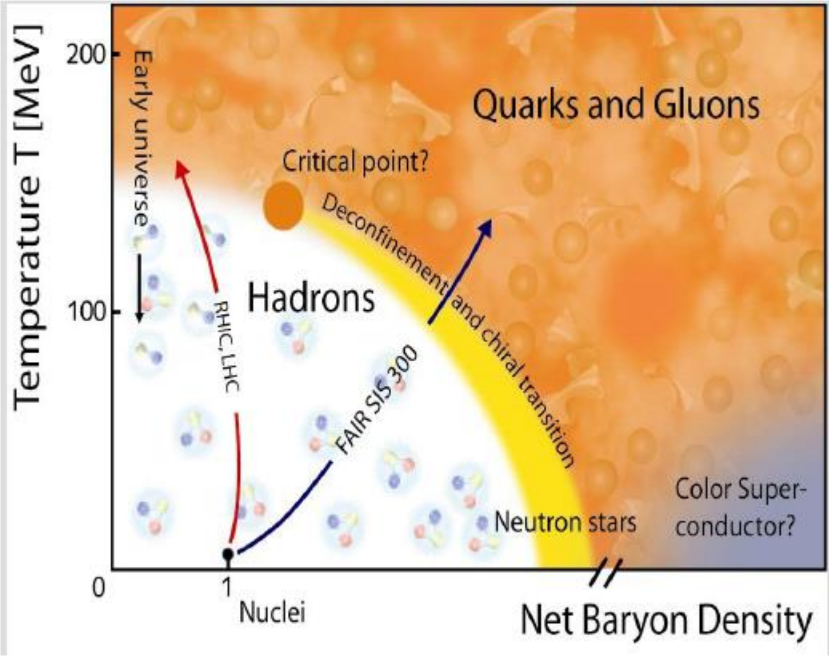
Status of AdS/QCD

SangJin Sin (Hanyang)
@YP-APCTP08

0707.0601 KY.Kim, SJS, I.Zahed

QCD phase diagram



Relevance of ads/qcd

- String scale ∼ 10^{18} GeV
- QCD ∼ 100 MeV
- Why string theory CAN be relevant to QCD at all?

answer

- Since the string (plank) scale decouple in a conformal AdS/CFT ;
  This happens since we are looking at a “Near horizon limit”.
- For non-conformal case, it comes with combination with other large number N.

caution

- AdS/nQCD
- Seeking for the Universality:
  Viscosity/entropy density

Hydrodynamic regime (high temperature small frequency /wave number regime.) is useful.

2nd message to particle physics from String theory

- Flavor is gauge symmetry in higher dim.
- Seeking for experimental evidence is important.

sQGP in RHIC

- RHIC found Unexpected strong nature of interaction in high energy collision.

- Only Lattice or other non-perturbative method can do something for it.

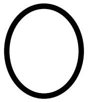
- String duality is one of such method.

Color/Flavor Unification

Open/closed duality

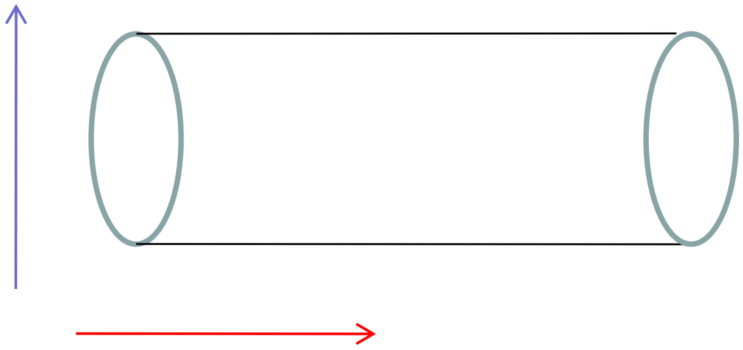
- Open string → gauge theory

- Closed string→ gravity



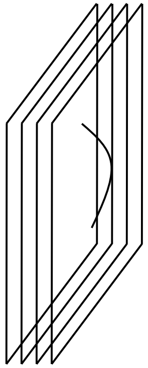
- Cylinder diagram

  →quantum gauge/classical gravity duality



D-brane AdS/CFT

- D-brane= closed string soliton whose vibration is restricted as open string vibration.
- Multiple D-branes :



- Open st.→U(N)
- Closed st.→extra-dim.→Holographic warped transverse space→ AdS

Remark: Color/Flavor Unification.

