

Pion observables calculated in Minkowski and Euclidean spaces with *Ansätze* for quark propagators

Talk presented at **ACHT 2023 workshop**

Non-Perturbative Aspects of Nuclear, Particle and Astroparticle Physics

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Overview

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Quark Propagator

Wick rotation

General properties of quark propagators

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Summary

Introduction

Talk = based on Kekez and Klabučar, Phys. Rev. D **107**, 094025 (2023).

While paying much attention to describing the pion phenomenologically as good as possible in the chosen model framework, this is secondary. This paper is above all concerned with formulating clear connections between Euclidean and Minkowski spacetime calculations of observables of $q\bar{q}$ bound states.

- Many QFT studies (on lattice, most of Schwinger–Dyson studies, etc.) are **not done in the physical, Minkowski spacetime, but in Euclidean.**
- \Rightarrow **Relating Minkowski and Euclidean spaces ("Wick rotation") must be under control. But, this is highly nontrivial in the nonperturbative case** – most importantly, the nonperturbative QCD.
- **Do nonperturbative Green's functions permit Wick rotation?**
- **For solving Bethe-Salpeter equation and calculation of processes, extrapolation to complex momenta is necessary. \Rightarrow Knowledge of the analytic behavior in the whole complex plane is needed.**
- **Very complicated matters \Rightarrow studies of Ansatz forms are instructive and can be helpful to *ab initio* studies of nonperturbative QCD Green's functions. ... and vice versa of course ...**

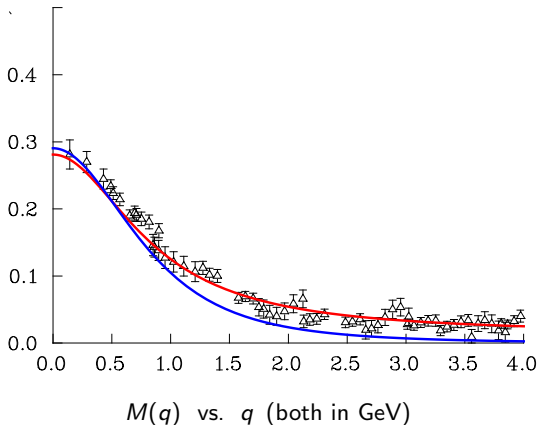
Lattice-inspired *Ansatz* for propagators in nonperturbative QCD

Lattice data: Parappilly *et al.*, Phys. Rev. D **73**, 054504 (2006)

compared with nonperturbatively dressed (DChSB-generated) constituent quark masses in the propagator *Ansätze* ('3R' and 'MMF') by

[1] Alkofer, Detmold, Fischer and Maris, Phys. Rev. D **70**, 014014 (2004).

[2] Mello, de Melo, and Frederico, *Minkowski space pion model inspired by lattice QCD running quark mass*, Phys. Lett. B **766**, 86 (2017).



How Schwinger–Dyson approach generates quark propagators

- Schwinger-Dyson (SD) approach: ranges from solving SD equations for Green's functions of non-perturbative QCD *ab initio*, to higher degrees of phenomenological modeling, *esp.* in applications including $T, \mu > 0$.

e.g., [Alkofer, v.Smekal Phys. Rept. 353 (2001) 281], and [Roberts, Schmidt Prog.Part.Nucl.Phys. 45 (2000)S1]

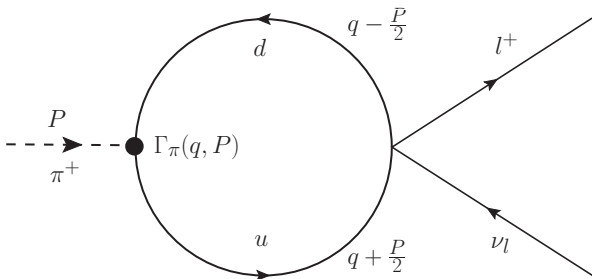
- SD approach to quark-hadron physics = nonperturbative, covariant bound state approach with strong connections with QCD.

The "gap" Schwinger–Dyson equation for the quark propagator:

$$S^{-1}(p) = \not{p} - m - iC_F g^2 \int \frac{d^4 k}{(2\pi)^4} \gamma^\mu S(k) \Gamma^\nu(k, p) G_{\mu\nu}(p - k). \quad (1)$$



Already just $S(p)$ enables calculating an observable - pion decay constant



... since **close to the chiral limit**, $\Gamma_{\pi}^{BS} \approx -\frac{2B(q^2)}{f_{\pi}} \gamma_5$ is a reasonable approximation,

$$\Rightarrow f_{\pi} \approx i \frac{N_c}{2} \frac{1}{M_{\pi}^2} \int \frac{d^4 q}{(2\pi)^4} \text{tr} \left(P \gamma_5 S(q + \frac{P}{2}) \left(-\frac{2B(q^2)}{f_{\pi}} \gamma_5 \right) S(q - \frac{P}{2}) \right)$$

