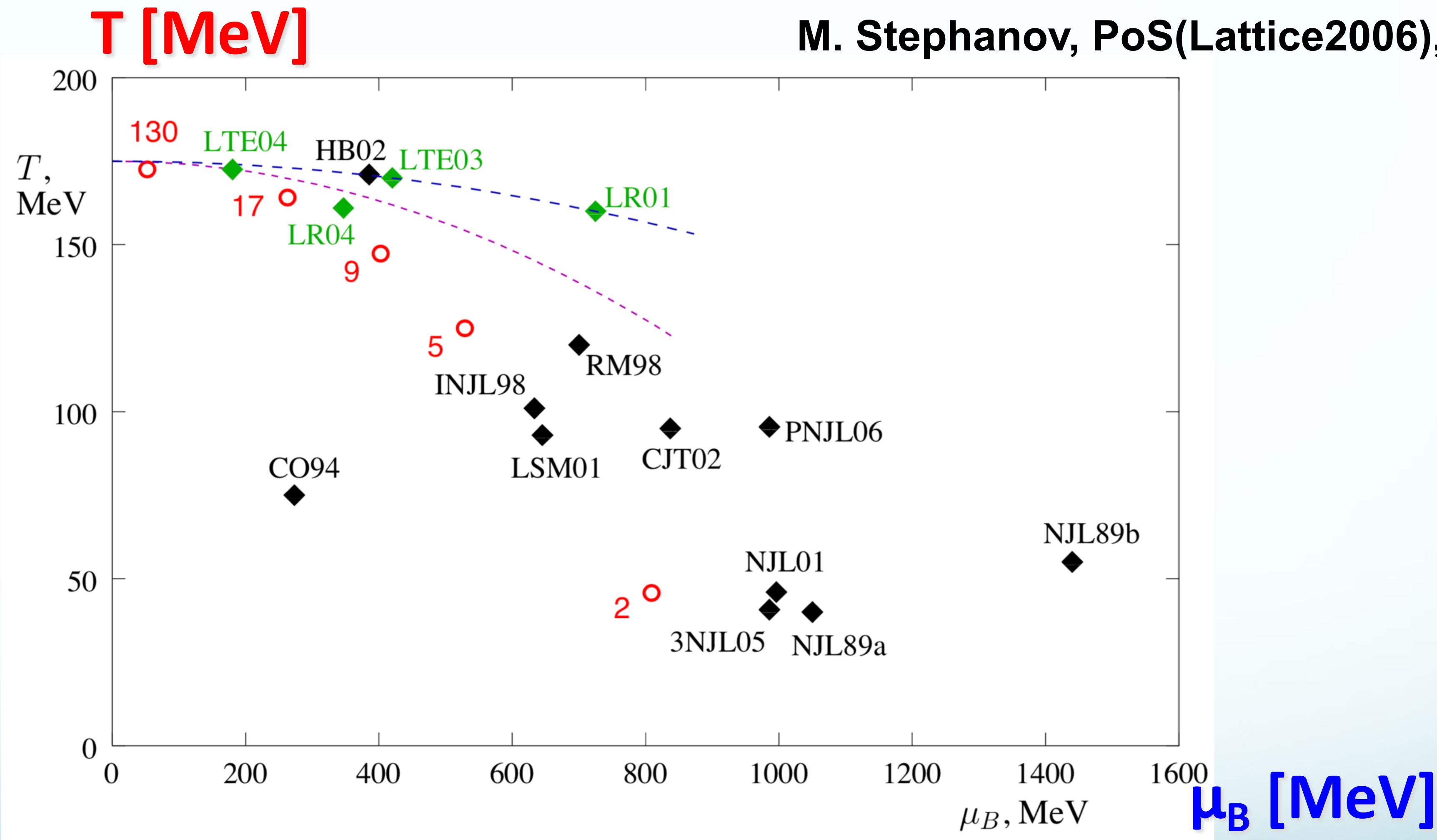
QCD Phase diagram (Prediction)



Where are the critical point and the phase transition line?

Predicted critical points

M. Stephanov, PoS(Lattice2006), 024



Where are the critical point and the phase transition line?
Lattice QCD at finite density: Existence of the sign problem

Monte Carlo Method

Grand canonical partition function

$$Z_{GC}(\mu_q) = \int \mathcal{D}U \, [\det D(\mu_q)]^{N_f} \, e^{-S_G}$$

S_G : Gauge action
$D(\mu_q)$: Fermion matrix
μ_q : Quark chemical potential
N_f : # of flavors

Expectation value of an arbitrary operator

$$\langle \mathcal{O} \rangle_{\mu_q} = \frac{1}{Z_{GC}(\mu_q)} \int \mathcal{D}U \, [\det D(\mu_q)]^{N_f} \, e^{-S_G} \, \mathcal{O}[U]$$

Degree of freedom for Gauge field "U": (lattice size)x4x8

very large number integrations => **impossible!**

Monte Carlo Method

$$\langle \mathcal{O} \rangle_{\mu_q} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}[U_i]$$ **with Probability:** $[\det D(\mu_q)]^{N_f} \, e^{-S_G}$

(importance sampling)

Sign Problem

S_G : Gauge action
$D(\mu_q)$: Fermion matrix
N_f : # of flavors

Monte Carlo Method

(importance sampling)

$$\langle \mathcal{O} \rangle_{\mu_q} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}\left[U_i\right]$$ **with Probability:** $$\left[\det D(\mu_q)\right]^{N_f} e^{-S_G}$$

Chemical potential	$\det D(\mu_q)$	Monte Carlo Method
$\mu_q = 0$	Real value	○
$\mu_q \neq 0$	Complex value	× **(Sign Problem)**

$$D(\mu_q) = D_\nu \gamma_\nu + m + \mu_q \gamma_0$$

$$D(\mu_q)^\dagger = -D_\nu \gamma_\nu + m + \mu_q^* \gamma_0 = \gamma_5 D(-\mu_q^*) \gamma_5$$

$$\left[\det D(\mu_q)\right]^* = \det \left[D(\mu_q)^\dagger\right] = \det \left[\gamma_5 D(-\mu_q^*) \gamma_5\right] = \det D(-\mu_q^*)$$

Sign Problem

S_G : Gauge action
$D(\mu_q)$: Fermion matrix
N_f : # of flavors

Monte Carlo Method

(importance sampling)

$$\langle \mathcal{O} \rangle_{\mu_q} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}\left[U_i\right]$$ **with Probability:** $$\left[\det D(\mu_q)\right]^{N_f} e^{-S_G}$$

Chemical potential	$\det D(\mu_q)$	Monte Carlo Method
$\mu_q = 0$	Real value	○
$\mu_q \neq 0$	Complex value	× **(Sign Problem)**
Pure Imag. $\mu_q = i\mu_{qI}$	**Real value**	○

$$D(\mu_q) = D_\nu \gamma_\nu + m + \mu_q \gamma_0$$

$$D(\mu_q)^\dagger = -D_\nu \gamma_\nu + m + \mu_q^* \gamma_0 = \gamma_5 D(-\mu_q^*) \gamma_5$$

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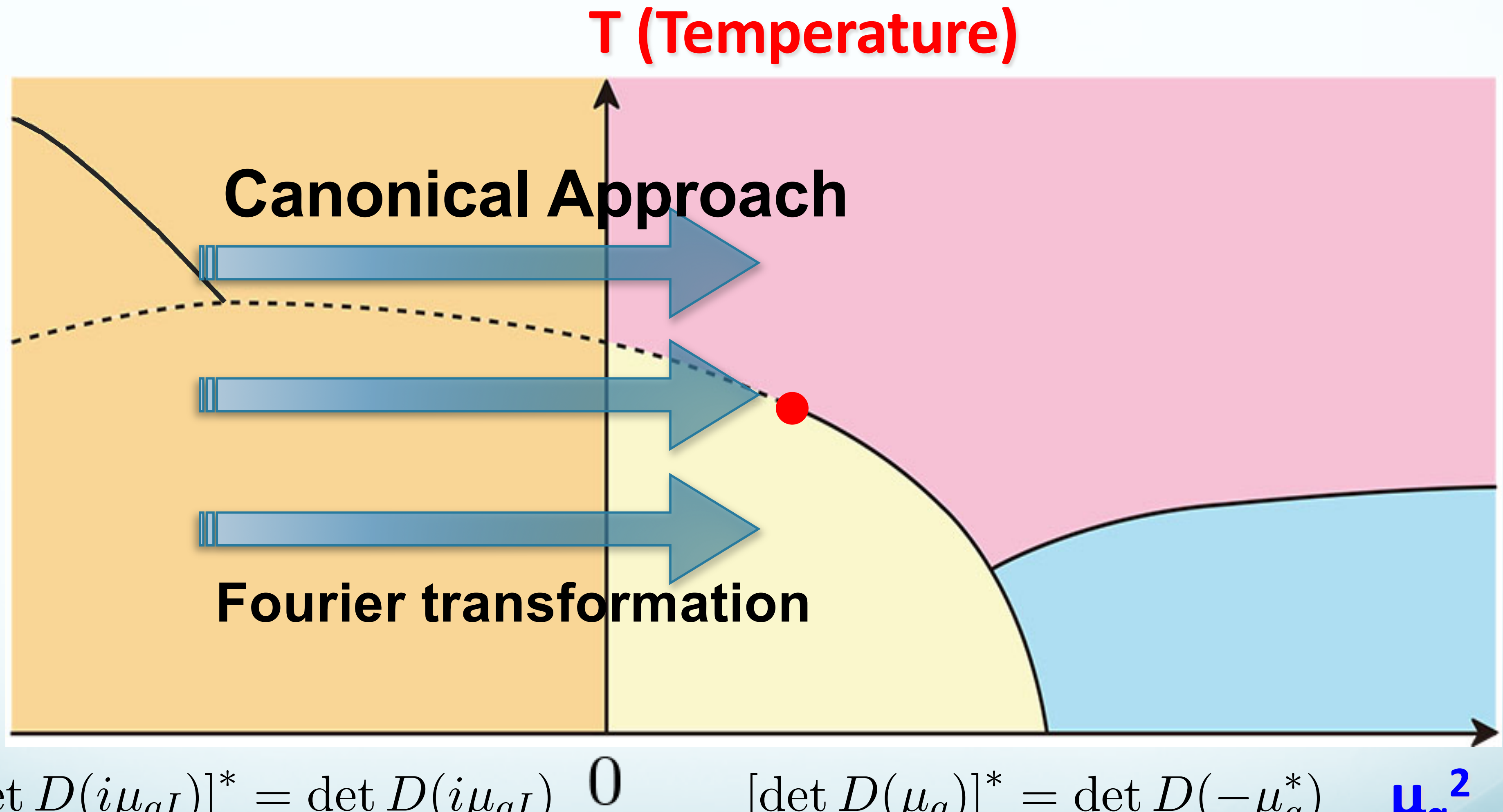
QCD Phase diagram (Prediction)



