LOW-ENERGY SCATTERING AND RESONANCES WITHIN THE NUCLEAR SHELL MODEL

Andrey Shirokov

Skobeltsyn Institute of Nuclear Physics

Lomonosov Moscow State University

Collaborators: J. Vary, P. Maris (Iowa State University)
V. Kulikov (Moscow State University)
A. Mazur, I. Mazur (Pacific National University)

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PLAN

Introduction to the nuclear shell model

HORSE (J-matrix) formalism as a natural extension of SM

SS-HORSE

How it works: Model problem

Application to n-α scattering within the no-core shell model

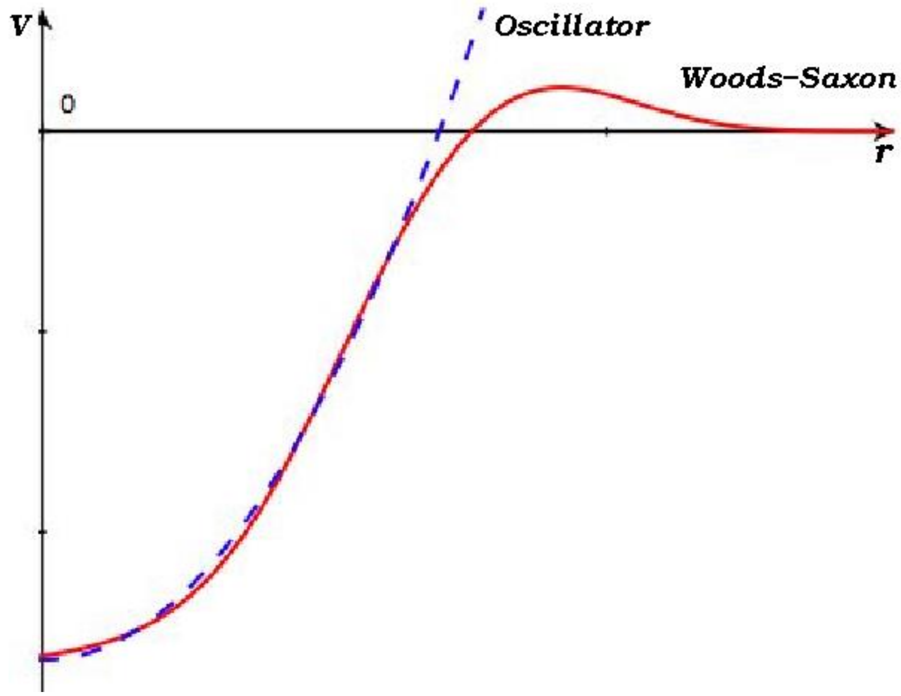
Summary

Shell Model

- SM is a standard traditional tool in nuclear structure theory
- Core SM: e.g., ^{19}F=core(^{16}O)+p+n+n − inert core ^{16}O times antisymmetrized function of 3 nucleons
- No-core SM: antisymmetrized function of all nucleons
- Wave function: $\Psi = \mathcal{A} \prod_i \phi_i(r_i)$
- Traditionally single-particle functions $\phi_i(r_i)$ are harmonic oscillator wave functions

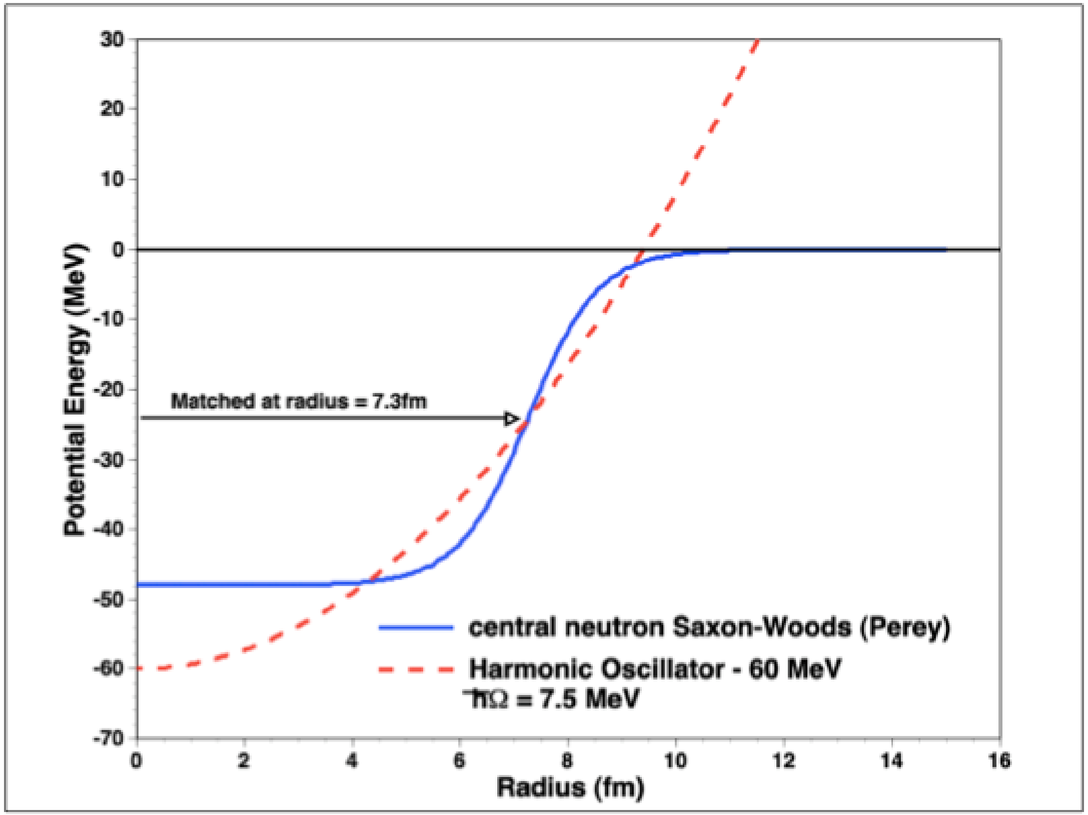
Why oscillator basis?

- Any potential in the vicinity of its minimum at $r=r_0$ has the form $V(r)=V_0+a(r-r_0)^2+b(r-r_0)^3+\ldots$,
  i.e., oscillator is the main term
- Oscillator is a good approximation for the standard Woods–Saxon potential for light nuclei
- Since Shell Model was introduced, oscillator become a language of nuclear physics; a well-developed technique for calculation of many-body matrix elements of various operators (kinetic and potential energy, EM transitions, etc.) has been developed for the harmonic oscillator; the spurious C.M. motion can be completely removed in the oscillator basis only, etc.



Why oscillator basis?

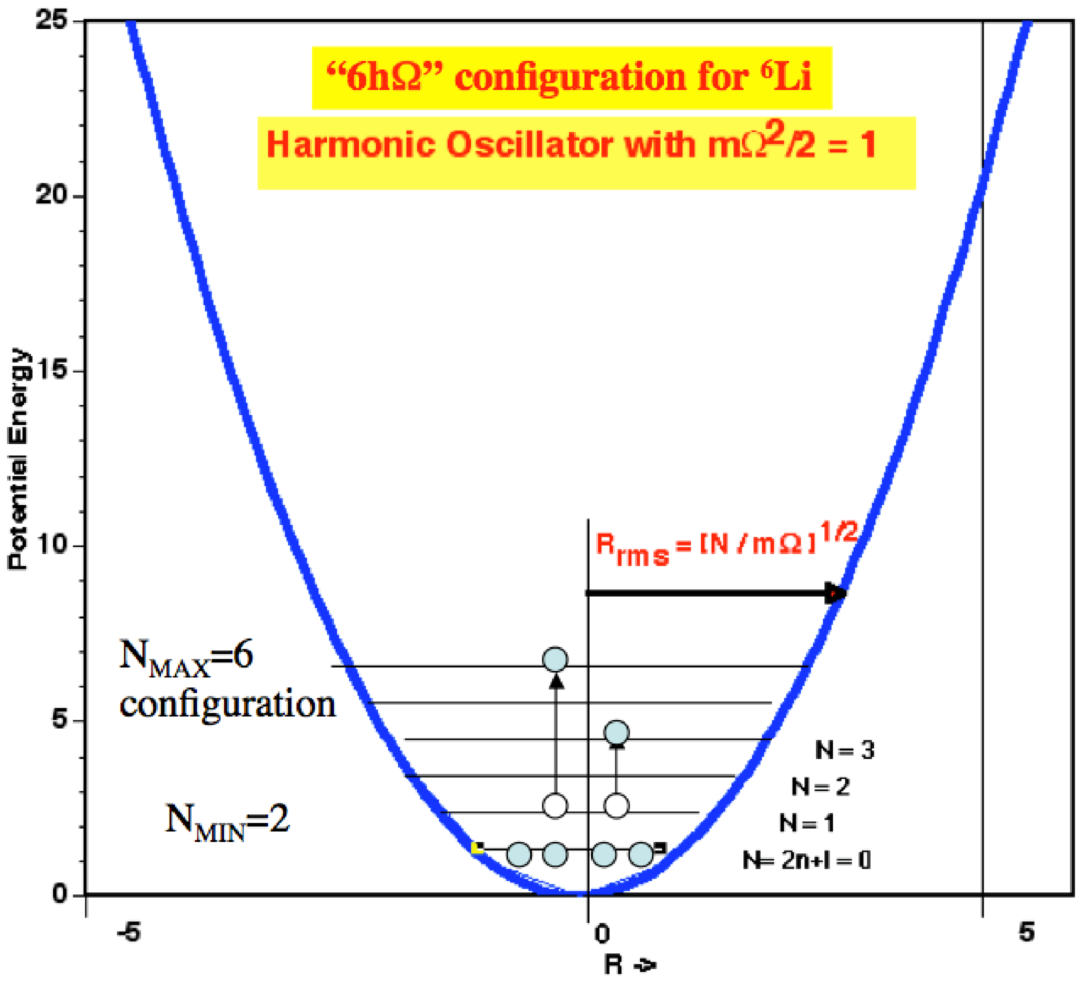
The situation is worse in heavy nuclei, but the harmonic oscillator remains a standard language of nuclear physics…



N_{max} truncation

All many-body states with total oscillator quanta up to some N_{max} are included in the basis space (N_{max} or $N\hbar\Omega$ truncation).

This truncation makes it possible to completely separate spurious CM excited states



Shell Model

- Shell model is a bound state technique, no continuum spectrum; not clear how to interpret states in continuum above thresholds − how to extract resonance widths or scattering phase shifts
- HORSE (*J*-matrix) formalism can be used for this purpose

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- Other possible approaches: NCSM+RGM; Gamov SM; Continuum SM; SM+Complex Scaling; …
- All of them make the SM much more complicated. Our aim is to interpret directly the SM results above thresholds obtained in a usual way without additional complexities and to extract from them resonant parameters and phase shifts at low energies.

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- **I will discuss a more general interpretation of SM results**

J-matrix (Jacobi matrix) formalism in scattering theory

- Two types of L^2 basises:
- Laguerre basis (atomic hydrogen-like states) — atomic applications
- Oscillator basis — nuclear applications
- Other titles in case of oscillator basis:

HORSE (harmonic oscillator representation of scattering equations),

Algebraic version of RGM

J-matrix formalism

- Initially suggested in atomic physics (E. Heller, H. Yamani, L. Fishman, J. Broad, W. Reinhardt) :

  H.A.Yamani and L.Fishman, J. Math. Phys **16**, 410 (1975). Laguerre and oscillator basis.

- Rediscovered independently in nuclear physics (G. Filippov, I. Okhrimenko, Yu. Smirnov):

  G.F.Filippov and I.P.Okhrimenko, Sov. J. Nucl. Phys. **32**, 480 (1980). Oscillator basis.

HORSE:
J-matrix formalism with oscillator basis

- Some further developments (incomplete list; not always the first publication but a more transparent or complete one):

Yu.I.Nechaev and Yu.F.Smirnov, Sov. J. Nucl. Phys. **35**, 808 (1982)

I.P.Okhrimenko, Few-body Syst. **2**, 169 (1987)

V.S.Vasievsky and F.Arickx, Phys. Rev. A **55**, 265 (1997)

S.A.Zaytsev, Yu.F.Smirnov, and A.M.Shirokov, Theor. Math. Phys. **117**, 1291 (1998)

J.M.Bang et al, Ann. Phys. (NY) **280**, 299 (2000)

A.M.Shirokov et al, Phys. Rev. C **70**, 044005 (2004)

HORSE:
J-matrix formalism with oscillator basis

- Active research groups:

  Kiev: G. Filippov, V. Vasilevsky, A. Nesterov et al

  Antwerp: F. Arickx, J. Broeckhove et al

  Moscow: A. Shirokov, S. Igashov et al

  Khabarovsk: S. Zaytsev, A. Mazur et al

  Ariel: Yu. Lurie

HORSE:

- Schrödinger equation:

$$H^l \Psi_{lm}(E, r) = E \Psi_{lm}(E, r)$$

- Wave function is expanded in oscillator functions:

$$\Psi_{lm}(E, \boldsymbol{r}) = \frac{1}{r} u_l(E, r)\, Y_{lm}(\hat{\boldsymbol{r}}),$$

$$u_l(E, r) = \sum_{n=0}^{\infty} a_{nl}(E)\, R_{nl}(r),$$

- Schrödinger equation is an infinite set of algebraic equations:

$$\sum_{n'=0}^{\infty} (H^l_{nn'} - \delta_{nn'}) a_{nn'}(E) = 0.$$

where H=T+V,

T — kinetic energy operator,

V — potential energy

HORSE:

- Kinetic energy matrix elements:

$$|nlm\rangle \equiv \phi_{nlm}(\boldsymbol{r}) = \frac{1}{r} R_{nl}(r) Y_{lm}(\hat{\boldsymbol{r}})$$

$$T^l_{nn'} \equiv \langle nlm|T|n'l'm'\rangle = \int \phi_{nlm}(\boldsymbol{r}) T \phi_{n'l'm'}(\boldsymbol{r})\, d^3\boldsymbol{r}$$
$$= \delta_{ll'}\delta_{mm'} \int R_{nl} T R_{n'l}\, dr$$

- Kinetic energy is tridiagonal:

$$T^l_{n,n-1} = -\frac{\hbar\omega}{2}\sqrt{n(n+l+1/2)},$$

$$T^l_{n,n} = \frac{\hbar\omega}{2}(2n+l+3/2),$$

$$T^l_{n,n+1} = -\frac{\hbar\omega}{2}\sqrt{(n+1)(n+l+3/2)}$$

- Note! Kinetic energy tends to infinity as *n* and *n'*=*n*, *n*±1 increases:

$$T^l_{nn'} \sim n, \quad n \to \infty, \quad n' = n, n \pm 1$$

HORSE:

- Potential energy matrix elements:

$$|nlm\rangle \equiv \phi_{nlm}(\boldsymbol{r}) = \frac{1}{r} R_{nl}(r)\, Y_{lm}(\hat{\boldsymbol{r}}),$$

$$V_{nn'}^{ll'} \equiv \langle nlm|V|n'l'm'\rangle = \int \phi_{nlm}(\boldsymbol{r})\, V\, \phi_{n'l'm'}(\boldsymbol{r})\; d^3\boldsymbol{r}$$

- For central potentials only

$$V_{nn'}^{ll'} = V_{nn'}^{l} = \delta_{mm'}\delta_{ll'} \int R_{nl}(r)\, V\, R_{n'l}(r)\; dr$$

- Note! Potential energy tends to zero as *n* and/or *n'* increases:

$$V_{nn'}^{ll'} \to 0, \quad n, n' \to \infty$$

- Therefore for large *n* or *n'*:

$$T_{nn'}^{l} \gg V_{nn'}^{ll'}, \quad n \text{ or/and } n' \gg 1$$

A reasonable approximation when *n* or *n'* are large

$$H_{nn'}^{l} = T_{nn'}^{l} + V_{nn'}^{l} \approx T_{nn'}^{l}, \quad n \text{ or/and } n' \gg 1.$$

HORSE:

- In other words, it is natural to truncate the potential energy:

$$\widetilde{V}^{l}_{nn'} = \begin{cases} V^{l}_{nn'} & \text{if } n \text{ and } n' \leq N; \\ 0 & \text{if } n \text{ or } n' > N. \end{cases}$$

- This is equivalent to writing the potential energy operator as

$$V = \sum_{n=0}^{N} \sum_{n'=0}^{N} \sum_{l,l',m,m'} |nlm\rangle \, V^{ll'}_{nn'} \, \langle n'l'm'|$$

- For large *n*, the Schrödinger equation

$$\sum_{n'=0}^{\infty} (H^{l}_{nn'} - \delta_{nn'}E)\, a_{n'l}(E) = 0$$

takes the form

$$\sum_{n'=0}^{\infty} (T^{l}_{nn'} - \delta_{nn'}E) a_{n'l}(E) = 0, \qquad n \geq N+1$$

General idea of the HORSE formalism

Infinite set of algebraic equations

$$\sum_{n'=0}^{\infty} (H^{l}_{nn'} - \delta_{nn'} E)\, a_{n'l}(E) = 0$$

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$$T^{l}_{n,\,n-1}\, a_{n-1,l}(E) + (T^{l}_{nn} - E)\, a_{nl}(E) + T^{l}_{n,\,n+1}\, a_{n+1,l}(E) = 0$$

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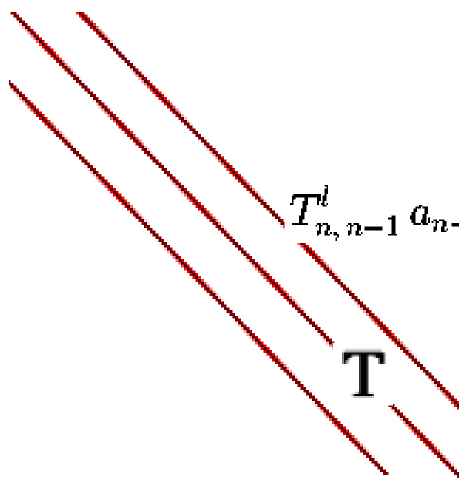

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$$\sum_{n'=0}^{\infty} (H^l_{nn'} - \delta_{nn'} E)\, a_{n'l}(E) = 0$$

The potential energy V^l is truncated:

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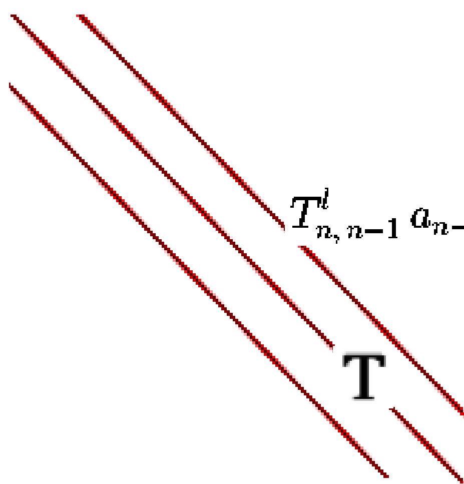

$$\sum_{n'=0}^{\infty} (T^l_{nn'} - \delta_{nn'} E)\, a_{n'l}(E) = 0, \quad n \geq N+1$$

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$$\sum_{n'=0}^{N} \left(T^l_{nn'} + V^l_{nn'} - \delta_{nn'} E\right) a_{n'l}(E) = 0, \quad n \le N$$

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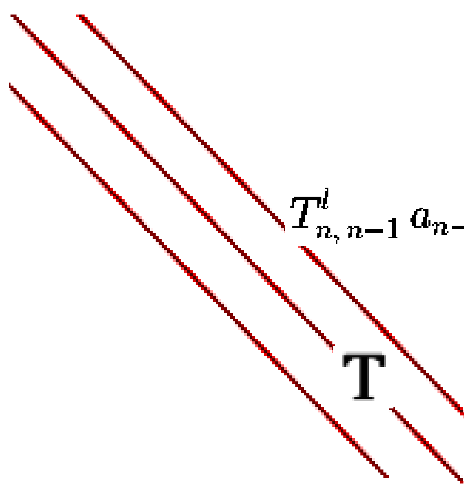