Dynamically generated nonperturbative quark propagator

In principle, SD Eq. (1) yields the nonperturbatively dressed quark propagator

$$S(q) = \frac{A(q^2)\not{q} + B(q^2)}{A^2(q^2)q^2 - B^2(q^2)} = Z(-q^2) \frac{\not{q} + M(-q^2)}{q^2 - M^2(-q^2)} = -\sigma_V(-q^2)\not{q} - \sigma_S(-q^2)$$

[$M(x)$ = dressed quark mass function, $Z(x)$ = wave-function renormalization]

- One usually gets just a model solution for the propagator $S(q)$, since one usually simplifies SD Eq. (1) by approximations & modeling! For example:
 - 1.) rainbow(-ladder) for the dressed quark-gluon vertex: $\Gamma^\mu(k, p) \rightarrow \gamma^\mu$,
 - 2.) $A(k^2) = 1$,
 - 3.) various model Ansätze for the dressed gluon propagator, $g^2 G^{\mu\nu}(k) \propto \alpha_s^{\text{eff}}(k^2)/k^2$, so that, e.g., Eq. (1) yields

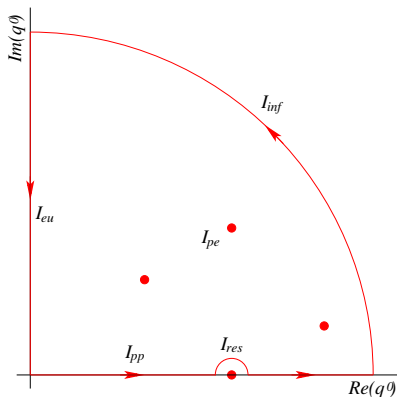
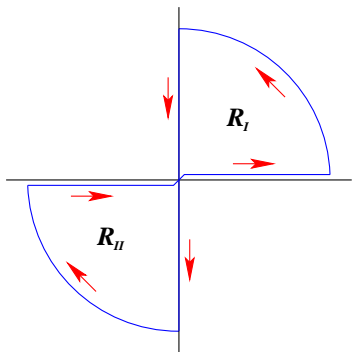
$$M(-p^2) = m + I(-p^2) = m + \int d^4k f(-k^2)g(-(k-p)^2)$$

$$g(-k^2) = 4\pi C_F \frac{\alpha_s^{\text{eff}}(-k^2)}{(-k^2)} \quad f(-k^2) = \frac{3i}{(2\pi)^4} \frac{M(k^2)}{k^2 - M^2(k^2)}$$

$$I(-p^2) = \int d^3k \int dk^0 f(-(k^0)^2 + |\mathbf{k}|^2)g(-(k^0 - p^0)^2 + |\mathbf{k} - \mathbf{p}|^2)$$

Wick rotation

- Even with such approxim. & modeling, **solving SD equations** like Eq. (1), **and related calculations** (e.g., of f_π) with Green's functions like $S(q)$, are **technically very hard to do in the physical, Minkowski space-time**.
- Thus, additional simplification is sought by transforming to QFT in 4-dim. Euclidean space by the Wick rotation to the imaginary time component:
 $q^0 \rightarrow i q^0$.



The issue of singularities is more problematic than in the perturbative case!

Example: simple separable approximation for low-E QCD

$$g^2 G^{\mu\nu} = g^{\mu\nu} D(p-k) \approx g^{\mu\nu} \left[D_0 f_0(-p^2) f_0(-k^2) - D_1(p \cdot k) f_1(-p^2) f_1(-k^2) \right]$$

produces complicated singularity structure in the complex plane $z = -k^2$

The "gap" SD equation \Rightarrow

$$A(p^2) = 1 + a f_1(-p^2)$$

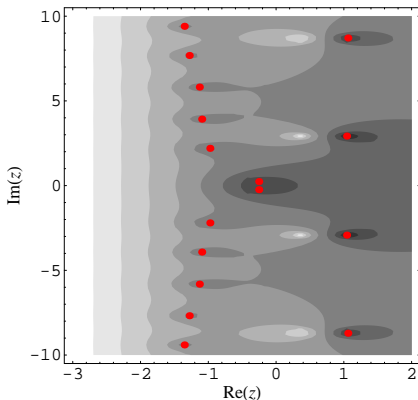
$$B(p^2) = m + b f_0(-p^2)$$

Typical phen.succes. Ansatz

$$f_0(x) = e^{-x/\Lambda_0^2}$$

$$f_1(x) = \frac{1 + e^{-x_0/\Lambda_1^2}}{1 + e^{-(x-x_0)/\Lambda_1^2}}$$

produces **poles** which are **obstacles to (any) Wick rotation!**



The contour plot of $z \mapsto \log |A(z)^2 z + B(z)^2|$ close to the origin. The red points are solutions of the equation $A(z)^2 z + B(z)^2 = 0$ and extend much further than the depicted region.