Holographic relation

YM→4d Boundary(global co-ord.)

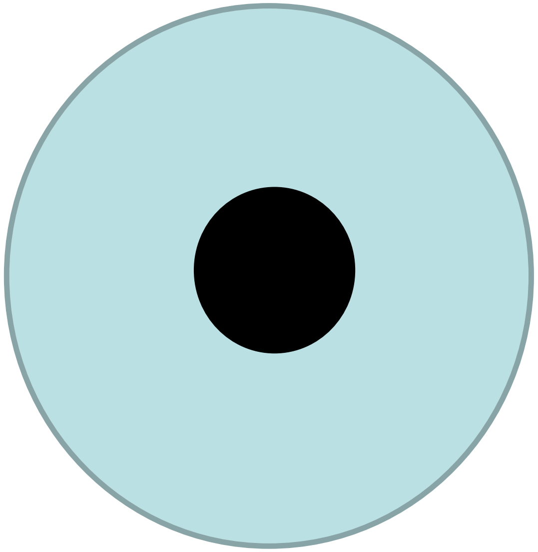
$$e^{-S_5[\phi_5]}|_{\phi_5(r=\infty)=\phi_4} \equiv \int d\phi_4 \exp(-S[\phi_4])$$

Transport coefficients
in
Expanding Medium

Idea of calculation

- Kubo-formula: TC $\sim$ 2pt fct.
- Use ads/cft to calculate 2pt fct.

Finite temperature YM ~ AdS Black hole

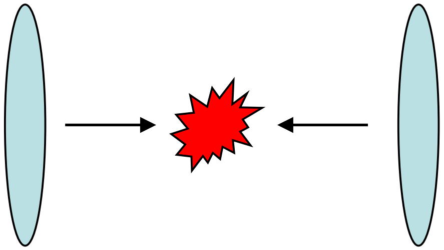


Expanding boundary Medium

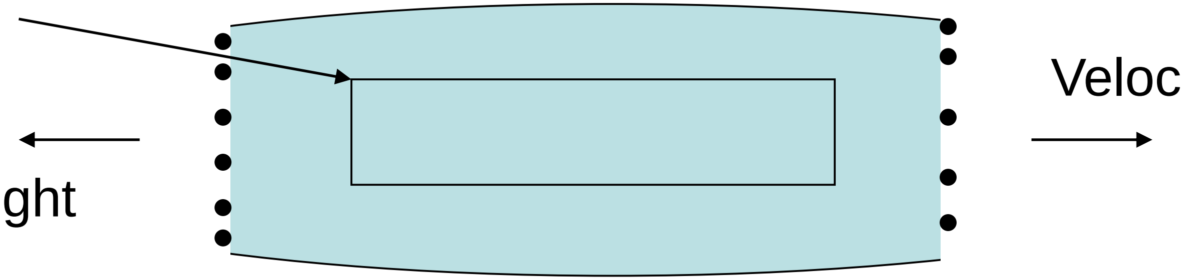
←→Falling horizon

(conformal invariance)

RHIC and Bjorken set up



Relativistically accelerated heavy nuclei



Central Rapidity Region

Velocity of light

Velocity of light

After collision

- one-dimensional expansion.

Bjorken System

Longitudinal Position ←→velocity.

All particles has common proper time

→ $choose \sim (\tau, y)$ as coordinate

$$(x^0, x^1, x^2, x^3) = (\tau \cosh y, \tau \sinh y, x^2, x^3).$$

$$(\tau, y, x^2, x^3)$$

Proper-time

Rapidity

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2$$

Bjorken frame

- a frame following the particle

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2$$

- Bjorken frame is comoving frame.
  : Milnor Universe

Relativistic Hydrodynamics

- Bjorken frame=local rest frame

where u=(1,0,0,0)

$T_{\mu\nu}$ simplies !