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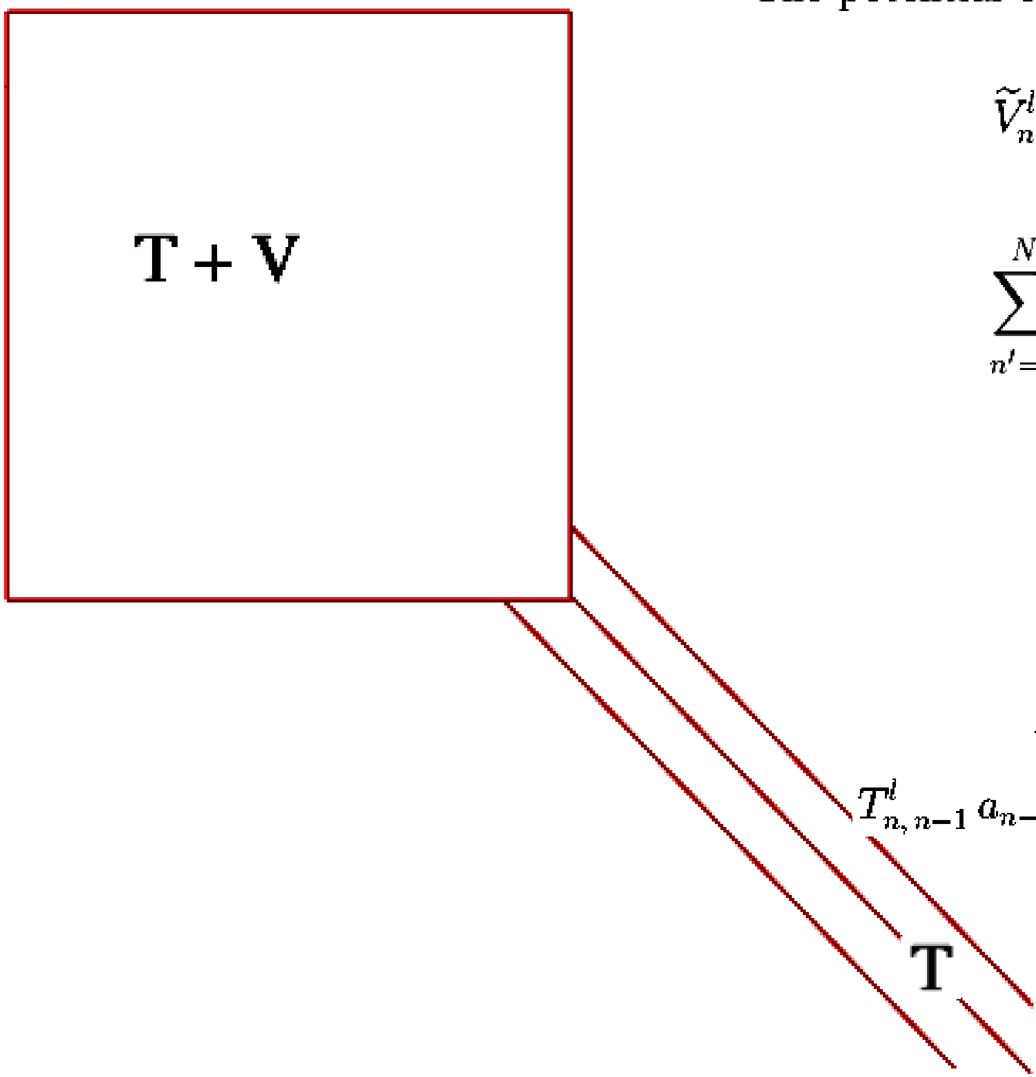

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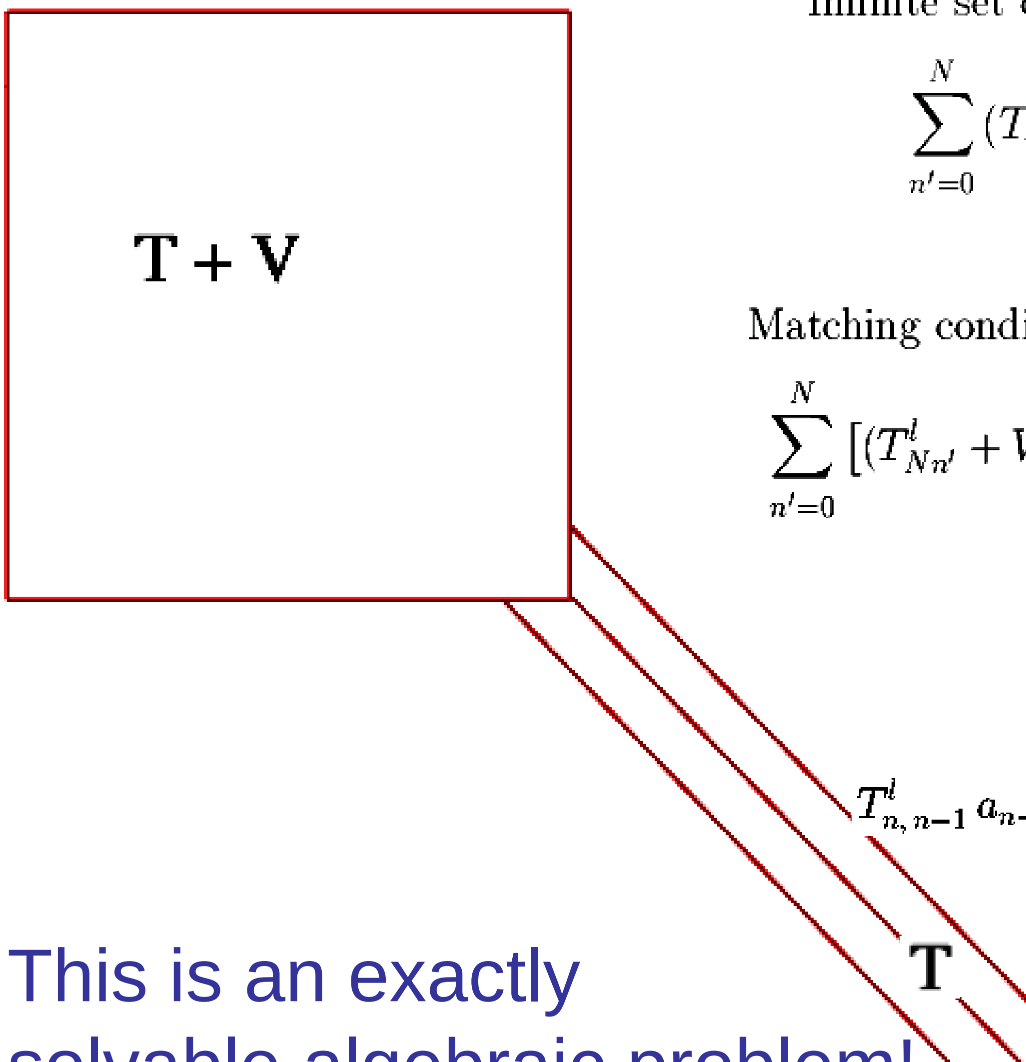


Infinite set of algebraic equations

$$\sum_{n'=0}^{N} (T^l_{nn'} + V^l_{nn'} - \delta_{nn'}E)\, a_{n'l}(E) = 0, \quad n \le N-1$$

Matching condition at $n = N$:

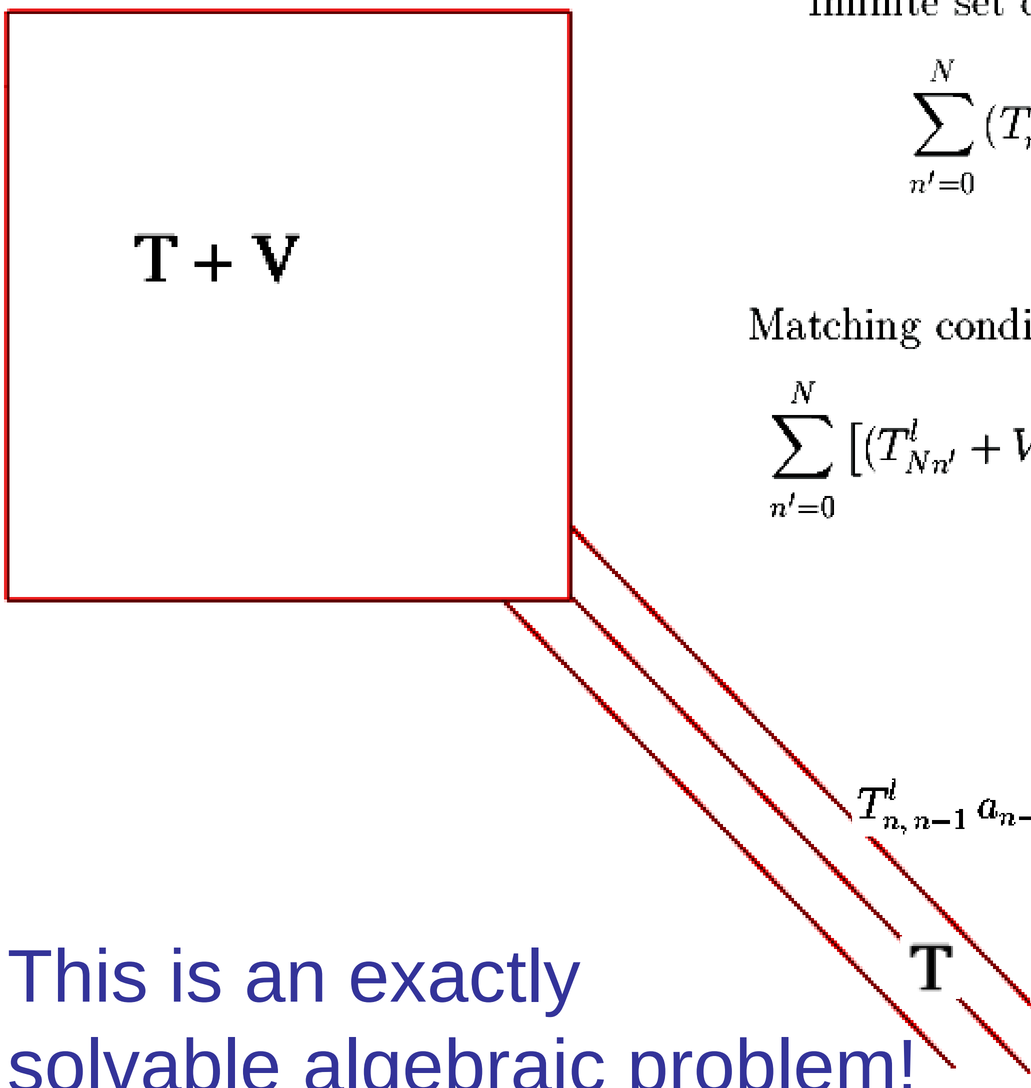
$$\sum_{n'=0}^{N} \left[(T^l_{Nn'} + V^l_{Nn'} - \delta_{Nn'}E)\, a_{n'l}(E)\right] + T^l_{N,N+1}\, a_{N+1,l}(E) = 0$$

$$\sum_{n'=0}^{\infty} (T^l_{nn'} - \delta_{nn'}E)\, a_{n'l}(E) = 0, \quad n \ge N+1$$

$$T^l_{n,\,n-1}\, a_{n-1,l}(E) + (T^l_{nn} - E)\, a_{nl}(E) + T^l_{n,\,n+1}\, a_{n+1,l}(E) = 0$$

This is an exactly solvable algebraic problem!

General idea of the HORSE formalism



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$$\sum_{n'=0}^{\infty} (T^{l}_{nn'} - \delta_{nn'}E)\, a_{n'l}(E) = 0, \quad n \geq N+1$$

$$T^{l}_{n,\,n-1}\, a_{n-1,l}(E) + (T^{l}_{nn} - E)\, a_{nl}(E) + T^{l}_{n,\,n+1}\, a_{n+1,l}(E) = 0$$

And this looks like a natural extension of SM where both potential and kinetic energies are truncated

This is an exactly solvable algebraic problem!

Asymptotic region $n \geq N$

- Schrödinger equation takes the form of three-term recurrent relation:

$$T^l_{n,\,n-1}\,a_{n-1,\,l}(E) + (T^l_{nn} - E)\,a_{nl}(E) + T^l_{n,\,n+1}\,a_{n+1,\,l}(E) = 0$$

- This is a second order finite-difference equation. It has two independent solutions:

$$S_{nl}(E) = \sqrt{\frac{\pi\, r_0\, n!}{\Gamma(n+l+3/2)}}\; q^{l+1}\, \exp\left(-\frac{q^2}{2}\right)\; L_n^{l+\frac{1}{2}}(q^2),$$

$$C_{nl}(E) = (-1)^l \sqrt{\frac{\pi\, r_0\, n!}{\Gamma(n+l+3/2)}}\; \frac{q^{-l}}{\Gamma(-l+1/2)}\, \exp\left(-\frac{q^2}{2}\right) \times \Phi(-n-l-1/2,\, -l+1/2;\, q^2)$$

where dimensionless momentum $q = \sqrt{\frac{2E}{\hbar\omega}}$

For derivation, see S.A.Zaytsev, Yu.F.Smirnov, and A.M.Shirokov, Theor. Math. Phys. **117**, 1291 (1998)

Asymptotic region $n \geq N$

- Schrödinger equation:

$$T^l_{n,\,n-1}\, a_{n-1,\,l}(E) + (T^l_{nn} - E)\, a_{nl}(E) + T^l_{n,\,n+1}\, a_{n+1,\,l}(E) = 0$$

- Arbitrary solution $a_{nl}(E)$ of this equation can be expressed as a superposition of the solutions $S_{nl}(E)$ and $C_{nl}(E)$, e.g.:

$$a_{nl}(E) = \cos\delta(E)\; S_{nl}(E) + \sin\delta(E)\; C_{nl}(E)$$

- Note that

$$\sum_{n=M}^{\infty} S_{Nl}(E) R_{nl}(r) \underset{r\to\infty}{\longrightarrow} j_l(qr) \sim \sin\left(qr - \frac{\pi l}{2}\right),$$

$$\sum_{n=M}^{\infty} C_{Nl}(E) R_{nl}(r) \underset{r\to\infty}{\longrightarrow} -n_l(qr) \sim \cos\left(qr - \frac{\pi l}{2}\right)$$

Asymptotic region $n \geq N$

- Therefore our wave function

$$u_l(E,r) = \sum_{n=0}^{\infty} a_{nl}(E)\, R_{nl}(r) \underset{r\to\infty}{\longrightarrow} \sin\left(qr + \delta - \frac{\pi l}{2}\right)$$

- Reminder: the ideas of quantum scattering theory.
- Cross section

$$\sigma \sim \sin^2 \delta$$

- Wave function

$$\Psi \underset{r\to\infty}{\longrightarrow} \sin\left(qr + \delta - \frac{\pi l}{2}\right)$$

- δ in the HORSE approach is the phase shift!

Internal region (interaction region) $n \leq N$

- Schrödinger equation

$$\sum_{n'=0}^{N} H^{l}_{nn'} \langle n' | \lambda \rangle = E_{\lambda} \langle n | \lambda \rangle, \qquad n \leq N$$

- Inverse Hamiltonian matrix:

$$(H - E)^{-1}_{nn'} \equiv -\mathscr{G}_{nn'} = \sum_{\lambda'=0}^{N} \frac{\langle n | \lambda' \rangle \langle \lambda' | n' \rangle}{E_{\lambda'} - E}$$

Matching condition at *n*=*N*

- Solution:

$$a_{nl}(E) = -(H-E)^{-1}_{nN}\, T^{l}_{N,N+1}\, a_{N+1,l}(E) = \mathcal{G}_{nN}\, T^{l}_{N,N+1}\, a_{N+1,l}(E)$$

- From the asymptotic region

$$a_{nl}(E) = \cos\delta(E)\, S_{nl}(E) + \sin\delta(E)\, C_{nl}(E), \qquad n \geq N$$

- Note, it is valid at $n=N$ and $n=N+1$. Hence

$$\tan\delta(E) = -\frac{S_{Nl}(E) - \mathcal{G}_{NN}\, T^{l}_{N,N+1}\, S_{N+1,l}(E)}{C_{Nl}(E) - \mathcal{G}_{NN}\, T^{l}_{N,N+1}\, C_{N+1,l}(E)}$$

- This is equation to calculate the phase shifts.
- The wave function is given by

$$\Psi_{lm}(E,\boldsymbol{r}) = \frac{1}{r}\, u_l(E,r)\, Y_{lm}(\hat{\boldsymbol{r}}),$$

$$u_l(E,r) = \sum_{n=0}^{\infty} a_{nl}(E)\, R_{nl}(r),$$

where

$$a_{nl}(E) = \cos\delta(E)\, S_{nl}(E) + \sin\delta(E)\, C_{nl}(E), \qquad n \geq N$$

$$a_{nl}(E) = \mathcal{G}_{nN}\, T^{l}_{N,N+1}\, a_{N+1,l}(E)$$

Problems with direct HORSE application

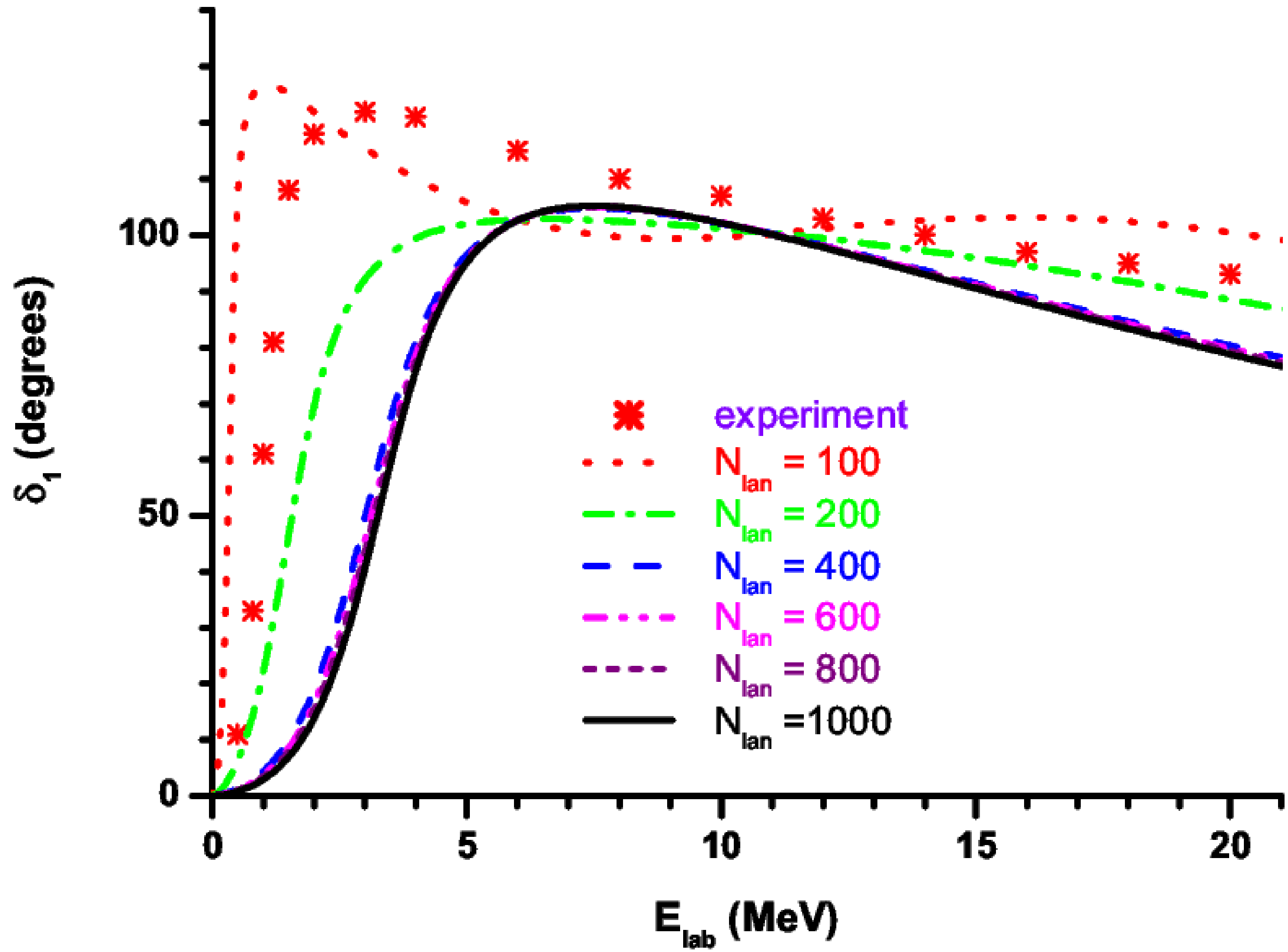
$$\tan\delta(E) = -\frac{S_{Nl}(E) - \mathcal{G}_{NN}\, T^{l}_{N,\,N+1}\, S_{N+1,\,l}(E)}{C_{Nl}(E) - \mathcal{G}_{NN}\, T^{l}_{N,\,N+1}\, C_{N+1,\,l}(E)}$$

$$(H-E)^{-1}_{nn'} \equiv -\mathcal{G}_{nn'} = \sum_{\lambda'=0}^{N} \frac{\langle n|\lambda'\rangle\langle\lambda'|n'\rangle}{E_{\lambda'} - E}$$

$$\sum_{n'=0}^{N} H^{l}_{nn'}\langle n'|\lambda\rangle = E_\lambda \langle n|\lambda\rangle, \qquad n \le N$$

- A lot of E_λ eigenstates needed while SM codes usually calculate few lowest states only
- One needs highly excited states and to get rid from CM excited states.
- $\langle n'|\lambda\rangle$ are normalized for all states including the CM excited ones, hence renormalization is needed.
- We need $\langle n'|\lambda\rangle$ for the relative *n*-nucleus coordinate r_{nA} but NCSM provides $\langle n'|\lambda\rangle$ for the *n* coordinate r_n relative to the nucleus CM. Hence we need to perform Talmi-Moshinsky transformations for all states to obtain $\langle n'|\lambda\rangle$ in relative *n*-nucleus coordinates.
- Concluding, the direct application of the HORSE formalism in *n*-nucleus scattering is unpractical.