General properties a quark propagator should have:

- $S(q) \rightarrow S_{\text{free}}(q)$ because of asymptotic freedom
 $\Rightarrow \sigma_{V,S}(-q^2) \rightarrow 0$ for $q^2 \in \mathbb{C}$ and $q^2 \rightarrow \infty$
- $\sigma_{V,S}(-q^2) \rightarrow 0$ cannot be analytic over the whole complex plane
- positivity violating spectral density \leftrightarrow confinement

Try **Ansätze of the form** (meromorphic parametrizations, like Alkofer&al. [1]):

$$S(p) = \frac{1}{Z_2} \sum_{j=1}^{n_p} r_j \left(\frac{\not{p} + a_j + ib_j}{p^2 - (a_j + ib_j)^2} + \frac{\not{p} + a_j - ib_j}{p^2 - (a_j - ib_j)^2} \right)$$

Dressing f' nctns are thus $\sigma_V(x) = \frac{1}{Z_2} \sum_{j=1}^{n_p} \frac{2r_j(x + a_j^2 - b_j^2)}{(x + a_j^2 - b_j^2)^2 + 4a_j^2 b_j^2}$

(e.g., see Ref. [1]) and $\sigma_S(x) = \frac{1}{Z_2} \sum_{j=1}^{n_p} \frac{2r_j a_j (x + a_j^2 + b_j^2)}{(x + a_j^2 - b_j^2)^2 + 4a_j^2 b_j^2}$

Constraints:

$$\sum_{j=1}^{n_p} r_j = \frac{1}{2} \qquad \sum_{j=1}^{n_p} r_j a_j = 0$$

3R quark propagator - 3 poles on the real axis

Parameters (yielding 3R propagator of Alkofer & al., and larger f_π): $n_p = 3$, $a_1 = 0.341$, $a_2 = -1.31$, $a_3 = -1.35919$, $b_1 = 0$, $b_2 = 0$, $b_3 = 0$, $r_1 = 0.365$, $r_2 = 1.2$, $r_3 = -1.065$, $Z_2 = 0.982731$.

⇒ **No obstacles to Wick rotation** ⇒ π decay constant calculated equivalently

* in Minkowski space:

$$f_\pi^2 = -i \frac{N_c}{4\pi^3 M_\pi^2} \int_0^\infty \xi^2 d\xi \int_{-\infty}^{+\infty} dq^0 B(q^2) \text{tr} \left(\not{P} \gamma_5 S(q + \frac{P}{2}) \gamma_5 S(q - \frac{P}{2}) \right)$$

where $q^2 = (q^0)^2 - \xi^2$, $\xi = |\mathbf{q}|$, and $q \cdot P = M_\pi q^0$, **OR**

* in Euclidean space:

$$f_\pi^2 = \frac{3}{8\pi^3 M_\pi^2} \int_0^\infty dx x \int_0^\pi d\beta \sin^2 \beta B(q^2) \text{tr} \left(\not{P} \gamma_5 S(q + \frac{P}{2}) \gamma_5 S(q - \frac{P}{2}) \right)$$

where $q^2 = -x$ and $q \cdot P = -iM_\pi \sqrt{x} \cos \beta$.

