Kinetic and Hydrodynamic Theory

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Suppose we have the TOE (Theory Of Everything). Are we done with PHYSICS? The answer is ABSOLUTELY NOT:

Think about biology, which is basically molecular physics, or chemistry, which is atomic and/or molecular physics. The fundamental theory of both of them is QED but incredible number of people (more than the number of physicists) are working on them.

TOE will make human proud of themselves but it does not improve human life, namely practical point of view it does not help much. Remember that in mathematics, an important question is raised. Lots of people working on the problem and one big guy will solve the problem. Once the solution is known, NOBODY works on that anymore. It of cause will be applied to TOE. One day, a guy named WITTENSTEIN will put the period and all of the rest lose their jobs. (After Prof. YM Kim's talk, I seriously reconsider the superstring theory. It seems to me the theory can produce almost all of physics).

Now coming back to bio-chemistry, what makes the chemistry and biology to survive even though the fundamental theory is well known? That is the phenomena coming from multiparticle interactions. Then how do we study them? The answer depends on the phenomena: If it is related to the static properties, we can use well developed Statistic Equilibrium Theory; i.e., we define ensembles (micro-, canonical-, grand canonical ensemble) and we can calculate all most all of the properties using them. How about nonequilibrium or dynamic properties depending on time? We follow the idea of our beloved physicists like NEWTON, BOLTZMANN, MAXWELL, SCHEODINGER, DIRAC and so on: The system we are studying is known at one incident time and hence forth looking for how will it evolve or how was it evolved to become the specific state. Say differently we look for the equations of motion. Luckly, the equations of motion of multiparticle system can be derived by using the fundamental theory of Newton, QED or QCD. We will think about this problem today. The problem can be looked at a variety of viewpoints: We try to answer to the question based on CLASSICAL PHYSICS first and move on to QUANTUM PHYSICS. We may go on to QFT.

I. CLASSICAL TRANSPORT THEORY

A. Micro Equation of Motion

Suppose we have a system which consists of N particles (could be same kind or mixture of a few different kinds). What is most general distribution, which has all the information of the system? As far as we know, it is a PHASE SPACE distribution since each particle is completely known once we have position and momentum (all the information can be calculated using them). Introduce N-particle phase space distribution,

$$F_N(\vec{r}_1, \vec{p}_1; \vec{r}_2, \vec{p}_2; ...; \vec{r}_N, \vec{p}_N; t).$$

This function has all information the system can have. We can visualize this distribution function in 6 dimension phase space (3 positions and 3 momentum variables) as the distribution of N points or in 6N phase space as one point. Some examples of this distribution are

$$F_1(\vec{r}, \vec{p}, t) = \delta(\vec{r} - \vec{r}_1)\delta(\vec{p} - \vec{p}_1)$$

$$F_N = \delta(\vec{r} - \vec{r}_1)\delta(\vec{p} - \vec{p}_1) + \delta(\vec{r} - \vec{r}_2)\delta(\vec{p} - \vec{p}_2) + ... + \delta(\vec{r} - \vec{r}_N)\delta(\vec{p} - \vec{p}_N),$$

where $\vec{r}_i$ and $\vec{p}_i$ are the position and momentum of i-th particle. In the first point of view, we can interprete the distribution as a probability distribution since if the number of particles is sufficiently large and we see the number of particles in a given phase space volume element as a number density. If we integrate over $\vec{r}_2, ..., \vec{r}_N$ variables and their momentum counterparts, we are left over $\vec{r}_1$ and $\vec{p}_1$. This is nothing but one particle distribution out of N-body. Or we can integrate over $(\vec{r}_3, \vec{p}_3), ..., (\vec{r}_N, \vec{p}_N)$ to give 2 particle distribution. How can we get the evolution equation? The answer is simple: each particle must satisfy Newton equation,

$$m_1 \frac{d\vec{p}_1}{dt} = \vec{F}_{ext} + \vec{F}_{int}$$

where $\vec{F}_{ext}$ is the force on the particle due to force other than the system, $\vec{F}_{int}$ is the internal force coming from the interaction within the system so that the time evolution of the system is described by

$$\frac{\partial F}{\partial t} = \{H, F\}_P, \tag{1.1}$$

where H is the microscopic Hamiltonian of the system under study. $\{H, F\}_P$ is the Poisson bracket,

$$\{H, F\}_P = \sum_{i=1}^{3N} \left(\frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} \right). \tag{1.2}$$

This is the well known LIOUVILLE EQUATION and it follows from the Hamilton equation of motion. This is most general microscopic equation of motion for N-body system. All of the classical transport equations can be derived from this equation.

B. Kinetic theory

All of the classical transport (kinetic) equations can be derived from the Lieuville equation. As an example, the transport equation of a one particle distribution function $f(\vec{p}, \vec{q}, t)$ can be obtained by integrating over allowed phase space all coordinates and momenta except the coordinates and momenta $(\vec{q}, \vec{p})$ of the particle considered. However, the resulting transport equation is not 'closed' in general. Namely, one particle distribution is coupled to the higher order distributions, such as for example, 2 particle and/or 3 particle distribution functions and so on. It is called BBGKY hierachy. Much of the effort in this field is devoted to finding suitable approximations which closes the system of transport equations. One most example of kinetic equation is Boltzmann equation,

$$\frac{\partial f}{\partial t} + \frac{d\vec{r}}{dt} \cdot \frac{\partial f}{\partial \vec{r}} + \frac{d\vec{p}}{dt} \cdot \frac{\partial f}{\partial \vec{p}} = \int\int\int dp_2 dp_3 dp_4 \tag{1.3}$$
$$[f_3(\vec{p}_3) f_4(\vec{p}_4)(1 \pm f(\vec{p}))(1 \pm f_2(\vec{p}_2)) W_{34\to12}(\vec{p}_3, \vec{p}_4; \vec{p}, \vec{p}_2)$$
$$- \; f_1(\vec{p}) f_2(\vec{p}_2)(1 \pm f_3(\vec{p}_3))(1 \pm f_4(\vec{p}_4)) W_{12\to34}(\vec{p}, \vec{p}_2; \vec{p}_3, \vec{p}_4)].$$

Note that we used the chotic assumption here, $F_2(\vec{r}, \vec{p}_1; \vec{r}, \vec{p}_2) = f_1(\vec{r}, \vec{p}_1) f_2(\vec{r}, \vec{p}_2)$ and the only short range two particle interactions be important and the collision time is negligible. The physics comes in through the collision matrix W. If the process is reversal, $W_{12->34} = W_{34->12}$. If we are considering classical cases, we can ignore the Bose enhancement or Pauli blocking.