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \tau^2 p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Hydro eq.

$$\nabla^{\mu} T_{\mu\nu} = 0 \rightarrow \frac{d\rho}{d\tau} = -\frac{4}{3}\frac{\rho}{\tau}$$

Gravity dual of Bjorken system

- Find a solution of Einstein eq. in AdS with zero 5d energy–momentum tensor. with falling horizon as BC.
- Use Hologrphic renormalization to find the relation of 5d metric and boundary energy momentum tensor.
- Such sol. found by Janik+Peschansky
Such sol. with viscousity found by
SJS +Nakamura

Janik-Peschansky sol.

$$\mathrm{d}s^2 = \frac{R^2}{z^2}\left[-\frac{\left(1-v^4\right)^2}{\left(1+v^4\right)}\mathrm{d}\tau^2 + \left(1+v^4\right)\left(\tau^2\mathrm{d}y^2 + \mathrm{d}x_\perp^2\right) + \mathrm{d}z^2\right]$$

$$v \equiv \frac{z}{(\tau/\tau_0)^{\frac{1}{3}}}\ \varepsilon^{\frac{1}{4}}\ , \qquad \varepsilon \equiv \frac{1}{4}(\pi T_0)^4\ ,$$

horizon is located at $v = 1$ or $z \sim \tau^{1/3}$

→ Falling Horizon solution as desired!

Quasi-static Form of metric

introducing the coordinate change

$$u(z,\tau) \equiv \frac{2v^2}{1+v^4}$$

$$\mathrm{d}s^2 = \frac{\pi^2 T_0^2 R^2}{u(\tau/\tau_0)^{2/3}}\left[-f(u)\mathrm{d}\tau^2 + \tau^2 \mathrm{d}y^2 + \mathrm{d}x_\perp^2\right] + \frac{R^2}{4f(u)}\frac{\mathrm{d}u^2}{u^2}$$

$$\boxed{+\frac{R^2}{9}\tau^{-2}\mathrm{d}\tau^2 - \frac{R^2}{3}\frac{\tau^{-1}}{u\sqrt{f(u)}}\mathrm{d}\tau\mathrm{d}u\ ,} \to 0$$

$\tau \to \infty$.

$f = 1 - u^2.$

New time

$$t/t_0 \equiv \frac{3}{2}(\tau/\tau_0)^{\frac{2}{3}} \ , \qquad \text{yields}$$

$$\mathrm{d}s^2 = \frac{\pi^2 T_0^2 R^2}{u}\left[-f(u)\mathrm{d}t^2 + \frac{4}{9}t^2\mathrm{d}y^2 + \frac{3}{2}\frac{t_0}{t}\mathrm{d}x_\perp^2)\right] + \frac{R^2}{4f(u)}\frac{\mathrm{d}u^2}{u^2} \ .$$

In this transformed metric

horizon is no longer moving away in the fifth direction

expanding in

the y direction and contracting in the transverse direction

Langevin eq.

$$\frac{dx_i}{dt} = \frac{p_i}{M} \,,$$

$$\frac{dp_i}{dt} = \xi_i(t) - \eta_D p_i \,, \qquad \langle \xi_i(t)\xi_j(t') \rangle = \kappa \delta_{ij} \delta(t - t') \,.$$

drag and fluctuation coefficients

Einstein relation : $\eta_D = \frac{\kappa}{2MT}$.