3R quark propagator Ansatz

Numerically, for $\xi = 0.5$

$$|I_{pp} + I_{res} + I_{inf} + I_{eu} - I_{pe}| \sim 10^{-8}$$

Particular integrals for $\xi = 0.5$

$$I_{pp} \approx 4 \cdot 10^{-12},$$

$$I_{res} = -0.0221314i$$

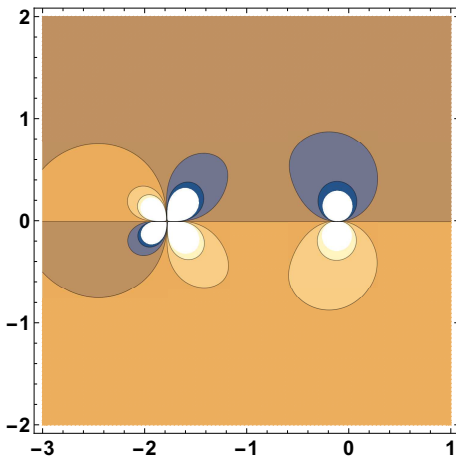
$$I_{inf} \approx 4 \cdot 10^{-12},$$

$$I_{eu} = 0.0221314i$$

$$I_{pe} = 0$$

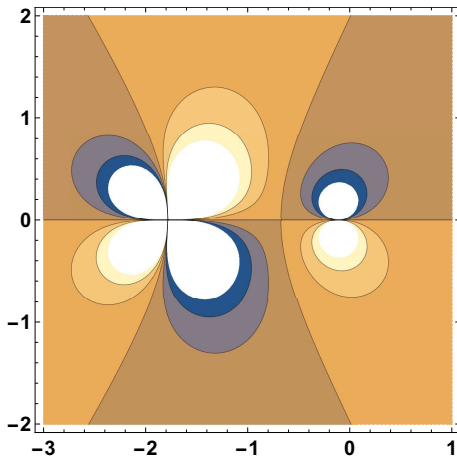
$\Rightarrow f_{\pi} = 0.072$ GeV in **both Euclidean and Minkowski space.**

3R Quark Propagator



Contour plot of $\text{Im}(\sigma_V(z))$ in the complex z -plane:
the first two poles on the real axis are very close - "glued together" .

3R Quark Propagator



A similar contour plot of $\text{Im}(\sigma_S(z))$ in the complex z -plane.

3R Quark Propagator

Spectral representation of the 3R quark propagator is defined by the spectral density

$$\rho(\sigma^2) = \sum_{j=1}^3 A_j^{-1} \delta(\sigma^2 - M_j^2)$$

where $A_1 = 1.35$, $A_2 = 0.41$, $A_3 = -0.46$, $M_j = B_j/A_j$, $j = 1, 2, 3$.

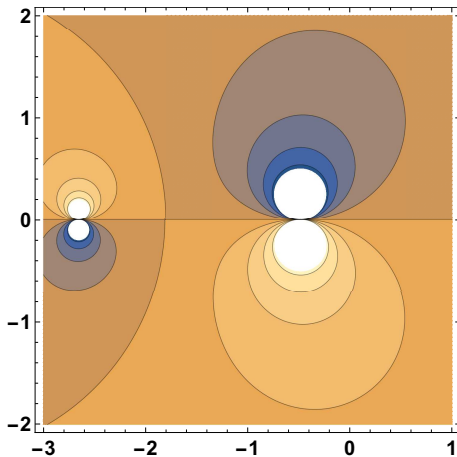
Although the propagator functions A and B from $S^{-1}(q) = A(-q^2)q\!\!\!/ - B(-q^2)$ exhibit different structure of the poles,

$$A(x) = \frac{\sigma_V(x)}{\sigma_S^2(x) + x \sigma_V^2(x)}$$

$$B(x) = \frac{\sigma_S(x)}{\sigma_S^2(x) + x \sigma_V^2(x)}$$

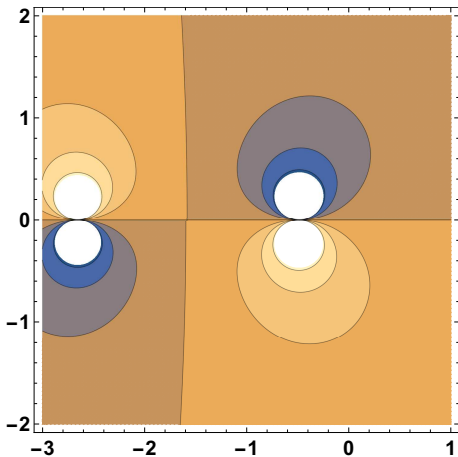
the poles are still on the real axis:

3R Quark Propagator



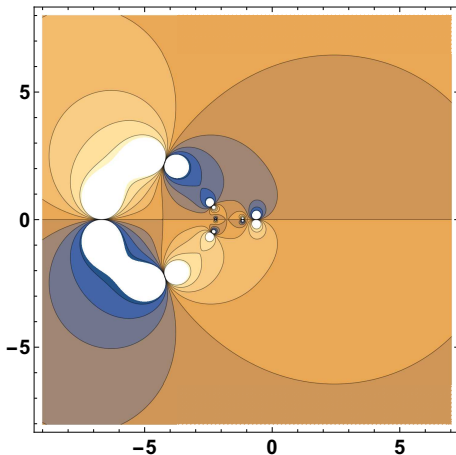
Contour plot of $\text{Im}(A(z))$ in the complex z -plane: all poles are still on the real axis.

Similarly with B in the 3R Quark Propagator:



Contour plot of $\text{Im}(B(z))$ in the complex z -plane: all poles are still on the real axis.

But a modest change of parameters spoils 3R Quark Propagator:



Contour plot of $\text{Im}(B(z))$ in the complex z -plane. Parameters changed to $a_1 = -1.31 \rightarrow a_1 = -2 \Rightarrow$ complicated analytic structure, some poles of $B(z)$ are now moved off the real axis. Wick rotation does not go any more!