C. Hydro Equations of Motion

Integrating the Boltzmann equation gives the continuity equation,

$$\int d^3p\frac{\partial f}{\partial t} + \int d^3p\frac{dr}{dt}\cdot\frac{\partial f}{\partial r} + \int d^3p\frac{dp}{dt}\cdot\frac{\partial f}{\partial p} \;=\; \int d^3p(\frac{df}{dt})_{coll} \qquad (1.4)$$

If the force is only a function of position, the third integration gives null and we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial J^i}{\partial r^i} \;=\; 0, \qquad (1.5)$$

where we assume the number of particle is conserved in collisions, and $\rho = \int f d^3p$ and $J^i = \int v^i f d^3p$. This is nothing but the famous continuity equation. If we multiply the mass of a particle, the equation is a mass continuity equation of motion.

Now multiplying by momentum and integrating over momentum to give momentum, we have

$$\int d^3pp^i\frac{\partial f}{\partial t} + \int d^3pp^iv^j\frac{\partial f}{\partial r^j} + \int d^3pp^iF^j\frac{\partial f}{\partial p^j} \;=\; \int d^3pp^i(\frac{df}{dt})_{coll} \quad (1.6)$$

If the momentum is conserved in collision, we have null on right hand side and give

$$\frac{\partial P^i}{\partial t} + \frac{\partial}{\partial r^j}\int d^3pmv^iv^jf - F^j\int d^3p\delta_{ij}f \;=\; 0. \qquad (1.7)$$

Now we set $v^i = \bar{v}^i + \delta v^i$ to give,

$$\frac{\partial P^i}{\partial t} + \frac{\partial}{\partial r^j}\int d^3pm\bar{v}^i\bar{v}^jf + \frac{\partial}{\partial r^j}\int d^3p\delta v^i\delta v^jf - F^i\int d^3pf \;=\; 0. \quad (1.8)$$

and

$$\frac{\partial P^i}{\partial t} + \frac{\partial}{\partial r^j}T^{ij} \;=\; F^i, \qquad (1.9)$$

where the stress tensor is $T^{ij} = [\rho_m \bar{v}^i \bar{v}^j + < m\delta v^i \delta v^j >]$ and $\mathcal{F}$ is the force density. This is the famous Navier-Stokes equation.

Now we multiply both side with particle energy and integrate over momentum,

$$\begin{aligned} \int d^3p \frac{1}{2}mv^2 \frac{\partial f}{\partial t} + \int d^3p \frac{1}{2}mv^2 \frac{p^j}{m}\frac{\partial f}{\partial r^j} \; + \int d^3p \frac{1}{2}mv^2 F^j \frac{\partial f}{\partial p^j} \\ = \int d^3p \frac{1}{2}mv^2 (\frac{df}{dt})_{coll} = 0, \end{aligned} \tag{1.10}$$

where the energy conservation was used. Again we seperate the velocity(or momentum) into two pieces

$$\begin{aligned} \frac{\partial f}{\partial t}(\frac{1}{2}m\bar{v}^2 + m < \delta v^2 >) + \int d^3p(\bar{v}^2 + 2\bar{v}^j\delta v^j + \delta v^2)(\bar{v}^i + \delta v^i)\frac{\partial f}{\partial r^j} \\ -F^i \int d^3p p^i f = 0, \end{aligned} \tag{1.11}$$

The final form is

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial r^j}[(\rho E + P)v^i + \pi^{ij}v^j + \rho < \delta v^i \frac{\delta v^2}{2} > \; = \; \rho < v^i > F^i, \tag{1.12}$$

where π^{ij} is the viscous stress tensor. Putting all together, we have hydrodynamic equations of motion,

$$\frac{\partial \rho}{\partial t} + \frac{\partial J^i}{\partial r^i} \; = \; 0, \tag{1.13}$$

$$\frac{\partial \vec{J}}{\partial t} + \vec{\nabla} \cdot [\vec{J} \times \vec{v} + \mathbf{\Pi}] \; = \; \vec{F}_{ext}, \tag{1.14}$$

$$\frac{\partial \rho E}{\partial t} + \vec{\nabla} \cdot [(\rho E + p)\vec{v}] + \vec{\nabla} \cdot \vec{h} + \vec{\nabla} \cdot (\vec{\pi} \cdot \vec{v}) \; = \; \vec{F}_{ext} \cdot \vec{v}, \tag{1.15}$$

where the press tensor is

$$\Pi^{ij} \; = \; P\delta^{ij} = \pi^{ij} = P\delta^{ij} - \eta(\frac{\partial v^i}{\partial r^j} + \frac{\partial v^j}{\partial r^i} - \frac{2}{3}\delta^{ij}) - \zeta(\vec{\nabla} \cdot \vec{v})\delta^{ij}, \tag{1.16}$$

where η is the dynamic viscosity and ζ the bulk viscosity.

II. QUANTUM TRANSPORT THEORY

A. From Schödinger Equation to Hydrodynamics

The Schödinger equation is

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{2.1}$$

where $H = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + U(\vec{r},t)$. The Born interpretation tells us that $|\psi|^2$ is the probability of the particle existance, (if you multiply by its mass, then it becomes mass density), and the probability current density is given by

$$\vec{J} = \frac{\hbar}{2im}[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*] \tag{2.2}$$

As well known, the probability is positive definite and add up to give 1. We put

$$f(\vec{r},t) = |\psi(\vec{r},t)|^2 \tag{2.3}$$

and we can set the complex function (Madelung transformation)

$$\psi = \sqrt{f}e^{iS/\hbar} \tag{2.4}$$

where f, S are named the quantum probability density and the quantum phase function respectively. Note that the function S has ambiguity of $2\pi n\hbar$. We may use branch cut to make the function a single valued. Just plug this in the Schödinger equation, we find

$$\frac{Df}{Dt} + f\vec{\nabla}\cdot\vec{J} = 0, \tag{2.5}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}|\vec{\nabla}S|^2 = -U_{QM} \tag{2.6}$$

where the convective derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{J}\cdot\vec{\nabla}$ and $U_{QM} = -\frac{\hbar^2}{2}(\frac{1}{2}\vec{\nabla}^2 ln\, f + \frac{1}{4}|\vec{\nabla}ln\, f|^2) + U$. Unfortunately, this formula is not quite satisfactory, i.e., S

and U_{QM} are not unique since the equations are invariant wrt Gauge transformation. We can go on to define the gauge invariant form. Please find references for more information.