Example: *nα* scattering



Single-state HORSE (SS-HORSE)

$$\sum_{n'=0}^{N} H^{l}_{nn'} \langle n'|\lambda\rangle = E_\lambda \langle n|\lambda\rangle, \qquad n \le N$$

$$(H - E)^{-1}_{nn'} \equiv -\mathcal{G}_{nn'} = \sum_{\lambda'=0}^{N} \frac{\langle n|\lambda'\rangle\langle\lambda'|n'\rangle}{E_{\lambda'} - E}$$

$$\tan\delta(E) = -\frac{S_{Nl}(E) - \mathcal{G}_{NN}\, T^{l}_{N,\,N+1}\, S_{N+1,l}(E)}{C_{Nl}(E) - \mathcal{G}_{NN}\, T^{l}_{N,\,N+1}\, C_{N+1,l}(E)}$$

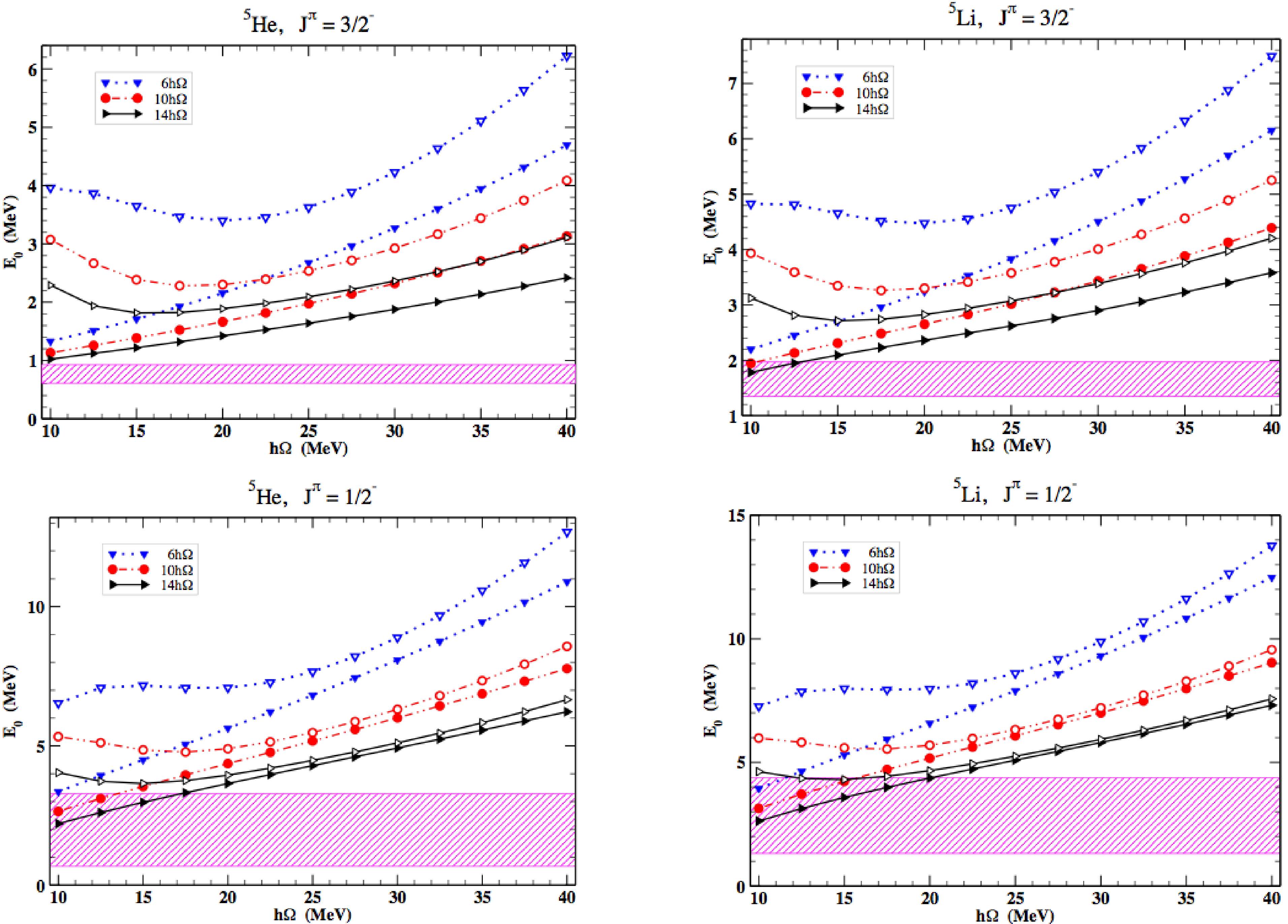
Suppose $E = E_\lambda$:

$$\tan\delta(E_\lambda) = -\frac{S_{N+1,\lambda}(E_\lambda)}{C_{N+1,\lambda}(E_\lambda)}$$

E_λ are eigenstates that are consistent with scattering information for given $\hbar\Omega$ and N_{max}; this is what you should obtain in any calculation with oscillator basis and what you should compare with your *ab initio* results.

*N*α scattering and NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$



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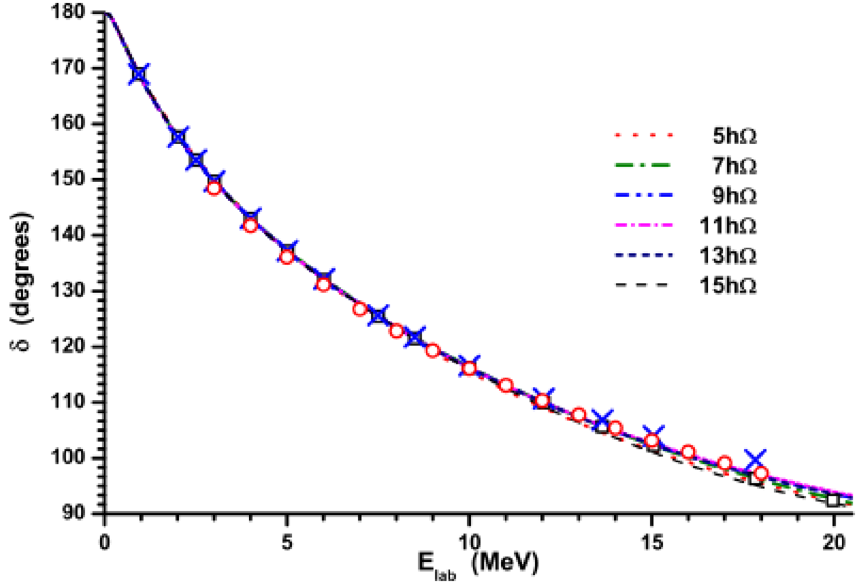

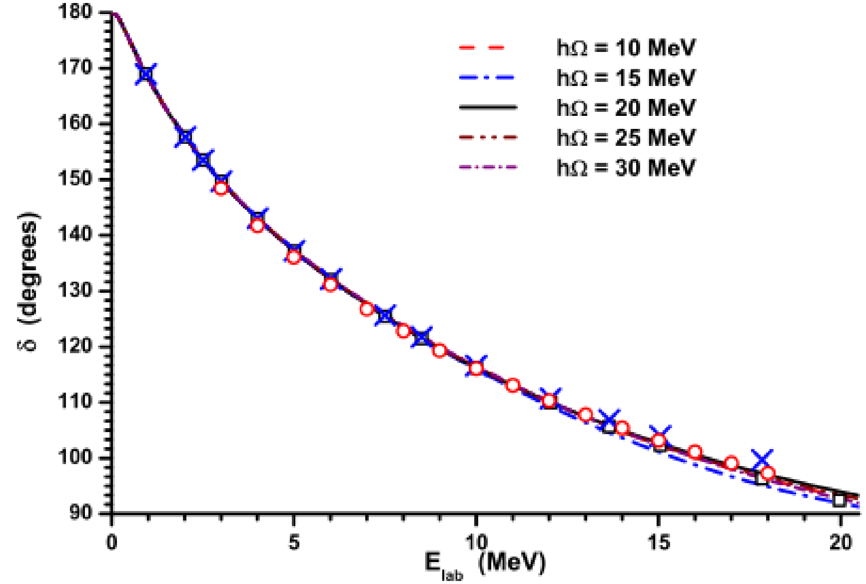

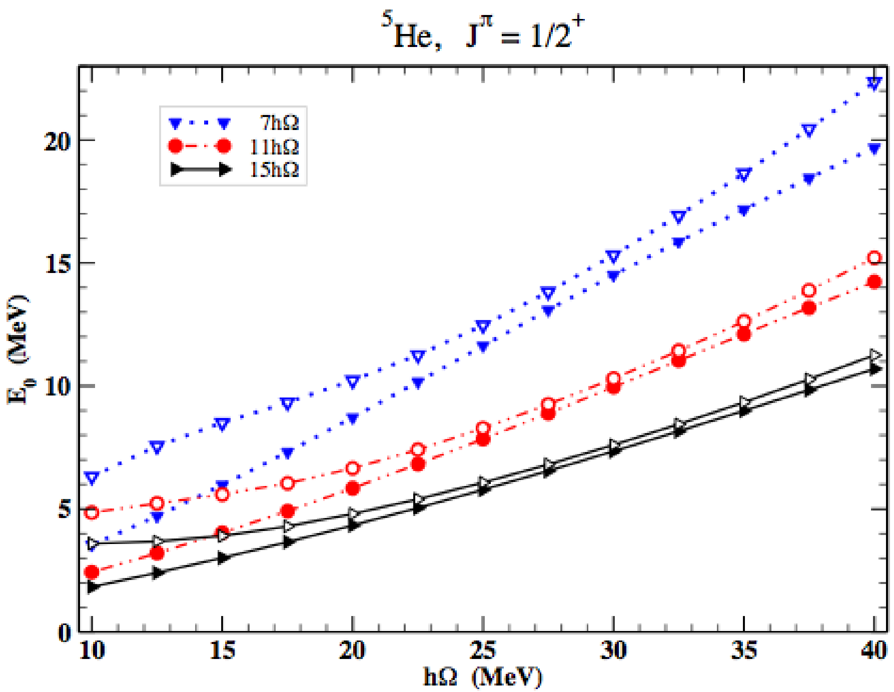

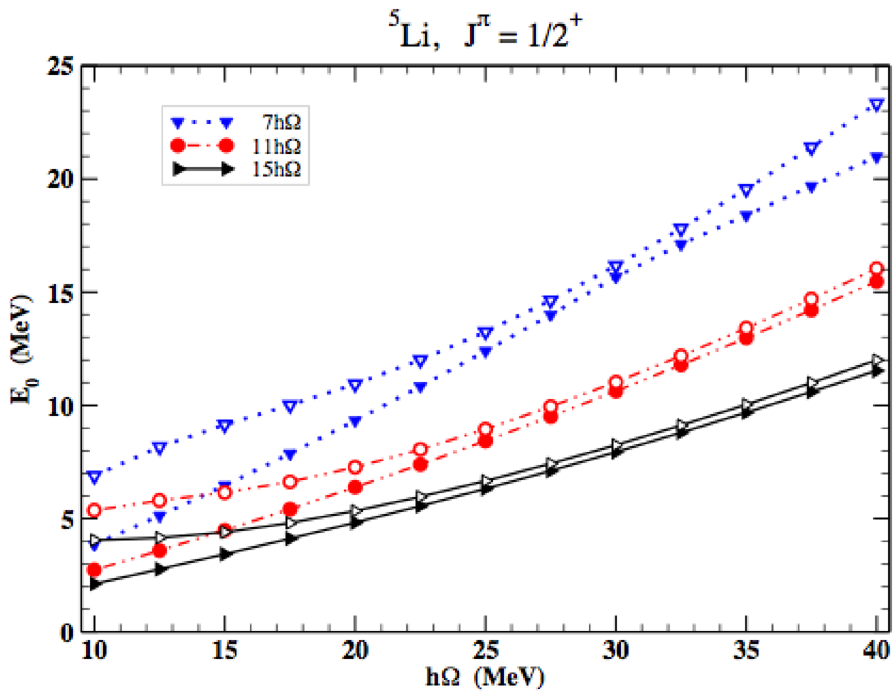

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$$\sum_{n'=0}^{N} H^{l}_{nn'} \langle n' | \lambda \rangle = E_\lambda \langle n | \lambda \rangle, \qquad n \le N$$

$$(H - E)^{-1}_{nn'} \equiv -\mathcal{G}_{nn'} = \sum_{\lambda'=0}^{N} \frac{\langle n | \lambda' \rangle \langle \lambda' | n' \rangle}{E_{\lambda'} - E}$$

$$\tan\delta(E) = -\frac{S_{Nl}(E) - \mathcal{G}_{NN}\, T^{l}_{N,\,N+1}\, S_{N+1,l}(E)}{C_{Nl}(E) - \mathcal{G}_{NN}\, T^{l}_{N,\,N+1}\, C_{N+1,l}(E)}$$

Suppose $E = E_\lambda$:

$$\tan\delta(E_\lambda) = -\frac{S_{N+1,\lambda}(E_\lambda)}{C_{N+1,\lambda}(E_\lambda)}$$

Calculating a set of E_λ eigenstates with different $\hbar\Omega$ and N_{max} within SM, we obtain a set of $\delta(E_\lambda)$ values which we can approximate by a smooth curve at low energies.

Single-state HORSE (SS-HORSE)

$$\sum_{n'=0}^{N} H^l_{nn'} \langle n'|\lambda\rangle = E_\lambda \langle n|\lambda\rangle, \qquad n \le N$$

$$(H-E)^{-1}_{nn'} \equiv -\mathscr{G}_{nn'} = \sum_{\lambda'=0}^{N} \frac{\langle n|\lambda'\rangle\langle\lambda'|n'\rangle}{E_{\lambda'}-E}$$

$$\tan\delta(E) = -\frac{S_{Nl}(E) - \mathscr{G}_{NN}\, T^l_{N,\,N+1}\, S_{N+1,\,l}(E)}{C_{Nl}(E) - \mathscr{G}_{NN}\, T^l_{N,\,N+1}\, C_{N+1,\,l}(E)}$$

Suppose $E = E_\lambda$:

$$\tan\delta(E_\lambda) = -\frac{S_{N+1,\lambda}(E_\lambda)}{C_{N+1,\lambda}(E_\lambda)}$$

Note, information about wave function disappeared in this formula, any channel can be treated

Calculating a set of E_λ eigenstates with different $\hbar\Omega$ and N_{max} within SM, we obtain a set of $\delta(E_\lambda)$ values which we can approximate by a smooth curve at low energies.

S-matrix at low energies

Symmetry property: $$S(-k) = \frac{1}{S(k)}$$

$$S(k) = \exp 2i\delta$$

Hence $$\delta(-k) = -\delta(k), \qquad k \sim \sqrt{E},$$

$$\delta \simeq C\sqrt{E} + D(\sqrt{E})^3 + F(\sqrt{E})^5 + \ldots$$

As $k \to 0$: $\delta_\ell \sim k^{2\ell+1} \sim (\sqrt{E})^{2\ell+1}$

Bound state: $$S_b^{(i)}(k) = \frac{k + ik_b^{(i)}}{k - ik_b^{(i)}},$$

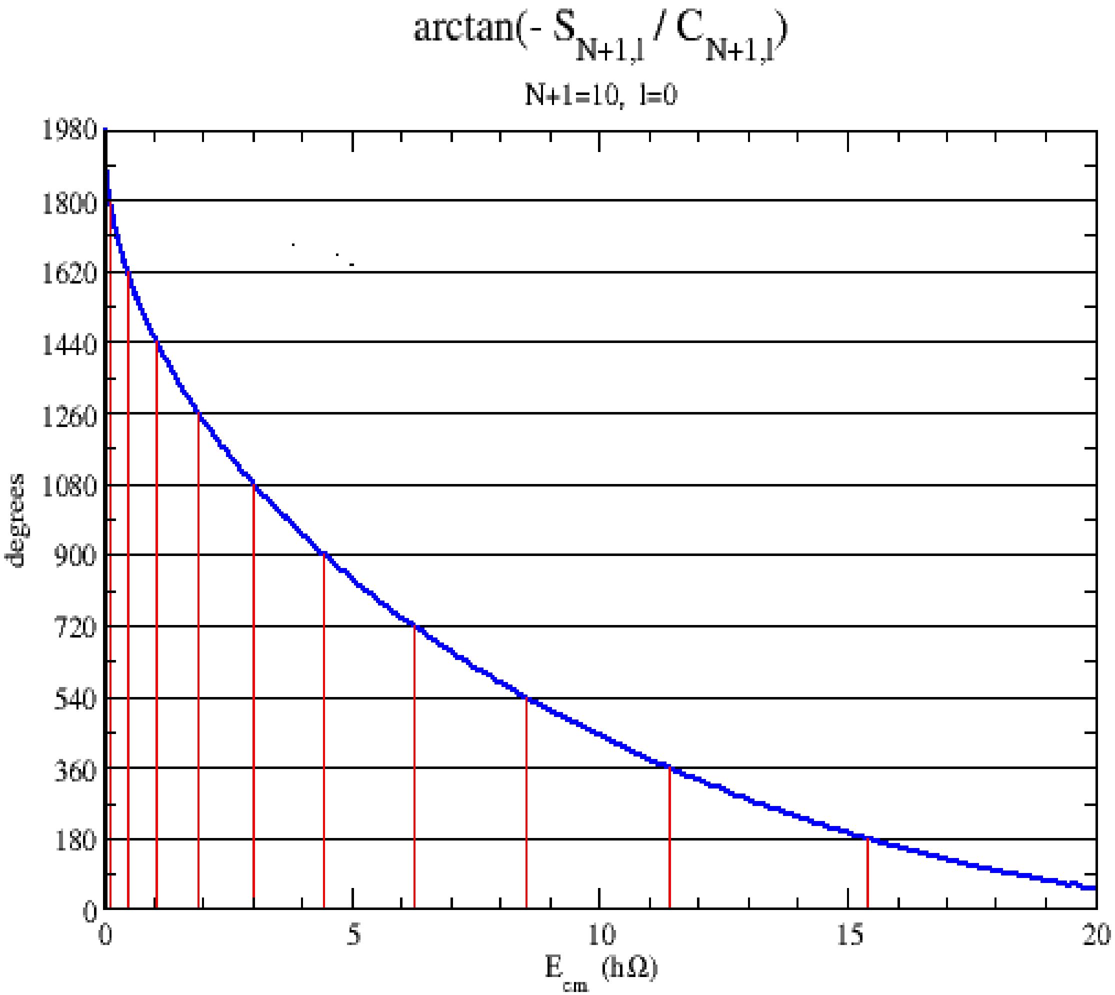
$$\delta_0 \simeq \pi - \arctan\sqrt{\frac{E}{|E_b|}} + c\sqrt{E} + d(\sqrt{E})^3 + f(\sqrt{E})^5 \ldots$$

Resonance: $$S_r^{(i)}(k) = \frac{(k + \kappa_r^{(i)})(k - \kappa_r^{(i)*})}{(k - \kappa_r^{(i)})(k + \kappa_r^{(i)*})}$$

$$\delta_1 \simeq -\arctan\frac{a\sqrt{E}}{E - b^2} + c\sqrt{E} + d(\sqrt{E})^3 + \ldots, \qquad c = -\frac{a}{b^2}.$$

Universal function

$$f_{nl}(E) = \arctan\left(-\frac{S_{nl}(E)}{C_{nl}(E)}\right)$$



S. Coon et al extrapolations

PHYSICAL REVIEW C **86**, 054002 (2012)

S. A. Coon, M. I. Avetian,
M. K. G. Kruse, U. van Kolck,
P. Maris, and J. P. Vary, PRC 86, 054002 (2012)

What is λ_{sc} dependence for resonances?

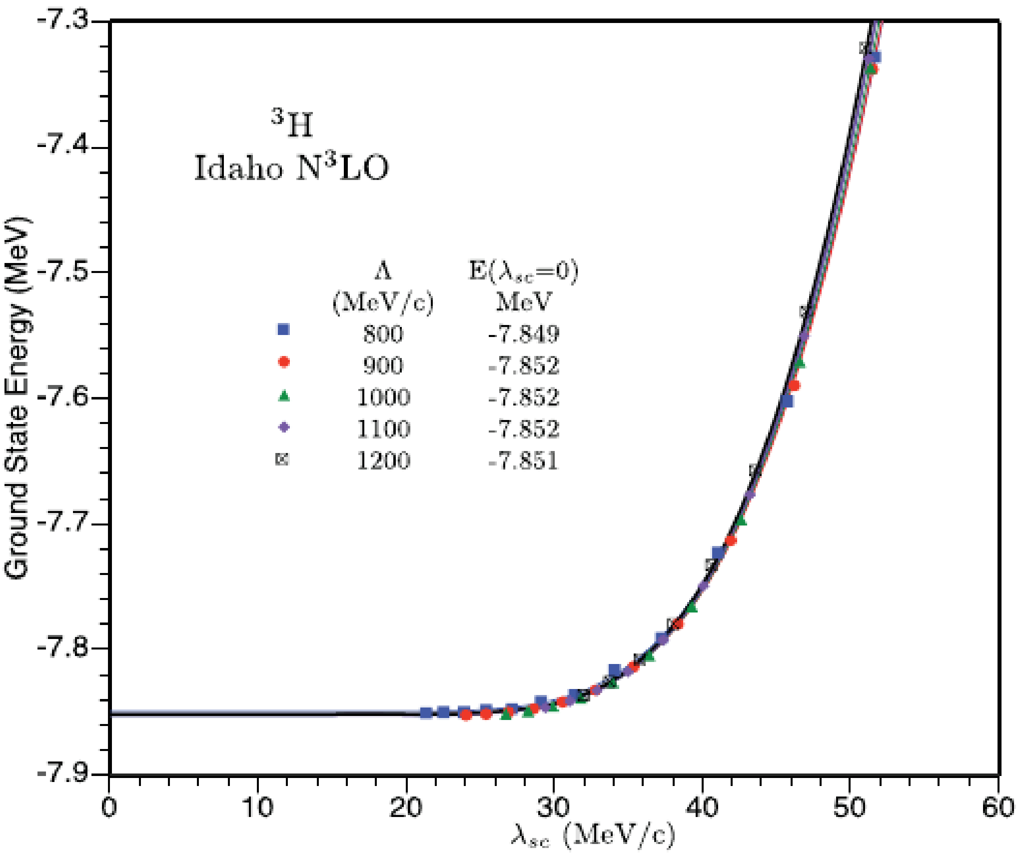


FIG. 7. (Color online) The ground-state energy of ^{3}H calculated at five fixed values of $\Lambda = \sqrt{m_N(N+3/2)\hbar\omega}$ and variable $\lambda_{sc} = \sqrt{(m_N\hbar\omega)/(N+3/2)}$. The curves are fits to the points and the functions fitted are used to extrapolate to the ir limit $\lambda_{sc} = 0$.