η_D can be related in turn to the diffusion coefficient

$$D = \frac{T}{M\eta_D} = \frac{2T^2}{\kappa} \,.$$

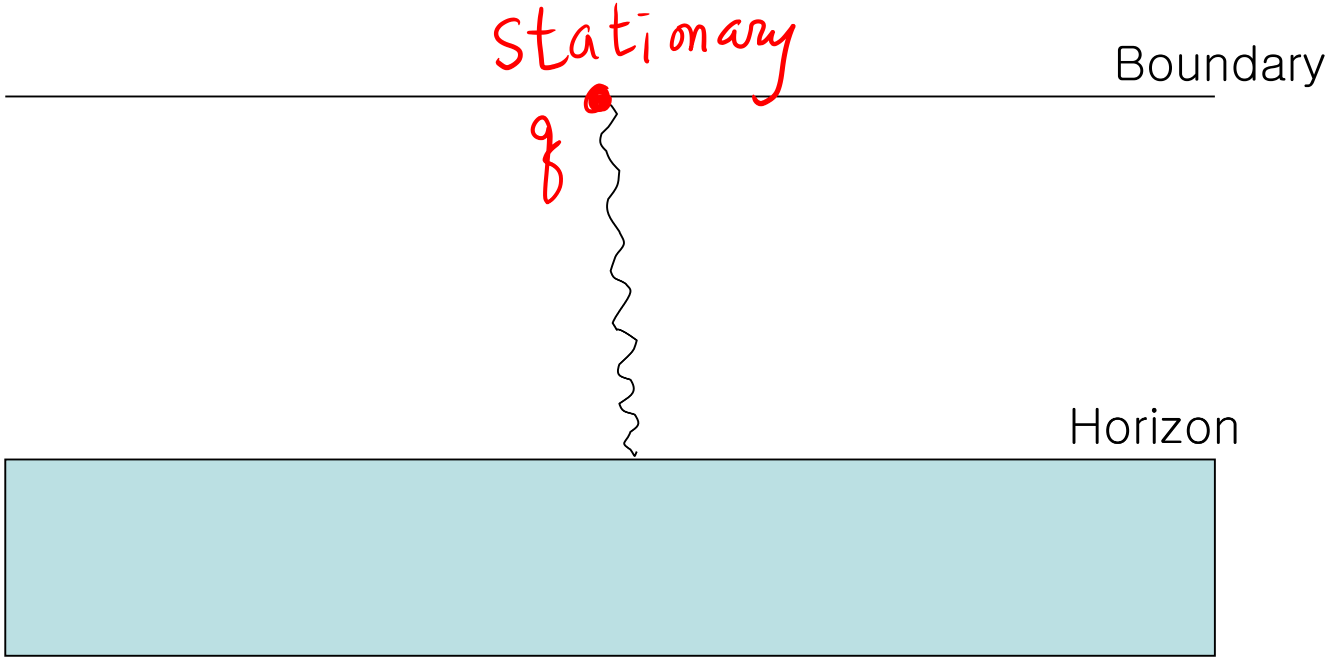
Noise v.s Force

$$\int dt \int dt' \, \langle \xi_i(t)\xi_j(t')\rangle = (\mathrm{time}) \times \kappa \, \delta_{ij} = \int dt \int dt' \, \langle \mathcal{F}_i(t)\mathcal{F}_j(t')\rangle$$

$$\kappa \; = \; \int dt \, \langle \mathcal{F}_y(t)\mathcal{F}_y(0)\rangle \quad .$$

String fluctuation in the frame following a particle

$$\delta X^1 = \xi(t,u) \qquad \langle \exp(i\int F(t)\xi(t))\rangle = \exp(iS_{cl}[\xi]).$$



Equation of Motion

Nambu-Goto action is

$$S = \frac{T_0\sqrt{\lambda}}{8}\int_0^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}u\left(\frac{3t_0}{2t}\right)\left[\frac{(\partial_t\xi)^2}{u^{\frac{2}{3}}f(u)} - \frac{4f(u)\pi^2T_0^2}{u^{\frac{1}{2}}}(\partial_u\xi)^2\right]$$

except for an overall factor of $\left(\frac{3t_0}{2t}\right)$

the same as the one in the static black hole metric

$$\partial_t^2\xi - \frac{1}{t}\partial_t\xi + 2\pi^4T_0^4f(u)(1+3u^2)\partial_u\xi - 4\pi^4T_0^4uf(u)^2\partial_u^2\xi = 0$$