Quark propagator model of Mello, de Melo, and Frederico - MMF

[2] Mello, de Melo, and Frederico, Phys. Lett. **B766**, 66 (2017)

$$M(x) = (m_0 - i\varepsilon) + m^3 \left[x + \lambda^2 - i\varepsilon \right]^{-1}$$

$$Z(x) = 1$$

Model parameters: $m_0 = 0.014$ GeV, $m = 0.574$ GeV, and $\lambda = 0.846$ GeV
Asymptotic expansions about ∞ and 0:

$$M(x) = m_0 + \frac{m^3}{x} - \frac{\lambda^2 m^3}{x^2} + \mathcal{O}\left(\left(\frac{1}{x}\right)^3\right), \quad \text{for } x \rightarrow \infty,$$

$$M(x) = \left(m_0 + \frac{m^3}{\lambda^2} \right) - \frac{m^3 x}{\lambda^4} + \frac{m^3 x^2}{\lambda^6} + \mathcal{O}(x^3), \quad \text{for } x \rightarrow 0,$$

Quark propagator model of Mello, de Melo, and Frederico - MMF

The quark dressing functions σ_V and σ_S :

$$\sigma_V(x) = \sum_{j=1}^3 \frac{b_{Vj}}{x + a_j}$$

$$\sigma_S(x) = \sum_{j=1}^3 \frac{b_{Sj}}{x + a_j}$$

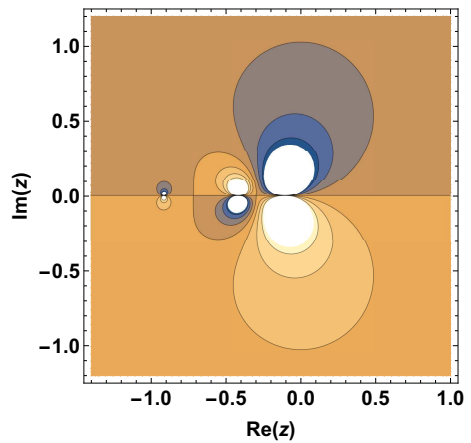
The mass parameters:

$$a_1 = 0.1046 \text{ GeV}^2$$

$$a_2 = 0.4160 \text{ GeV}^2$$

$$a_3 = 0.9110 \text{ GeV}^2$$

Quark propagator model of Mello, Melo, and Frederico - MMF



Contour plot of $\text{Im}(\sigma_V(z))$ in the complex z -plane.

Pion Decay Constant in the MMF model for the quark propagator

$$f_\pi = i \frac{N_c}{2} \frac{1}{M_\pi^2} \int \frac{d^4 q}{(2\pi)^4} \text{tr} \left(\not{P} \gamma_5 S(q + \frac{P}{2}) \left(-\frac{2B(-q^2)}{f_\pi} \gamma_5 \right) S(q - \frac{P}{2}) \right)$$

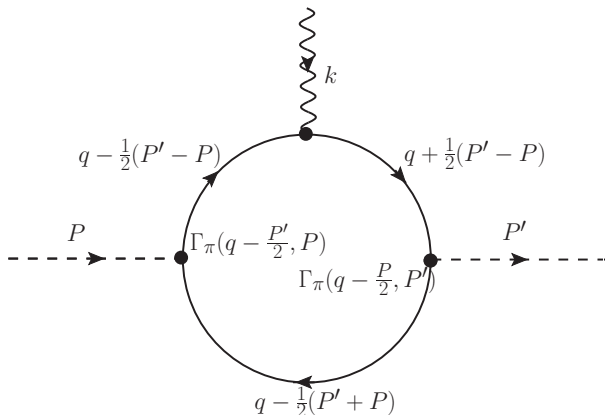
(1) Calculation using FeynCalc [?, ?] and Package-X [?, ?] (or LoopTools [?])

$$f_\pi = 87.5599 \text{ MeV}$$

(2) Euclidean integration

- “naïve” Wick rotation ($q^0 \rightarrow -iq^0$) is correct here
- two nontrivial integration (red variables)
 $q = (q^0, \xi \sin(\vartheta) \cos(\varphi), \xi \sin(\vartheta) \sin(\varphi), \xi \cos(\vartheta))$
- numerical integration using Mathematica
- the same result for f_π

Electromagnetic form factor



Generalized impulse approximation to the charged pion electromagnetic form factor $F_\pi(Q^2)$.