B. Wigner Function

In quantum physics, we cannot measure a coordinate and its momentum simultaneously with an arbitrary precision because of the Heisenberg uncertainty principle, $\Delta x \Delta p \geq \hbar$. This makes impossible to define the quantum phase space distribution function. Nonetheless, it has been known to define a function which has almost all the features expected from a phase space distribution function. Note first that the quantum expectation value of an observable $\hat{O}$ can be calculated by

$$< \hat{O} > \, = \, \mathrm{Tr} \hat{O} \hat{\rho}, \tag{2.7}$$

where Tr is the "trace" on any complete basis, $\hat{O}$ the quantum operator of the observable and $\hat{\rho}$ the density matrix of the quantum system. The density matrix can be expressed in general as follows,

$$\hat{\rho} = \sum_i w_i |i\rangle\langle i|, \tag{2.8}$$

where the weighing factor has following properties:

1. $w_i \geq 0$,
2. $\sum_i w_i = 1$,
3. the set $\{|i >\}$ is a complete ortho-normal basis.

The corollary of the second property of the weighing factor is

$$\mathrm{Tr} \hat{\rho} \, = \, 1, \tag{2.9}$$

If all w_i but one of them are zero, the system is in a PURE STATE. Otherwise, the system is in a MIXED STATE.

Now I take the coordinate space $|\vec{x}>$ as the quantum basis so that the Eq.(2.7) becomes

$$
\begin{aligned}
<\hat{O}> &= \int d\vec{x} < \vec{x}|\hat{\rho}\hat{O}|\vec{x}> \\
&= \int d\vec{x}d\vec{y} < \vec{x}|\hat{\rho}|\vec{y}>< \vec{y}|\hat{O}|\vec{x}> \\
&= \int d\vec{R}d\vec{r} < \vec{R}+\frac{1}{2}\vec{r}|\hat{\rho}|\vec{R}-\frac{1}{2}\vec{r}>< \vec{R}-\frac{1}{2}\vec{r}|\hat{O}|\vec{R}+\frac{1}{2}\vec{r}> \\
&= \int d\vec{R}d\vec{y}_1d\vec{y}_2 < \vec{R}+\frac{1}{2}\vec{y}_1|\hat{\rho}|\vec{R}-\frac{1}{2}\vec{y}_1> \delta(\vec{y}_1-\vec{y}_2) < \vec{R}-\frac{1}{2}\vec{y}_2|\hat{O}|\vec{R}+\frac{1}{2}\vec{y}_2> \\
&= \int d\vec{x}d\vec{p}\, W(\vec{x},\vec{p},t)O(\vec{x},\vec{p}), \qquad (2.10)
\end{aligned}
$$

where

$$
\begin{aligned}
W(\vec{x},\vec{p}) &= \int \frac{d\vec{y}}{(2\pi\hbar)^3} < \vec{x}+\frac{\vec{y}}{2}|\hat{\rho}|\vec{x}-\frac{\vec{y}}{2}> e^{-i\vec{p}\cdot\vec{y}/\hbar}, \qquad (2.11)\\
O(\vec{x},\vec{p}) &= \int d\vec{y} < \vec{x}+\frac{\vec{y}}{2}|\hat{O}|\vec{x}-\frac{\vec{y}}{2}> e^{-i\vec{p}\cdot\vec{y}/\hbar}. \qquad (2.12)
\end{aligned}
$$

This is so–called the WEYL TRANSFORMATION. In the derivation, I have used the relation

$$
\begin{aligned}
1 &= \int d\vec{y}\,|\vec{y}>< \vec{y}|, \qquad (2.13)\\
\delta(\vec{q}) &= \frac{1}{(2\pi\hbar)^3}\int d\vec{p}e^{-i\vec{p}\cdot\vec{q}/\hbar}. \qquad (2.14)
\end{aligned}
$$

Note that not all quantum observables $\hat{O}$ can be transformed from an operator to a single scalar function by the Weyl transformation. Spin is a typical counter-example and its representation requires several functions. I shall address this issue further below. Noticing close resemblance of the forms between the quantum expectation value Eq.(2.10) and the classical one, it is customary to call the *WIGNER* FUNCTION Eq.(2.11) also *the quantum distribution function*. The function W was first introduced by Wigner to study quantum corrections to classical statistical mechanics. Ever since, the Wigner function has found numerous applications in many fields of modern physics. The power of this approach to quantum mechanics is not limited to being only a calculational tool but it also provides fundamental insights into the relations between classical and quantum physics.

Let us consider some properties of the Wigner function of a pure state, i.e. we consider the Weyl transform of any quantum eigenstate $\psi_i(\vec{r}) = \langle \vec{x}|i\rangle$:

$$W_i(\vec{r},\vec{p}) = \int \frac{d\vec{s}}{(2\pi\hbar)^3}\, e^{-i\vec{p}\cdot\vec{s}/\hbar}\psi_i^*(\vec{r}-\vec{s}/2)\psi_i(\vec{r}+\vec{s}/2). \tag{2.15}$$

Note that with Eqs.(2.8,2.11) I have in general $W = \sum_i w_i W_i$. W_i has the following properties:

1. $W_i(\vec{r},\vec{p},t)$ is a Hermitian, i.e. $W_i^\dagger(\vec{r},\vec{p},t) = W_i(\vec{r},\vec{p},t)$. Therefore, $W_i(\vec{r},\vec{p},t)$ is real.

2. The Wigner function satisfies the following relations

$$\int d\vec{p}\, W_i(\vec{r},\vec{p}) = |\psi_i(\vec{r})|^2, \tag{2.16}$$

$$\int d\vec{r}\, W_i(\vec{r},\vec{p}) = |\tilde{\psi}_i(\vec{p})|^2, \tag{2.17}$$

$$\int d\vec{r}\, d\vec{p}\, W_i(\vec{r},\vec{p}) = 1, \tag{2.18}$$

   where $\tilde{\psi}_i(\vec{p})$ is the wavefunction in momentum space.

3. The quantum expectation value of an observable $\hat{O}$ is

$$< i|\hat{O}|i >= \int d\vec{r}\, d\vec{p}\, W_i(\vec{r},\vec{p},t) O(\vec{r},\vec{p}), \tag{2.19}$$

   where $O(\vec{r},\vec{p})$ is the Weyl transform of the observable, self–adjoint operator $\hat{O}$.