Reduced Boundary action

$$S_{\mathrm{boundary}} = \frac{3\pi^2\sqrt{\lambda}T_0^3 t_0}{4}\int \mathrm{d}t\ \frac{f(u)}{\sqrt{u}t}\xi(t,u)\partial_u\xi(t,u)|_{u=0}^{u=1}$$

$$= \int \frac{\mathrm{d}\omega}{2\pi}\ \tilde{\xi}_0(-\omega)\left[\left(\frac{3\pi^2\sqrt{\lambda}T_0^3 t_0}{4}\right)\frac{f(u)}{\sqrt{u}}\Psi_{-\omega}(u)\partial_u\Psi_\omega(u)\right]_{u=0}^{u=1}\tilde{\xi}_0(\omega)$$

Eq.of M for $\Psi_\omega(u)$

$$\partial_u^2 \Psi_\omega(u) - \frac{3u^2+1}{2uf(u)} \partial_u \Psi_\omega(u) + \frac{\mathfrak{w}^2}{4uf(u)^2} \Psi_\omega(u) = 0$$

where $\mathfrak{w} \equiv \frac{\omega}{\pi T_0}$.

Near the horizon the solution behaves as

$$\Psi_\omega \sim (1-u)^{\pm i\mathfrak{w}/4} \ ,$$

minus choice corresponds to the infalling boundary condition.

Retarded Green Function
And Boundary condition

$$G_R(\omega) \equiv \left[-\frac{3\pi^2 \sqrt{\lambda} T_0^3 t_0}{2} \right] \left[\frac{f(u)}{\sqrt{u}} \Psi_{-\omega}(u) \partial_u \Psi_\omega(u) \right]_{u=0} .$$

Need Infalling boundary condition for $\Psi_\omega(u)$

C. P. Herzog and D. T. Son,

Scheme of Calculation

AdS/CFT correspondence $\langle \exp(i \int F(t)\xi(t)) \rangle = \exp(iS_{cl}[\xi]).$

Wightman function $G(t_1, t_2) \equiv \frac{1}{2}\langle F(t_1)F(t_2) + F(t_2)F(t_1) \rangle$

$$G(\omega) = -\coth\frac{\omega}{2T_0}\mathrm{Im}G_R(\omega)$$

C. P. Herzog and D. T. Son, hep-th/0212072

The key problem

For the retarded Green's function,

we need the wave function near zero *satisfying*

infalling boundary condition at the horizon.

u=0

Infalling fcts are
Needed here

horizon

Infalling fcts are
easily found here

Stratege of work

First we find two independent solutions near the horizon

$$\Psi^H_{\omega,in} \equiv (1-u)^{-i\mathfrak{w}/4}\left[1-(1-u)\left(\frac{i\mathfrak{w}^2}{8i+4\mathfrak{w}}\right)\right]+\cdots\ ,$$
$$\Psi^H_{\omega,out} \equiv (\Psi^H_{\omega,i})^*.$$

$\Psi^H_{\omega,in}$ is the infalling solution

these solutions are valid for all $\mathfrak{w}$.

Near the boundary ($u \sim 0$)

$$\begin{aligned}
\Psi^{B}_{\omega,0} &\equiv u^{3/2} - \frac{\mathfrak{w}^2}{10} u^{5/2} + \left(\frac{3}{7} + \frac{\mathfrak{w}^4}{280} \right) u^{7/2} + \cdots \ , \\
\Psi^{B}_{\omega,1} &\equiv 1 + \frac{\mathfrak{w}^2}{2} u - \frac{\mathfrak{w}^4}{8} u^2 + \left(\frac{\mathfrak{w}^2}{9} + \frac{\mathfrak{w}^6}{144} \right) u^3 + \cdots
\end{aligned}$$

take the near-horizon wave-function $\Psi^{H}_{\omega,in}(u)$

as the initial data

numerically integrate it to the boundary

solution is expressed as a linear sum of basis $\Psi^{B}_{\omega,0}$ and $\Psi^{B}_{\omega,1}$

$$\Psi^{H}_{\omega,in}(u) \xrightarrow{(14)} \mathcal{A}\Psi^{B}_{\omega,1}(u) + \mathcal{B}\Psi^{B}_{\omega,0}(u)$$

$\mathcal{A}$ and $\mathcal{B}$ are complex numbers determined numerically.

