Non-Invertible Symmetries

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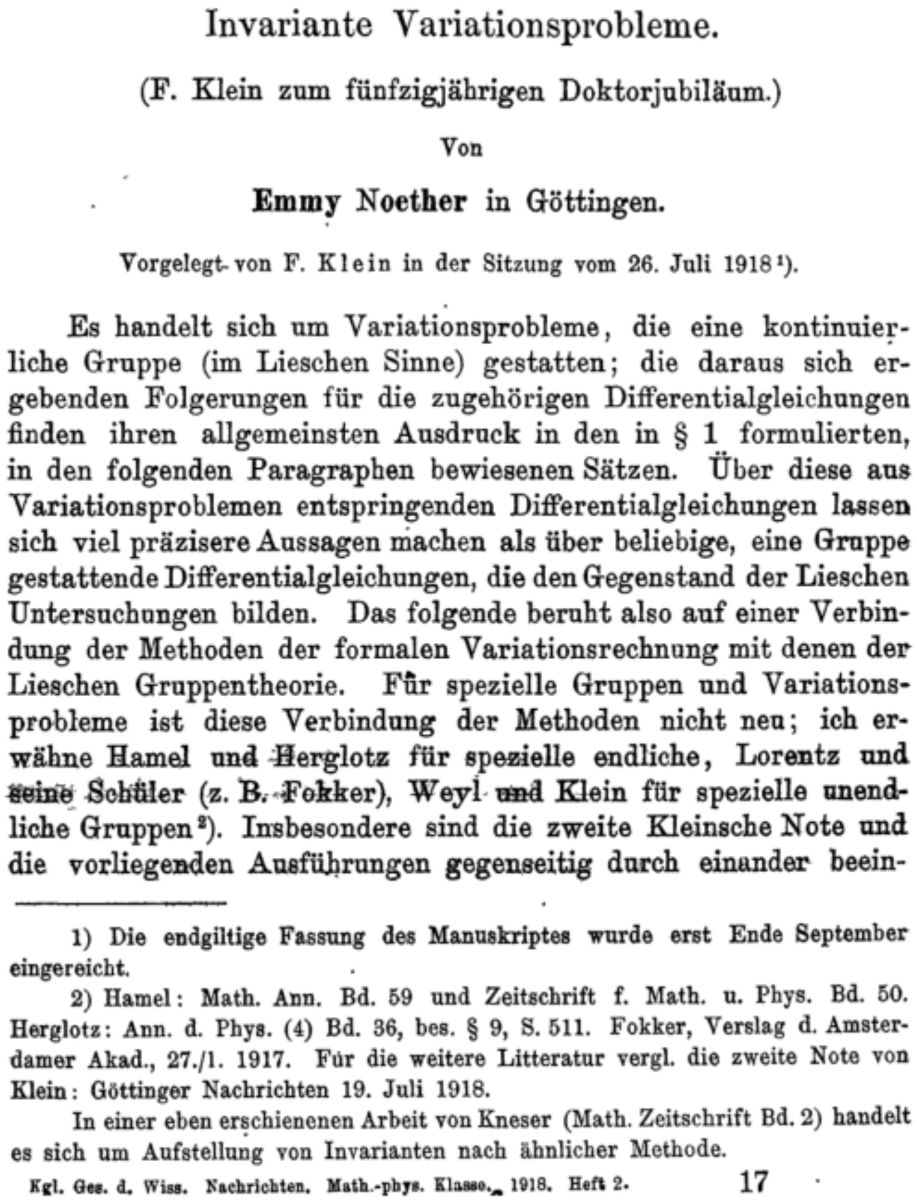
Symmetries from Topological Operators

1918: This is a long way from Noether's continuous "Lieschen" type symmetries, though the core idea is the same:



Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

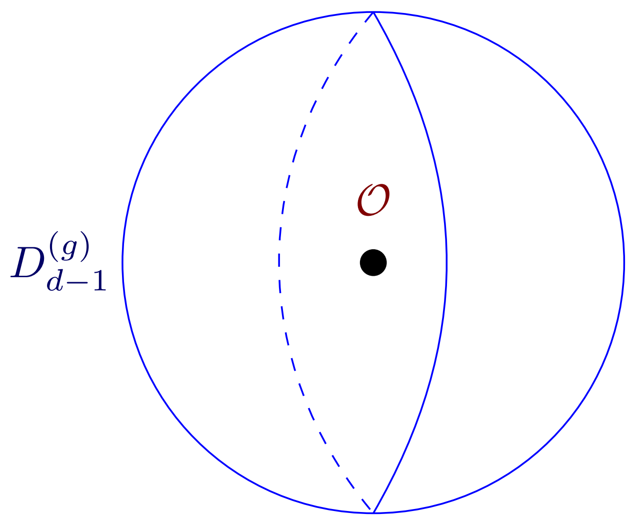
Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2. 17



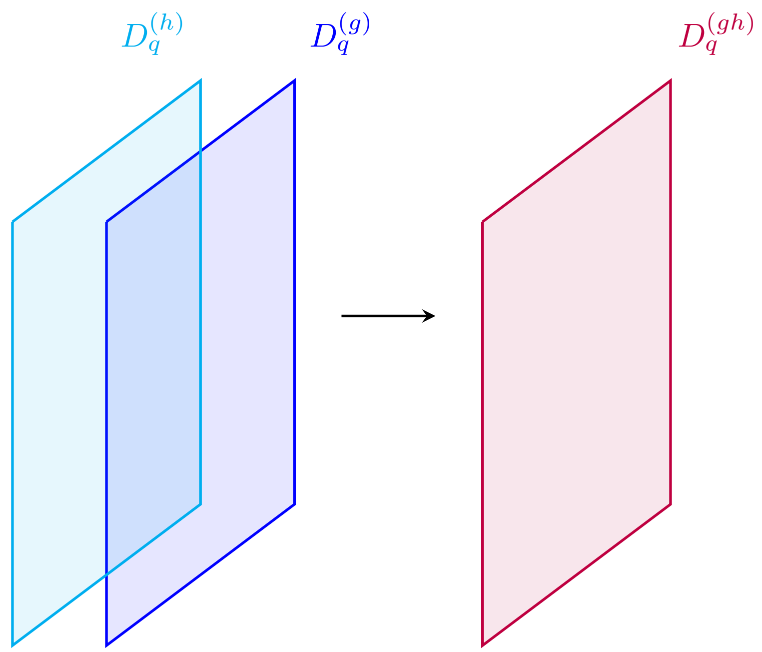
2023: The symmetries are topological operators in a QFT.

Generalized Global Symmetries

Motto: Any topological operator in a QFT is a symmetry generator.