$$f_{nl}(E) = \arctan\left(-\frac{S_{nl}(E)}{C_{nl}(E)} \right)$$

scaling with $\lambda_{SC} = \sqrt{(m_N \hbar\Omega)/(2n+l+3/2)}$

Limit $n \to \infty$:

$$n \gg \sqrt{\frac{2E}{\hbar\Omega}}$$

$$S_{nl}(q) \approx q\sqrt{r_0}\,(n+l/2+3/4)^{\frac{1}{4}}\, j_l(2q\sqrt{n+l/2+3/4})$$

$$\approx \sqrt{r_0}\,(n+l/2+3/4)^{-\frac{1}{4}}\, \sin[2q\,\sqrt{n+l/2+3/4} - \pi l/2]$$

$$C_{nl}(q) \approx -\,q\sqrt{r_0}\,(n+l/2+3/4)^{\frac{1}{4}}\, n_l(2q\sqrt{n+l/2+3/4})$$

$$\approx \sqrt{r_0}\,(n+l/2+3/4)^{-\frac{1}{4}}\, \cos[2q\,\sqrt{n+l/2+3/4} - \pi l/2]$$

$$q = \sqrt{\frac{2E}{\hbar\Omega}} \qquad q\sqrt{n+l/2+3/4} = \frac{\sqrt{m_N E}}{\lambda_{SC}}$$

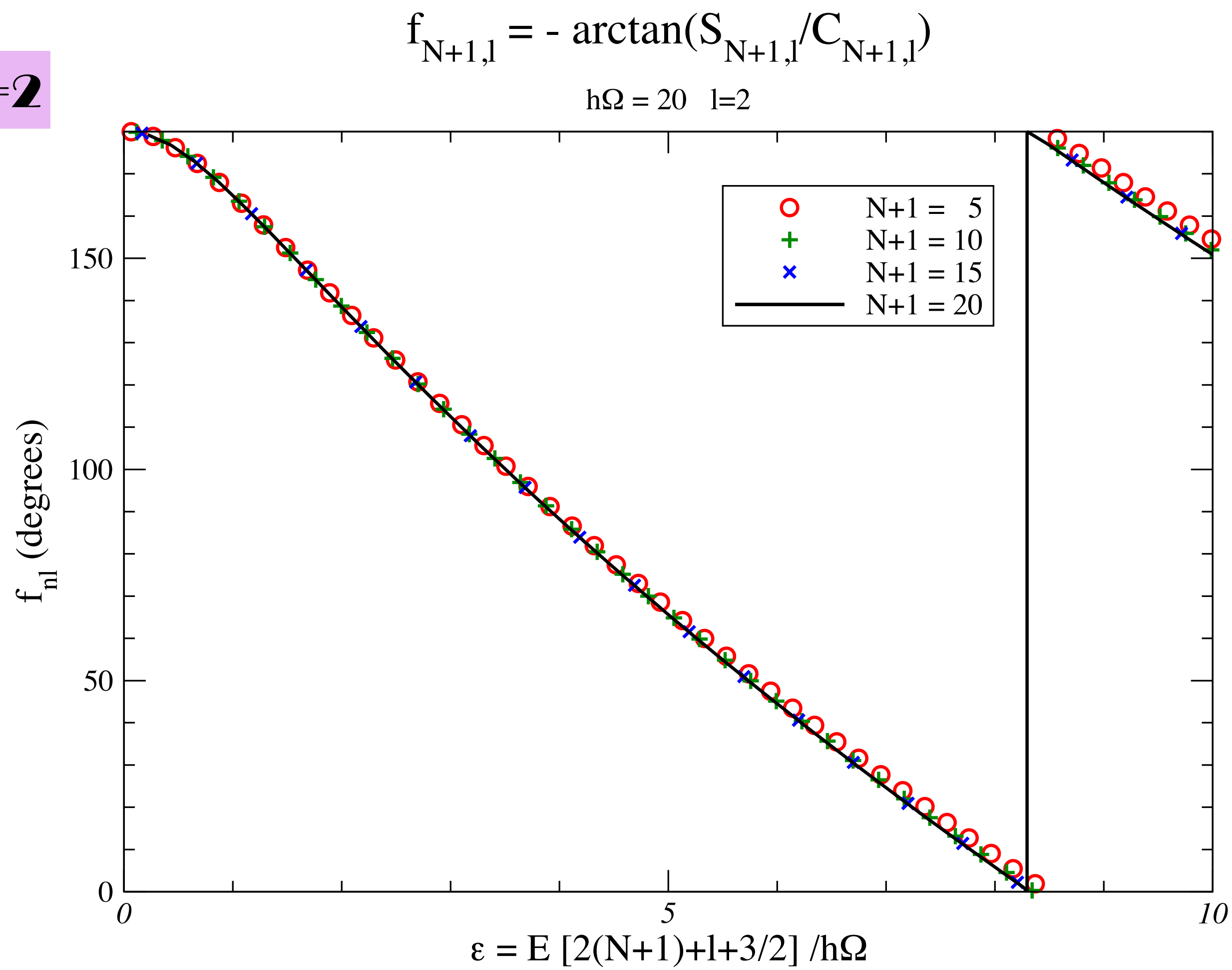
Universal function scaling

$$E_{cm}(\text{MeV}) \Rightarrow e = \frac{E_{cm}[2(N+1)+l+3/2]}{\hbar\Omega}$$

S.Coon et al (ir cutoff)

$$\lambda_{SC} = \sqrt{(m_N \hbar\Omega)/(N_{tot}+3/2)}$$

l=2

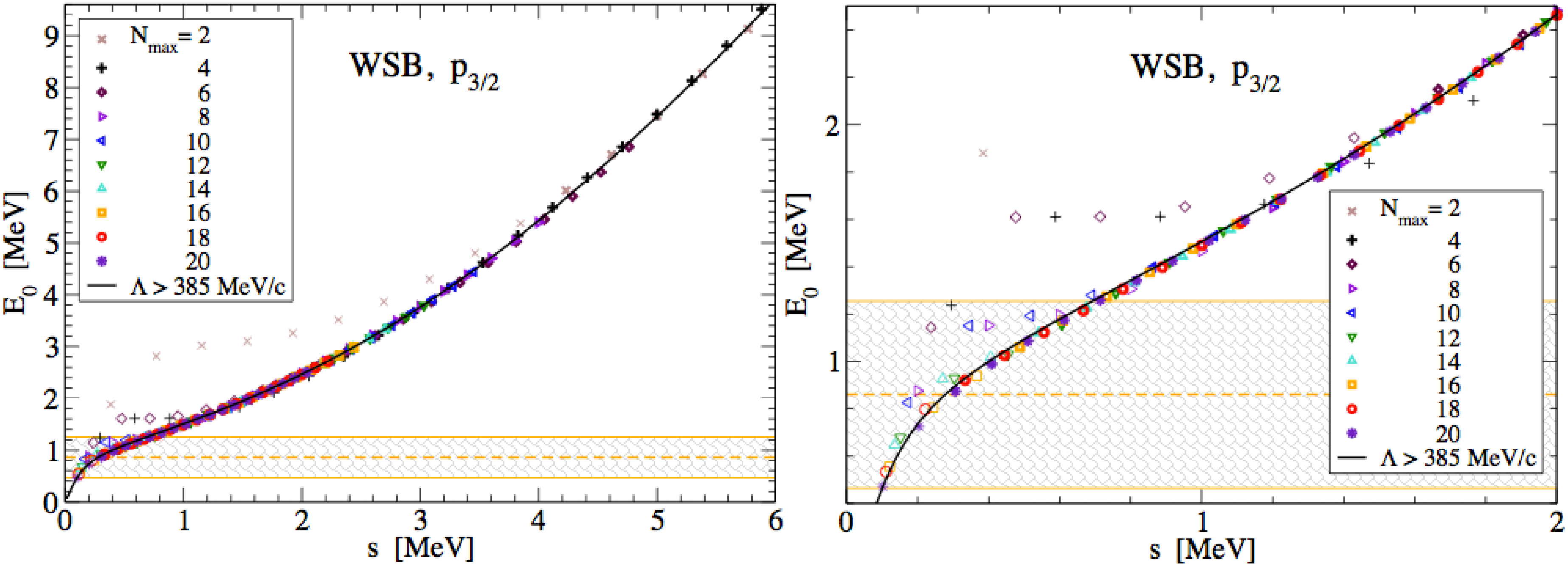


How it works

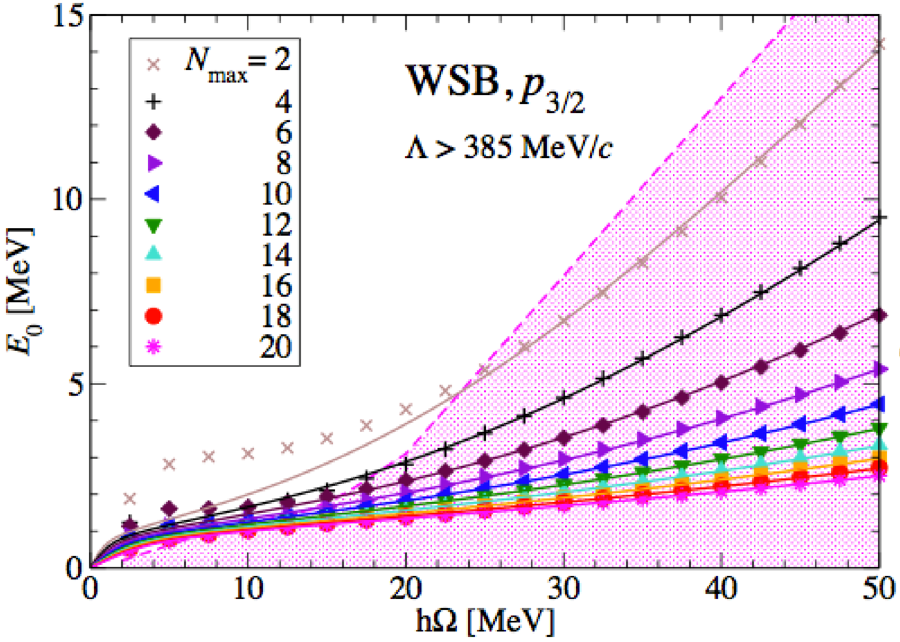
- Model problem: *n*α scattering by Woods-Saxon potential
  J. Bang and C. Gignoux, Nucl. Phys. A, 313 , 119 (1979).
- UV cutoff of S. A. Coon, M. I. Avetian, M. K. G. Kruse, U. van Kolck, P. Maris, and J. P. Vary, PRC 86, 054002 (2012) to select eigenvalues:

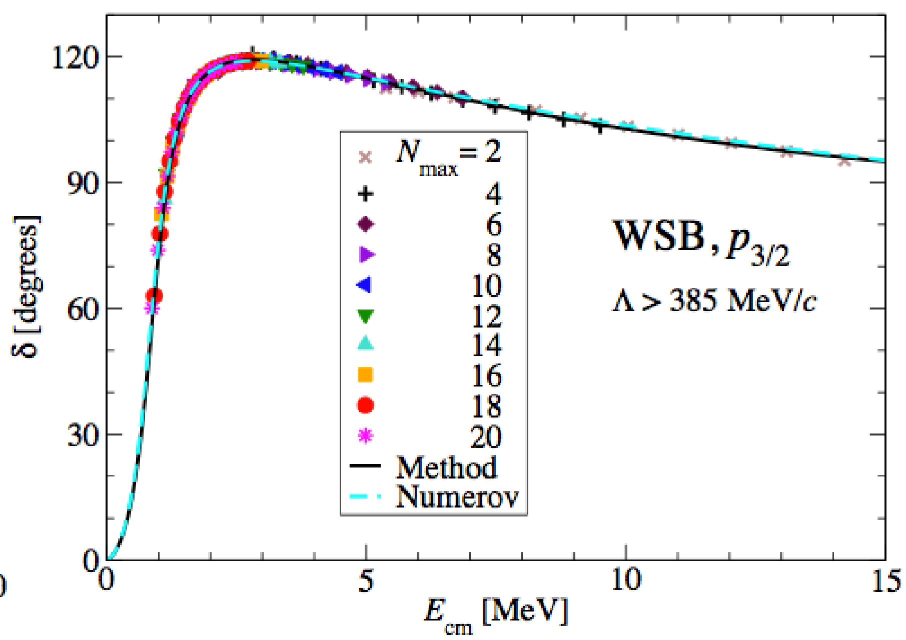
$$\Lambda = \sqrt{m_{nucl}\hbar\Omega(N_{\max} + 2 + \ell + 3/2)}$$

Model problem

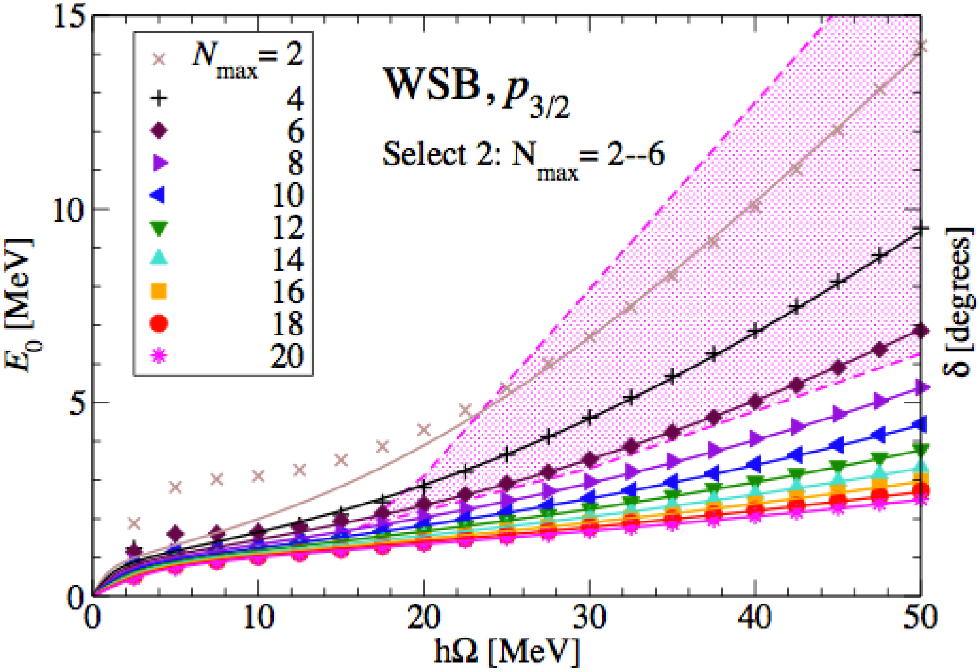


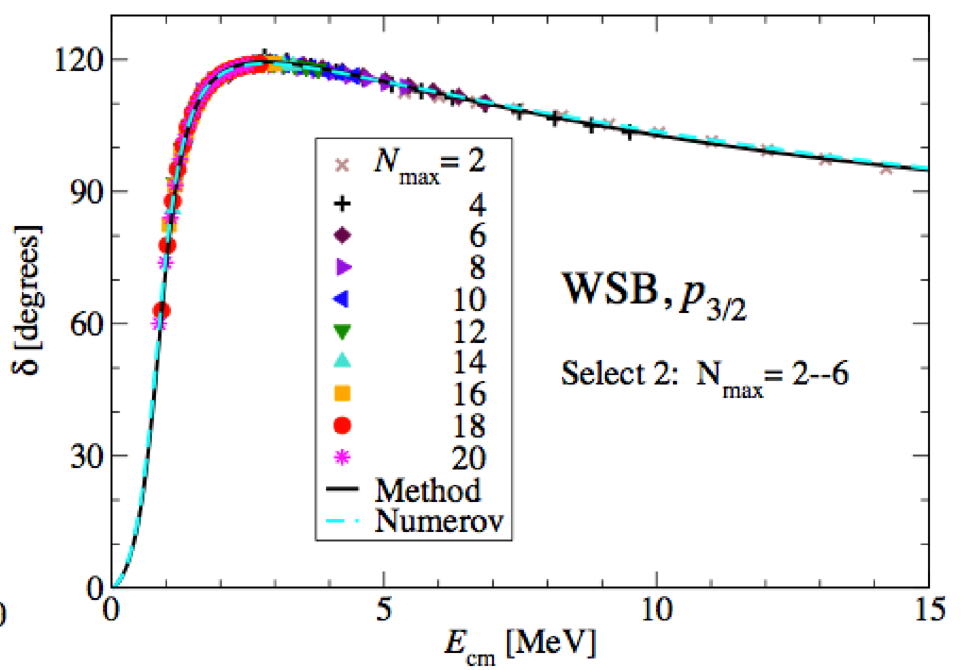
Model problem





Model problem

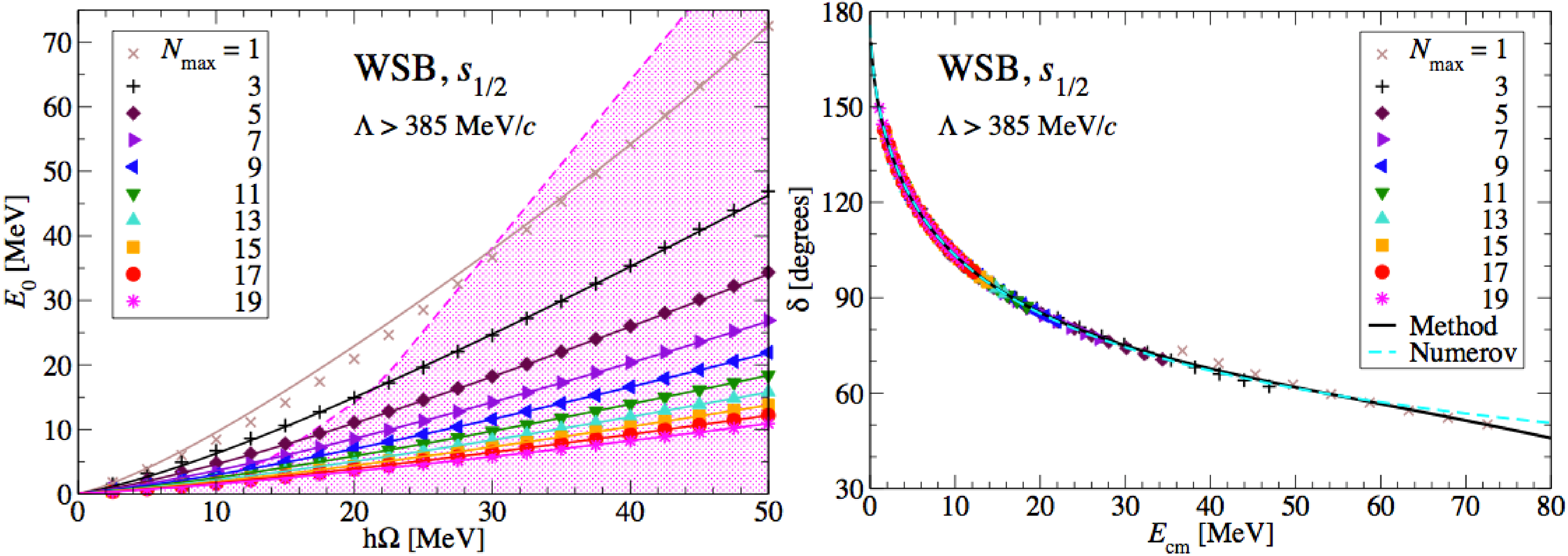




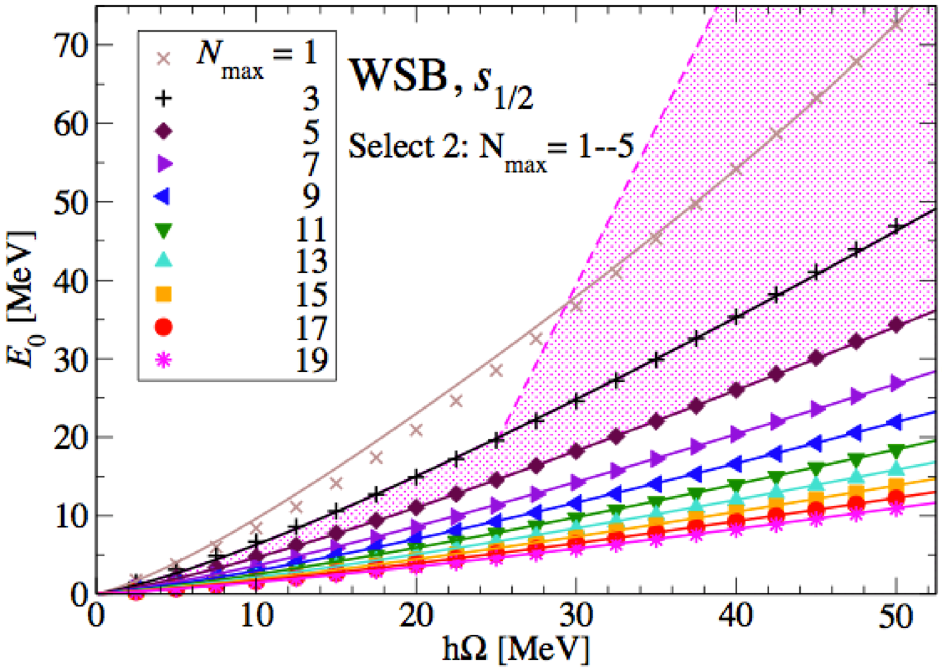
	a, MeV$^{\frac{1}{2}}$	b^2, MeV	d, MeV$^{-\frac{3}{2}}$	E_r, MeV	Γ, MeV	$\sqrt{\frac{\chi^2}{datum}}$, MeV	# pts.
$\Lambda > 385$, $N_{max} = 2 \div 20$	0.412	0.948	0.00541	0.863	0.785	0.037	156
Select 2: $N_{max} = 2 \div 6$	0.411	0.948	0.00530	0.863	0.782	0.070	39
exact (J-matrix)				0.836	0.780		