$[\det D(i\mu_{qI})]^* = \det D(i\mu_{qI})$ $[\det D(\mu_q)]^* = \det D(-\mu_q^*)$

Pure imaginary chemical potential: $\mu_q = i\mu_{qI}$

Canonical Approach

Fugacity expansion

Grand Canonical partition function

$$
\begin{aligned}
Z_{\mathrm{GC}}(\mu_q, T, V) &= \mathrm{Tr}\left(e^{-(\hat{H}-\mu_q \hat{N})/T}\right) \\
&= \sum_n \langle n|\, e^{-(\hat{H}-\mu_q \hat{N})/T}\, |n\rangle \\
&= \sum_n \langle n|\, e^{-\hat{H}/T}\, |n\rangle\, e^{n\mu_q/T} \\
&= \sum_n Z(n, T, V)\xi^n
\end{aligned}
$$

Fugacity: $\xi = e^{\mu_q/T}$

Canonical partition function

Canonical Approach

Fugacity expansion

Grand Canonical partition function

$$Z_{\mathrm{GC}}(\mu_q, T, V) = \sum_{n=-\infty}^{\infty} Z(n, T, V)\xi^n$$

Fugacity: $\xi = e^{\mu_q/T}$

Canonical partition function

Fourier transformation

A. Hasenfrantz & D. Toussaint, Nucl. Phys. B371 (1992)

$$Z(n, T, V) = \int_0^{2\pi} \frac{d(\mu_{qI}/T)}{2\pi} e^{-in\mu_{qI}/T} Z_{\mathrm{GC}}(\mu_q = i\mu_{qI}, T, V)$$

We can calculate Z_{GC} with Monte Carlo Method at pure imaginary μ_q.

$$[\det D(i\mu_{qI})]^* = \det D(i\mu_{qI})$$

History

Basic Idea of Canonical Approach

A. Hasenfrantz, D. Toussaint, Nucl. Phys. B371 **(1992)**

❌ **Numerical instability of (discrete) Fourier transformation**

Sign Problem ? ⇒ No, this is caused by cancelation of significant digits !

R.Fukuda, A.Nakamura, S.Oka, PRD93 **(2016)**

1.234567890123456 – 1.234567890123455 = 0.000000000000001
(16 significant digits) (1 significant digits)

Multiple-precision arithmetic

1.23456789012345666666666
– 1.23456789012345555555555 = 0.00000000000000111111111
(24 significant digits) (10 significant digits)

History

Basic Idea of Canonical Approach

A. Hasenfrantz, D. Toussaint, Nucl. Phys. B371 **(1992)**

❌ **Numerical instability of (discrete) Fourier transformation**

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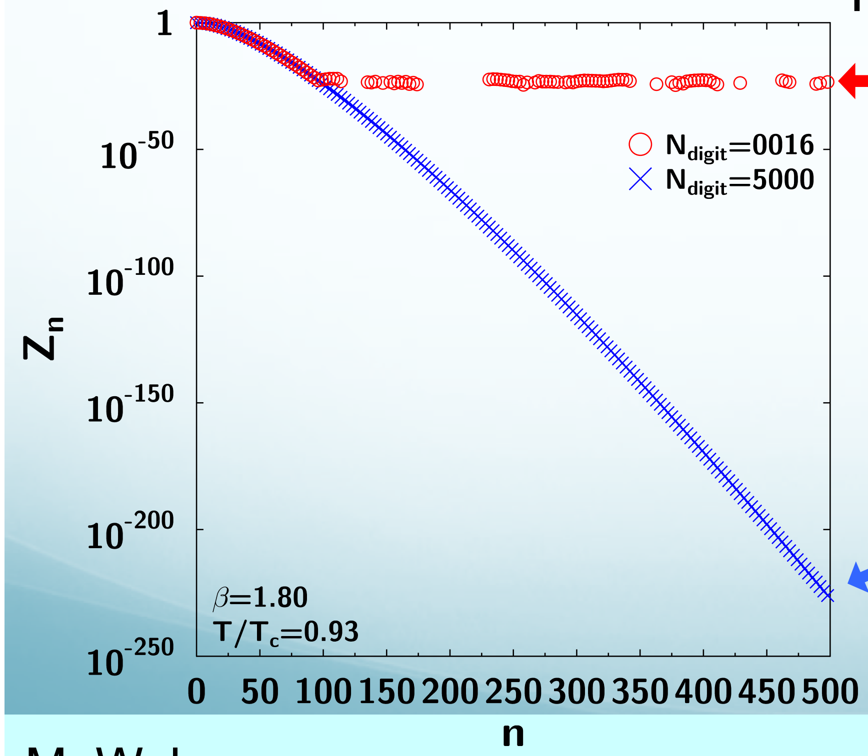


In double-precision arithmetic, cancelation of significant digits occurs at high n region.

In multiple-precision arithmetic, we can evaluate Zn up to high n region with accuracy.

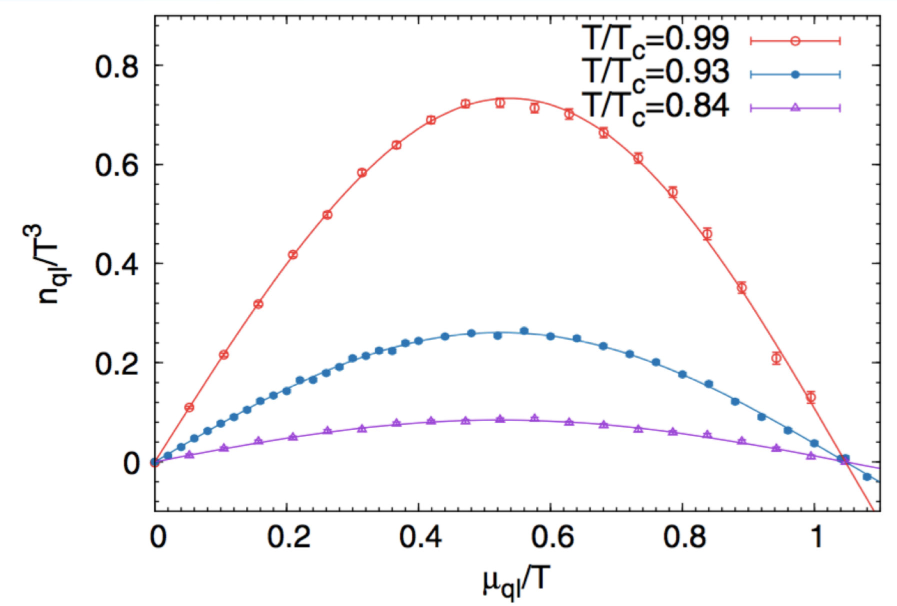
Number density formulation

How to calculate $Z_{\mathrm{GC}}(\mu_q = i\mu_{qI}, T, V)$

V.G. Bornyakov et al., PRD95, 094506 (2017)

Quark number density

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\mathrm{GC}}$$
$$= \frac{1}{VT^3}\frac{1}{Z_{\mathrm{GC}}}\int \mathcal{D}U\ \det D(\mu_q)\, e^{-S_G}\ \mathrm{Tr}\left[D^{-1}\frac{\partial D}{\partial(\mu_q/T)}\right]$$



$$n_q = i n_{qI} \qquad \theta = \frac{\mu_{qI}}{T}$$

Approximated by a Fourier series.

$$\frac{n_{qI}}{T^3}(\theta) \sim \sum_{k=1}^{N_{\sin}} f_k \sin(k\theta)$$

Outline

Lattice QCD

$$n_q(\mu_q = i\mu_{qI}, T, V)$$

Number density formulation
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\mathrm{GC}}$$

$$Z_{\mathrm{GC}}(\mu_q = i\mu_{qI}, T, V)$$

Fourier transformation

$$Z(n, T, V)$$

$$Z_{\mathrm{GC}}(\mu_q, T, V) = \sum_{n=-\infty}^{\infty} Z(n, T, V)\xi^n$$

$$\xi = e^{\mu_q/T}$$

If we get Z_n for all n, we can search at ANY density!