4. For given wave functions $\psi_i(\vec{r})$ and $\psi_j(\vec{r})$, I have

$$|\int d\vec{r}\, \psi_i^*(\vec{r})\psi_j(\vec{r})|^2 = (2\pi\hbar)^3 \int d\vec{r} \int d\vec{p}\, W_i(\vec{r},\vec{p},t)W_j(\vec{r},\vec{p},t) \tag{2.20}$$

For $i = j$ I obtain:

$$\int d\vec{r} \int d\vec{p}\, [W_i(\vec{r},\vec{p},t)]^2 = \frac{1}{(2\pi\hbar)^3}. \tag{2.21}$$

Taking the wave functions ψ_i and ψ_j to be orthogonal to each other, I obtain

$$\int d\vec{r} \int d\vec{p}\, W_i(\vec{r},\vec{p},t)W_j(\vec{r},\vec{p},t) = 0. \tag{2.22}$$

Hence we see that in general $W_i(\vec{r},\vec{p})$ cannot be in everywhere positive, i.e. it must also assume negative values. For this reason one calls W_i a pseudo–probability density. However, there exists a simple way to make W_i positive definite by using the coarse graining (smearing) function:

$$G(\vec{r},\vec{p};\lambda_r,\lambda_p) = \frac{1}{(\pi\lambda_r\lambda_p)^3} e^{-\vec{p}^2/\lambda_p^2 - \vec{r}^2/\lambda_r^2}. \tag{2.23}$$

The coarse grained distribution is

$$W_{s,i}(\vec{r},\vec{p},t;\lambda_r,\lambda_p) = \frac{1}{(\pi\lambda_r\lambda_p)^3} \int d\vec{r}\,'d\vec{p}\,'\; e^{-(\vec{p}-\vec{p}\,')^2/\lambda_p^2 - (\vec{r}-\vec{r}\,')^2/\lambda_r^2} W_i(\vec{r}\,',\vec{p}\,',t). \tag{2.24}$$

5. From the Cauchy–Schwartz inequality and the normalization condition of the wavefunction it further follows:

$$\begin{aligned} |W_i(\vec{r},\vec{p},t)|^2 &\leq \int \frac{d\vec{s}}{(2\pi\hbar)^3} |\psi_i(\vec{r}+\vec{s}/2,t)|^2 \int \frac{d\vec{s}}{(2\pi\hbar)^3} |\psi_i(\vec{r}-\vec{s}/2,t)|^2 \\ &= \left(\frac{2}{2\pi\hbar}\right)^6. \end{aligned} \tag{2.25}$$

Therefore, I get

$$|W_i(\vec{r},\vec{p},t)| \le \left(\frac{2}{2\pi\hbar}\right)^3 . \tag{2.26}$$

This is another form of the uncertainty principle. Namely, since the probability to find the particle at point $(\vec{r},\vec{p})$ is less than $(\frac{2}{h})^3$, the phase volume needed to find the particle should be larger than $(\frac{h}{2})^3$.

One can introduce a Wigner function in a gauge invariant way. The idea is that a wave function can be written by using the translation operator as follows;

$$\psi(\vec{r} \pm \vec{s}/2) = e^{\pm\vec{s}\cdot\vec{\nabla}/2}\psi(\vec{r}), \tag{2.27}$$

and one then replaces the derivative $\vec{\nabla}$ by the covariant derivative $\vec{D}$ for a gauge transformation, i.e.

$$\vec{\nabla} \to \vec{D} = \vec{\nabla} - ie\vec{A}(\vec{r},t)/\hbar \tag{2.28}$$

for the electromagnetic fields. The final result for the Wigner function in electromagnetic fields is

$$W(\vec{r},\vec{p}) = \int \frac{d\vec{s}}{(2\pi\hbar)^3}\, \exp\left(-i\frac{\vec{s}}{\hbar}\cdot[\vec{p}+e\int_{-1/2}^{1/2} d\lambda\, \vec{A}(\vec{r}+\lambda\vec{s})]\right)\psi^*(\vec{r}-\vec{s}/2)\psi(\vec{r}+\vec{s}/2). \tag{2.29}$$

It is possible to generalize this definition further to allow for non–abelian gauge theory.

I note that there are many other definitions for a quantum distribution in addition to the one (Wigner) presented here. I will not consider these further. Also, I will mostly address pure state Wigner functions.

C. Quantum Transport Equation

The evolution equation of a Wigner function $W(\vec{r},\vec{p},t)$ follows directly from the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t) \;=\; \left[-\frac{\hbar^2}{2m}\nabla^2+V(\vec{r})\right]\psi(\vec{r},t). \tag{2.30}$$

Differentiating the Wigner function Eq.(2.15) with respect to time gives

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}W(\vec{r},\vec{p},t) \;=\; & \int\frac{d^3s}{(2\pi\hbar)^3}\;e^{-i\vec{p}\cdot\vec{s}/\hbar}[-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}+\vec{s}/2,t)\;\psi^*(\vec{r}-\vec{s}/2,t)\\ & +\frac{\hbar^2}{2m}\psi(\vec{r}+\vec{s}/2,t)\;\nabla^2\psi^*(\vec{r}-\vec{s}/2,t)\\ & +[V(\vec{r}+\vec{s}/2)-V(\vec{r}-\vec{s}/2)]\psi(\vec{r}+\vec{s}/2,t)\;\psi^*(\vec{r}-\vec{s}/2,t)] \end{aligned} \tag{2.31}$$

First of all, I replace the derivative with respect to $\vec{r}$ by the derivative with respect to $\vec{s}$ and integrate by parts and then go from $\vec{s}$ back to $\vec{r}$–derivative to obtain

$$\begin{aligned} \left[\partial_t+\frac{1}{m}\vec{p}\cdot\nabla\right]W(\vec{r},\vec{p},t) \;=\; & \frac{1}{i\hbar}\int\frac{d^3s}{(2\pi\hbar)^3}\;e^{-i\vec{p}\cdot\vec{s}/\hbar}\,[V(\vec{r}+\vec{s}/2)-V(\vec{r}-\vec{s}/2)]\\ & \cdot\psi(\vec{r}+\vec{s}/2,t)\;\psi^*(\vec{r}-\vec{s}/2,t). \end{aligned} \tag{2.32}$$

Assuming the potential V can be expanded in Taylor series around point $\vec{r}$, I obtain finally,

$$\left[\partial_t+\frac{1}{m}\vec{p}\cdot\vec{\nabla}\right]W(\vec{r},\vec{p},t) \;=\; \sum_{k=0}^{\infty}\frac{(i\hbar)^{2k}}{2^{2k}(2k+1)!}(\vec{\nabla}'\cdot\vec{\partial})^{2k+1}V(\vec{r})W(\vec{r},\vec{p},t),$$

or, moving the classical force term $\vec{F}=-\vec{\nabla}V$ to the left hand side:

$$\left[\partial_t+\frac{1}{m}\vec{p}\cdot\vec{\nabla}+\vec{F}(\vec{r})\cdot\vec{\partial}\right]W(\vec{r},\vec{p},t)=\sum_{k=1}^{\infty}\frac{(-\hbar^2/4)^k}{(2k+1)!}(\vec{\nabla}'\cdot\vec{\partial})^{2k+1}V(\vec{r})W(\vec{r},\vec{p},t), \tag{2.33}$$

where $\vec{\nabla}'$ is the derivative with respect to coordinate $\vec{r}$ on the potential V very next to it and $\vec{\partial}$ the derivative with respect to momentum $\vec{p}$. This equation

is the quantum transport equation for a one particle Wigner function.
In the classical limit $\hbar \to 0$, I have

$$\partial_t W(\vec{r},\vec{p},t) = -\left[\frac{1}{m}\vec{p}\cdot\vec{\nabla} + \vec{F}(\vec{r})\cdot\vec{\partial}\right] W(\vec{r},\vec{p},t) = \{H, W\}, \quad (2.34)$$

which is just a Liouville equation, as indicated. We will see that the classical limit ($\hbar \to 0$) of the *relativistic* quantum transport equation is actually a much more subtle matter, and I will study it in the context of the relativistic classical limit in more detail. Note further that in the classical limit the evolution of a Wigner function is determined by the solution of Hamilton's equations of motion, i.e.