Electromagnetic form factor

$$\begin{aligned} \langle \pi^+(P') | J^\mu(0) | \pi^+(P) \rangle &= (P^\mu + P'^\mu) F_\pi(Q^2) = i(Q_u - Q_d) \frac{N_c}{2} \int \frac{d^4 q}{(2\pi)^4} \times \\ &\times \text{tr} \left\{ \bar{\Gamma}(q - \frac{P}{2}, P') S(q + \frac{1}{2}(P' - P)) \Gamma^\mu(q + \frac{1}{2}(P' - P), q - \frac{1}{2}(P' - P)) \right. \\ &\quad \left. \times S(q - \frac{1}{2}(P' - P)) \Gamma(q - \frac{1}{2}P', P) S(q - \frac{1}{2}(P + P')) \right\} \end{aligned}$$

- $\Gamma^\mu(p', p)$ – dressed quark γ vertex, modeled by Ball-Chiu vertex
- The proper perturbative QCD asymptotics:

$$F_\pi(Q^2) = 16\pi \frac{\alpha_s(Q^2)}{Q^2} f_\pi^2 \propto \frac{1}{Q^2 \ln(Q^2)} \quad \text{for } Q^2 \rightarrow \infty$$

cannot be expected with the *present* Ansätze, but presently available experimental data are anyway well above the pQCD predictions.

* In the case of the MMF quark propagator, we used three different methods of calculation. All three yielded the same results.

Various calculation methods employed by Kekez and Klabučar, Phys. Rev. D **107**, 094025 (2023)

- (1) Calculation using FeynCalc [?, ?] and Package-X [?, ?]
- (2) “Euclidean” integration
 - 3 nontrivial integrations (over q^0 , ξ , ϑ)
- (3) Minkowski space integration utilizing light-cone momenta

Dressed quark–quark–photon vertex $\Gamma^\mu(p', p)$

Quark–quark–photon vertex: Ball–Chiu vertex

$$\Gamma^\mu(p', p) = \frac{1}{2} [A(-p'^2) + A(-p^2)] \gamma^\mu + \frac{(p' + p)^\mu}{(p'^2 - p^2)} \left\{ [A(-p'^2) - A(-p^2)] \frac{(p' + p)^\mu}{2} - [B(-p'^2) - B(-p^2)] \right\}$$

Ward–Takahashi identity: $(p' - p)_\mu \Gamma^\mu(p', p) = S^{-1}(p') - S^{-1}(p)$

For the model of Mello, Melo & Frederico, $\Gamma^\mu(p', p) = \gamma^\mu - \frac{m^3(p'^\mu + p^\mu)}{(p'^2 - \lambda^2)(p^2 - \lambda^2)}$

Electromagnetic form factor

Poles of the integrand:

$$(q_0)_{1,2} = \mp \sqrt{M_q^2 + \xi^2 - \xi \sqrt{Q^2} \cos \vartheta + Q^2/4}$$

$$(q_0)_{3,4} = \mp \sqrt{M_q^2 + \xi^2 + \xi \sqrt{Q^2} \cos \vartheta + Q^2/4}$$

$$(q_0)_{5,6} = \frac{1}{2} \left(\sqrt{4M_\pi^2 + Q^2} \mp 2\sqrt{M_q^2 + \xi^2} \right)$$

$$(q_0)_{7,8} = \frac{1}{4} \left(\sqrt{4M_\pi^2 + Q^2} \mp \sqrt{16M_q^2 + 16\xi^2 + 8\xi \sqrt{Q^2} \cos \vartheta + Q^2} \right)$$

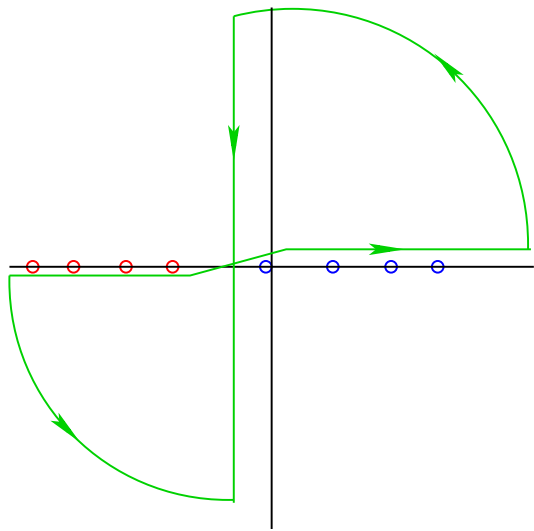
$$(q_0)_{9,10} = \frac{1}{4} \left(\sqrt{4M_\pi^2 + Q^2} \mp \sqrt{16M_q^2 + 16\xi^2 - 8\xi \sqrt{Q^2} \cos \vartheta + Q^2} \right)$$