Outline

Lattice QCD

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Number density formulation
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

$$Z_{\rm GC}(\mu_q = i\mu_{qI}, T, V)$$

Fourier transformation

$$Z(n, T, V)$$

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

In numerical calculations, n is finite.

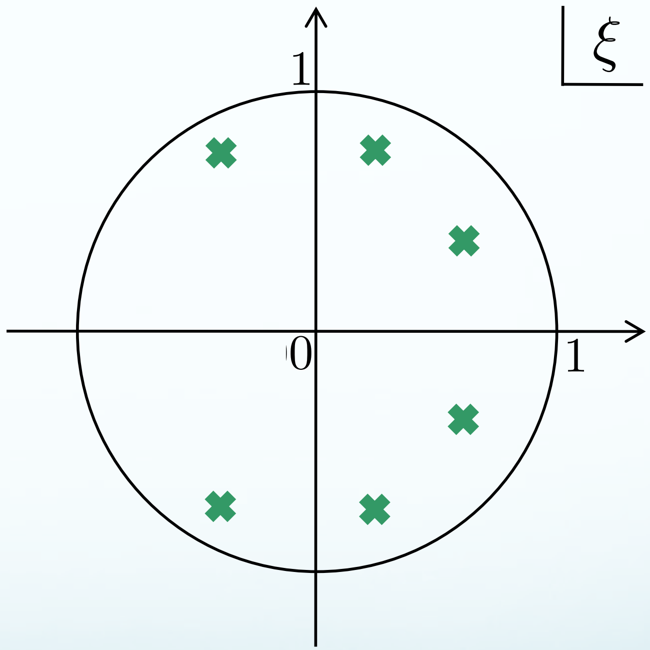
Lee-Yang Zeros

Zeros of Z_{GC} called Lee-Yang Zeros contain a valuable information on the phase transitions of a system.

T.D. Lee & C.N. Yang, Phys. Rev. 87, 404&410 (1952)

$$Z_{\mathrm{GC}}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n = 0$$

There are $2N_{max}$ LYZs in the complex $\xi = e^{\mu_q/T}$ plane.



N_{max} ~ small

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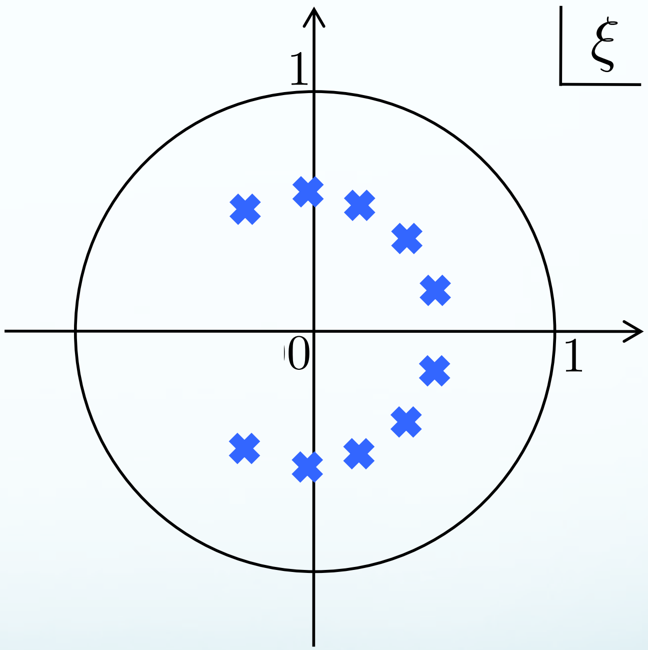

N_{max} ~ large

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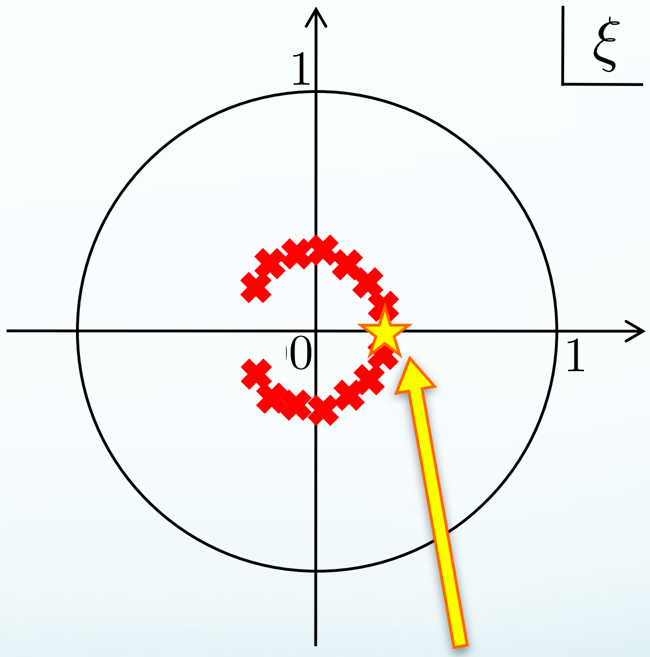

Phase Transition
N_{max} ~ infinity
(V ~ infinity)

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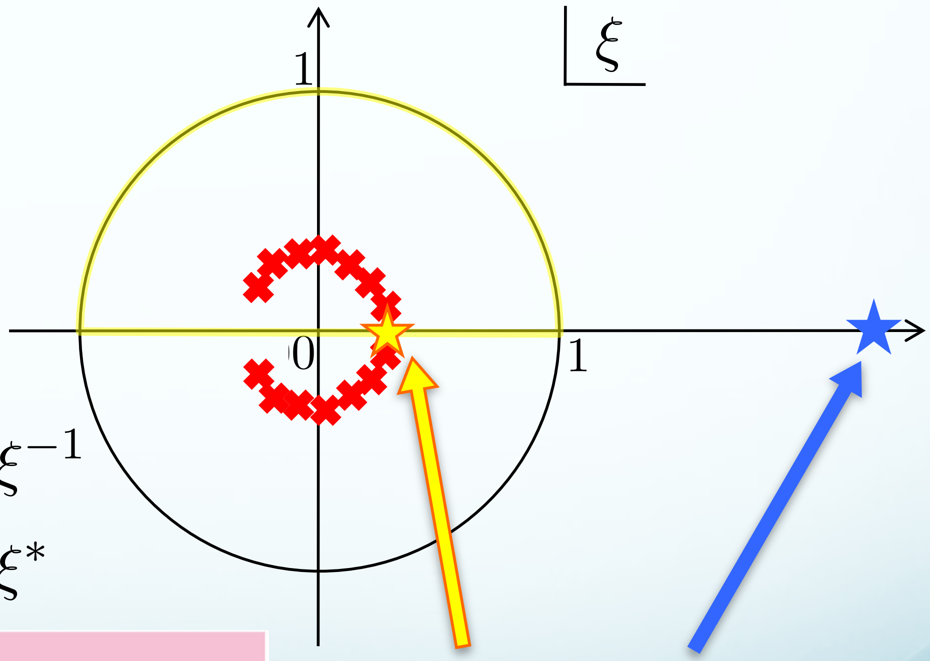
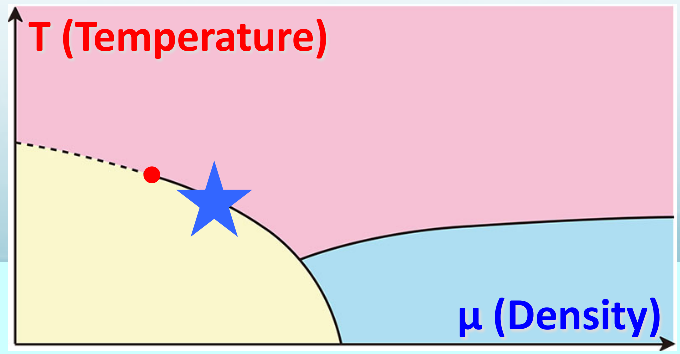


Z(n) properties

$Z(n,T,V) = Z(-n,T,V)$ ⇒ $\xi \leftrightarrow \xi^{-1}$

$Z(n,T,V)$: **Real values** ⇒ $\xi \leftrightarrow \xi^{*}$

$\xi = e^{\mu_q/T} =$ ★



Phase Transition
N_{max} ~ infinity
(V ~ infinity)