$$W(\vec{r},\vec{p},t) = W(\vec{r}(\vec{r}_0,\vec{p}_0,t_0|t),\vec{p}(\vec{r}_0,\vec{p}_0,t_0|t)), \quad (2.35)$$

where $\vec{r}(\vec{r}_0,\vec{p}_0,t_0|t)$ and $\vec{p}(\vec{r}_0,\vec{p}_0,t_0|t)$ are the solution of Hamilton's equations with initial condition $(\vec{r}_0,\vec{p}_0)$ at time t_0. This $(\vec{r}(t),\vec{p}(t))$ is the so–called Wigner trajectory.
It is often useful to express the quantum transport equation in a integral equation. To that end, I define the Green's function which satisfies the equation,

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\cdot\vec{\nabla}\right) G(\vec{r},\vec{p},t) = \delta(\vec{r})\delta(t). \quad (2.36)$$

The integral form of Eq.(2.33) is then

$$W(\vec{r},\vec{p},t) = W_0(\vec{r},\vec{p},t) + \int d\vec{r}\,'dt'\; G(\vec{r}-\vec{r}\,',\vec{p},t-t')K(\vec{r}\,',\vec{p},t')W(\vec{r}\,',\vec{p},t'), \quad (2.37)$$

where $W_0(\vec{r},\vec{p},t)$ is the solution of a field free equation. The kernel K is defined by

$$K(\vec{r},\vec{p},t) = \sum_{k=0}^{\infty} \frac{(i\hbar)^{2k}}{2^{2k}(2k+1)!}(\vec{\nabla}'\cdot\vec{\partial})^{2k+1}V(\vec{r}). \quad (2.38)$$

The form of the Green's function is most easily obtained considering the integral equation Eq.(2.37) in the $(\vec{q}, \vec{p}, \omega)$ space: consider the Fourier transformations with respect to $\vec{r}$ and t

$$\tilde{f}(\vec{q}, \vec{p}, \omega) = \int d\vec{r}\, dt\, e^{-i\vec{q}\cdot\vec{r}/\hbar + i\omega t/\hbar}\, f(\vec{r}, \vec{p}, t). \tag{2.39}$$

The Green's function Eq.(2.36) becomes after this transformation,

$$\tilde{G}_{ret}(\vec{q}, \vec{p}, \omega) = \frac{i}{\omega + i\varepsilon - \vec{p}\cdot\vec{q}/m}, \tag{2.40}$$

where ε is an infinitesimal positive definite. Note that I explicitly incorporate the causality principle. I so obtain the integral equation in this representation,

$$\tilde{W}(\vec{q}, \vec{p}, \omega) = \tilde{W}_0(\vec{q}, \vec{p}, \omega) + \tilde{G}(\vec{q}, \vec{p}, \omega) \int \frac{d\vec{q}'d\omega'}{(2\pi\hbar)^4} \tilde{K}(\vec{q} - \vec{q}', \vec{p}, \omega - \omega') \tilde{W}(\vec{q}', \vec{p}, \omega'). \tag{2.41}$$

$\tilde{W}_0(\vec{q}, \vec{p}, \omega)$ is the solution of a homogeneous equation and the kernel $\tilde{K}(\vec{q}, \vec{p}, \omega)$ is the Fourier transform of $K(\vec{r}, \vec{p}, t)$,

$$\begin{aligned} \tilde{K}(\vec{q}, \vec{p}, \omega) &= \sum_{k=0}^{\infty} \frac{-i(\hbar)^{2k}}{2^{2k}(2k+1)!} (\vec{q}\cdot\vec{\partial})^{2k+1} \tilde{V}(\vec{q}, \omega) \\ &= \frac{1}{i\hbar} \tilde{V}(\vec{q}, \omega) \left(e^{+\frac{\hbar}{2}\vec{q}\cdot\vec{\partial}} - e^{-\frac{\hbar}{2}\vec{q}\cdot\vec{\partial}} \right). \end{aligned} \tag{2.42}$$

Since the exponential term on the right hand side is a translation operator, one obtains:

$$\begin{aligned} \tilde{W}(\vec{q}, \vec{p}, \omega) = \tilde{W}_0(\vec{q}, \vec{p}, \omega) &+ \frac{1}{i\hbar} \tilde{G}(\vec{q}, \vec{p}, \omega) \int \frac{d\vec{q}'d\omega'}{(2\pi\hbar)^4}\, \tilde{V}(\vec{q} - \vec{q}', \omega - \omega') \\ &\cdot \left[\tilde{W}(\vec{q}', \vec{p} + \frac{\hbar}{2}(\vec{q} - \vec{q}'), \omega') - \tilde{W}(\vec{q}', \vec{p} - \frac{\hbar}{2}(\vec{q} - \vec{q}'), \omega') \right] \end{aligned} \tag{2.43}$$

If the potential $V(\vec{r}, t)$ does not depend on time, the Fourier transform of the potential is

$$\tilde{V}(\vec{q}, \omega) = \tilde{V}(\vec{q})(2\pi\hbar)\delta(\omega), \tag{2.44}$$

so that the integral equation simplifies to

$$\begin{aligned}\tilde{W}(\vec{q}, \vec{p}, \omega) = \tilde{W}_0(\vec{q}, \vec{p}, \omega) + \frac{1}{i\hbar}\tilde{G}(\vec{q}, \vec{p}, \omega) \int \frac{d^3q\,'}{(2\pi\hbar)^3}\, \tilde{V}(\vec{q} - \vec{q}\,') \\ \cdot \left[\tilde{W}(\vec{q}\,', \vec{p} + \frac{\hbar}{2}(\vec{q} - \vec{q}\,'), \omega) - \tilde{W}(\vec{q}\,', \vec{p} - \frac{\hbar}{2}(\vec{q} - \vec{q}\,'), \omega)\right]\end{aligned} \tag{2.45}$$

This integral form is often useful in quantum physics.