$$M_q^2 \in \{a_1, a_2, a_3, \lambda^2\}$$

Electromagnetic form factor

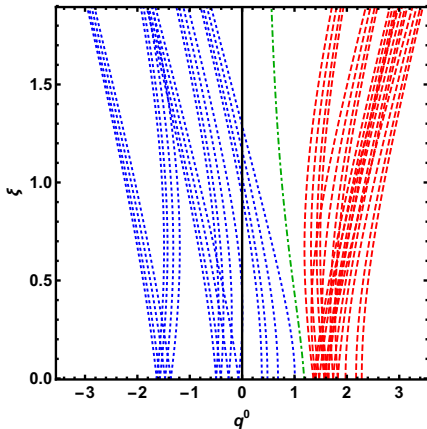
Loop integration: q^0 complex plane Wick rotation:

$$q^0 = (q^0)_c - iq_4, \quad -\infty < q_4 < +\infty$$



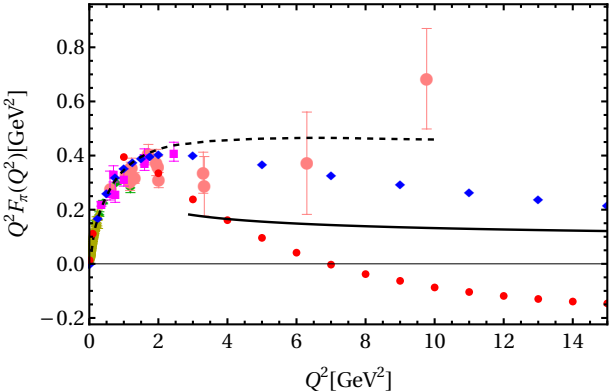
Electromagnetic form factor

$$\vartheta = \frac{\pi}{3}$$



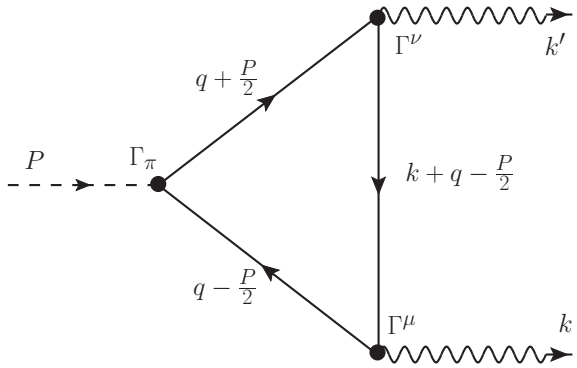
All in units of GeV. Dot-dashed green line represents $(q^0)_c$, blue dotted lines represent odd-indexed poles (set \mathcal{A}), and red dashed lines represents even-indexed poles (set \mathcal{B}).

Electromagnetic form factor - data and calculations compared



Charged π EM form factor $\times Q^2$. Experimental points are shown by olive triangles [102], green diamonds [103], pink circles [104-106], and magenta squares [107-110]. Solid **red circles** and **blue diamonds** are our results from 3R and MMF *Ansätze*, respectively. The **black dashed line** is the result of Mello et al. [19]. The **black solid line** corresponds to the perturbative QCD result with asymptotic PDA.

Transition form factor for flavorless pseudoscalar mesons



- Diagram for $\pi^0 \rightarrow \gamma\gamma$ decay, and for the $\gamma^*\pi^0 \rightarrow \gamma$ process if $k'^2 \neq 0$
- ... also for η and η' , but even just π^0 is challenging enough for now

Transition form factor

$$S_{fi} = (2\pi)^4 \delta^{(4)}(P - k - k') e^2 \varepsilon^{\alpha\beta\mu\nu} \varepsilon_\mu^*(k, \lambda) \varepsilon_\nu^*(k', \lambda') T_{\alpha\beta}(k^2, k'^2)$$

$$T^{\mu\nu}(k, k') = \varepsilon^{\alpha\beta\mu\nu} k_\alpha k'_\beta T(k^2, k'^2)$$

$$= -N_c \frac{Q_u^2 - Q_d^2}{2} \int \frac{d^4 q}{(2\pi)^4} \text{tr} \left\{ \Gamma^\mu \left(q - \frac{P}{2}, k + q - \frac{P}{2} \right) S \left(k + q - \frac{P}{2} \right) \right.$$

$$\times \Gamma^\nu \left(k + q - \frac{P}{2}, q + \frac{P}{2} \right) S \left(q + \frac{P}{2} \right) \left(-\frac{2B(q^2)}{f_\pi} \gamma_5 \right) S \left(q - \frac{P}{2} \right) \left. \right\}$$

$$+ (k \leftrightarrow k', \mu \leftrightarrow \nu) .$$

The π^0 transition form factor: $F_{\pi\gamma}(Q^2) = |T(-Q^2, 0)|$

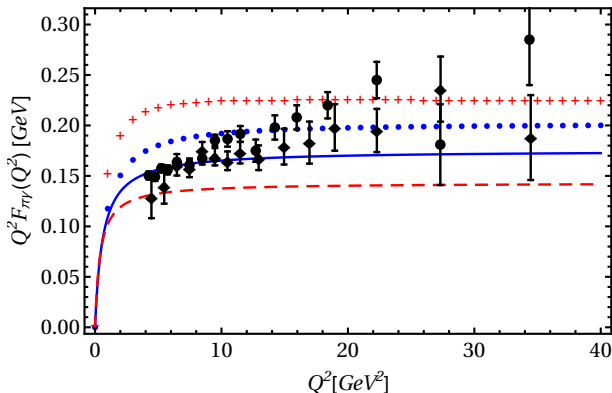
* **UV limit: asymptotically**, $F_{\pi\gamma}(Q^2) \rightarrow 2f_\pi/Q^2$ for $Q^2 \rightarrow \infty$