Outline

Lattice QCD

$$n_q(\mu_q = i\mu_{qI}, T, V)$$

Number density formulation
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

$$Z_{\rm GC}(\mu_q = i\mu_{qI}, T, V)$$

Fourier transformation

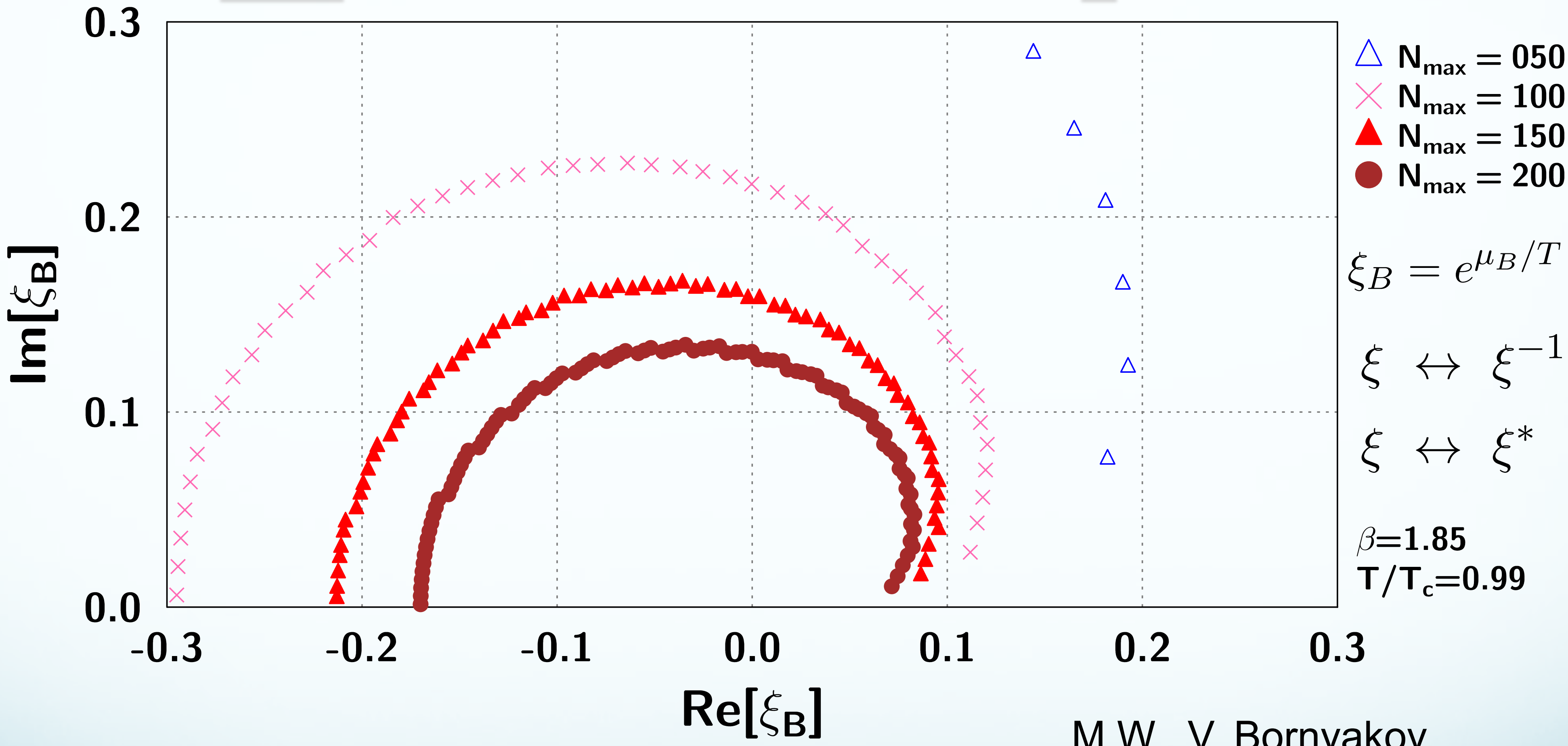
$$Z(n, T, V)$$

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

Lee-Yang zeros

Phase transition point

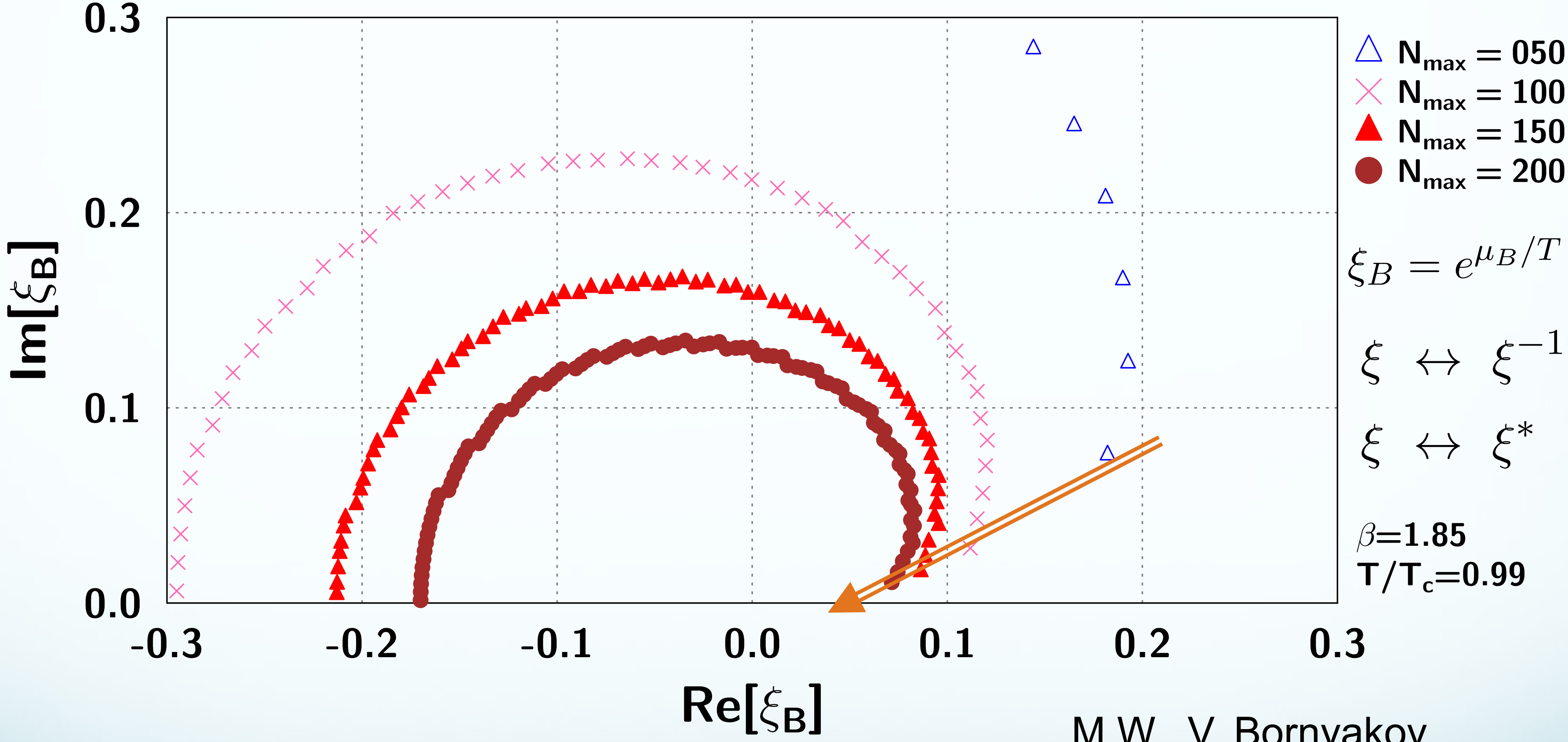
N_{max} dependence (T/T_c=0.99)



As N_{max} increases, right edges of LYZs approach to the real positive axis.
Phase transition point: $\mu_B/T \sim 3\text{-}3.5$ **?**

M.W., V. Bornyakov, D. Boyda, V. Goy, H. Iida, A. Molochkov, A. Nakamura, V. Zakharov, PLB793, 227 (2019)

N_{max} dependence (T/T_c=0.99)



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A. Nakamura, V. Zakharov,
PLB793, 227 (2019)

Outline

NJL model

$n_q(\mu_q = i\mu_{qI}, T)$

Lattice QCD

$n_q(\mu_q = i\mu_{qI}, T, V)$

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q} \ln Z_{\mathrm{GC}}$$

$Z_{\mathrm{GC}}(\mu_q = i\mu_{qI}, T, V)$

Fourier transformation

$Z(n, T, V)$

We can check whether the canonical approach works well or not from the NJL model.