Normalization of wave function

we have to normalize Ψ such that it goes to 1

normalized wave function with correct boundary conditions is $\Psi_\omega = \mathcal{A}^{-1}\Psi^H_{\omega,in}(u)$

$$\Psi_\omega(u) = \Psi^B_{\omega,1}(u) + \frac{\mathcal{B}}{\mathcal{A}}\Psi^B_{\omega,0}(u),$$

which readily yields $\mathrm{Im}\Big[\frac{f(u)}{\sqrt{u}}\Psi_{-\omega}(u)\partial_u\Psi_\omega(u)\Big]_{u=0} = \frac{3}{2}\,\mathrm{Im}\tilde{\mathcal{B}}$

with $\tilde{\mathcal{B}} = \frac{\mathcal{B}}{\mathcal{A}}$.

Now the Wightman function $G(\omega)$ is given by

$$G(\omega) = \left[\frac{3\pi\sqrt{\lambda}T_0^3\tau_0}{2}\right]\left(\pi\coth\frac{\omega}{2T_0}\right)\left(\frac{3}{2}\ \mathrm{Im}\tilde{\mathcal{B}}(\omega)\right)$$

while $\mathcal{A}$ is easily accessible : $\mathcal{B}$ is not.

taking the imaginary part of $\Psi_\omega(u) = \Psi^B_{\omega,1}(u) + \frac{\mathcal{B}}{\mathcal{A}}\Psi^B_{\omega,0}(u),$

$$\mathrm{Im}\tilde{\mathcal{B}} = \left[\frac{\mathcal{A}^{-1}\Psi^H_{\omega,in}(u)}{\Psi^B_{\omega,0}(u)}\right]$$

then we evaluate it at any point, say, $u = 1$.

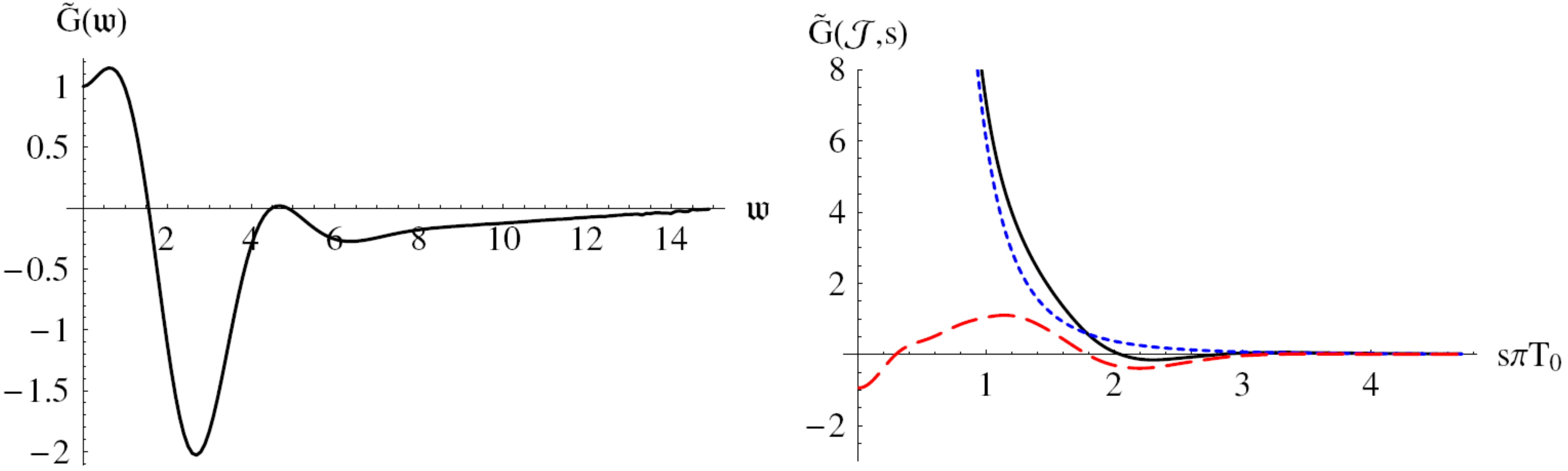
$\Psi^{B}_{\omega,0}(u) \quad u \, = \, 1$