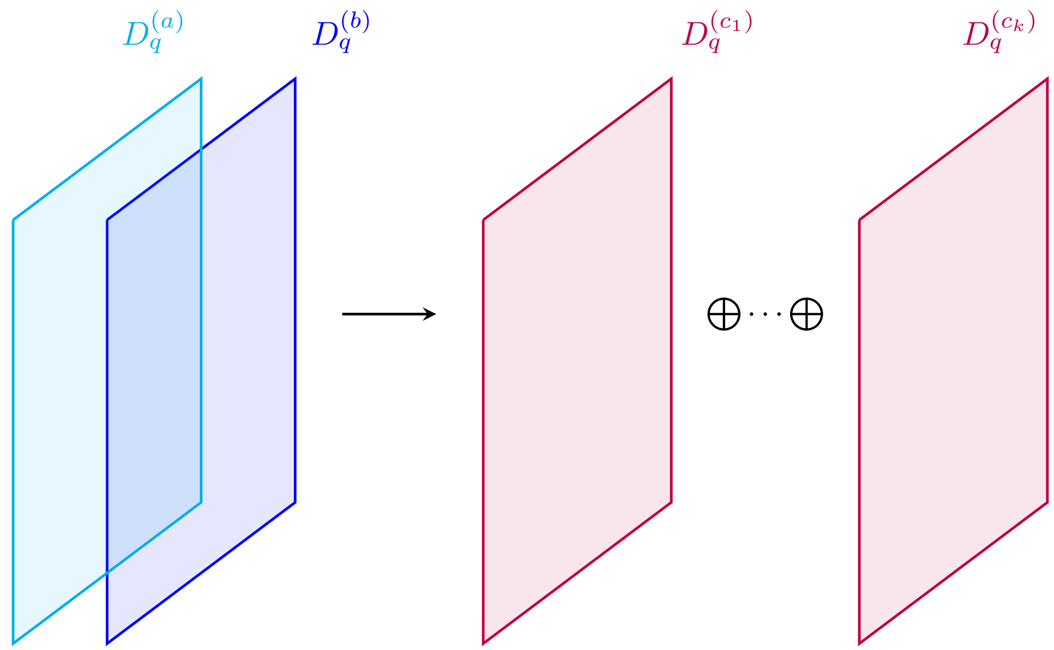
1. Higher-form symmetries $\Gamma^{(p)}$: [Gaiotto, Kapustin, Seiberg, Willett, 2014]
   charged objects are p-dimensional defects, whose charge is measured by codimension $p+1$ topological operators $D^{g}_{q=d-(p+1)}$, $g \in \Gamma^{(p)}$:

$$D_q^g \otimes D_q^h \; = \; D_q^{gh} \, , \qquad g, h \in \Gamma^{(p)}$$

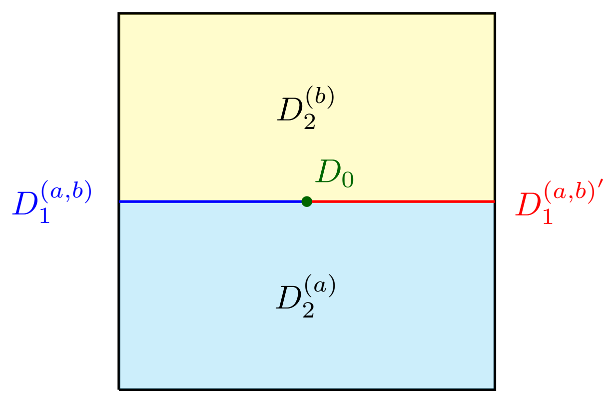


2. Higher-group symmetries:
   $\{p$-form symmetries$\}$ might not form product groups

3. Non-invertible symmetries:
group $\Rightarrow$ algebra



4. Higher-categorical symmetries: topological operators of dimensions $0, \cdots, d-1$, with non-invertible fusion:



Generalized Symmetry "Revolution"

Generalized Global Symmetries

Davide Gaiotto (Perimeter Inst. Theor. Phys.), Anton Kapustin (Stony Brook U.), Nathan Seiberg (Princeton, Inst. Advanced Study), Brian Willett (Princeton, Inst. Advanced Study)

Dec 16, 2014

77 pages
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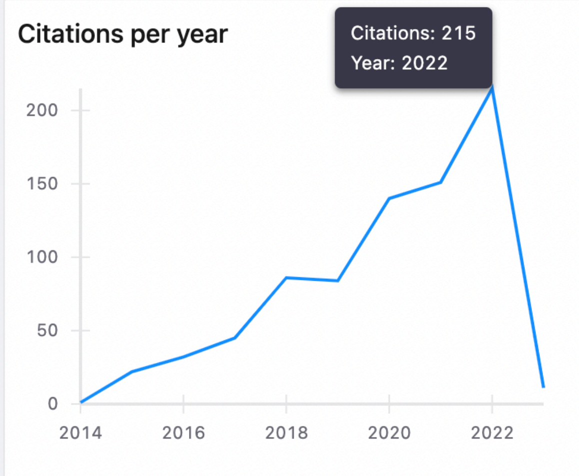


Abstract: (Springer)
A q-form global symmetry is a global symmetry for which the charged operators are of space-time dimension q, e.g. Wilson lines, surface defects, etc., and the charged excitations have q spatial dimensions, e.g. strings, membranes, etc. Many of the properties of ordinary global symmetries (q = 0) apply here. They lead to Ward identities and hence to selection rules on amplitudes. Such global symmetries can be coupled to classical background fields and they can be gauged by summing over these classical fields. These generalized global symmetries can be spontaneously broken (either completely or to a sub-group). They can also have 't Hooft anomalies, which prevent us from gauging them, but lead to 't Hooft anomaly matching conditions. Such anomalies can also lead to anomaly inflow on various defects and exotic Symmetry Protected Topological phases. Our analysis of these symmetries gives a new unified perspective of many known phenomena and uncovers new results.

Note: 49 pages plus appendices. v2: references added

Plan

- Lightning review of generalized symmetries
- Constructions of non-invertible symmetries
- Physical applications of non-invertible symmetries

Based on work in collaboration

- 2208.05973 with Lakshya Bhardwaj, Jingxiang Wu
- 2204.06564, 2212.06159, 2212.06842
  with Lakshya Bhardwaj, +-+ Lea Bottini, Apoorv Tiwari
- 2208.07373 with Fabio Apruzzi, Ibou Bah, Federico Bonetti
- Work in progress with **Lakshya Bhardwaj**

1. Higher-form symmetries $\Gamma^{(p)}$

p-dimensional extended operators/defects, carry charge by linking with topological operators of codimension $p+1$:

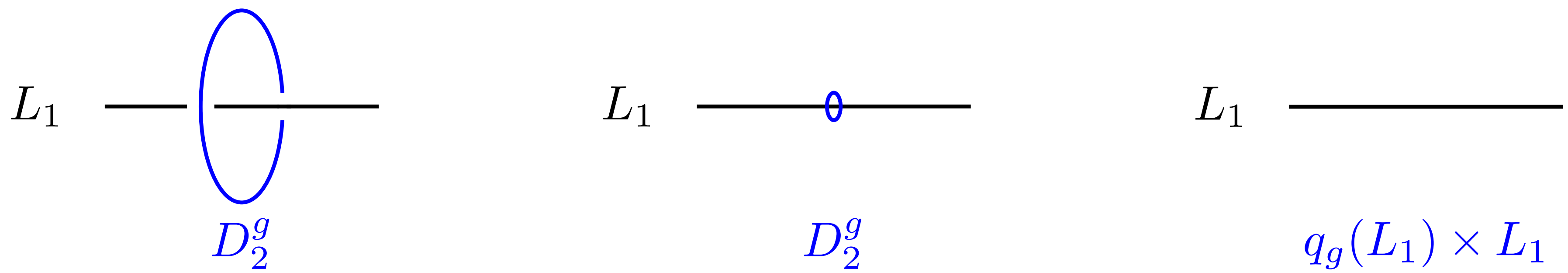
$$D^g_{q=d-(p+1)}\,, \qquad g \in \Gamma^{(p)}$$

which form the p-form symmetry group [Gaiotto, Kapustin, Seiberg, Willett, 2014]