(Caveat: persistent nonperturb. effects by Eichmann & *al*, PLB 774 (2017) 425)

* The current data (up to 35 GeV²) do not show agreement with this limit yet.

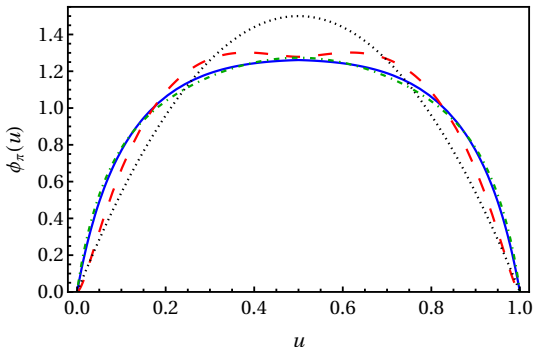
In the chiral limit, the π^0 decay amplitude to two real photons: $T(0, 0) = \frac{1}{4\pi f_\pi}$

Transition form factor



Blue dots represent π^0 transition form factor from MMF quark-propagator Ansatz. Red pluses are calculated from 3R Ansatz. The blue solid line and red dashed line represent the Brodsky-Lepage interpolation formula with f_π from MMF and 3R quark-propagator models, respectively. Solid circles and diamonds (with error bars) are the data points of BABAR [116] and Belle [117] Collaborations, respectively.

Pion distribution amplitude



Blue and red lines = obtained from MMF and 3R *Ansätze*, respectively. Black dotted line represents the asymptotic form, $\phi_\pi^{\text{as}}(u) = 6u(1-u)$. Dash-dotted green line (very close to the solid blue one) is the PDA from the state-of-the-art SDE pion bound state, Eq. (22) in *Roberts, Symmetry 12 (2020) 9, 1468*.

$$\phi_\pi(u) = i \frac{N_c}{8\pi f_\pi} \text{tr} \left(\gamma_+ \gamma_5 \int \frac{dq_-}{2\pi} \int \frac{d^2 q_\perp}{(2\pi)^2} S(q + \frac{P}{2}) \Gamma_\pi(q, P) S(q - \frac{P}{2}) \right)$$

Pion distribution amplitude

- For high values of Q^2 :

$$F_{\pi\gamma}(Q^2) = \frac{2f_\pi}{3Q^2} \int_0^1 \frac{du \phi_\pi(u)}{1-u}$$

- The asymptotic form of the pion distribution amplitude gives

$$\int_0^1 \frac{du \phi_\pi^{\text{as}}(u)}{1-u} = 3$$

- This actual ϕ_π gives

$$\int_0^1 \frac{du \phi_\pi(u)}{1-u} = 3.07$$

- Asymptotic form of the transition form factor:

$$F_{\pi\gamma}(Q^2) \sim \frac{2f_\pi}{Q^2}$$

Summary

- The analytic structure of quark propagators and some observables has been investigated in the nonperturbative regime of QCD for two Ansätze/propagator models which permitted us formulating a clear connections between Euclidean and Minkowski spacetime calculations for all observables mentioned below.
- The propagator model of Mello, de Melo, and Frederico [2] was found relatively successful phenomenologically. Already they calculated pion decay constant and electromagnetic form factor $F_\pi(Q^2)$ of the charged pion. However, we found their result increasingly inaccurate for $Q^2 > 2 \text{ GeV}^2$, since their $Q^2 F_\pi(Q^2)$ was not falling noticeably, but remained almost a constant. We checked the correctness of our $Q^2 F_\pi(Q^2)$ by redoing the calculations in three independent ways.
- Our present work (PRD **107** (2023) 094025) also found empirically reasonable values, for both Ansätze MMF and 3R, of the pion charge radius r_π and the slope parameter a of the transition form factor for small timelike momenta.
- We also calculated, for both considered quark propagator models, the $\pi^0 \gamma$ transition form factor $F_{\pi\gamma}(Q^2)$ up to 40 GeV^2 .
- Our pion distribution amplitudes, especially the one from MMF Ansatz, are close to PDA obtained from state-of-the-art SDE pion bound state.



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