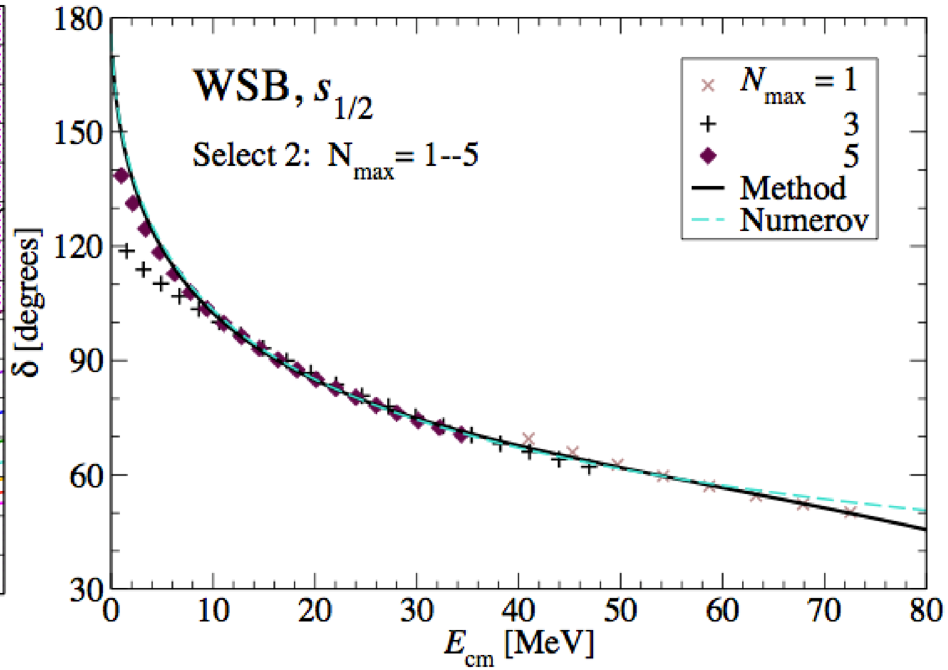
$$\delta_1 \simeq -\arctan\frac{a\sqrt{E}}{E-b^2} + c\sqrt{E} + d(\sqrt{E})^3 + ..., \quad c = -\frac{a}{b^2}.$$

Model problem



Model problem

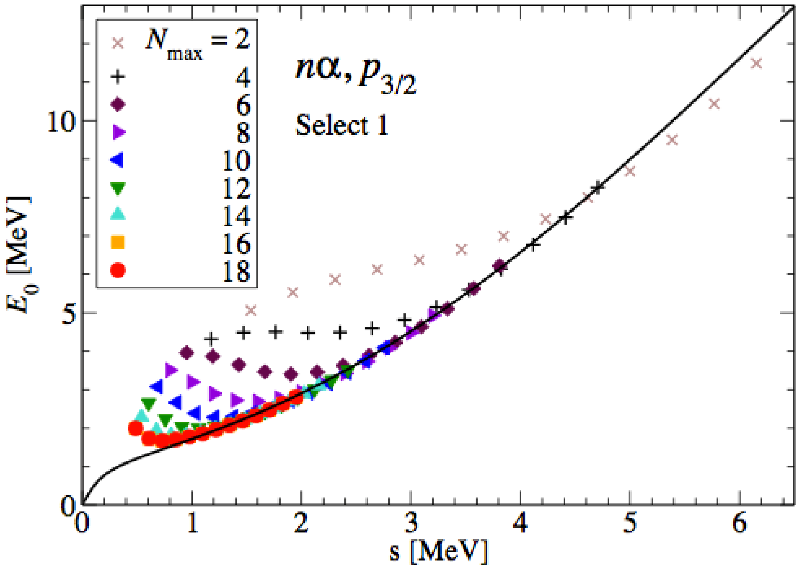




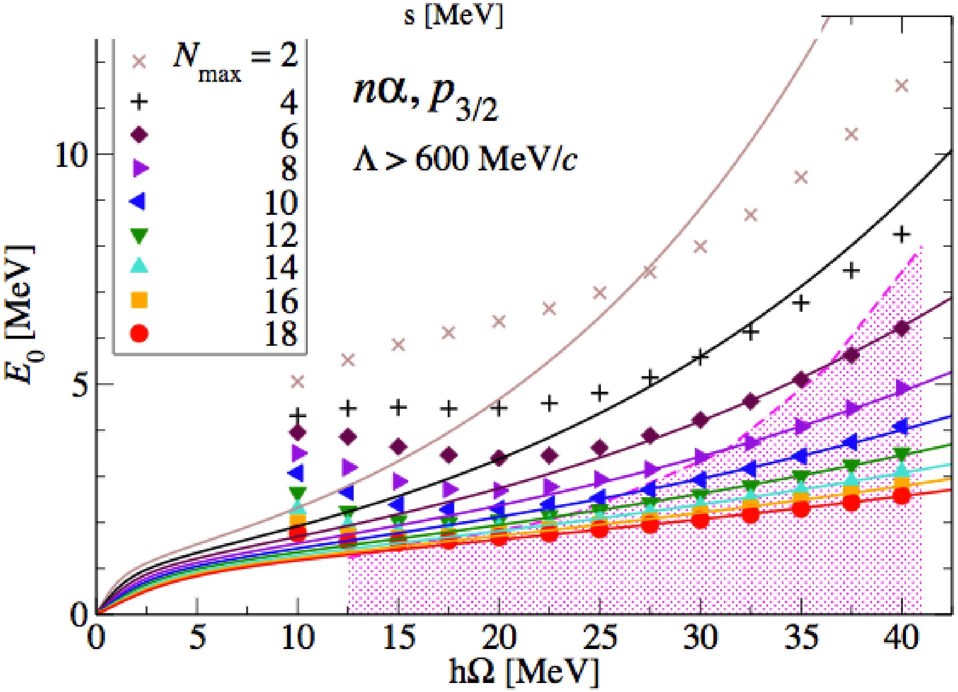
	E_b, MeV	c, MeV$^{-\frac{1}{2}}$	d, MeV$^{-\frac{3}{2}}$	f, MeV$^{-\frac{5}{2}}$	$\sqrt{\frac{\chi^2}{datum}}$, MeV	# pts.
$\Lambda > 385$, $N_{max} = 1 \div 19$	-7.084	-0.159	$+1.22 \cdot 10^{-3}$	$-1.0 \cdot 10^{-5}$	0.183	78
$\Lambda > 385$, $N_{max} = 1 \div 5$	-6.865	-0.158	$+1.24 \cdot 10^{-3}$	$-1.0 \cdot 10^{-5}$	0.332	35
Select 1, $N_{max} = 1 \div 19$	-6.865	-0.158	$+1.21 \cdot 10^{-3}$	$-0.8 \cdot 10^{-5}$	0.139	188
Select 2, $N_{max} = 1 \div 5$	-6.370	-0.155	$+1.21 \cdot 10^{-3}$	$-1.0 \cdot 10^{-5}$	0.264	48
diag. Ham.	-9.85					

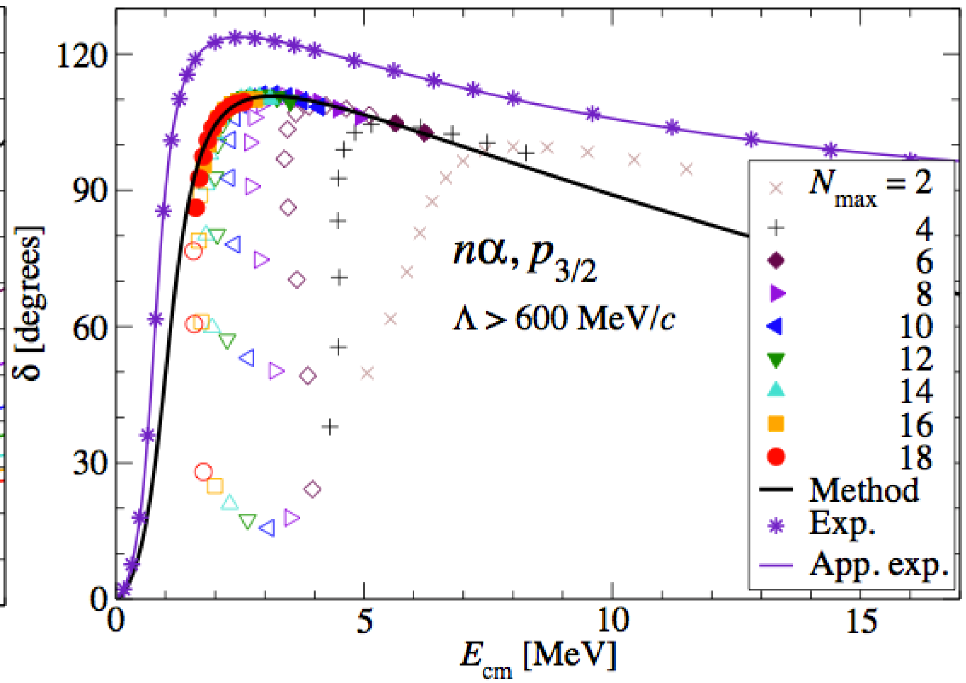
*n*α scattering: NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$



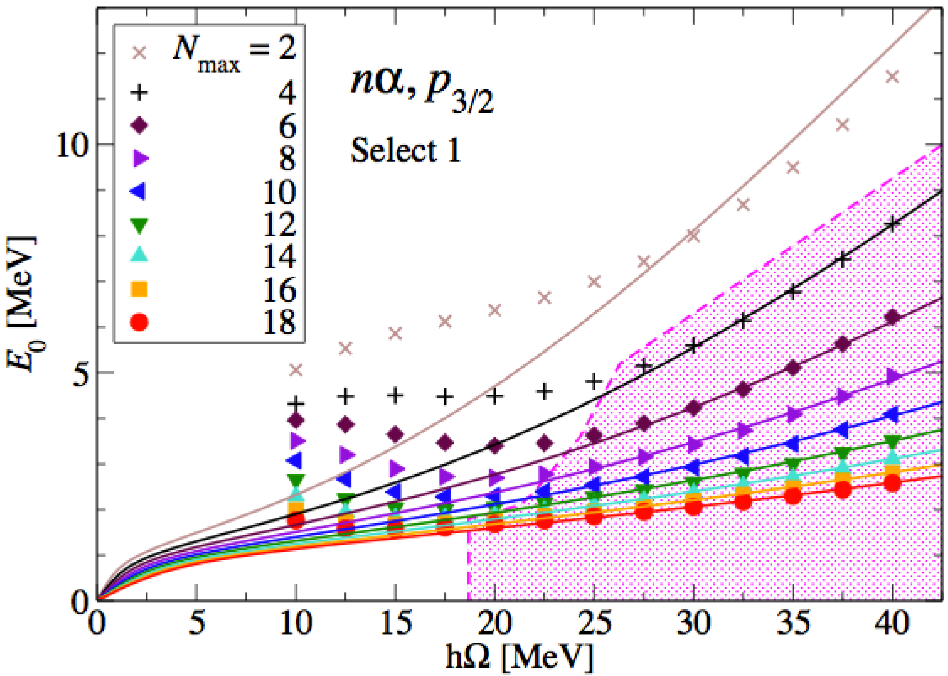
$$s = \frac{\hbar\Omega}{(N_{\max} + 2 + \ell + 3/2)}.$$

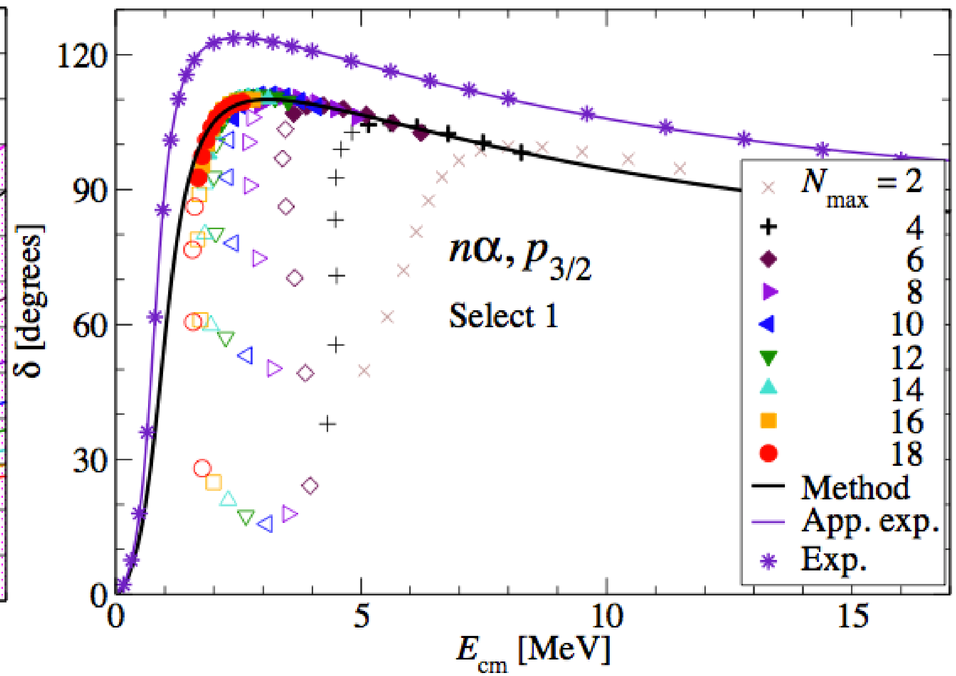




$n\alpha$ scattering: NCSM, JISP16

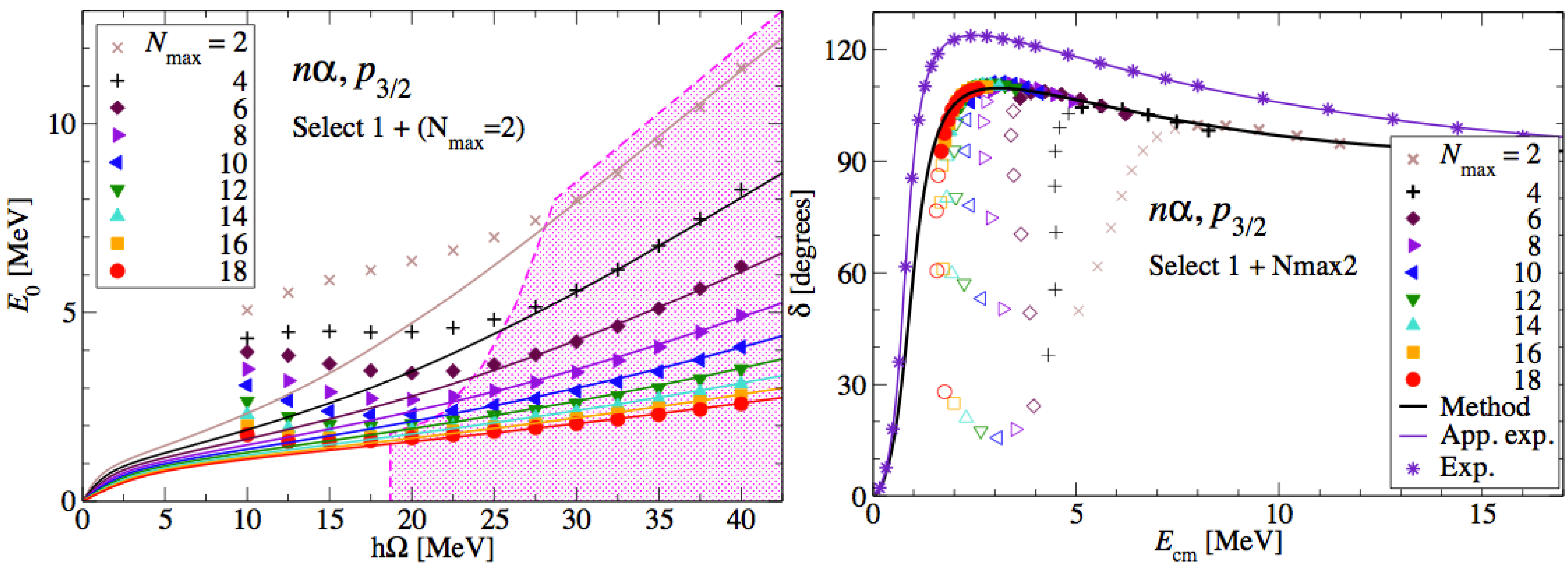
$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$





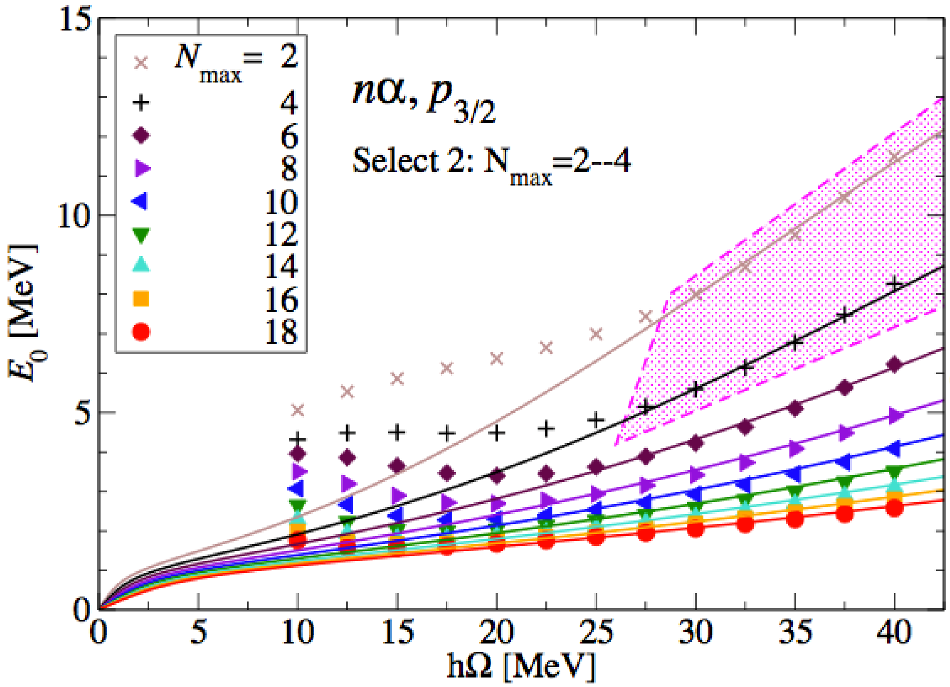
*n*α scattering: NCSM, JISP16

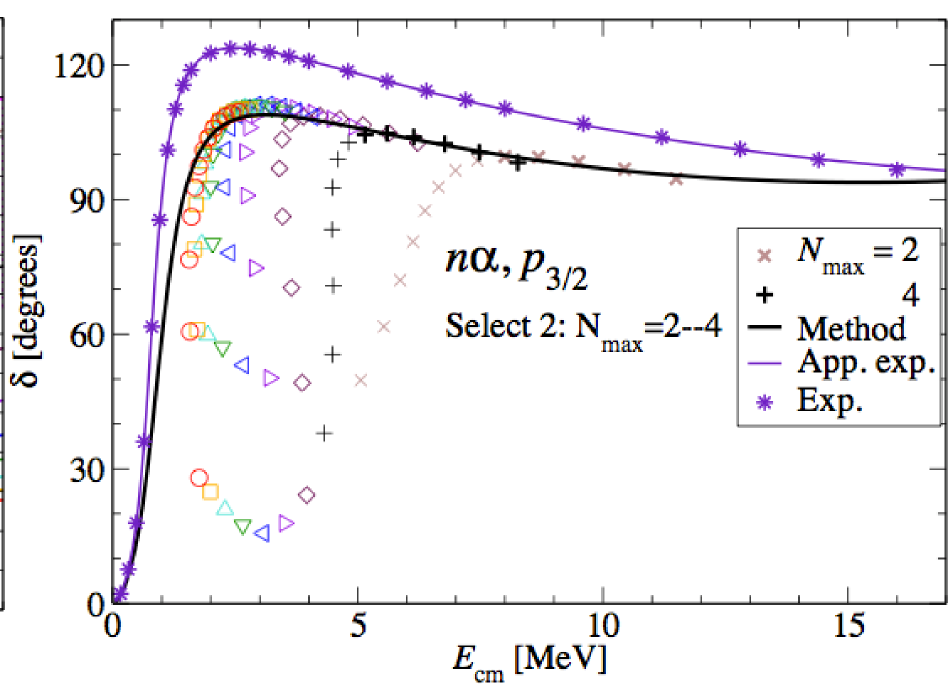
$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$



$n\alpha$ scattering: NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$





nα scattering: NCSM, JISP16

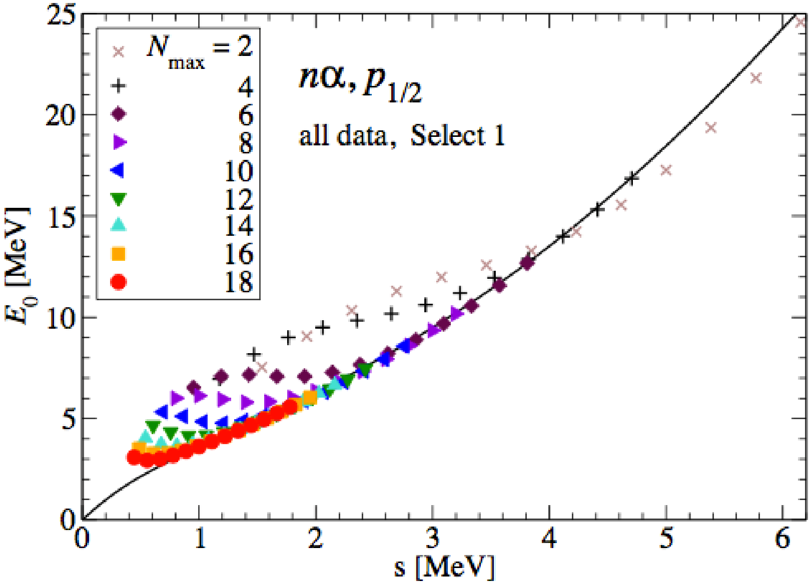
$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$

$$\delta_1 \simeq -\arctan\frac{a\sqrt{E}}{E-b^2} + c\sqrt{E} + d(\sqrt{E})^3 + \ldots, \quad c = -\frac{a}{b^2}.$$