$$D^g_q \otimes D^h_q = D^{gh}_q\,, \qquad g, h \in \Gamma^{(p)}$$

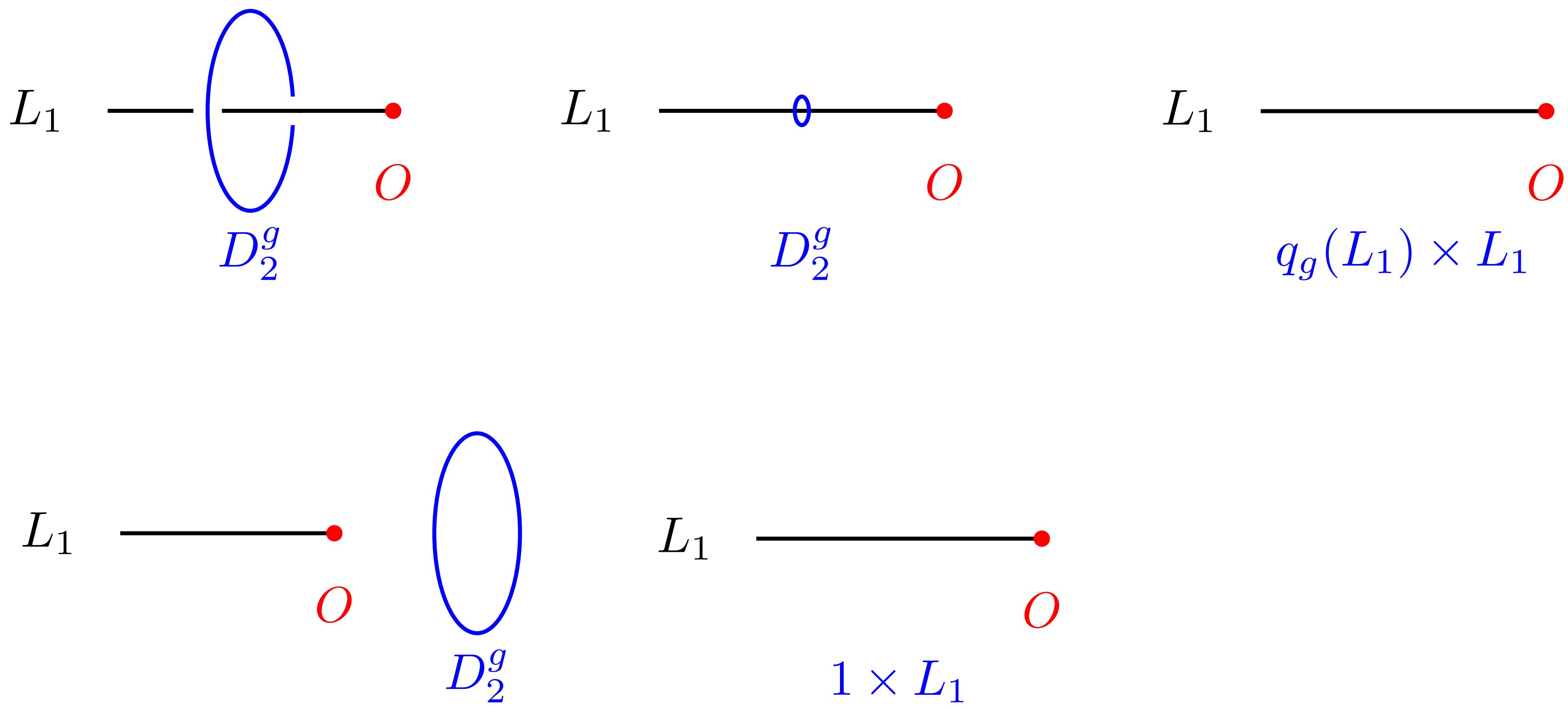
<u>Example: Line operators in 4d:</u>

$p=1$ acting on line operators in 4d: D^g_2 generate the 1-form symmetry:



Screening of Higher-Form Symmetries

1-form symmetries can be screened by local operators: ″endable″



Higher-Form Symmetries

Examples:

1. Maxwell theory: $G = U(1)$

$$D_2^\alpha = e^{i\alpha \int_{M_2} \star F_2}$$

   generate a $U(1)^{(1)}$. Line operators are charged under these.
   Adding matter with charge N, breaks the 1-form symmetry to $\mathbb{Z}_N^{(1)}$.

2. Pure Yang Mills with gauge group G:

$$\Gamma^{(1)} = Z_G = \text{Center}(G)$$

   Screening by matter, depends on charge of reps under center.

3. Confinement: $\Gamma^{(1)}$ provides an order parameter for confinement.
   E.g. for $G = SU(N)$: Wilson lines are charged under the center $\Gamma^{(1)} = \mathbb{Z}_N$.
   Confining vacua (area law) preserve this 1-form symmetry.

2. Higher-Group Symmetries

Higher-form symmetries might not form product groups, e.g. $\Gamma^{(1)} \times \Gamma^{(0)}$, but a group extension. [Sharpe][Tachikawa][Benini, Cordova, Hsin....]

Basic Group Theory:

There are two finite groups of order 4:

$$\text{Klein} = \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad \text{and} \qquad \mathbb{Z}_4 \,.$$

$\mathbb{Z}_4$ can be thought of as a non-trivial extension:

$$1 \to \mathbb{Z}_2^A \to \mathbb{Z}_4 \to \mathbb{Z}_2^B \to 1$$

Extensions are classified by the group cohomology $H^2(\mathbb{Z}_2^B, \mathbb{Z}_2^A) = \mathbb{Z}_2$.

2-group symmetry:

$\mathbb{Z}_2^A$ be $\Gamma^{(1)}$, and $\mathbb{Z}_2^B$ part of a the 0-form flavor symmetry F.

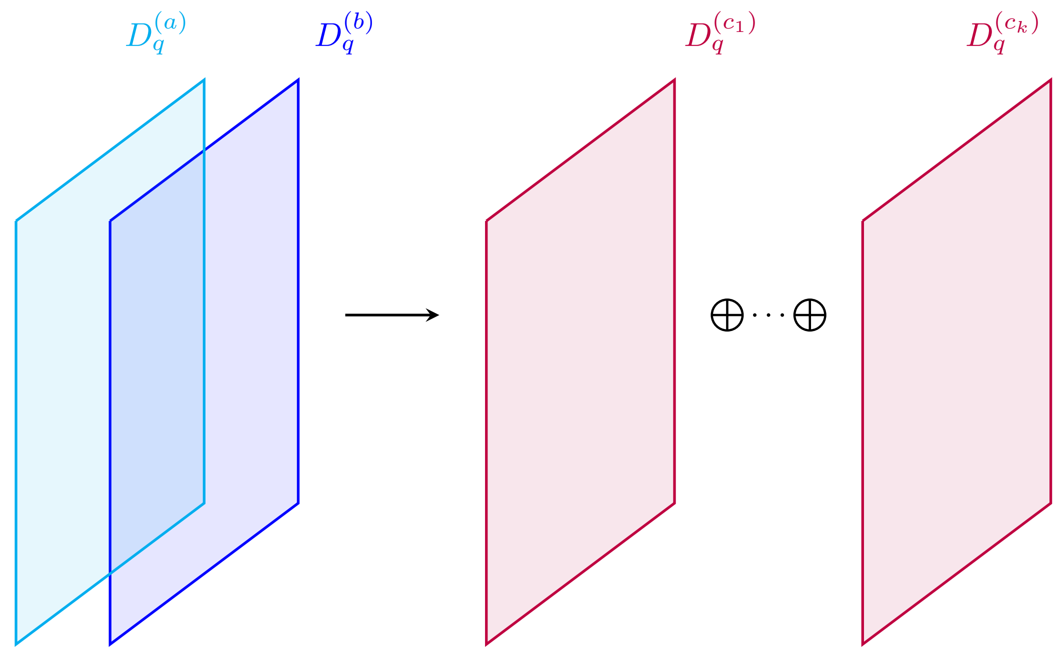
Examples:

by now loads of examples in any dimensions (e.g. Spin$(4N+2)$ gauge groups with matter).