$$Z_{\mathrm{GC}}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

NJL model

$n_q(\mu_q, T)$

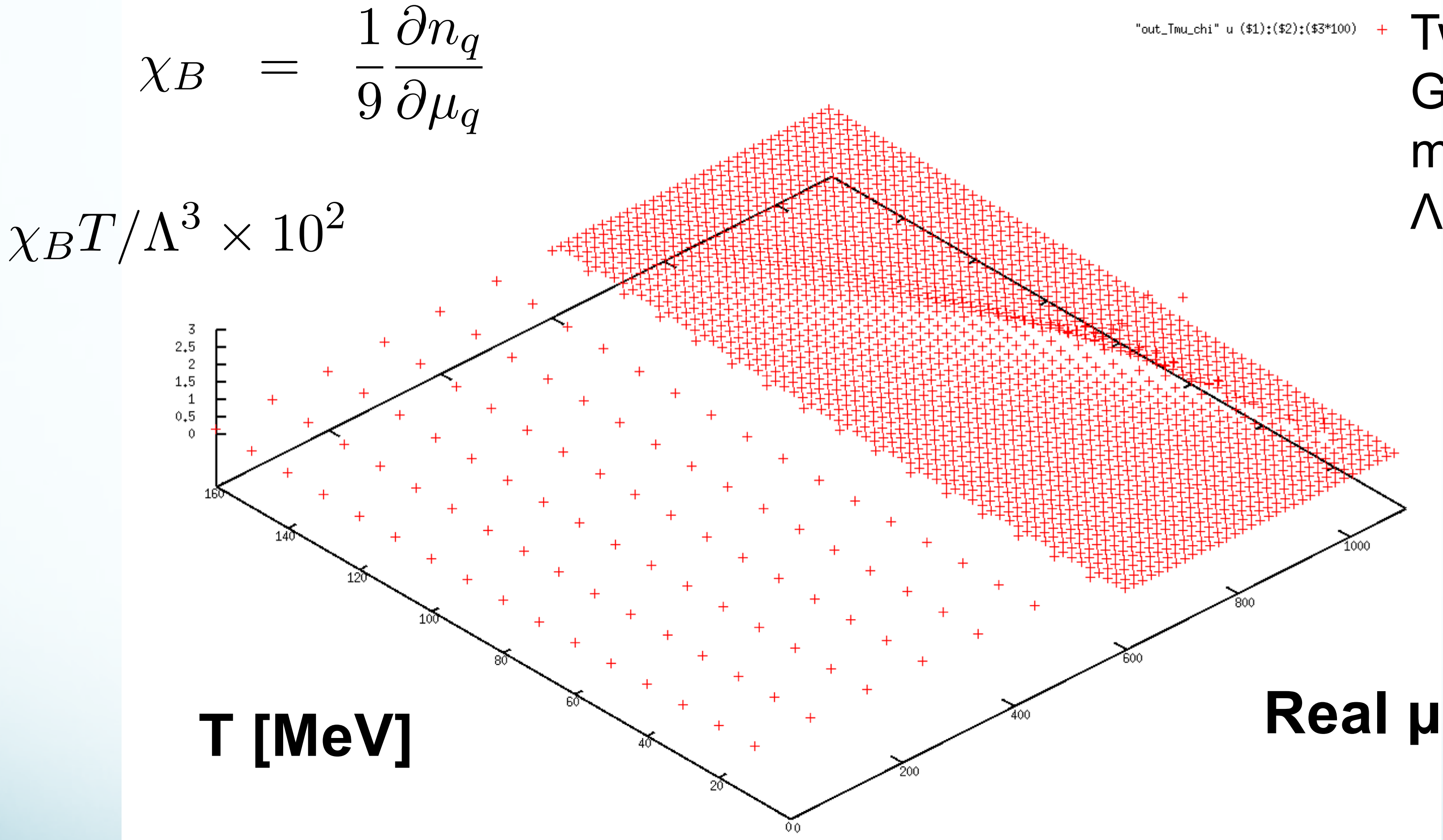
Lee-Yang zeros

Phase transition point

Susceptibility in the NJL model

$$\chi_B = \frac{1}{9}\frac{\partial n_q}{\partial \mu_q}$$

Two-flavor NJL
$G=5.5\text{GeV}^{-2}$
$m_q=5.5\text{MeV}$
$\Lambda=631\text{MeV}$



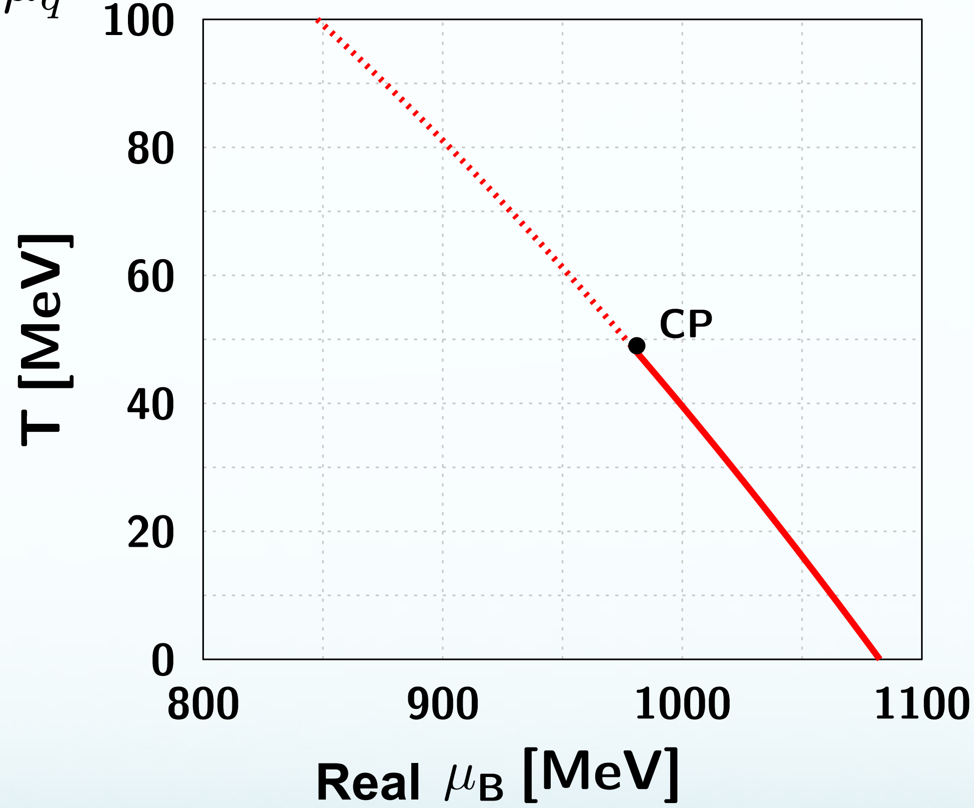
Critical Point: (T,μ_B) ~ (49,981) [MeV]

Phase transition of the NJL model

$$\chi_B = \frac{1}{9}\frac{\partial n_q}{\partial \mu_q}$$

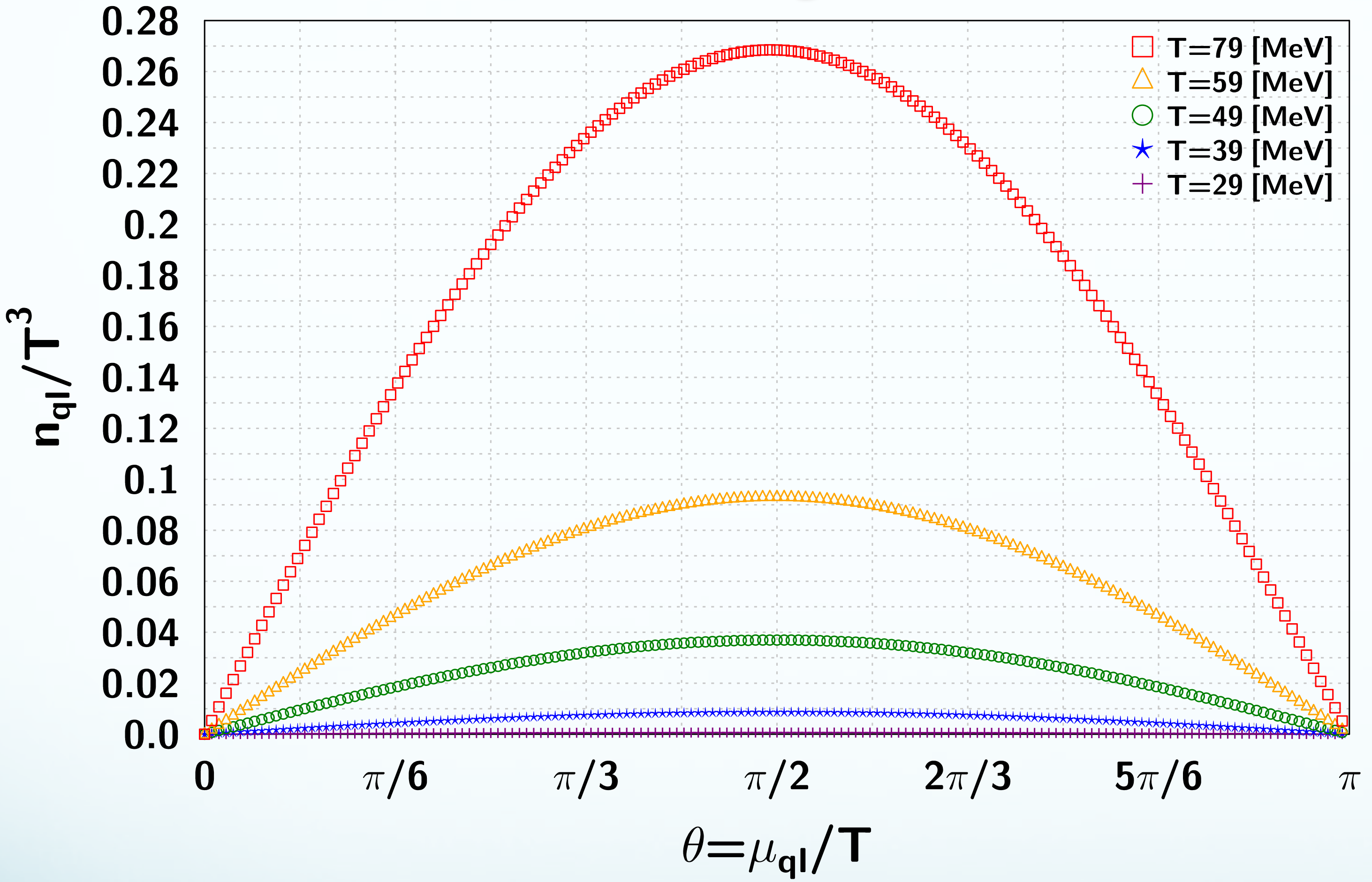
$\chi_B T/\Lambda^3 \times 10^2$

Two-flavor NJL
G=5.5GeV^{-2}
m_q=5.5MeV
Λ=631MeV



Critical Point: (T,μ_B) ~ (49,981) [MeV]