we need to numerically integrate from the boundary

Therefore we get the numerical recipe:

$$\mathrm{Im}\tilde{\mathcal{B}} = \mathrm{Im}\left[\frac{\Psi^{H}_{\omega,in}(u=1-\epsilon)}{\Psi^{H}_{\omega,in}(u \xrightarrow{(14)} 0)\cdot\Psi^{B}_{\omega,0}(u \xrightarrow{(14)} 1)}\right]$$

Result



(a)

(b)

Figure 1: Force-Force decorrelator: (a) $\widetilde{G}(\mathfrak{w}) = \dfrac{G(\mathfrak{w}) - \frac{\pi}{2}|\mathfrak{w}^3|}{\left[\frac{3\pi\sqrt{\lambda}T_0^3\tau_0}{2}\right]}$ (b) $\widetilde{G}(\mathfrak{T}, s) = \dfrac{G(\mathfrak{T}, s)}{\left[\frac{3\pi^2\sqrt{\lambda}T_0^4\tau_0}{2\mathfrak{T}}\right]}$

[The dashed red line: discrete Fourier transform of (a). The dotted blue line: the divergent contribution alone. The solid line: the total result.]

Decorrelation time

decorrelation time follows readily from the dashed red curve

$$t_F \sim \frac{2}{\pi T_0} \ .$$

This time compares favorably with

lowest quasi-normal mode $\mathfrak{w}_1^{qn}$ associated to string fluctuations

$$\mathfrak{w}_1^{qn} \approx 2.69 - 2.29i.$$

This yields a decorrelation time of order $0.44/T_0$ which is comparable to our $0.64/T_0$

force-force decorrelation time, denoted by δt, is

$$\delta t \sim \frac{1}{T_0} \ .$$

Using the relation $t/t_0 \equiv \frac{3}{2}(\tau/\tau_0)^{\frac{2}{3}}$

$$t_F = \delta t = (\tau_0/\tau)^{1/3}\delta\tau$$

$$\delta\tau \sim \frac{(\tau/\tau_0)^{1/3}}{T_0} \equiv \frac{1}{T(\tau)},$$

which is the natural time dependent temperature, $T(\tau)$

Momentum correlation and Diffusion constant

$$\langle \Delta p(t)^2 \rangle \equiv \langle (p(t+\Delta t) - p(t))^2 \rangle$$

$$\begin{aligned} \langle \Delta p(t)^2 \rangle &= \int_t^{t+\Delta t} \mathrm{d}t_1 \int_t^{t+\Delta t} \mathrm{d}t_2 \langle F(t_1) F(t_2) \rangle \approx \int_t^{t+\Delta t} \mathrm{d}\mathfrak{T} \int_{-\infty}^{\infty} \mathrm{d}s G(\mathfrak{T}, s) \\ &\approx \frac{3}{2} \pi \sqrt{\lambda} T_0^3 t_0 \frac{\Delta t}{t} \end{aligned}$$

$$\langle \Delta p(\tau)^2 \rangle = \pi \sqrt{\lambda} T_0^3 t_0 \frac{\Delta \tau}{\tau} := \kappa(\tau) \Delta \tau$$

$$\boxed{\kappa(\tau) = \frac{\pi \sqrt{\lambda} T_0^3}{\tau / \tau_0} = \pi \sqrt{\lambda} T^3(\tau)}$$

$\kappa(\tau)$ is the time-dependent momentum diffusion constant.

equilibration in the diffusion regime.