3. Non-Invertible Symmetries

So far the topological defects had group-like composition (in particular there was an inverse to each generator).

Non-invertible symmetries:
group $\Rightarrow$ algebra



Perhaps surprisingly:
These are ubiquitous in higher dimensional QFTs, e.g. 4d pure Yang Mills.

Science-sociological bonus:
Provides a really exciting connection between hep-th, hep-ph, cond-mat, and math.

Examples:

2d: Verlinde lines (tological lines) in a 2d rational conformal field theory (RCFT). They fuse according to

$$D_1^i \otimes D_1^j = \bigoplus_k N_k^{ij} D_1^k$$

N_{ij}^k = RCFT fusion coefficients obtained by the Verlinde formula

3d: Classification of topological order: modular tensor categories

4d: 4d Yang-Mills with disconnected gauge groups:
e.g. gauging outer automorphisms or charge conjugation:
(i) $O(2) = U(1) \rtimes \mathbb{Z}_2$
(ii) Spin$(4N)$: outer automorphism exchanges the spinor and cospinor Wilson lines. Gauging this results in a non-invertible symmetry in the Pin$^+(4N)$ theory.

Non-invertible Symmetries in $d > 3$:

In the context of QFTs in $d > 3$ within the last year

[Heidenreich, McNamara, Monteiro, Reece, Rudelius, Valenzuela]

[Koide, Nagoya, Yamaguchi]

[Kaidi, Ohmori, Zheng]

[Choi, Cordova, Hsin, Lam, Shao]

[Bhardwaj, Bottini, SSN, Tiwari]^3

[Roumpedakis, Seifnashri, Shao]

[Antinucci, Galati, Rizi]

[Choi, Cordova, Hsin, Lam, Shao]

[Kaidi, Zafrir, Zheng]

[Choi, Lam, Shao]

[Cordova, Ohmori]

[Bhardwaj, SSN, Wu]

[Bartsch, Bullimore, Ferrari, Pearson]

[Cordova, Hong, Koren, Ohmori]

. . .

Global Symmetries and their Utility

Standard 0-form symmetries G in QFTs there are numerous applications:

- UV Constraints:
  Symmetry constrains the spectrum of local operators, which have to form representations of G
- IR Constraints:
  # Spontaneous symmetry breaking (SSB)
  # Goldstone's theorem
  # 't Hooft anomaly matching
  # Classification of phases of matter (Ginzburg Landau theory)

Higher-Form Global Symmetries and their Utility

Similarly: Higher-form (or higher-group) symmetries $\Gamma^{(p)}$ have UV and IR imprints:

- UV Constraints:
  - # Constraints on the spectrum of p-dimensional extended operators: representations of $\Gamma^{(p)}$, e.g. Wilson lines in $SU(N)$ are charged under $\mathbb{Z}_N$ center.
  - # Constraints on which extended operators can end, e.g. local operators at ends of line operators.
- IR Constraints:
  - Spontaneous symmetry breaking (SSB): constraints on vacuum structure through higher-form symmetry SSB
    Example: SSB of $\Gamma^{(1)}$ corresponds to deconfinement (Wilson loops have perimeter law)
  - 't Hooft anomalies: mixed and pure p-form symmetry 't Hooft anomaly matching:
    Example: mixed chiral symmetry-1-form symmetry anomaly $A_1 B_2^2$ (more of that later)

Non-Invertible Symmetries and their Utility

Non-invertible symmetries = topological defects (of any dimension $< d$), with a not necessarily group-like fusion.

1. UV Implications: What generalizes representations?
   Higher-representations [Bhardwaj, SSN - to appear].

2. IR implications: Many applications – but clearly only scratching the surface:

   (a) 2d Ising model: Kramers-Wannier duality defects.

   (b) 2d Constraints on gapped phases (more later)

   (c) Confinement/Deconfinement: in 4d QFT and holography constrained by non-invertible defects in $\mathcal{N} = 1$ pure Yang-Mills (more later)

   (d) Applications to hep-ph: e.g. neutrino mass generation from non-invertible symmetry breaking see e.g. Clay Cordova's talk in the `Symmetry Seminar, Feb 14, 2023`.

Constructions of Non-Invertible Symmetries in $d > 2$

Unified perspective in [Bhardwaj, SSN, Tiwari 12/22].

Let me illustrate this with two concepts that are relatively easy to understand:

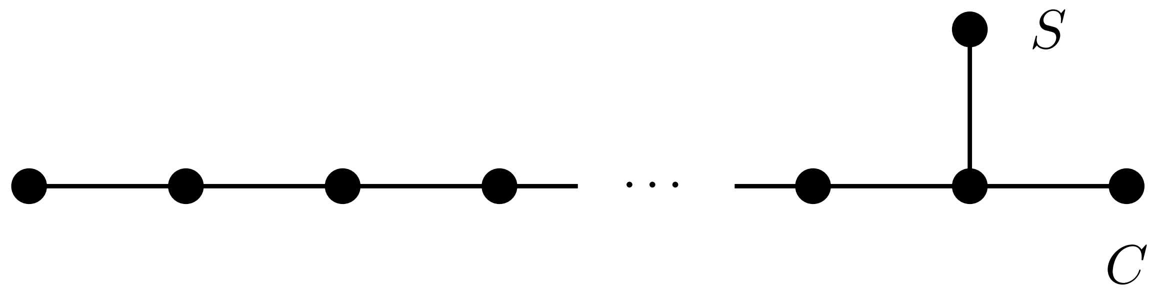
1. Yang-Mills with disconnected gauge group: Outer automorphism gauging
2. Theta-Defects: generalize the notion of a theta angle
3. Holographic/Stringy Realizations

Gauging Outer Automorphisms

Consider an outer-automorphism 0-form symmetry acting on a pure gauge theory.

Examples.
$U(1)$ Maxwell [Heidenreich, McNamara, Montero,Reece, Rudelius, Valenzuela]
Outer automorphism of Spin$(4N)$ gauge groups [Bhardwaj, Bottini, SSN, Tiwari]

Spin$(4N)$:



4d Spin$(4N)$ Yang-Mills, and the outer automorphism $G^{(0)} = \mathbb{Z}_2^{(0)}$ that exchanges the two factors of the 1-form symmetry

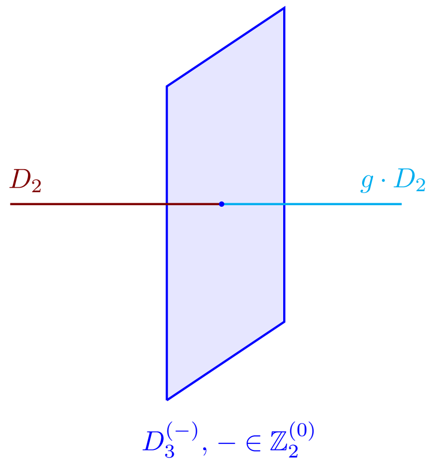
$$\text{Center}(\text{Spin}(4N)) = \Gamma^{(1)} = \mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(C)}$$

Generated by topological surface operators:

$$D_2^{(S)},\ D_2^{(C)}, \quad D_2^{(V)} = D_2^{(S)} \otimes D_2^{(C)}, \qquad D_2^{(i)} \otimes D_2^{(i)} = D_2^{(\text{id})}$$

What is the symmetry of the theory after gauging this outer automorphism?