III. QUANTUM FIELD TRANSPORT THEORY

The Wigner transport formulation has been extended to allow for relativistic kinematics and particle production next to the matter flow processes.

A. Introduction

In the previous section, we introduced the one–particle Wigner function as the Weyl transform of the density matrix. While this Wigner function found applications in a variety of fields such as the chemical reactions, nuclear physics, quantum optics and solid state physics, this theory cannot describe the particle production process. Since the process of particle production is unavoidable in relativistic formulation and/or at sufficiently high energy, one has to develop a transport theory which has room for the process. Nearly 30 years ago Carruthers and Zachariasen introduced a relativistic 8D Wigner function for spinless neutral particle fields:

$$F(p,x) = \int d^4y\, e^{ip\cdot y} \, < \Psi|\hat{\varphi}(x-y/2)\hat{\varphi}(x+y/2)|\Psi >, \qquad (3.1)$$

where $|\Psi>$ is the state vector. $\hat{\varphi}(x)$ is the 'second-quantized' Klein–Gordon field obeying the equation of motion

$$(\partial_\mu \partial^\mu + m^2)\hat{\varphi}(x) = \hat{j}(x), \qquad (3.2)$$

where $\hat{j}(x)$ is the source fixed in its form by the model under consideration. The equation of motion of this Wigner function can be obtained by applying $(\partial_\mu \partial^\mu + m^2)$ on Eq.(3.1) and using the field equation. Since in general the source $\hat{j}(x)$ is a function of the field itself and/or other fields, the equation of motion cannot be closed unless one makes some kind of approximation.

This is one of the general properties of transport theory and is known as the BBGKY hierarchy problem — the dynamics of a two-point Wigner function is determined by a four– or higher point Wigner function and that of a four–point Wigner function by six or higher–point Wigner function, and so on. To break this hierarchy a suitable approximation, for example, MFA(mean field approximation), is made. It is important to remember that this transport theory inherits the intrinsic infinities from the relativistic field theory. Therefore one needs to renormalize the theory to produce meaningful quantities. This problem has been addressed by Cooper et. al. for the scalar field.
This field theoretical approach has been extended to the Dirac field by Hakim who was interested to study strongly interacting particles forming relativistic dense matter. While the scalar field Wigner function is in principle 2×2 matrix due to the particle and antiparticle sector, the spinor field Wigner function is 4×4 matrix coming from the spinor structure,

$$F(p,x) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-ip\cdot y} < \Psi|\hat{\bar{\psi}}(x+y/2) \otimes \hat{\psi}(x-y/2)|\Psi > . \quad (3.3)$$

This 4×4 matrix Wigner function has been decomposed into 16 functions on the basis of 16 linear independent matrices,

$$1_4, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5, \gamma^\mu\gamma^5. \quad (3.4)$$

The dynamics of this Wigner function can be determined using the Dirac field equation.
While the formulations presented above are in general manifestly Lorentz covariant and can describe the particle production process, they miss one important ingredient, namely, a gauge covariance. It is well–known that the observable is gauge invariant and the physical process must be described by

a gauge covariant theory. In particular, the gauge symmetry resides in the heart of modern physics, as for example in QED, QCD or standard model. Heinz and Elze et al were able to propose Wigner functions which had full gauge symmetry. To this end, they consider the Wigner operator,

$$\begin{aligned}\hat{W}(x,p) &= \int \frac{d^4y}{(2\pi\hbar)^4} e^{-ip\cdot y} \hat{\bar{\psi}}(x+y/2) \otimes \hat{\psi}(x-y/2) \\ &= \int \frac{d^4y}{(2\pi\hbar)^4} e^{-ip\cdot y} \hat{\bar{\psi}}(x) e^{+\frac{y}{2}\partial_x^\dagger} \otimes e^{-\frac{y}{2}\partial_x} \hat{\psi}(x), \end{aligned} \tag{3.5}$$

where $\partial^\dagger$ operates on the function to the left of it and the relation $f(x \pm y) = e^{\pm y\cdot\partial_x} f(x)$. $\otimes$ is the tonsorial product in spinor space (4×4) as well as the internal quantum number such as the color. To make this function gauge covariant, it is only necessary to replace the derivative by the covariant derivative, i.e.

$$\partial_\mu \to D_\mu = \partial_\mu - igA_\mu, \tag{3.6}$$

where g is the coupling constant and A_μ is the gauge field. Thus, the gauge covariant Wigner operator is

$$\hat{W}(x,p) = \int d^4y \; \hat{\bar{\psi}}(x+y/2) U(x+y/2,x) \otimes U(x,x-y/2) \hat{\psi}(x-y/2) \tag{3.7}$$

with

$$U(a,b) = \exp\left[ig \int_b^a dx_\mu A^\mu\right], \tag{3.8}$$

where the path of the link operator U must be straight line in order to interpret p as the physical 4–momentum.

Under the gauge transformation,

$$\psi(x) \to S(x)\psi(x), \quad S(x) = e^{i\theta_a(x)t_a}, \tag{3.9}$$

$$A_\mu^a \to A_\mu^a - \frac{1}{g}\partial_\mu\theta^a - f_{bc}^a \theta_b A_{c,\mu}, \tag{3.10}$$

the Wigner operator transforms covariantly, akin to the operator D^μ:

$$\hat{W}(x,p) \rightarrow S(x)\hat{W}(x,p)S^{-1}(x). \tag{3.11}$$

Here t_a is the generator of the gauge group and f_{abc} the structure constant of the gauge group. Of cause, the Wigner function is the quantum expectation of the Wigner operator in a given state $|\Psi>$. The dynamics of this function can be obtained by the field equation and it requires tedious operator ordering especially in the case of non–abelian gauge theory. This formulation is manifestly Lorentz covariant, and the dynamics described occur also off the mass–shell. Thus in order to calculate a physical observable one should project the results on the mass–shell, which is a (complex) constraint of the 8-dimensional dynamical motion.