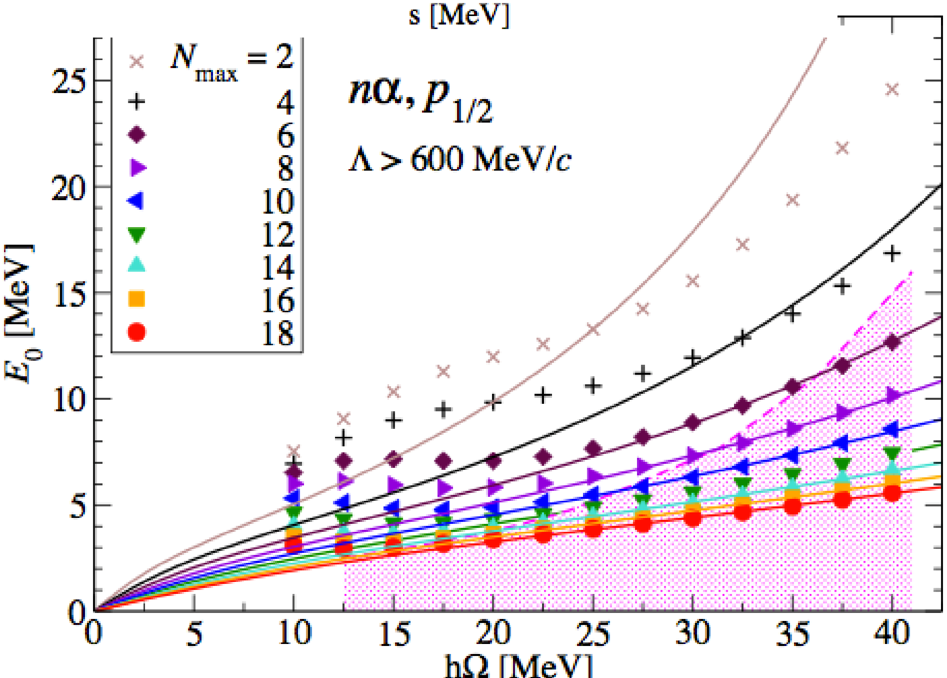
$3/2^-$, $n-{}^4$He	a, MeV$^{\frac{1}{2}}$	b^2, MeV	d, MeV$^{-\frac{3}{2}}$	E_r, MeV	Γ, MeV	$\sqrt{\frac{\chi^2}{datum}}$, MeV	# pts.
$\Lambda > 600$, $N_{max} = 6 \div 18$	0.505	1.135	−0.00009	1.008	1.046	0.031	46
Select 1	0.506	1.054	+0.00647	0.926	1.008	0.053	63
Select 1 + ($N_{max} = 2$)	0.506	1.019	+0.00932	0.891	0.989	0.070	68
Select 2: $N_{max} = 2 \div 4$	0.515	1.025	+0.0101	0.892	1.008	0.106	11
Select 2: $N_{max} = 2 \div 6$	0.512	1.022	+0.00988	0.891	1.002	0.097	18
Select 3: $N_{max} = 12$	0.469	1.307	−0.0265	1.197	1.050	0.011	8
R-matrix [3]				0.80	0.65		
J-matrix [4]				0.772	0.644		
Fit with exp. data	0.358	0.839	+0.00559	0.774	0.643	0.21 [deg]	26

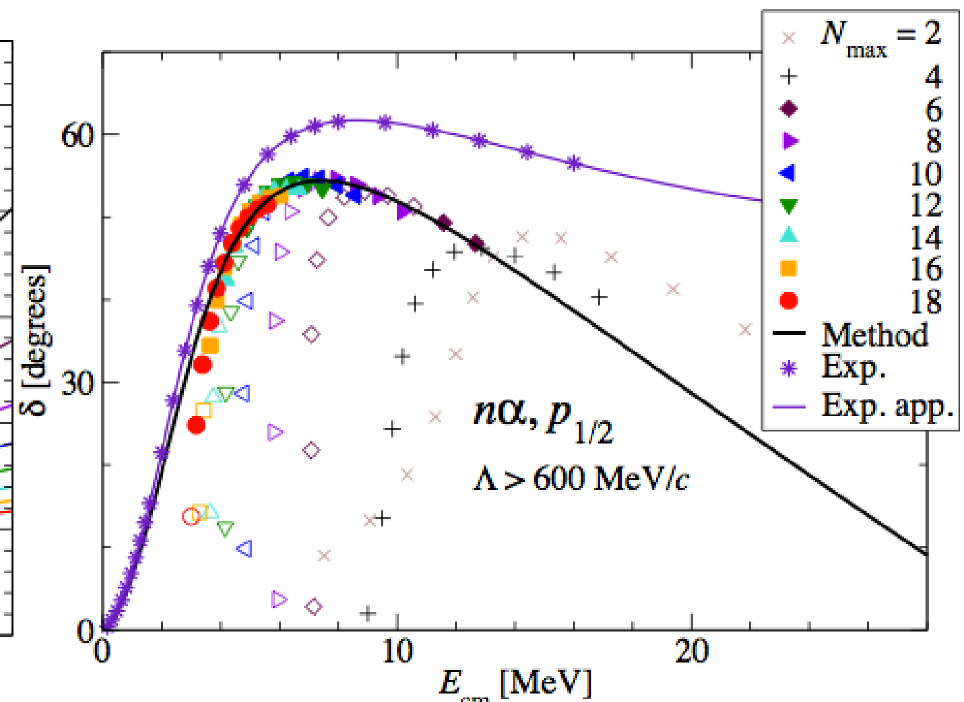
*n*α scattering: NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$



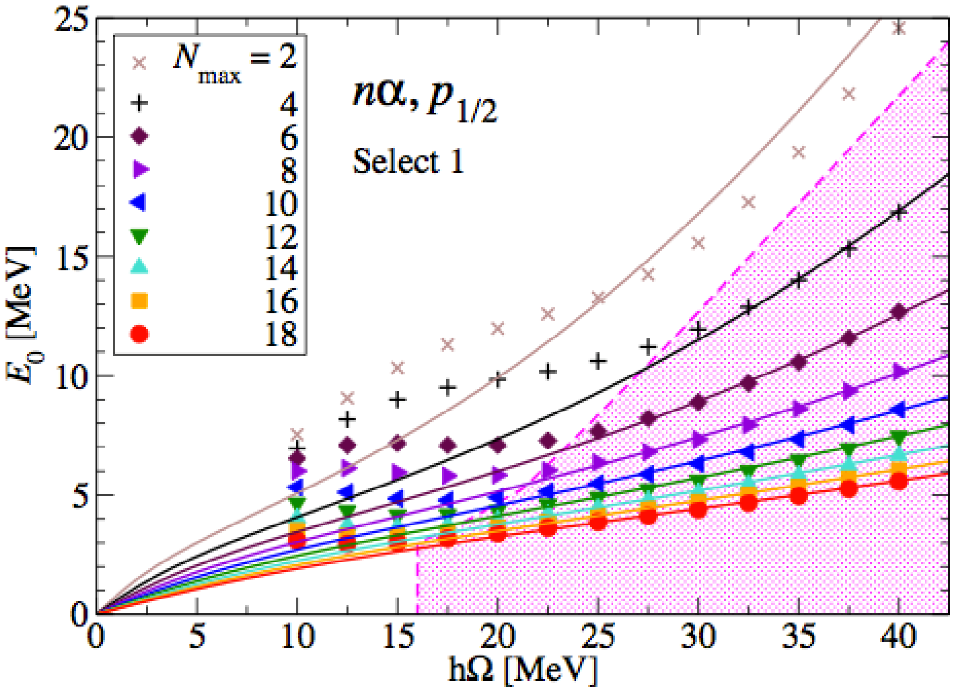
$$s = \frac{\hbar\Omega}{(N_{\max} + 2 + \ell + 3/2)}.$$

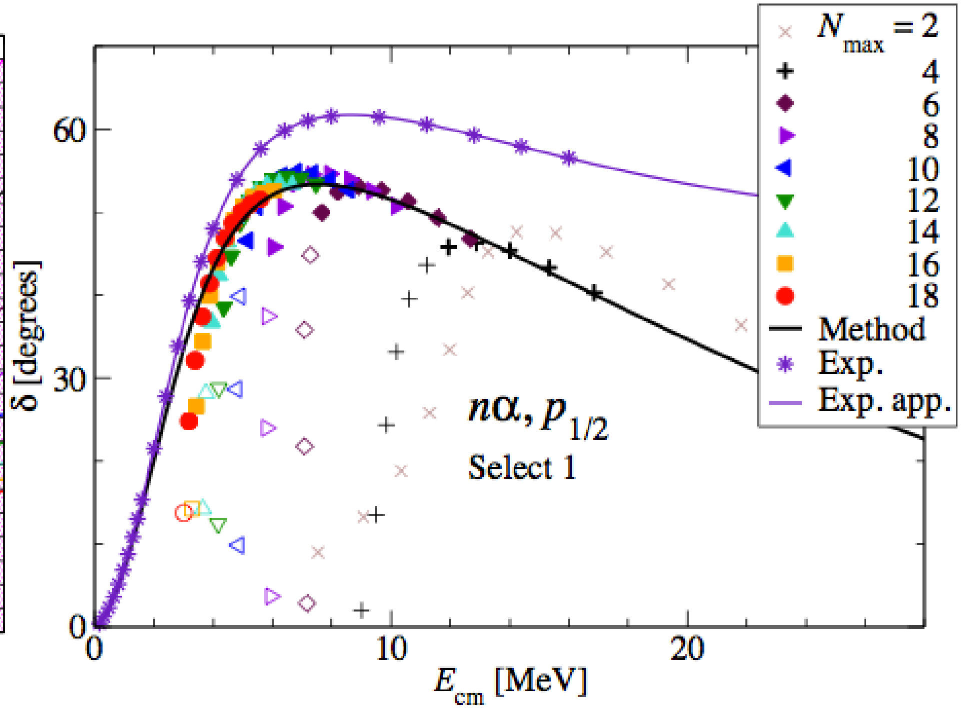




$n\alpha$ scattering: NCSM, JISP16

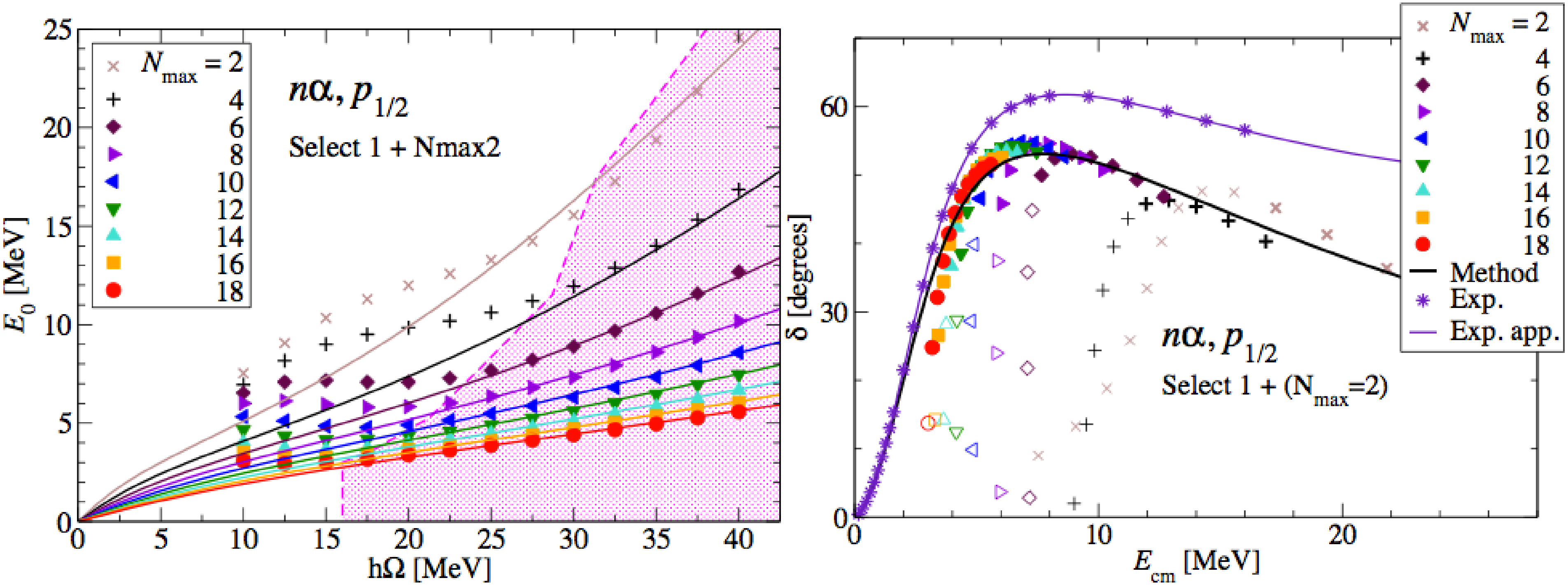
$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$





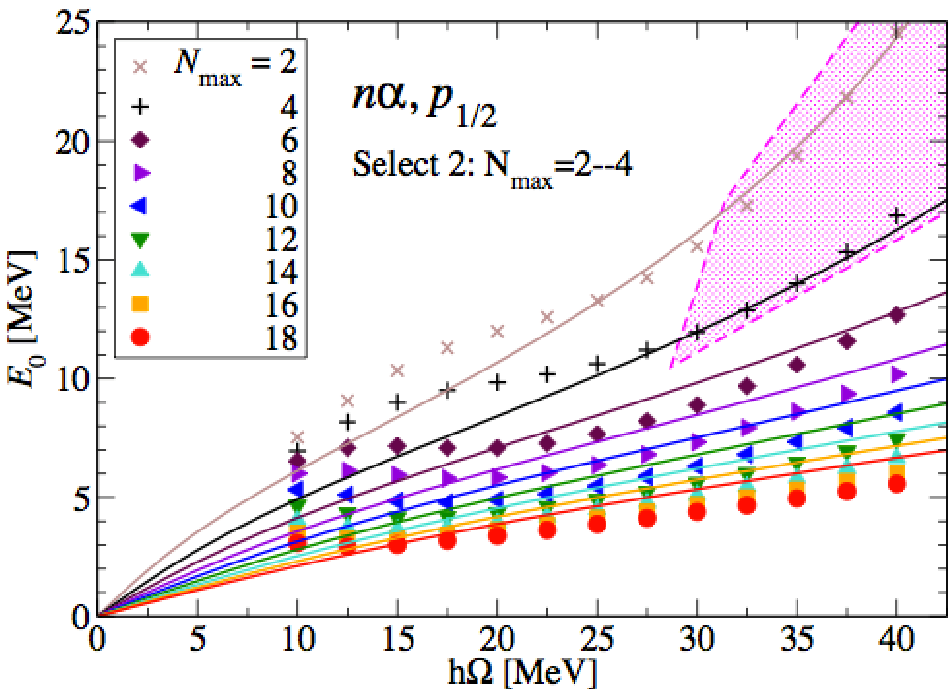
*n*α scattering: NCSM, JISP16

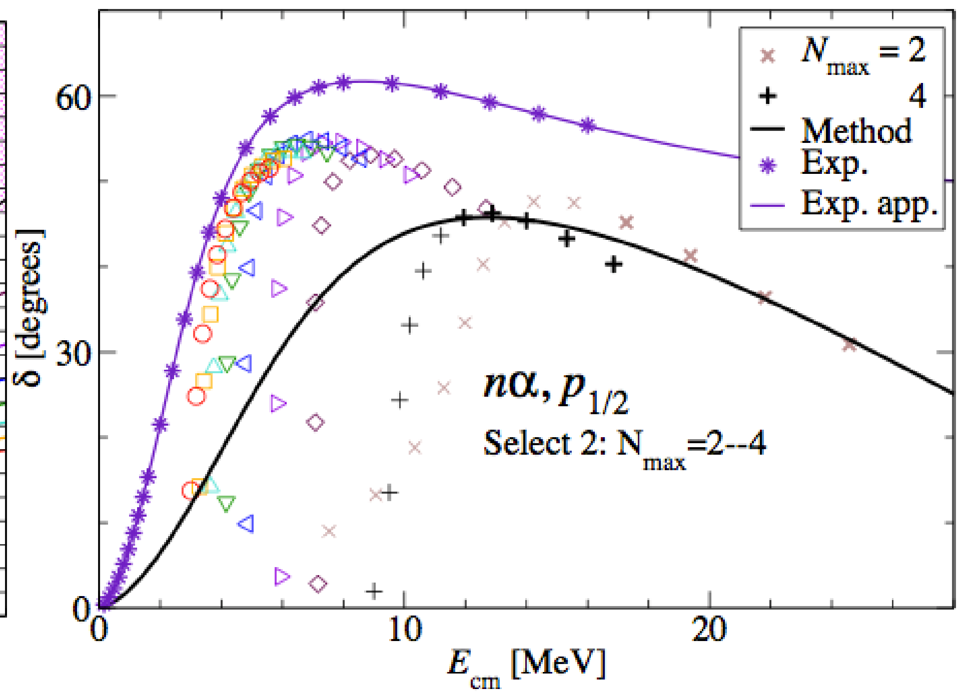
$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$



$n\alpha$ scattering: NCSM, JISP16

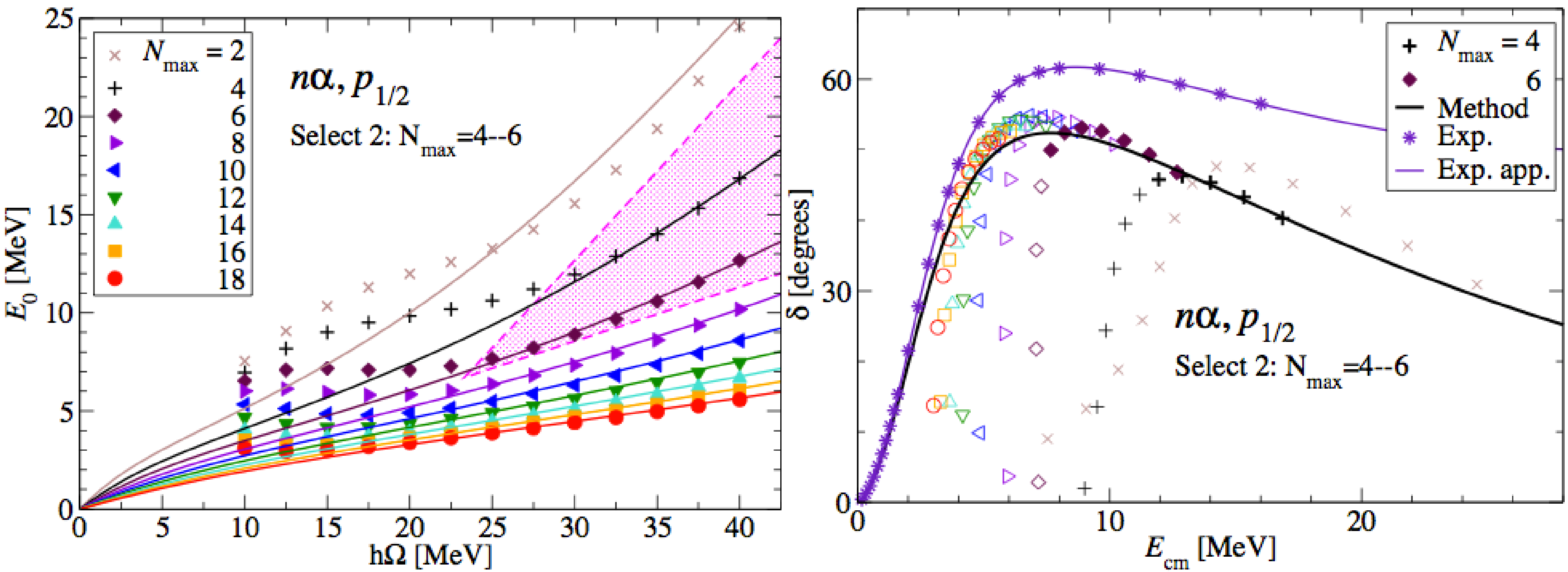
$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$





*n*α scattering: NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$



*n*α scattering: NCSM, JISP16

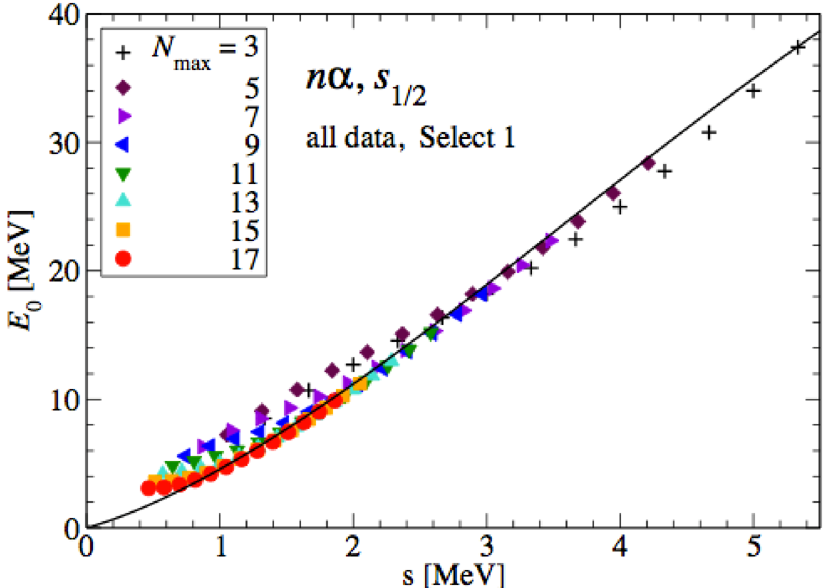
$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$

$$\delta_1 \simeq -\arctan\frac{a\sqrt{E}}{E-b^2} + c\sqrt{E} + d(\sqrt{E})^3 + \dots, \quad c = -\frac{a}{b^2}.$$