The 0-form symmetry exchanges two one-form symmetry generators:



Maxwell: $D_2^{(\alpha)} \leftrightarrow D_2^{(-\alpha)}$
Spin$(4N)$: $D_2^{(S)} \leftrightarrow D_2^{(C)}$

After gauging the outer automorphism we get to $O(2)$ or $\mathrm{Pin}^+(4N)$, which has non-invertible fusion. The invariant combination is

$$D_2^{(\mathrm{inv})} = D_2 \oplus g \cdot D_2$$

with fusion (disclaimer: this is a simplified analysis)

$$\begin{aligned} O(2): \qquad & D_2^{(\mathrm{inv})} \otimes D_2^{(\mathrm{inv})} = D_2^{(\mathrm{id})} \oplus D_2^{(2\alpha)} \\ \mathrm{Pin}^+(4N): \qquad & D_2^{(\mathrm{inv})} \otimes D_2^{(\mathrm{inv})} = D_2^{(\mathrm{id})} \oplus D_2^{(V)} \end{aligned}$$

Non-Invertible Symmetries from Theta-Defects

Lets start with 4d Maxwell:

$$\mathcal{L}_{U(1)} = \frac{1}{g^2} \int F \wedge \star F + \theta \int F \wedge F$$

We can think of this theory as follows:
Consider a 4d trivial theory with a trivial $U(1)$ global symmetry, background A, but a symmetry protected phase (SPT)

$$\mathcal{L}_T = \mathcal{L}_{\text{trivial}} + \text{SPT}\,, \qquad \text{SPT} = \theta \int F \wedge F$$

Gauging $U(1)$ we obtain Maxwell and the SPT becomes the theta-angle

$$\mathcal{L}_{T/U(1)} = \frac{1}{g^2} \int F \wedge \star F + \theta \int F \wedge F$$

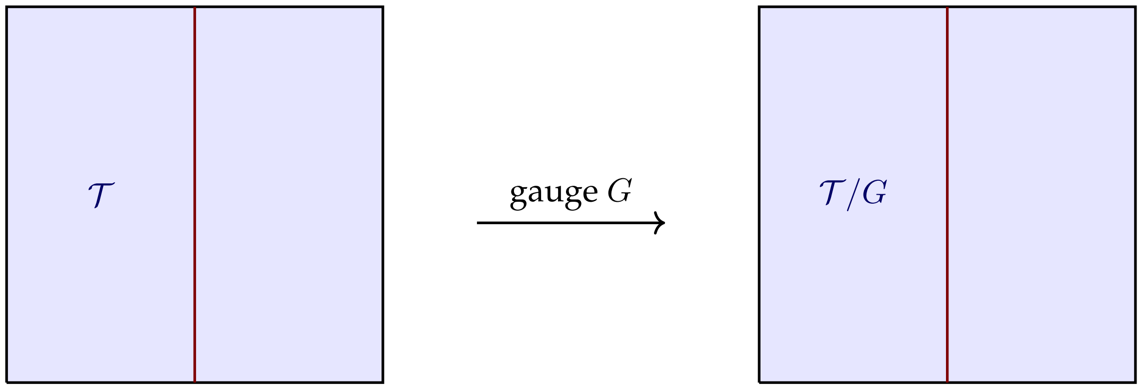
This can be generalized to any theory with $U(1)$ global symmetry that can be gauged:

Stacking with $U(1)$-SPT and gauging adds a θ-angle.

Theta Defects

[Bhardwaj, SSN, Wu][Bhwardwaj, SSN, Tiwari]

Let $\mathcal{T}$ be a d-dim QFT with a $G^{(0)}$ symmetry.
Consider a G-symmetric p-dimensional TQFT. Gauge the diagonal $G^{(0)}$:



p-dim G-TQFT

Topological line defect D_p

In the gauged theory, the TQFT is now a topological defect of the theory.
Generically: the fusion of these defects is non-invertible.

Example: Dual Symmetry in 2d

Let $\mathcal{T}$ be a 2d theory with a 0-form symmetry G, generated by topological lines $D_1^{(g)}$, which fuse according to the group multiplication in G

$$D_1^{(g)}\,,\; g \in G\,, \qquad D_1^{(g)} \otimes D_1^{(h)} = D_1^{(gh)}$$

Gauging G means introducing a dynamical G gauge field:

- There is a dynamical G gauge field a and Wilson lines in G-representations $\mathbf{R}$

$$D_1^{(\mathbf{R})} = \mathrm{Tr}_{\mathbf{R}}\, e^{\int a}$$

- These Wilson lines fuse according to the representations of G, $\mathrm{Rep}(G)$:

$$D_1^{(\mathbf{R}_1)} \otimes D_1^{(\mathbf{R}_2)} = \bigoplus_{\mathbf{R}_3} N_{\mathbf{R}_3}^{\mathbf{R}_1 \mathbf{R}_2} D_1^{(\mathbf{R}_3)}$$

For abelian groups $\mathrm{Rep}(G)$ is encoded in the characters i.e. the Pontryagin dual group

$$\mathrm{Rep}(\mathbb{Z}_N) \equiv \mathrm{Hom}(\mathbb{Z}_N, U(1)) \cong \mathbb{Z}_N$$

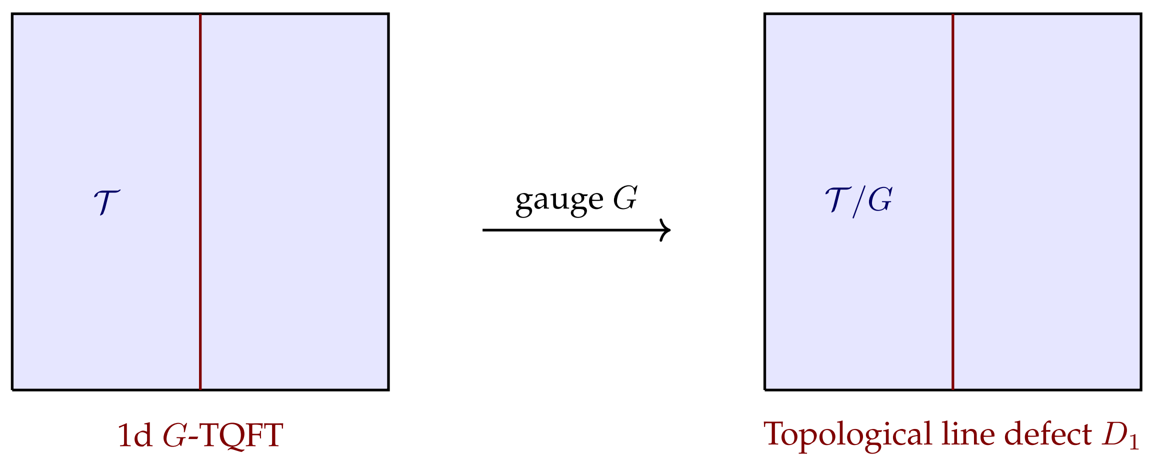
NB: this is an instance of a "categorical symmetry".

Complementary perspective: Theta-Defects

We propose a more physical perspective on gauging – which naturally generalizes to higher-dimensions.