$$\frac{\mathrm{d}p(\tau)}{\mathrm{d}\tau} = -\eta_D(\tau)p(\tau) + F(\tau) \ , \quad \langle F(\tau)\rangle = 0$$

$$\eta(\tau) = \frac{1}{2MT(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\tau \mathrm{d}\tau_1 \int_0^\tau \mathrm{d}\tau_2 \langle F(\tau_1)F(\tau_2)\rangle = \frac{\kappa(\tau)}{2MT(\tau)}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\langle x^2\rangle + \eta(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\langle x^2\rangle - 2\langle v(\tau)^2\rangle = 0$$

$$\langle x(\tau)F(\tau)\rangle = \langle x(\tau)\rangle\langle F(\tau)\rangle = 0,$$

Diffusion Rate

$$D(\tau) \equiv \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\langle x^2\rangle$$

$$\dot{D}(\tau) + \eta(\tau)D(\tau) - \langle v(\tau)^2\rangle = 0\ .$$

we need two inputs: $\eta(\tau)$ and $\langle v(\tau)^2\rangle$.

$$\eta(\tau) = \frac{\pi\sqrt{\lambda}T(\tau)^2}{2M} \qquad\qquad < v(\tau)^2 >= \frac{T(\tau)}{M}$$

$$\dot{D}(\tau) + a\ \tau^{-2/3}D(\tau) - b\ \tau^{-1/3} = 0,$$

with $a = \eta_0\tau_0^{2/3}$ and $b = T_0\tau_0^{1/3}/M$

Solution

$$D(\tau) = \frac{b}{a}\tau^{1/3} + D(0)e^{-3a\tau^{1/3}}$$

it shows how the diffusion rate for a quark changes in an expanding

At short times it is $D(0)$

at large times $D(\tau) = \frac{2}{\pi\sqrt{\lambda}T(\tau)}$

with an adiabatically changing temperature.

Cross over is Exponential

Conclusion

- We considerred Diffusion of heavy quark in a expanding medium
- In comoving frame time dependent diffusion problem is captured in the retarded Green function, which is calculated by AdS/CFT
- Equilibrium is reached exponentially fast. With time scale $\tau \sim 1/\eta_0^3$

$$\tau/\tau_0 = (1/3\eta_0\tau_0)^3.$$

At RHIC $\tau_0 \approx 1$ fm so that $\tau/\tau_0 \approx 1/\eta_0^3$

Hankel transform

$$\xi(t,u) \;\; \equiv \;\; \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\sqrt{\frac{i\pi\omega}{2}}\; tH_1^{(2)}(\omega t)\Psi_\omega(u)\tilde{\xi}_0(\omega)$$

$\Psi_\omega(u)$ is normalized such that $\Psi_\omega(0)=1$. $tH_1^{(2)}(\omega t)$

assume the following 'completeness relation'

$$-\frac{1}{4}\int_{-\infty}^{\infty}\mathrm{d}t\; tH_1^{(2)}(\omega t)H_1^{(2)}(-\omega' t) \simeq \frac{1}{\omega}\delta(\omega-\omega').$$

$$\int_0^\infty \mathrm{d}t\; tJ_\nu(\omega t)J_\nu(\omega' t) = \frac{1}{\omega}\delta(\omega-\omega')$$

Hydrodynamic Limit $\omega \to 0$

$$\Psi_\omega = (1-u)^{-i\mathfrak{w}/4}\left[1 + \frac{i\mathfrak{w}}{2}\left(-\tan^{-1}\sqrt{u} + \ln(1+\sqrt{u})\right)\right] + \mathcal{O}(\mathfrak{w}^2)$$

$$\lim_{\omega\to 0}\left(\pi\coth\frac{\pi\mathfrak{w}}{2}\right)\mathrm{Im}\Big[\frac{f(u)}{\sqrt{u}}\Psi_{-\omega}(u)\partial_u\Psi_\omega(u)\Big]_{u=0} \to 1$$

WKB Limit $\omega \to \infty.$

$$\lim_{\omega\to 0}\left(\pi\coth\frac{\pi\mathfrak{w}}{2}\right)\mathrm{Im}\Big[\frac{f(u)}{\sqrt{u}}\Psi_{-\omega}(u)\partial_u\Psi_\omega(u)\Big]_{u=0} \to \frac{\pi|\mathfrak{w}|^3}{2}$$

For general ω, we have to resort to numerical methods.