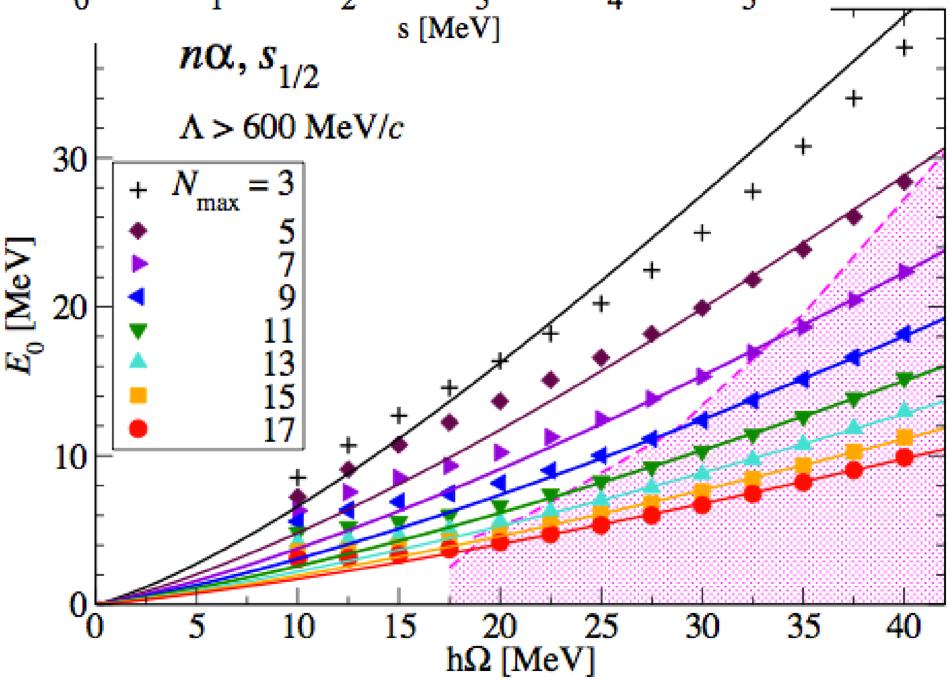
$1/2^-$, n–^{4}He	a, MeV$^{\frac{1}{2}}$	b^2, MeV	d, MeV$^{-\frac{3}{2}}$	E_r, MeV	Γ, MeV	$\sqrt{\frac{\chi^2}{datum}}$, MeV	# pts.
$\Lambda > 600$, $N_{max} = 6 \div 18$	1.680	3.443	−0.00036	2.031	5.559	0.061	46
Select 1	1.711	3.307	+0.00231	1.843	5.491	0.120	66
Select 1 + ($N_{max} = 2$)	1.735	3.302	+0.00342	1.798	5.540	0.208	70
Select 2: $N_{max} = 2 \div 4$	2.460	6.734	−0.00150	3.710	11.241	0.326	9
Select 2: $N_{max} = 2 \div 6$	1.817	3.534	+0.00314	1.884	5.981	0.368	16
Select 2: $N_{max} = 4 \div 6$	1.746	3.340	+0.00285	1.817	5.606	0.151	12
Select 3: $N_{max} = 12$	1.238	4.283	−0.0297	3.516	4.890	0.037	9
R-matrix [3]				2.07	5.57		
J-matrix [4]				1.97	5.20		
Fit with exp. data	1.622	3.276	+0.00463	1.960	5.249	0.038 [deg]	26

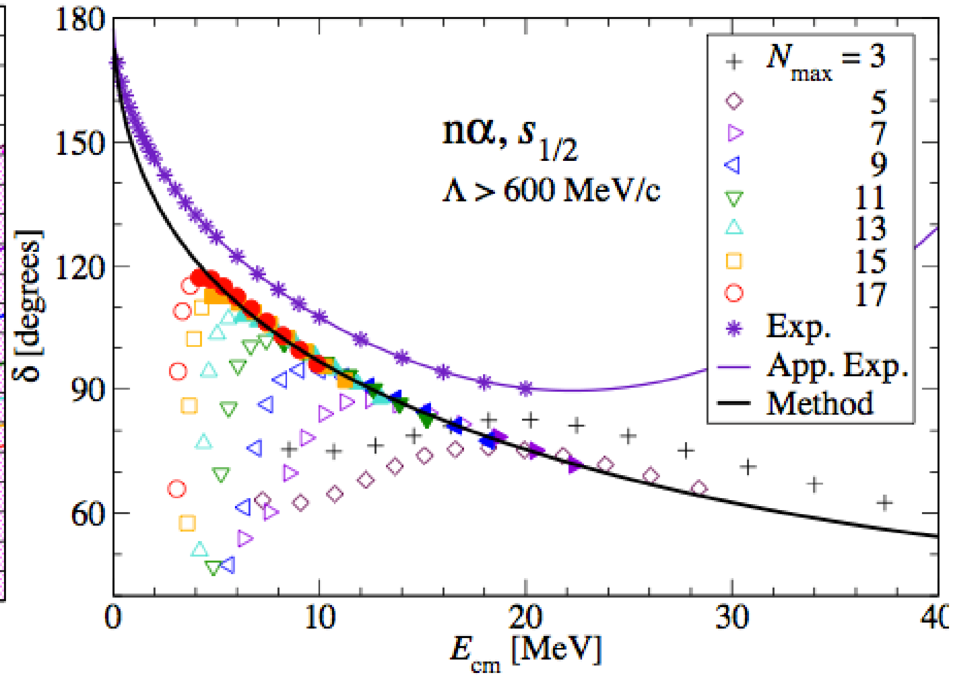
$n\alpha$ scattering: NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$



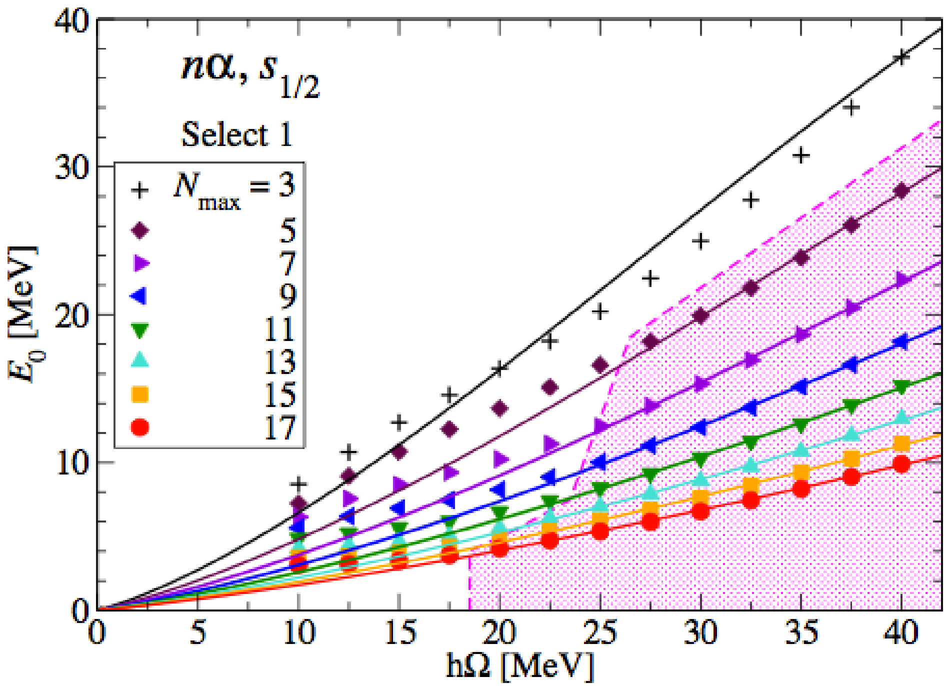
$$s = \frac{\hbar\Omega}{(N_{\max} + 2 + \ell + 3/2)}.$$

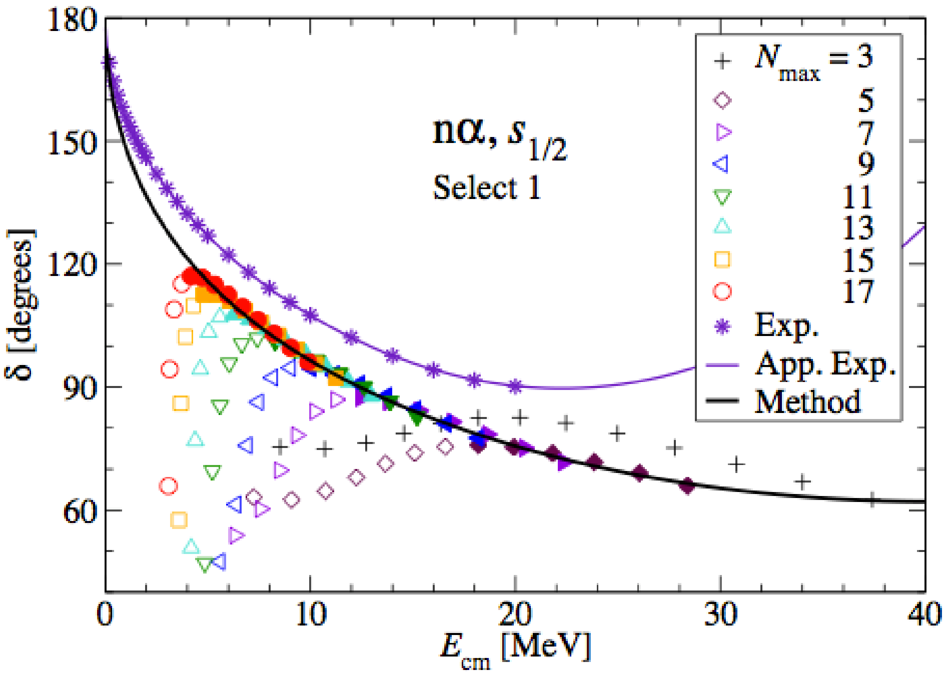




$n\alpha$ scattering: NCSM, JISP16

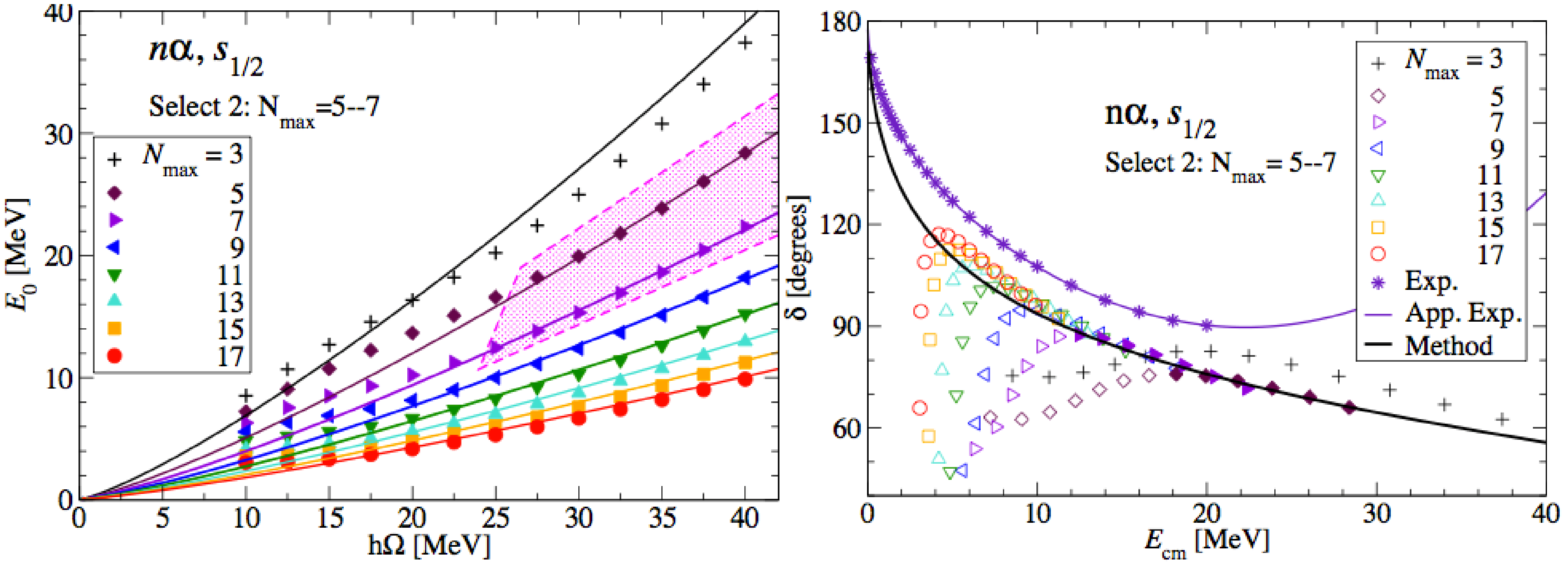
$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$





$n\alpha$ scattering: NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$



*n*α scattering: NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$

$$\delta_0 \simeq \pi - \arctan\sqrt{\frac{E}{|E_b|}} + c\sqrt{E} + d(\sqrt{E})^3 + f(\sqrt{E})^5 \ldots$$

$1/2^+$, $n-{}^4$He	E_b, MeV	c, MeV$^{-\frac{1}{2}}$	d, MeV$^{-\frac{3}{2}}$	f, MeV$^{-\frac{5}{2}}$	$\sqrt{\frac{\chi^2}{datum}}$, MeV	# pts.
$\Lambda > 600$, $N_{max} = 5 \div 17$	-5.996	-0.171	$-8.02 \cdot 10^{-5}$	$6.48 \cdot 10^{-6}$	0.085	39
Select 1	-6.072	-0.172	$-1.42 \cdot 10^{-4}$	$1.05 \cdot 10^{-5}$	0.115	47
Select 2: $N_{max} = 5 \div 7$	-3.052	-0.138	$+5.50 \cdot 10^{-4}$	$-3.0 \cdot 10^{-5}$	0.037	7
Select 2: $N_{max} = 5 \div 9$	-4.538	-0.156	$+2.21 \cdot 10^{-4}$	$-1.3 \cdot 10^{-5}$	0.159	14
Fit with exp. data	-13.75	-0.1556	$-4.29 \cdot 10^{-3}$	$2.2 \cdot 10^{-4}$	0.018 [deg]	26

Coulomb + nuclear interaction

$$V^{Sh} = \begin{cases} V^{Nucl} + V^{Coul}, & r \le R'; \\ 0, & r > R'. \end{cases} \quad R' \ge R_{Nucl}.$$

$$\tan\delta_\ell = -\frac{W_{R'}(j_\ell, F_\ell) - W_{R'}(n_\ell, F_\ell)\tan\delta_\ell^{Sh}}{W_{R'}(j_\ell, G_\ell) - W_{R'}(n_\ell, G_\ell)\tan\delta_\ell^{Sh}}.$$

$$W_{R'}(j_\ell, F_\ell) = \left(\frac{d}{dr}\Big[j_\ell(kr)\Big]F_\ell(\eta, kr) - j_\ell(kr)\frac{d}{dr}\Big[F_\ell(\eta, kr)\Big]\right)\Bigg|_{r=R'}, \quad \eta = \frac{\mu Z_1 Z_2}{k} = Z_1 Z_2 \alpha\sqrt{\frac{\mu c^2}{2E}}$$

- SS-HORSE:

$$\tan\delta_\ell(E_\nu) = -\frac{W_{R'}(n_\ell, F_\ell)S_{2N+2,\ell}(E_\nu) + W_{R'}(j_\ell, F_\ell)C_{2N+2,\ell}(E_\nu)}{W_{R'}(n_\ell, G_\ell)S_{2N+2,\ell}(E_\nu) + W_{R'}(j_\ell, G_\ell)C_{2N+2,\ell}(E_\nu)}.$$

- Scaling at $N + 1 \gg \sqrt{\frac{2E}{\hbar\Omega}}$:

$$\delta_\ell(E_\nu) = -\arctan\frac{F_\ell(\eta(E_\nu), 2\sqrt{E_\nu/s})}{G_\ell(\eta(E_\nu), 2\sqrt{E_\nu/s})}.$$

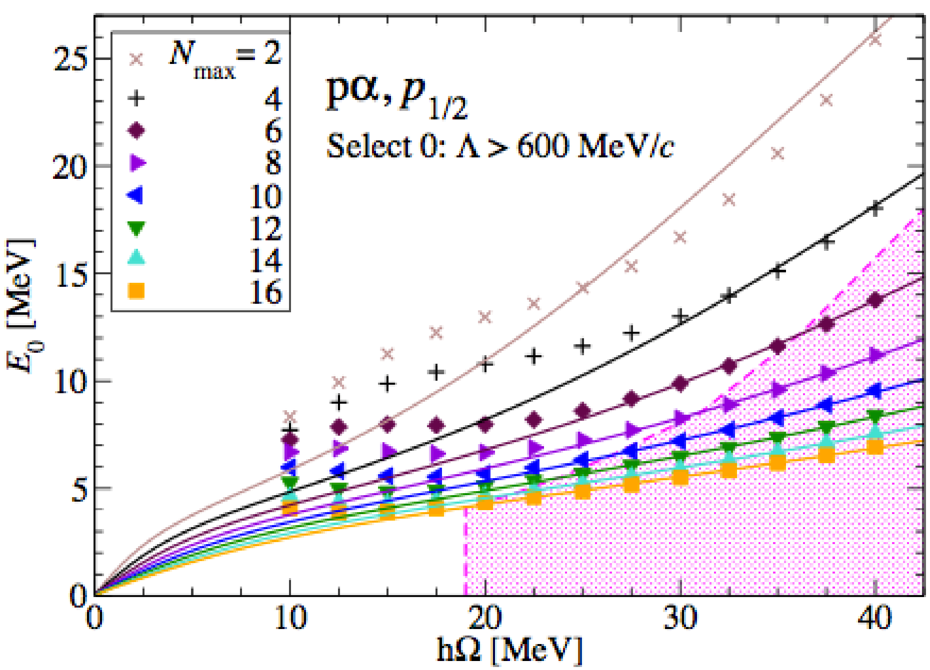
S-matrix and phase shift

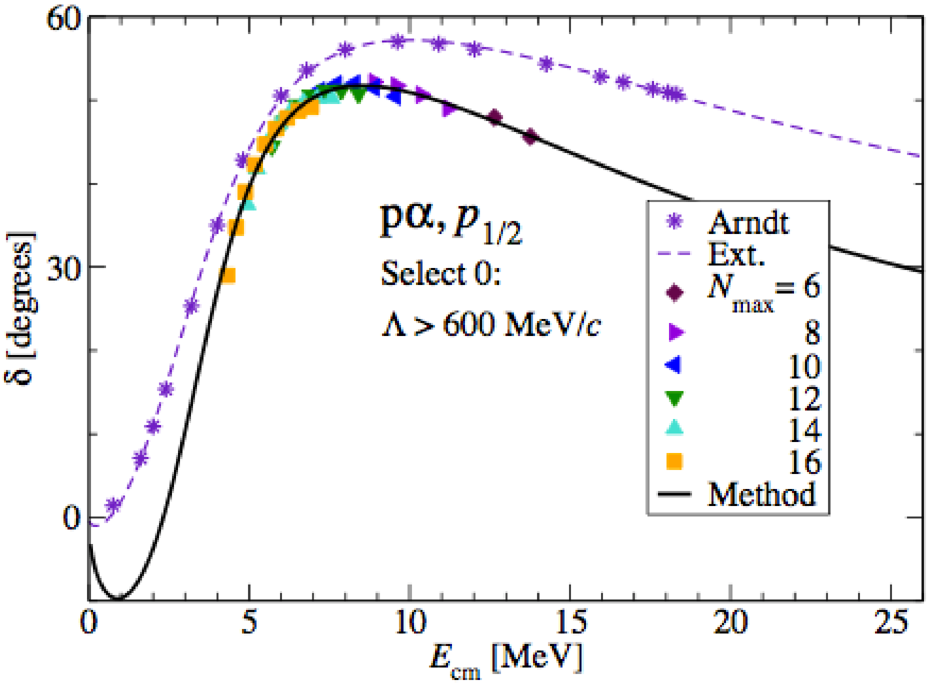
$$\delta_\ell(E) = -\arctan\frac{a\sqrt{E}}{E-b^2} + c\sqrt{E} + d(\sqrt{E})^3.$$

- No relation between *a*, *b* and *c.*

*p*α scattering: NCSM, JISP16

$$E_\lambda(\hbar\Omega, N_{\max}) = E_\lambda^{A=5}(\hbar\Omega, N_{\max}) - E_\lambda^{A=4}(\hbar\Omega, N_{\max})$$