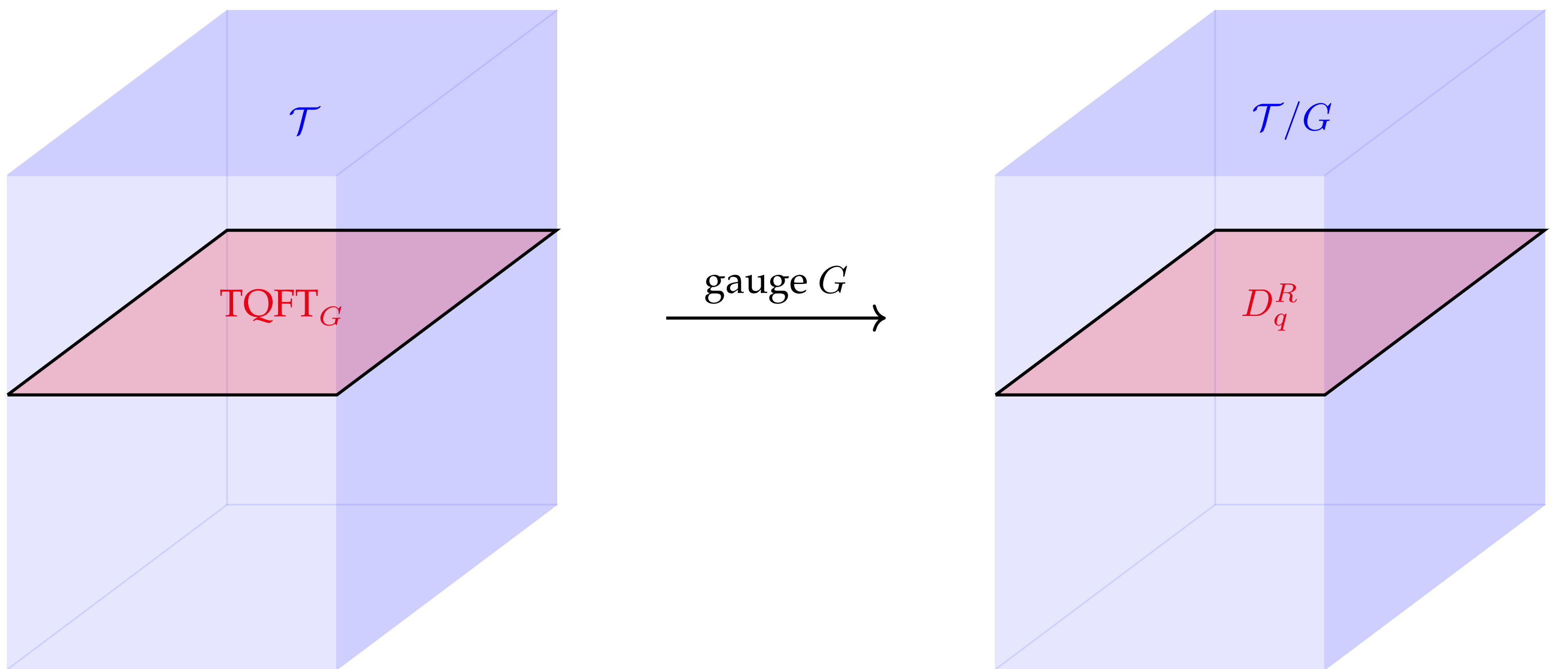
Consider a 2d theory $\mathcal{T}$, finite 0-form symmetry G.

Stacking a 1d TQFT with G-symmetry, and gauging the diagonal G results in topological lines $D_1^{(\mathbf{R})}$, $\mathbf{R}$ rep of G in the gauged theory:



1d G-TQFTs:
Characterized by the number of vacua and G action on them, i.e. a G-representation. They form a subset $\mathrm{Rep}(G)$ of the symmetry of $\mathcal{T}/G$.

Theta-Defects: higher dimensions



2d G-TQFTs to 2d Defects

The first non-trivial application is for surface defects, in $d \geq 3$.
What are 2d G-TQFTs?

- Spontaneous Symmetry Breaking to a subgroup H in G.
- 2d H-SPTs, which are classified by

$$\alpha \in H^2(H, U(1))$$

$G = \mathbb{Z}_2$ then $H = 1$ or $\mathbb{Z}_2$ and α trivial.
Objects:

- $D_2^{(H=1)} \equiv D_2^{(-)}$: TQFT with two vacua $|\pm\rangle$, which is a non-trivial defect.
- $D_2^{(H=\mathbb{Z}_2)} \equiv D_2^{(\mathrm{id})}$: TQFT with 1 vacuum $|0\rangle$, trivial defect (identity).

$$D_2^{(-)} \otimes D_2^{(-)} = 2D_2^{(-)}$$

Extensions and Generalizations

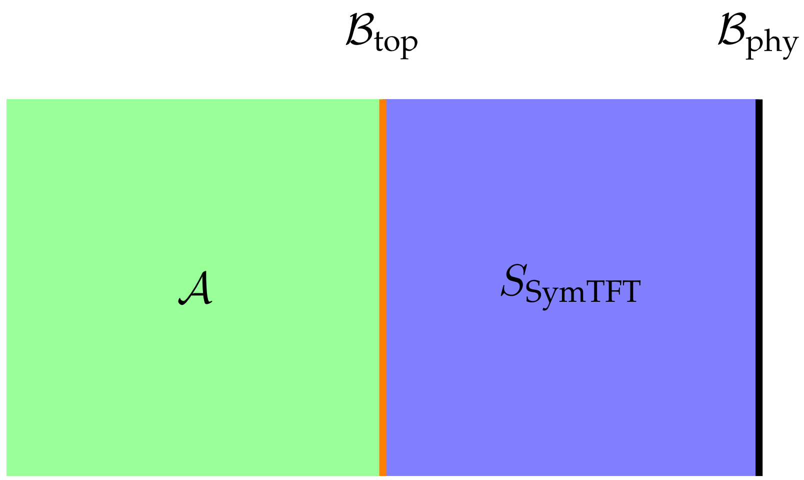
(a) Generalized gauging beyond stacking SPTs and SSB

$\Rightarrow$ 3d G-TQFTs can be modular tensor categories [Moore, Seiberg] i.e. may not necessarily have gapped boundaries. Ex: 4d $\mathcal{N}=1$ SYM

(b) Symmetry TFT

[Gaiotto, Kulp][Apruzzi, Bonetti, Garcia-Extebarria, Hosseini, SSN] [Freed, Moore, Teleman]

Different global forms correspond to different b.c. on the so-called Symmetry TFT, which is a $(d+1)$dim TQFT that admits gapped boundary conditions:



$\mathcal{A}$ is the anomaly theory; $\mathcal{T}$ is obtained after collapsing the interval. B.c. result in different "global forms".

Naturally realized in holography. [Apruzzi, Bah, Bonetti, SSN]

Physical Implications of Non-Invertible Symmetries Part 1: Constraining IR in 2d

[Zamolodchikov][Chan, Lin, Shao, Wang, Yin][Bhardwaj, SSN, wip].

Start with a 2d rational CFT, This has a symmetry given by the Verlinde lines

$$\mathcal{C} = \{D_1^a\,,\ a = \text{chiral primaries}\}$$

with fusion determined by the fusion coefficients of the RCFT.

Turn on a relevant deformation that flows to a gapped theory (also studied using TBA by Zamolodchikov)
Denote the symmetries commuting with this deformation by $\mathcal{C}_{\text{flow}}$.