B. BGR Functions and Equations

This projection requirement makes it difficult to extract the physical information from the transport functions, which are even more difficult to obtain. Consequently little progress was made regarding practical applications of the eight dimensional formulation. However, a different approach has been also recently proposed by Białynicki–Birula, Górnicki and Rafelski, who introduced the so–called Dirac–Heisenberg–Wigner (DHW) function, which is the Weyl transform of Dirac–Heisenberg density matrix. In many regards this formulation is similar to the conventional nonrelativistic Wigner theory. The DHW function for the matter field of the abelian gauge theory (QED), is introduced as follows,

$$W_{\alpha\beta}(\vec{r},\vec{p},t) = -\frac{1}{2}\int d^3s\ e^{-i\vec{p}\cdot\vec{s}} < \Psi | e^{-ie\int d\lambda \vec{s}\cdot\vec{A}(\vec{r}+\lambda\vec{s},t)} [\hat{\psi}_\alpha(\vec{r}+\vec{s}/2,t), \hat{\psi}^\dagger_\beta(\vec{r}-\vec{s}/2,t)] | \Psi > \quad (3.12)$$

where $|\Psi>$ is a state vector and $\vec{A}$ is the gauge field. α and β are the spinor index. This DHW function has following properties:

1) since the DHW function is gauge invariant, I can fix the gauge in a most convenient way which is here the temporal gauge ($A_0 = 0$);

2) the DHW function is not manifestly Lorentz covariant because it has only one time t which is a laboratory time (this is a reason why it is called a single time formulation). However, it has full Poincare symmetry;

3) the field operators in Eq.(3.12) have been combined such that $W_{\alpha\beta}$ possesses the charge conjugation symmetry;

4) the transformation variable $\vec{p}$ is the physical kinetic momentum, a consequence of choosing the straight line integral in the phase factor which makes the function gauge invariant;

5) $W_{\alpha\beta}$ is a Hermitian by construction so that there are 16 linearly independent real functions defining the matrix.

One can decompose this 4×4 matrix **W** on the complete set of 4×4 Hermitian matrices,

$$\mathbf{W}(\vec{r},\vec{p},t) \;=\; \frac{1}{4}\left(f_0 + \sum_{i=1}^{3}\rho_i f_i + \vec{\sigma}\cdot\vec{g}_0 + \sum_{i=1}^{3}\rho_i\vec{\sigma}\cdot\vec{g}_i\right), \qquad (3.13)$$

where the complete set of 4×4 Hermitian matrices is given in Appendix B. This decomposition allows a direct physical interpretation of the coefficient functions which are called 'BGR' functions. The physical meanings of the 16 component functions can be inferred from their momentum integrals:

$$\int \frac{d^3p}{(2\pi\hbar)^3}\, f_0(\vec{r},\vec{p},t) \;=\; \mathrm{Tr}[\bar{\psi}(\vec{r},t)\gamma^0\psi(\vec{r},t)], \qquad (3.14)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\, f_1(\vec{r},\vec{p},t) \;=\; \mathrm{Tr}[\bar{\psi}(\vec{r},t)i\gamma^0\gamma^5\psi(\vec{r},t)], \qquad (3.15)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\, f_2(\vec{r},\vec{p},t) \;=\; \mathrm{Tr}[\bar{\psi}(\vec{r},t)\gamma^5\psi(\vec{r},t)], \qquad (3.16)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\, f_3(\vec{r},\vec{p},t) \;=\; \mathrm{Tr}[\bar{\psi}(\vec{r},t)\psi(\vec{r},t)], \qquad (3.17)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\, \vec{g}_0(\vec{r},\vec{p},t) \;=\; -\mathrm{Tr}[\bar{\psi}(\vec{r},t)i\gamma^5\vec{\gamma}\psi(\vec{r},t)], \qquad (3.18)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\, \vec{g}_1(\vec{r},\vec{p},t) \;=\; \mathrm{Tr}[\bar{\psi}(\vec{r},t)\vec{\gamma}\psi(\vec{r},t)], \qquad (3.19)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\, \vec{g}_2(\vec{r},\vec{p},t) \;=\; -\mathrm{Tr}[\bar{\psi}(\vec{r},t)i\gamma^0\vec{\gamma}\psi(\vec{r},t)], \qquad (3.20)$$

$$\int \frac{d^3p}{(2\pi\hbar)^3}\, g_3^k(\vec{r},\vec{p},t) \;=\; \mathrm{Tr}[\bar{\psi}(\vec{r},t)i\epsilon^{ijk}\gamma^{ij}\psi(\vec{r},t)], \qquad (3.21)$$

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, and $\gamma^{ij} = \gamma^i\gamma^j$. Tr stands here for the trace over the spinor space only. Thus $(f_0, \vec{g}_1)$ form the current four vector phase space distributions, f_3 is the mass density, $\vec{g}_0$ the spin density, $\vec{g}_3$ the magnetic moment density, etc. These interpretations can be further justified by the conservation laws I shall discuss below.

The time evolution of this DHW function, thus the relativistic quantum transport equation, can be obtained by differentiating Eq.(3.12) with respect to time and using Dirac field equations,

$$i\partial_t\psi_\mu = [\vec{\alpha}\cdot(-i\vec{\nabla} - e\vec{A}) + \beta m]_{\mu\nu}\psi_\nu(\vec{r},t), \tag{3.22}$$

$$-i\partial_t\psi^\dagger_\mu = \psi^\dagger_\nu(\vec{r},t)[\vec{\alpha}\cdot(i\nabla - e\vec{A}) + \beta m]_{\nu\mu}. \tag{3.23}$$

Keeping track of arguments $(\vec{r} - \vec{y}/2)$ and $(\vec{r} + \vec{y}/2)$ of the field operators carefully, one obtains the time evolution,

$$D_t\mathbf{W} = -\frac{c}{2}\vec{D}\cdot\{\rho_1\vec{\sigma},\mathbf{W}\} - \frac{ic}{\hbar}[\rho_1\vec{\sigma}\cdot\vec{P} + \rho_3 mc, \mathbf{W}], \tag{3.24}$$

where the integro-differential operators are

$$D_t = \partial_t + e\int_{-1/2}^{1/2} d\lambda \vec{E}(\vec{r} + i\hbar\lambda\vec{\partial}_p, t)\cdot\vec{\partial}_p, \tag{3.25}$$

$$\vec{D} = \vec{\nabla} + \frac{e}{c}\int_{-1/2}^{1/2} d\lambda \vec{B}(\vec{r} + i\hbar\lambda\vec{\partial}_p, t)\times\vec{\partial}_p, \tag{3.26}$$

$$\vec{P} = \vec{p} - \frac{ie\hbar}{c}\int_{-1/2}^{1/2} d\lambda\lambda \vec{B}(\vec{r} + i\hbar\lambda\vec{\partial}_p, t)\times\vec{\partial}_p. \tag{3.27}$$

Since the formulation is constructed on the temporal gauge, the electric and magnetic fields are given by

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t}, \tag{3.28}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}. \tag{3.29}$$