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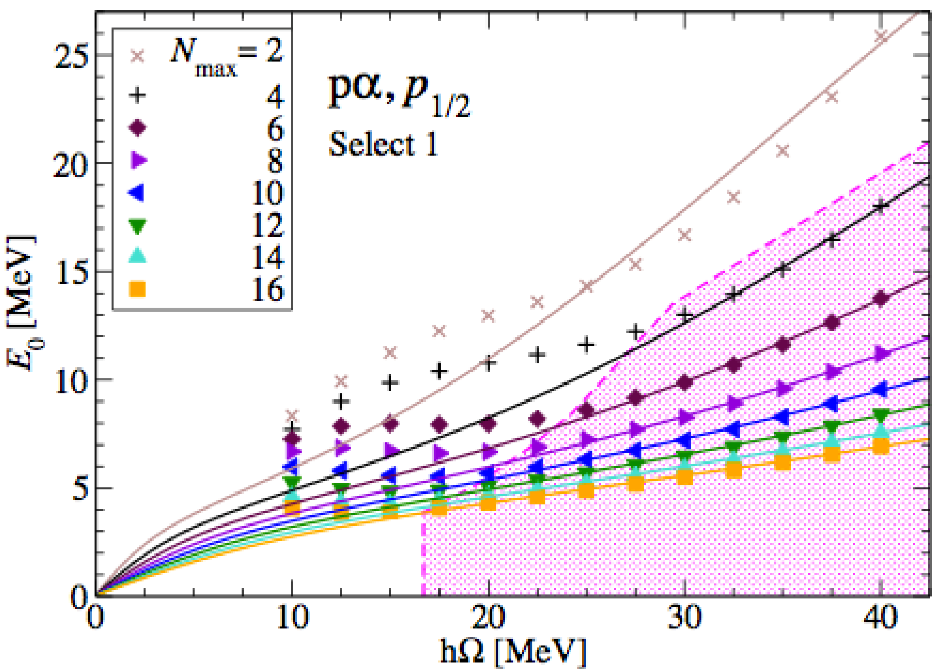

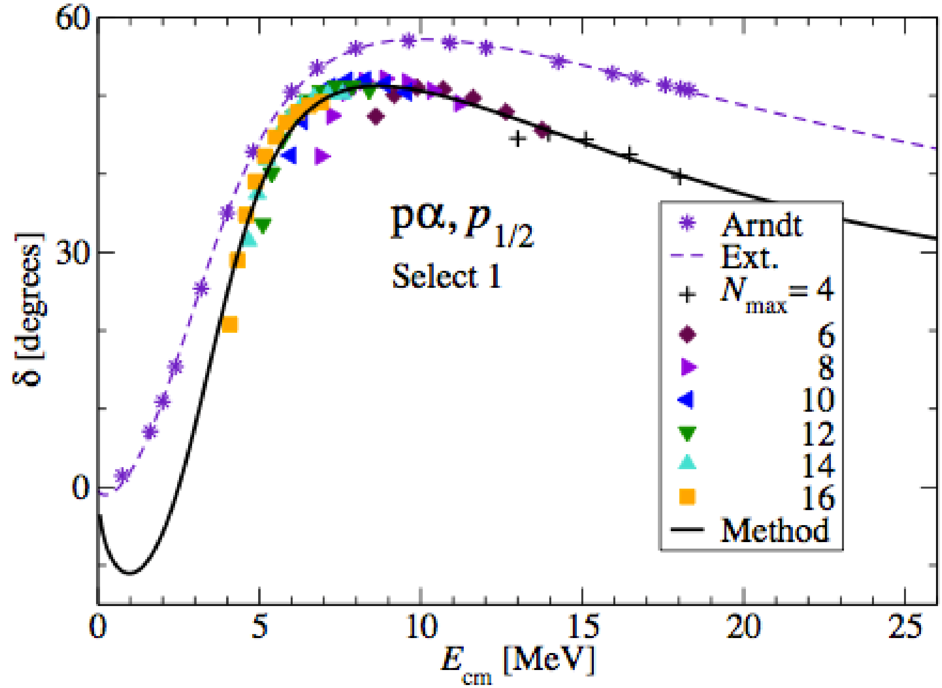

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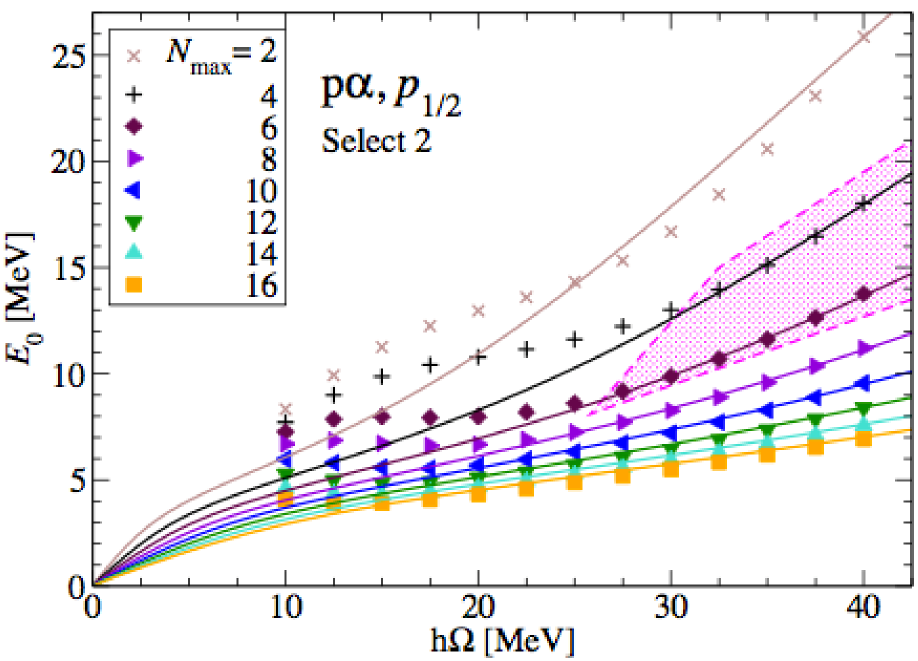

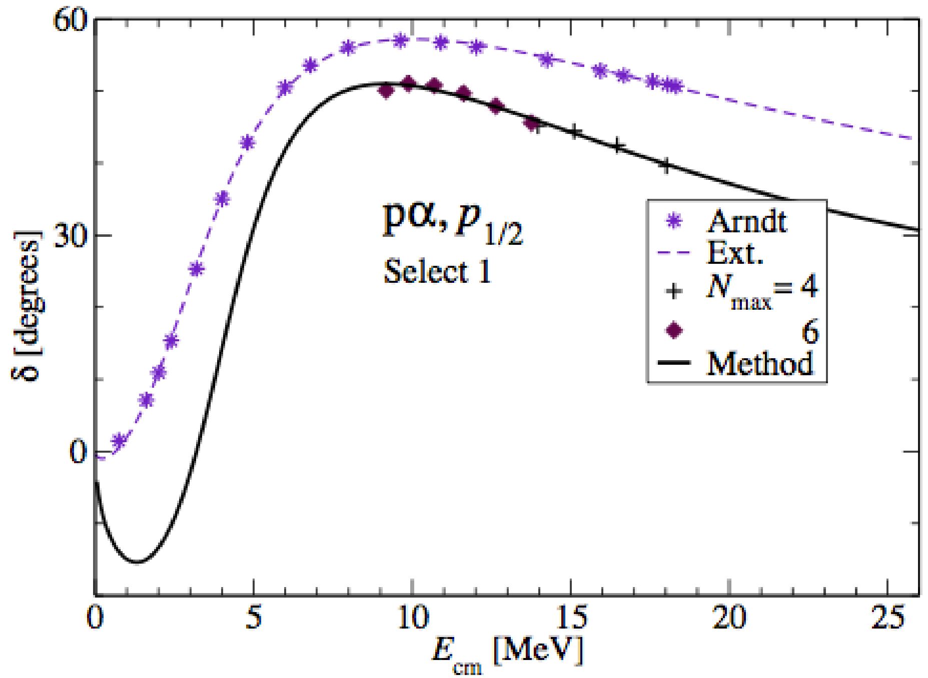

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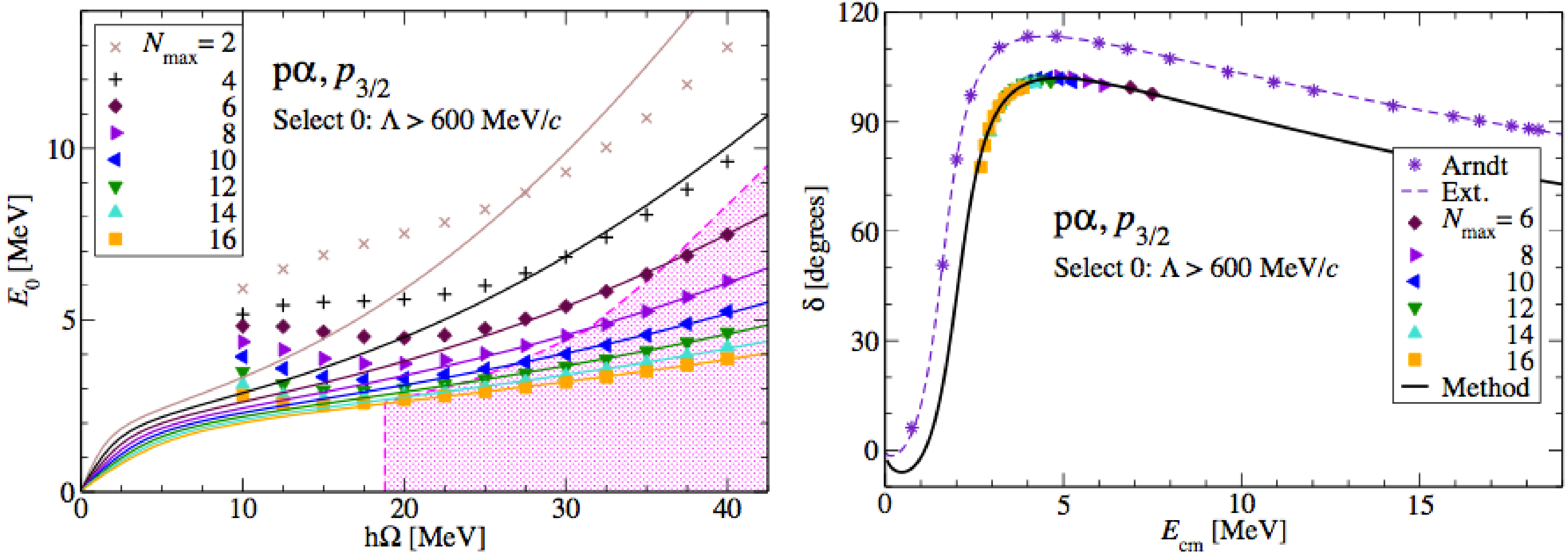

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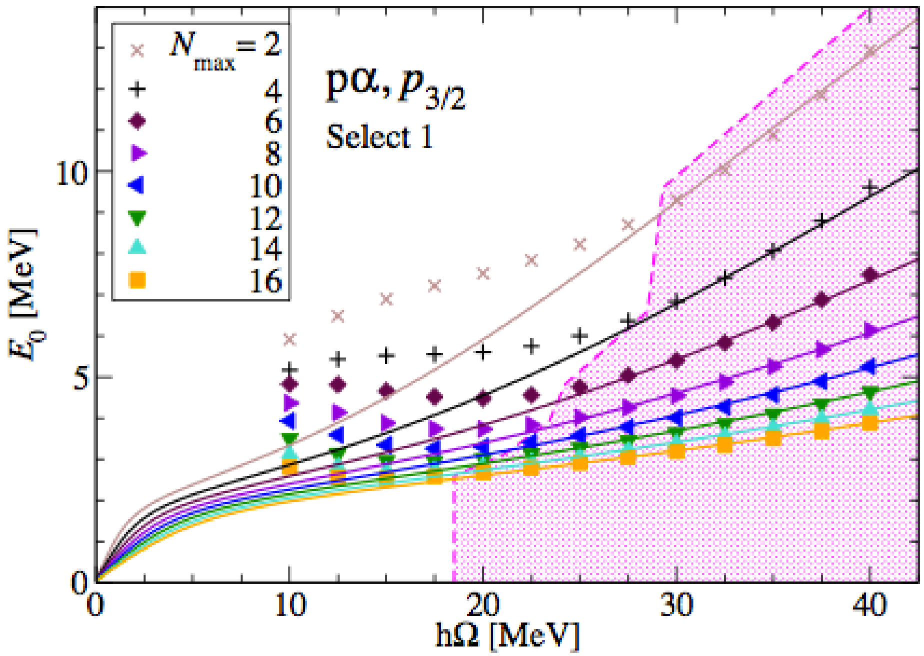

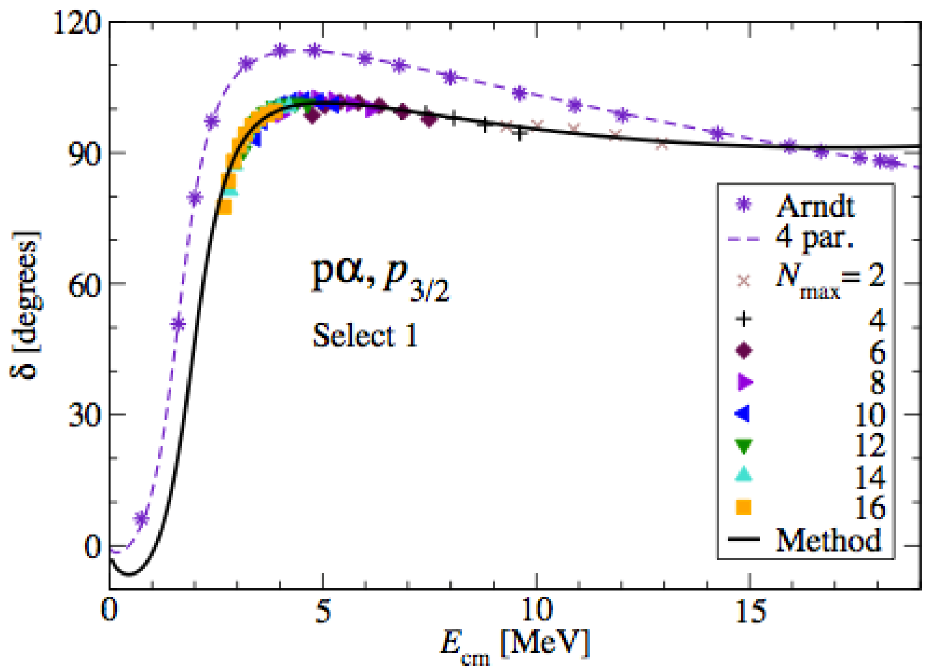

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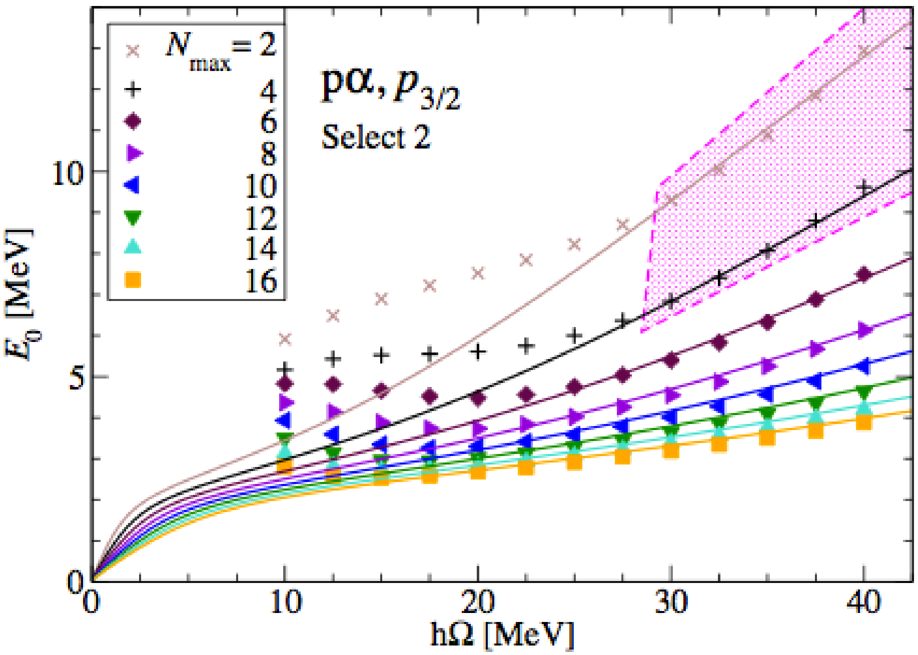

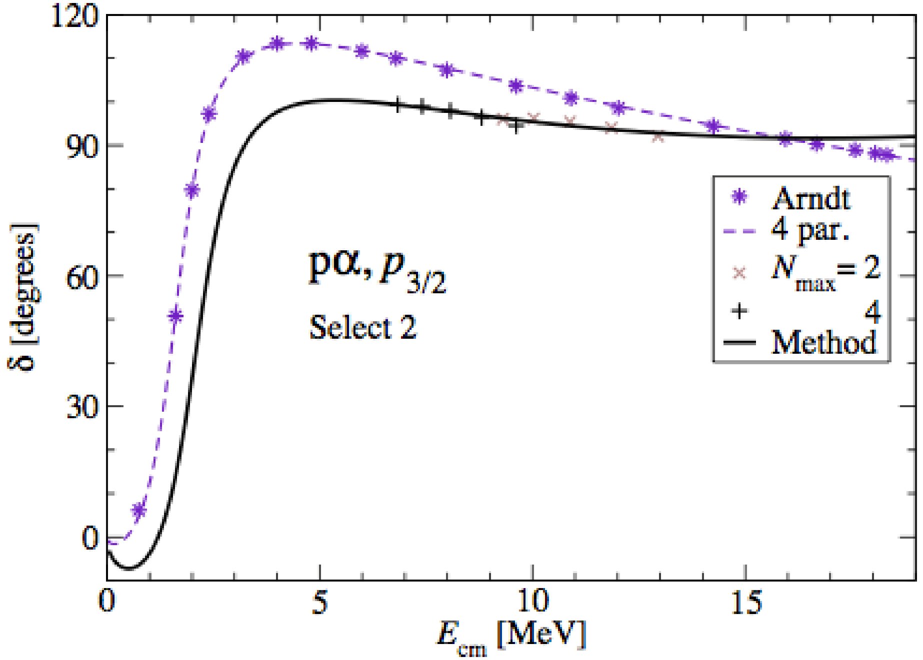

$p\alpha$ scattering: NCSM, JISP16

Select	a, $\mathrm{MeV}^{\frac{1}{2}}$	b^2, MeV	c, $\mathrm{MeV}^{-\frac{1}{2}}$	$d \cdot 10^3$, $\mathrm{MeV}^{-\frac{3}{2}}$	E_r, MeV	Γ, MeV	$\sqrt{\frac{\chi^2}{dat.}}$, keV	#
$p_{1/2}$								
0	1.254	3.981	−0.571	4.26	3.195	4.750	39	36
1	1.228	4.044	−0.580	4.90	3.290	4.702	109	56
2	1.122	4.456	−0.580	4.59	3.827	4.565	75	10
Arndt	1.725	4.146	−0.470	2.96	2.658	6.362	0.317°	19
$p_{3/2}$								
0	0.507	2.147	−0.454	2.90	2.018	1.464	26	36
1	0.515	2.063	−0.490	8.74	1.931	1.454	67	58
2	0.528	2.206	−0.488	8.77	2.067	1.543	12	10
Arndt	0.502	1.736	−0.384	2.04	1.610	1.299	0.455°	19

Summary

- SM states obtained at energies above thresholds can be interpreted and understood.
- Parameters of low-energy resonances (resonant energy and width) and low-energy phase shifts can be extracted from results of conventional Shell Model calculations
- Generally, one can study in the same manner *S*-matrix poles associated with bound states and design a method for extrapolating SM results to infinite basis. However this is a more complicated problem that is not developed yet.

Thank you!

Why oscillator basis?

- Any potential in the vicinity of its minimum at $r=r_0$ has the form $V(r)=V_0+a(r-r_0)^2+b(r-r_0)^3+\dots$,
  i.e., oscillator is the main term

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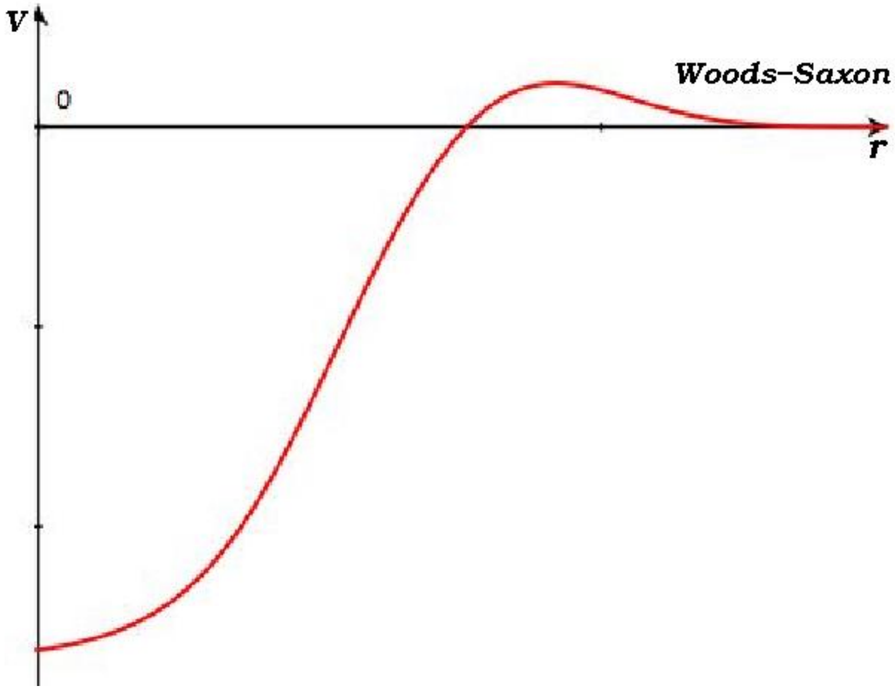

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