This symmetry allow 't Hooft-like anomaly matching to the IR and we can determine the number of vacua, and the action of the symmetry.

Example: IR application:
If $\mathcal{C}_{\text{flow}} \supset$ Ising= $\{1, S, P\}$ with fusion $P^2 = 1$, $PS = SP = S$, $S^2 = 1 + P$. Then the number of vacua has to be a multiple of 3.

Physical Implications of Non-Invertible Symmetries 2: De-/Confinement in 4d SYM

Beyond Stacking SPTs: Non-Invertibles in 4d $\mathcal{N}=1$ SYM.
For 3d topological defects D_3 we can also consider G-TQFTs which do not necessarily admit gapped boundary conditions.

4d $SU(M)$ pure SYM has M confining vacua, and invertible symmetries.

$\Rightarrow$ What about $PSU(M)$?

4d $\mathcal{N}=1$ $SU(M)$ SYM has global symmetries:

- Chiral Symmetry: $G^{(0)}=\mathbb{Z}_{2M}$, with background field A_1
- Center Symmetry: $\Gamma^{(1)}=\mathbb{Z}_M$, with background field B_2

These enjoy a mixed anomaly:

$$\mathcal{A}=-\frac{2\pi}{M}\int A_1\cup\frac{(B_2\cup B_2)}{2}$$

Gauging the 1-form Symmetry: $SU(M)$ to $PSU(M)$

The $PSU(M)$ SYM theory is obtained by gauging $\Gamma^{(1)}$.

The chiral symmetry generator $D_3^{(g)}$ is not invariant under background gauge field transformations in presence of background fields for $\Gamma^{(1)}$:

$$D_3^{(g)}(M_3) \to D_3^{(g)}(M_3) \exp\left(\int_{M_4} -\frac{2\pi i}{M} \frac{(B_2 \cup B_2)}{2} \right)$$

for $\partial M_4 = M_3$. Gauging the 1-form symmetry is thus not consistent.
This can be remedied by stacking the defect $D_3^{(g)}$ with a 3d TQFT that has

- 1-form symmetry
- anomaly $\mathcal{A}$

There are many such TQFTs, with boundaries that are not gapped. The minimal choice was determined in [Hsin, Lam, Seiberg]. This is precisely the construction [Kaidi, Ohmori, Zheng], (similar duality defects [Choi, Cordova, Hsin, Lam, Shao]) and an example of a (twisted) theta-defect.

Non-Invertible Symmetries in 4d SYM

Same philosophy as before: we stack the defect with a TQFT – in this case the TQFT needs to cancel the anomaly. For $\mathbb{Z}_M^{(1)}$ the minimal such TQFT is [Hsin, Lam, Seiberg].

$$\mathcal{A}^{M,1} = U(1)_M$$

(Twisted) Theta-defect is

$$\mathcal{N}_3^{(1)} = D_3^{(1)} \otimes \mathcal{A}^{M,1}$$

Non-invertible 0-form symmetry in the $PSU(M)$ theory

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)} = \mathcal{A}^{M,2} \mathcal{N}_3^{(2)}$$

Defining the conjugate $\mathcal{N}_3^{(1)\dagger} = D_3^{-1} \otimes \mathcal{A}^{M,-1}$ results in

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)\dagger} = \sum_{M_2 \in H_2(M_3, \mathbb{Z}_M)} \frac{(-1)^{Q(M_2)} D_2(M_2)}{|H^0(M_3, \mathbb{Z}_M)|}$$

which is the condensation defect of the 1-form symmetry on M_3 with $D_2(M_2) = e^{i2\pi \int_{M_2} b_2/M}$, where b_2 is the gauge field for the 1-form symmetry.

Physical Implications of Non-Invertible Symmetries 2: Holographic Dual Description of Non-Invertible Symmetries

SymTFT from Holography:

Conjectural observation: [Apruzzi, van Beest, Gould, SSN][Apruzzi, Bah, Bonetti, SSN]
The bulk topological couplings in the supergravity on AdS_{d+1} or M_{d+1} (e.g. for Klebanov Strassler) give rise to the SymTFT of the holographically dual boundary theory.

Passes the sniff test:
$\text{AdS}_5 \times S^5$ dual to 4d $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{su}(N)$ has

$$N \int_{M_5} B_2 \wedge dC_2$$

The gapped, topological b.c. are given in terms of B_2 and C_2. E.g. Dirichlet b.c. for B_2. This results in line operators (e.g. Wilson, 't Hooft). [Witten '98].

By now there is a huge amount of evidence for this:
including the construction of non-invertible symmetries in holography
[Apruzzi, Bah, Bonetti, SSN], [Gracia-Extebarria][Antinucci, Benini, Copetti, Galati, Rizzi]

Application to Holographic Confinement

[Apruzzi, Bah, Bonetti, SSN]

Holographic Dual description to 4d pure $\mathfrak{su}(M)$ SYM:
Consider the [Klebanov-Strassler] (KS) solution:

- N D3s at the conifold $C(T^{1,1})$ have a holographic dual in type IIB on $\mathrm{AdS}_5 \times T^{1,1}$, $\int F_5 = N$.
- $T^{1,1} \sim S^3 \times S^2$: wrap D5-branes on S^2, inducing $\int_{S^3} F_3 = M$
  $\Rightarrow$ breaks conformal invariance
- Dual to a cascade of Seiberg dualities, which for $N = kM$ end in pure $\mathfrak{su}(M)$ $\mathcal{N} = 1$ SYM:

$$ds^2 = \underbrace{\frac{r^2}{R^2} d\mathbf{x}^2 + \frac{R^2}{r^2} dr^2}_{M_5} + R^2 ds^2_{T^{1,1}} \,.$$

r= radial direction, RG-flow; $R(r) \sim \ln(\frac{r}{r_s})^{1/4}$, $r_s = r_0 e^{-N/gM^2 - 1/4}$.
The near horizon limit is $r \to r_0$.

The global form of gauge group is not fixed by this data alone.

SymTFT for the Klebanov-Strassler Solution

The full 5d topological action for the KS solution is, in the near horizon limit:

[Cassani, Faedo][Apruzzi, van Beest, Gould, SSN][Apruzzi, Bah, Bonetti, SSN]

$$S_{\text{SymTFT}} = 2\pi \int_{M_5} \left(\frac{1}{2} N(b_2 dc_2 - c_2 db_2) + M(A_1 dc_3 + c_3 dA_1) + N b_2 f_3^b + A_1 (g_2^b)^2 \right)$$

$\Rightarrow$ structure of BF-couplings for 1-form fields and 0-/2-form symmetries, as well as mixed anomalies.

A_1 R-symmetry background
b_2, c_2 come from H_3 and F_3
f_1^b, g_2^b, integral lifts of classes in $H^1(M, \mathbb{Z}_{2M})$ and $H^2(M, \mathbb{Z}_M)$: gauge fields for 0-form and 1-form symmetries.

Symmetry Generators from Gauss Laws

Symmetries can be extracted using the Gauss law constraints, generalizing [Belov, Moore].

- Consider radial direction as Euclidean time
- Gauss law constraints corresponds to small gauge transformations on spatial slice:

For the KS solution, one of the Gauss law constraints is

$$\mathcal{G} = 2M dc_3 + (g_2^b)^2$$

- g_2^b Dirichlet: the symmetry defect simply becomes const. $\times\, e^{2\pi i \int_{M_3} c_3}$ $\Rightarrow SU(M)$ invertible 0-form symmetry.
- g_2^b Neumann: take the Mth root by rewriting as

$$\mathcal{N}_3^{(1)}(M_3) = \int Da e^{2\pi i \int_{M_3} \left(c_3 + \frac{1}{2} M a da + a g_2^b\right)}$$

This operator has precisely the non-invertible fusion expected for $PSU(M)$ SYM.