Number density in the NJL model



Number density

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

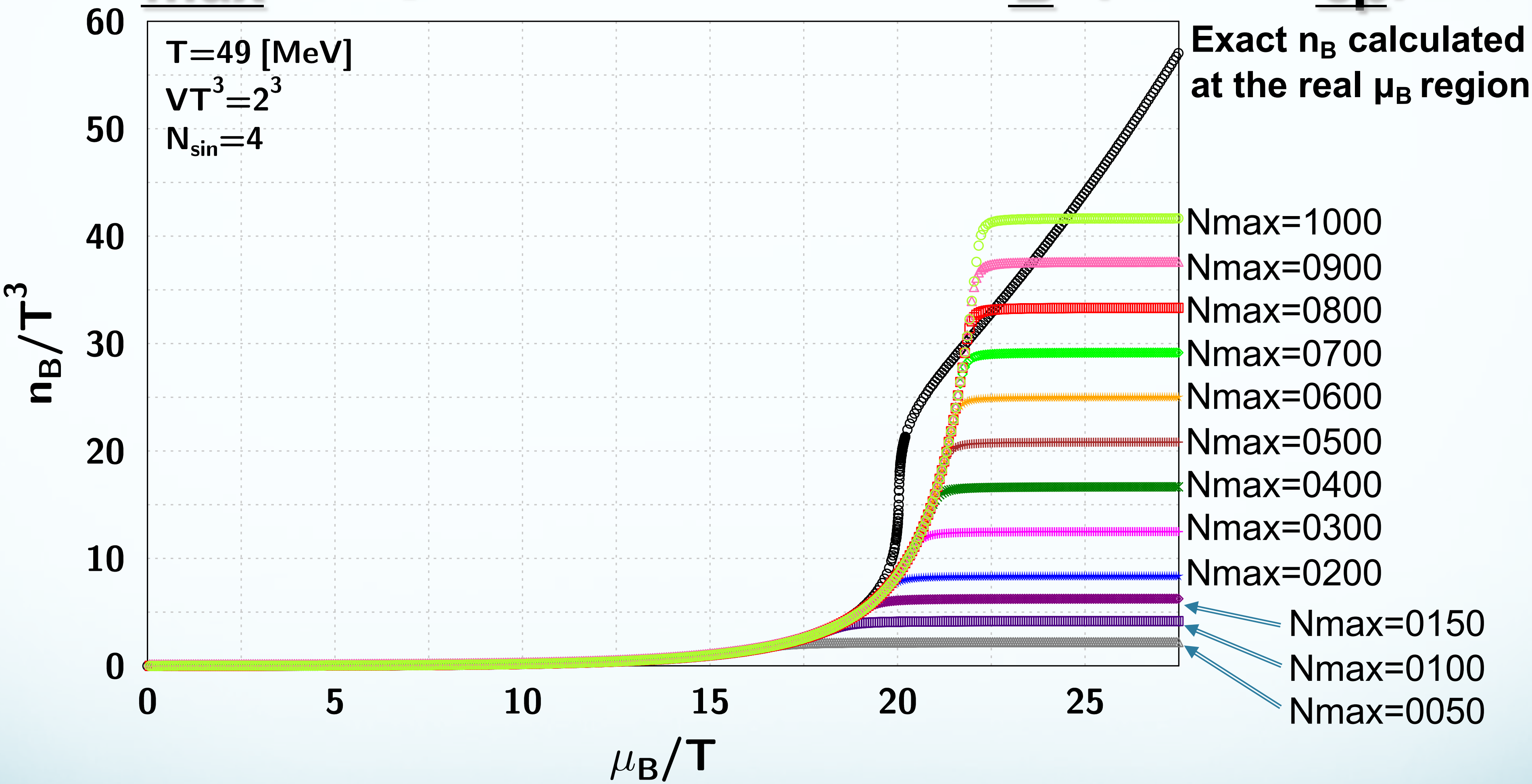
$$n_q = i n_{qI}$$
$$\mu_q = i \mu_{qI}$$

Number density formulation (V. Bornyakov et al., PRD95(2017))

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC} \sim \sum_{k=1}^{N_{\rm sin}} f_k \sin(k\theta)$$

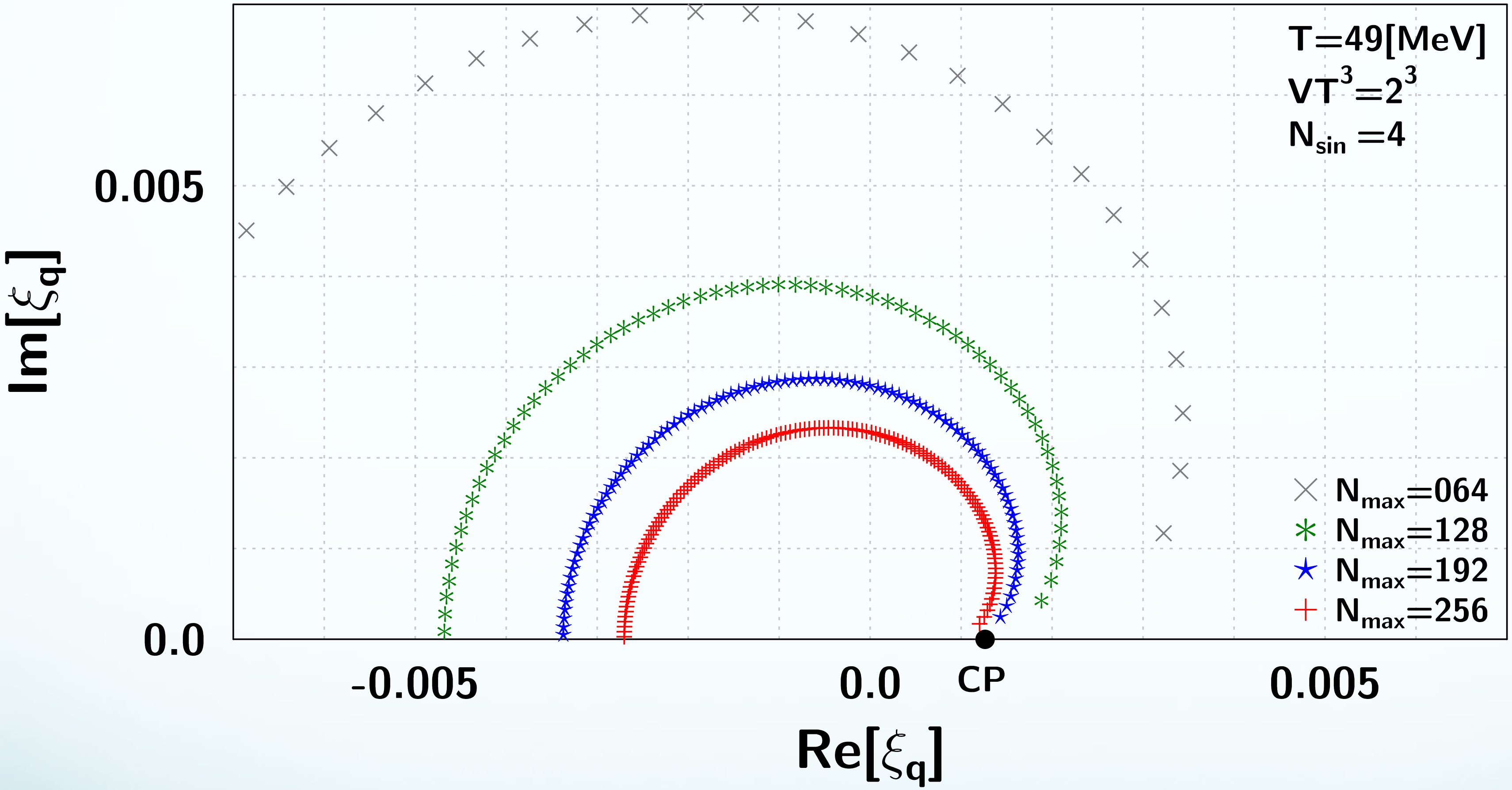
We fit the number density as it was done in the lattice simulations.

N_{max} dependence of n_B ($T = T_{cp}$)



Under the phase transition density, exact n_B calculated at the real μ_B region can be reconstructed from the results of the canonical approach of $N_{max} \geqq 200$.