One further assumption was made to obtain these equations: The expectation value of the products of Dirac field operators with gauge field strength was replaced by the product of corresponding expectation values,

$$< \Psi|\vec{E}\ \exp\left(-ie\int d\lambda\vec{s}\cdot\vec{A}(\vec{r}+\lambda\vec{s},t)\right)[\psi(\vec{r}+\vec{s}/2,t),\psi(\vec{r}-\vec{s}/2,t)]|\Psi> \rightarrow$$

$$< \Psi|\vec{E}|\Psi> < \Psi|\exp\left(-ie\int d\lambda\vec{s}\cdot\vec{A}(\vec{r}+\lambda\vec{s},t)\right)[\psi(\vec{r}+\vec{s}/2,t),\psi(\vec{r}-\vec{s}/2,t)]|\Psi>, \quad (3.30)$$

$$< \Psi|\vec{B}\ \exp\left(-ie\int d\lambda\vec{s}\cdot\vec{A}(\vec{r}+\lambda\vec{s},t)\right)[\psi(\vec{r}+\vec{s}/2,t),\psi(\vec{r}-\vec{s}/2,t)]|\Psi> \rightarrow$$

$$< \Psi|\vec{B}|\Psi> < \Psi|\exp\left(-ie\int d\lambda\vec{s}\cdot\vec{A}(\vec{r}+\lambda\vec{s},t)\right)[\psi(\vec{r}+\vec{s}/2,t),\psi(\vec{r}-\vec{s}/2,t)]|\Psi>. \quad (3.31)$$

If one does not make this approximation, the equation of motion cannot be closed since the gauge field strength will introduce the Dirac field as a source. This is similar to the BBGKY hierarchy as mentioned before. In this approximation one neglects the fluctuation in the number of the photons, while all fluctuations of the matter field are retained. Consequently, this approach is particularly suitable to the study of the matter field in the presence of strong gauge fields.

After substituting the expansion of DHW function **W**, Eq.(3.13), into the evolution equation, one obtains BGR equations,

$$
\begin{aligned}
D_t f_0 + c\vec{D}\cdot\vec{g}_1 &= 0, &(3.32)\\
D_t f_1 + c\vec{D}\cdot\vec{g}_0 &= -\frac{2mc^2}{\hbar} f_2, &(3.33)\\
D_t f_2 + \frac{2c}{\hbar}\vec{P}\cdot\vec{g}_3 &= +\frac{2mc^2}{\hbar} f_1, &(3.34)\\
D_t f_3 - \frac{2c}{\hbar}\vec{P}\cdot\vec{g}_2 &= 0, &(3.35)\\
D_t \vec{g}_0 + c\vec{D} f_1 - \frac{2c}{\hbar}\vec{P}\times\vec{g}_1 &= 0, &(3.36)\\
D_t \vec{g}_1 + c\vec{D} f_0 - \frac{2c}{\hbar}\vec{P}\times\vec{g}_0 &= -\frac{2mc^2}{\hbar}\vec{g}_2, &(3.37)\\
D_t \vec{g}_2 + c\vec{D}\times\vec{g}_3 + \frac{2c}{\hbar}\vec{P} f_3 &= +\frac{2mc^2}{\hbar}\vec{g}_1, &(3.38)\\
D_t \vec{g}_3 - c\vec{D}\times\vec{g}_2 - \frac{2c}{\hbar}\vec{P} f_2 &= 0. &(3.39)
\end{aligned}
$$

To close the set of equations, Maxwell equations must be added,

$$
\begin{aligned}
\partial_t \vec{B} &= -\vec{\nabla}\times\vec{E}, &(3.40)\\
\vec{\nabla}\cdot\vec{B} &= 0, &(3.41)\\
\partial_t \epsilon_0 \vec{E} &= \vec{\nabla}\times\mu_0^{-1}\vec{B} - \vec{j}_t, &(3.42)\\
\vec{\nabla}\cdot\epsilon_0\vec{E} &= \rho_t, &(3.43)
\end{aligned}
$$

where charge and current density including a back reaction are

$$
\begin{aligned}
\rho_t(\vec{r},t) &= e\int d\vec{p}\, f_0(\vec{r},\vec{p},t) + \rho_{ext}(\vec{r},t), &(3.44)\\
\vec{j}_t(\vec{r},t) &= e\int d\vec{p}\, \vec{g}_1(\vec{r},\vec{p},t) + \vec{j}_{ext}(\vec{r},t), &(3.45)
\end{aligned}
$$

and where ρ_{ext} and $\vec{j}_{ext}$ is the external charge and current density. The total charge, energy, momentum and angular momentum for the closed system are given by, respectively,

$$Q = e \int d\Gamma\ f_0(\vec{r}, \vec{p}, t), \tag{3.46}$$

$$E = \int d\Gamma\ [c\vec{p} \cdot \vec{g}_1(\vec{r}, \vec{p}, t) + mc^2 f_3(\vec{r}, \vec{p}, t)] + \frac{1}{2} \int d^3r\ [\epsilon_0 \vec{E}^2(\vec{r}, t) + \mu_0^{-1} \vec{B}^2(\vec{r}, t)], \tag{3.47}$$

$$\vec{P} = \int d\Gamma\ \vec{p} f_0(\vec{r}, \vec{p}, t) + \int d^3r\ [\epsilon_0 \vec{E}(\vec{r}, t) \times \vec{B}(\vec{r}, t)], \tag{3.48}$$

$$\vec{M} = \int d\Gamma\ [\vec{r} \times \vec{p} f_0(\vec{r}, \vec{p}, t) + \frac{\hbar}{2} \vec{g}_0(\vec{r}, \vec{p}, t)] + \int d^3r\ \vec{r} \times [\epsilon_0 \vec{E}(\vec{r}, t) \times \vec{B}(\vec{r}, t)], \tag{3.49}$$

where $d\Gamma$ is a phase space volume element, $d\Gamma = d^3r d^3p/(2\pi\hbar)^3$. ϵ_0 and μ_0 is the electric permittivity and magnetic permeability. It is straightforward to prove that those quantities are constants of motion. Note that equations (3.46-3.49) give further motivation for the interpretation of the distributions $f_0, f_3, \vec{g}_0, \vec{g}_3$ presented above.

C. Closed Time Path Method: Schwinger-Keldysh formalism

Why do we need this formalism? Think about a quantum field theory. One major object is to calculate the transition matrix element,

$$i\mathcal{M} = < out|T\phi(x)\phi(y)|in > . \tag{3.50}$$

We know how to handle this expression perturbatively. Now consider statistical problem to obtain expectation,

$$< \psi, t_0|T\phi(x)\phi(y)|\psi, t_0 > . \tag{3.51}$$

namely the quantum state is at same time, which is huge difference from standard formalism. One way to go around is to define a new time path and see the time t_0 different; the time t_0 on the right hand side is starting and goes to + infinity and comes back to the time t_0 on the left side. We can use all the machinery of the standard quantum field theory. But one problem, which is good and bad, is that the time order of field operators is much more: actually 4 of them. See the figures: However all of them have physical meanings. See the further information in Phys. Rep. 118, 1 (1985) by G. Zhou, Z. Su, B. Hao and L. Yu.