Alternative: Symmetries from Branes

[Apruzzi, Bah, Bonetti, SSN]

Proposal: in the near horizon limit, branes inserted in a holographic setup furnish symmetry generators. Close to the boundary $r \to \infty$:

$$T_{\mathrm{Dp}} \sim r^p, \qquad p > 0$$

such that the DBI part of the action decouples, and only topological couplings from the WZ term remain.

In the KS setup: D5-branes on $S^3 \times M_3 \subset T^{1,1} \times M_4$ have topological couplings in the near horizon limit $r \to r_0 \to \infty$

$$S_{\mathrm{D5}} = 2\pi \int_{M_3} \left(c_3 + \frac{M}{2} a_1 da_1 + a_1 db_1 \right)$$

The fields are c_3, from C_6 on S^3, b_1 from C_4 on S^3, $U(1)$ gauge field a_1 on the brane.

- b_1 Neumann: $SU(M)$: the second term is a trivial DW theory
- b_1 Dirichlet: $PSU(M)$: precisely the dressing with $U(1)_M$.

Fusion from Branes

What happens when we fuse two branes?

Naively: $U(2)$ non-abelian gauge theory on the world-volume. However, in the presence of B_2 flux in the transverse S^2 gives rise to the Myers effect: D5s puff up to a D7 with $\int_{S^2} f_2 = 2$

$$2 \times \text{D5} \; + B_2 \quad \rightarrow \quad \text{D7 with flux} \int_{S^2} f_2 = 2$$

The theory on the 7-brane on $S^2 \times S^3 \times M_3$ is

$$S_{\text{D7}} = 2\pi \int_{M_3} (2c_3 + M a_1 da_1 + 2 a_1 db_1)$$

This precisely reproduces the non-invertible fusion in the field theory

$$\mathcal{N}_3^{(1)} \otimes \mathcal{N}_3^{(1)} = A^{M,2} \mathcal{N}_2^{(2)}$$

The $\mathcal{N}_3^{(1)} \otimes (\mathcal{N}_3^{(1)})^\dagger$ = condensation defect for the 1-form symmetry on M_3, is D5-$\overline{\text{D5}}$ via tachyon condensation [Sen] resulting in the condensation defect!

Disorder Operators from Hanany-Witten transition

[Apruzzi, Bah, Bonetti, SSN]

How do the non-invertible symmetry generators act on ’t Hooft lines?

- Charged line operators:
  D3s stretching along the radial direction and wrapped on $S^2 \times S^1$ give rise to ’t Hooft lines.

- Topological defects:
  D5s on $S^3 \times M_3$ generate the non-invertible codim 1 topological defects.

Brane	x_0	x_1	x_2	x_3	r	z_1	z_2	w_1	w_2	w_3
D3	X				X	X	X			
D5	X	X	X					X	X	X

Brane	x_0	x_1	x_2	x_3	r	z_1	z_2	w_1	w_2	w_3
D3	X				X	X	X			
D5	X	X	X					X	X	X

Charge conservation implies that the total linking of the branes is conserved – in particular when we exchange the position of the D3 and D5: The linking is

$$\int_{\mathbb{R}^2_{x_1,x_2}\times S^3} F_5 = -\int_{\mathbb{R}_r\times S^2} F_3$$

which evaluates to

$$\int_{\mathbb{R}^2_{x_1,x_2}} db_1 = -\int_{\mathbb{R}_r} dc_0$$

On the D5:

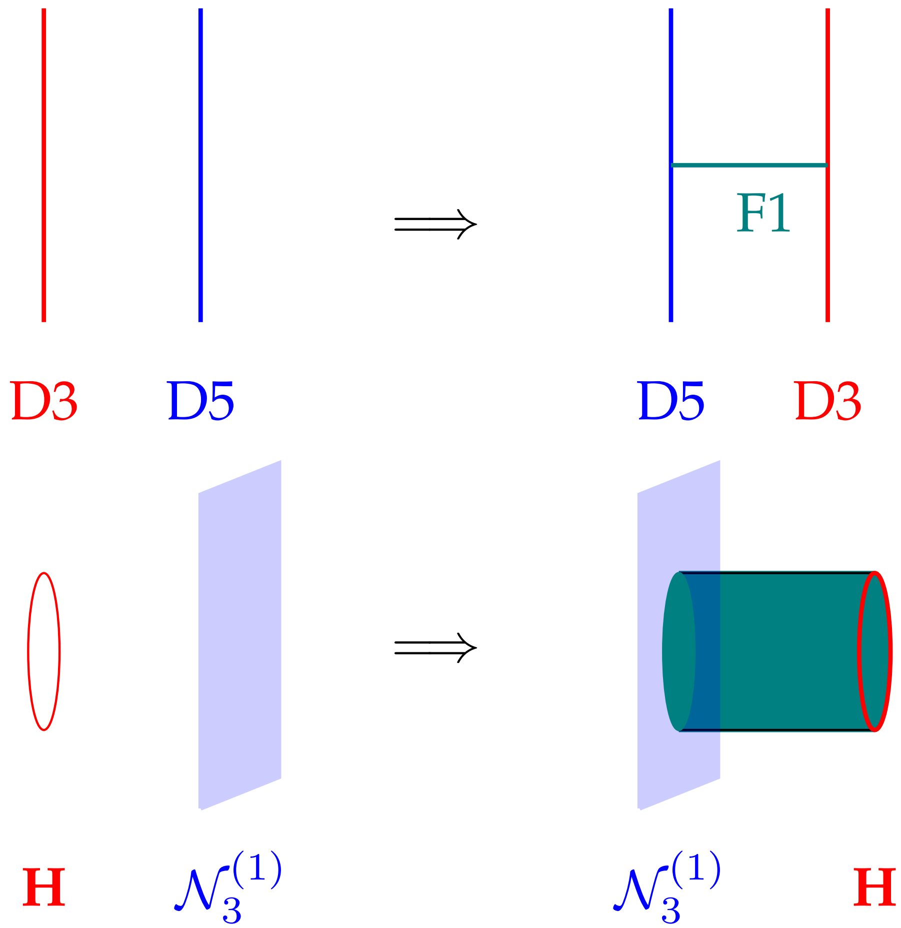
$$db_1 = -M da_1$$

As we pass the D3 through the D5:

$$db_1 = -M da_1 + \delta(p \in \mathbb{R}^2_{x_1,x_2})$$

which mean there is an additional object that intersects long x_3.

Preserving the linking requires the creation of an F1:



’t Hooft loop gets flux attachment when it crosses the non-invertible defect – similar to disorder operator in Kramers-Wannier duality.

Summary

- Generalized Symmetries are ubiquitous:
  starting from higher-form symmetries, higher-groups and now non-invertibles in QFTs in various dimensions. ($d \geq 2$).
- We are only now uncovering their fundamental implications on UV and IR physics.
- Many exciting cross-connections between these developments in the fields of hep-th, hep-ph, cond-mat, and math.
  Example:
  Symmetry TFT in addition to hep-th, was studied in cond-mat in 2d/3d theories, in holography as bulk topological couplings, and in math as math as the Drinfeld Center of a higher category. All these are deeply interconnected and tell us about the physics of non-invertible symmetries.