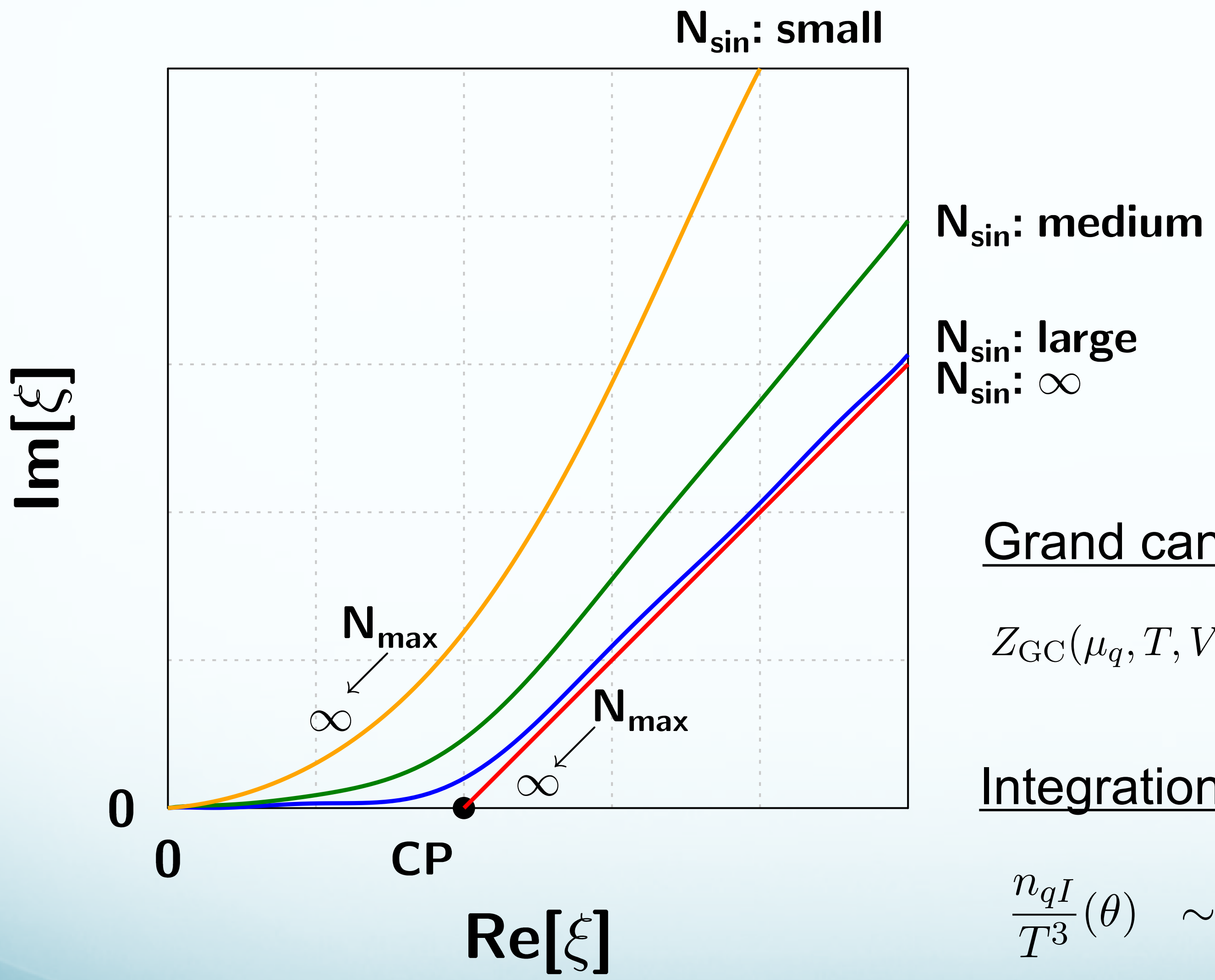
N_{max} dependence ($T = T_{cp}$)



$$\xi = e^{\mu_q/T}$$

As N_{max} increases, edges of LYZs approach to the real axis. But we find that for the finite Nsin, edges of LYZs pass over the expected CP.

Schematic flows of the edges of LYZs



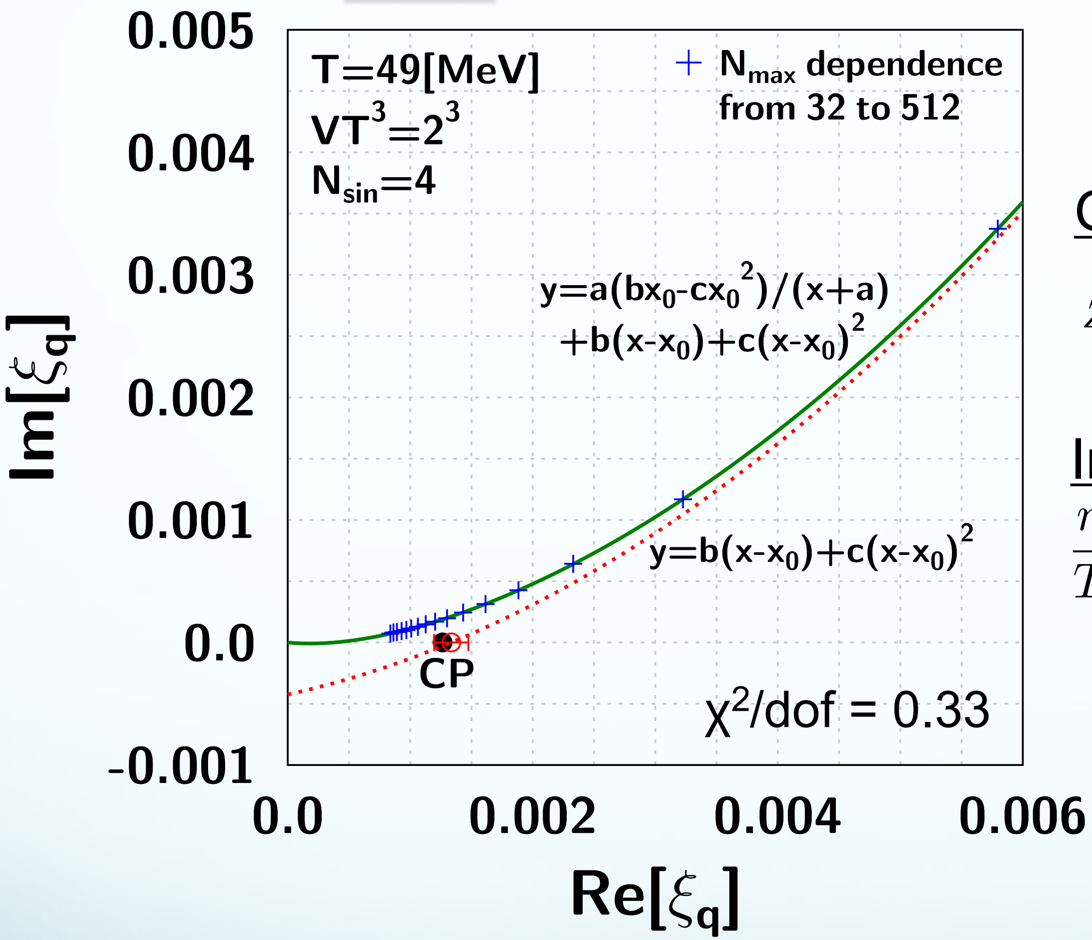
Grand canonical partition function

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n$$

Integration method

$$\frac{n_{qI}}{T^3}(\theta) \sim \sum_{k=1}^{N_{\rm sin}} f_k \sin(k\theta)$$

N_{max} dependence ($T = T_{cp}$)



$$\xi = e^{\mu_q/T}$$

Grand canonical partition function

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n$$

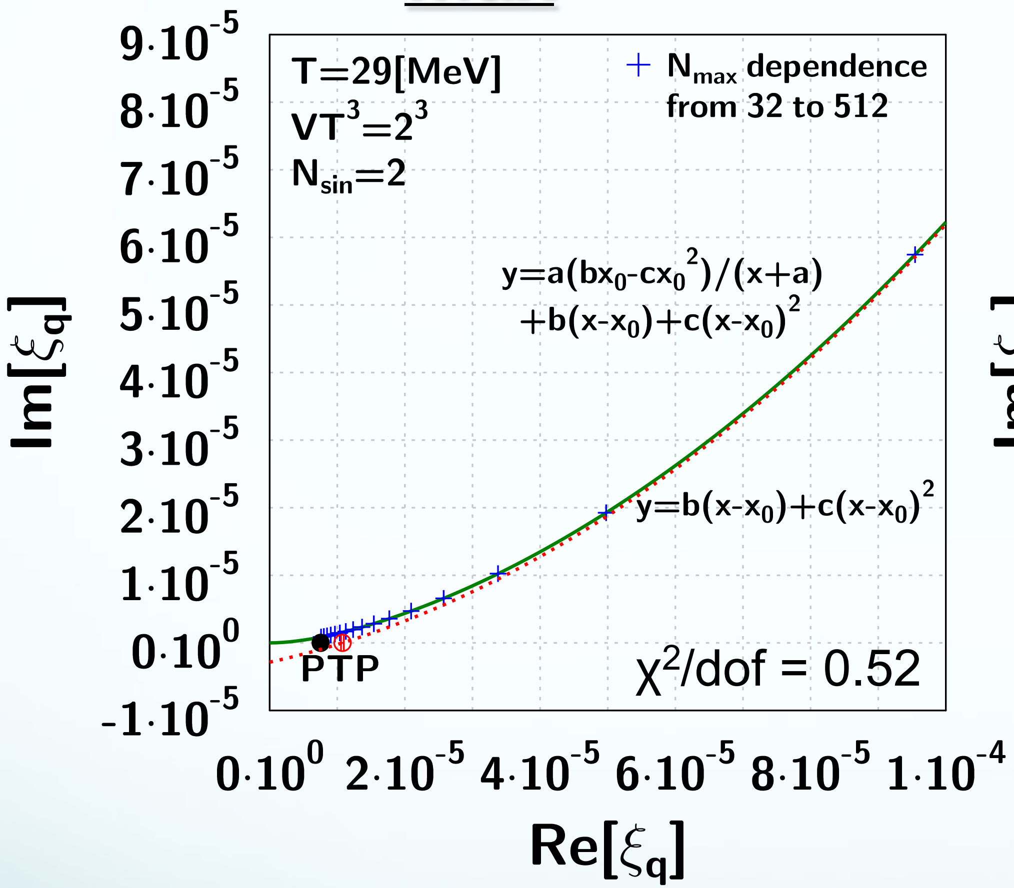
Integration method

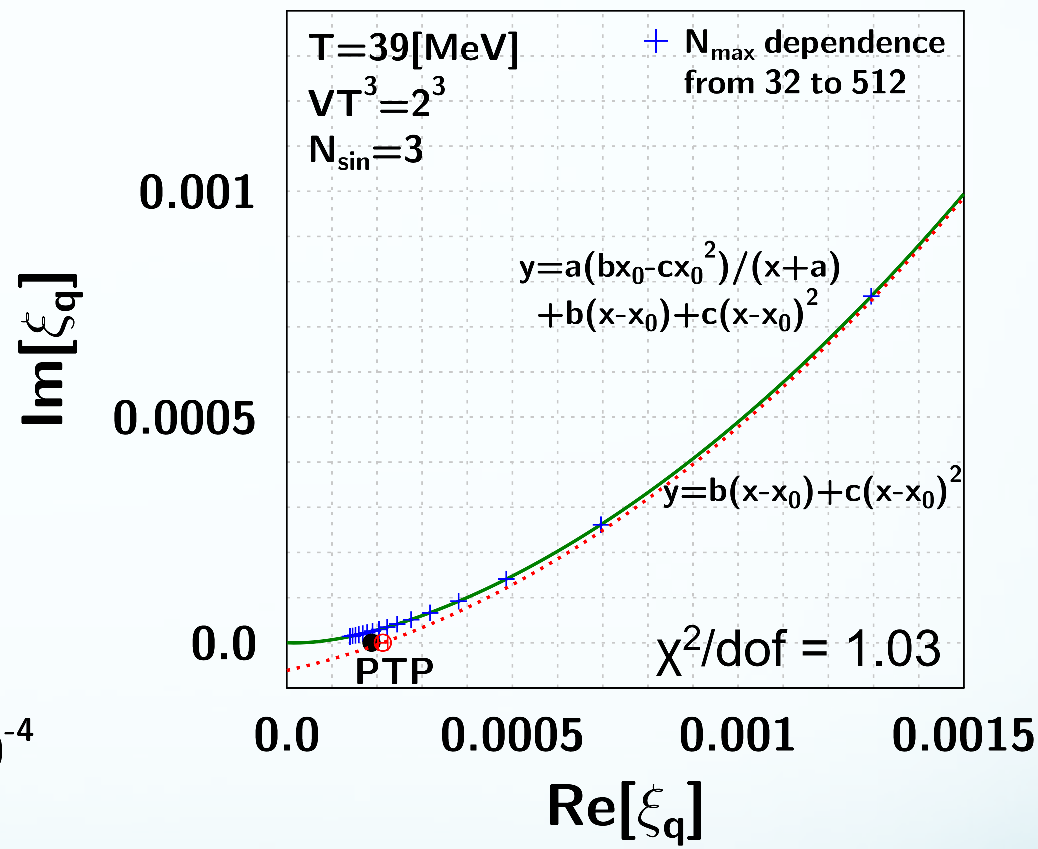
$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC} \sim \sum_{k=1}^{N_{\rm sin}} f_k \sin(k\theta)$$

M.W., A. Hosaka, PLB975, 548 (2019)

We have succeeded in subtracting a term associated with finite N_{sin} effect from the fitted function. The resulting curve represented the dotted curve nicely reproduces the expected critical point (CP) in the NJL model.

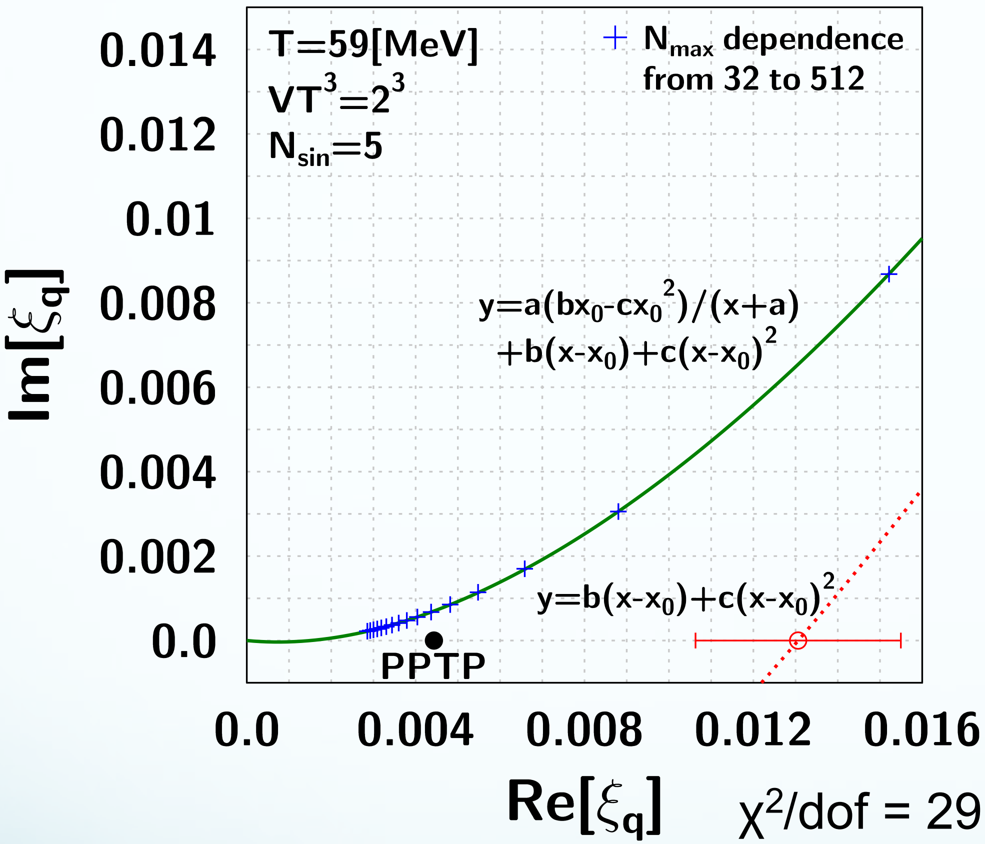
N_{max} dependence ($T < T_{cp}$)

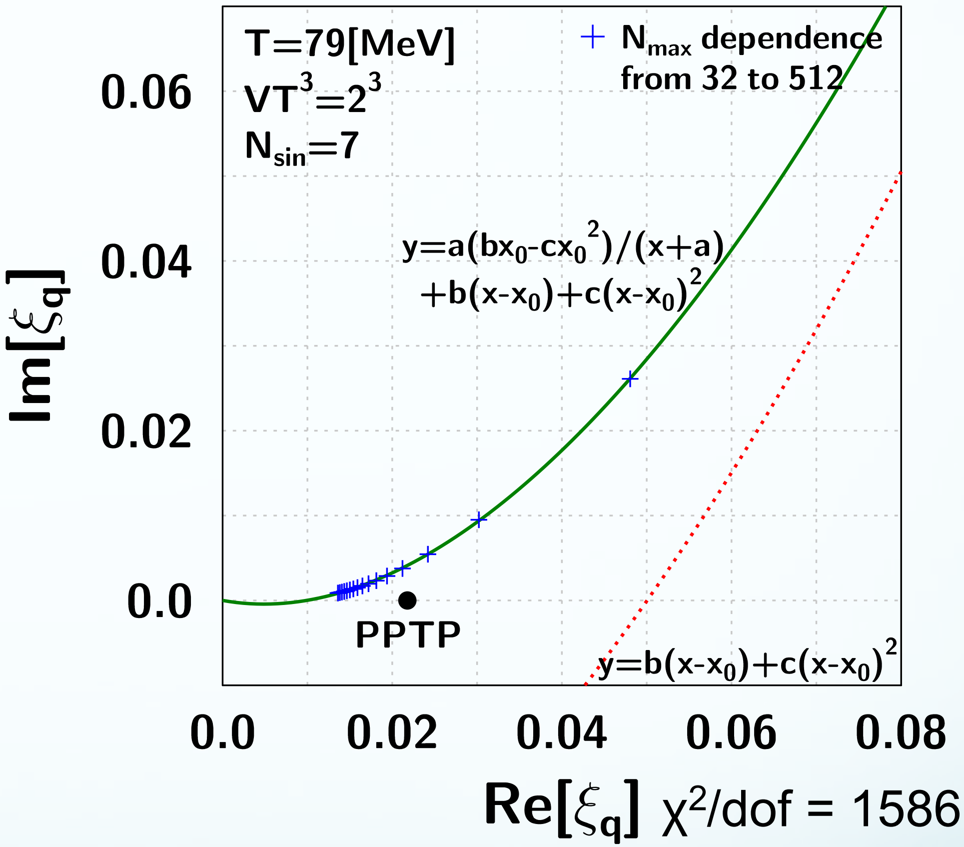




This extrapolation procedure works well to obtain the expected phase transition points (PTP).

N_{max} dependence ($T > T_{cp}$)





This extrapolation procedure does not work well to obtain the expected pseudo phase transition points (PPTP), which is consistent with the disappearance of PTP.

Summary

- We studied Lee-Yang zeros for Z_n obtained from the canonical approach in lattice QCD and the NJL model.
- The phase transition points can be roughly estimated from lattice QCD.
- We found the reasonable extrapolation procedure of the edge of LYZs in the NJL model.

Future work

- Other Example: Polyakov-loop-extended NJL model
  -- It has the Roberge-Weiss symmetry.
- Realistic lattice QCD calculations and determine the QCD phase transitions