Correcting Complex Langevin in full QCD

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UNI GRAZ

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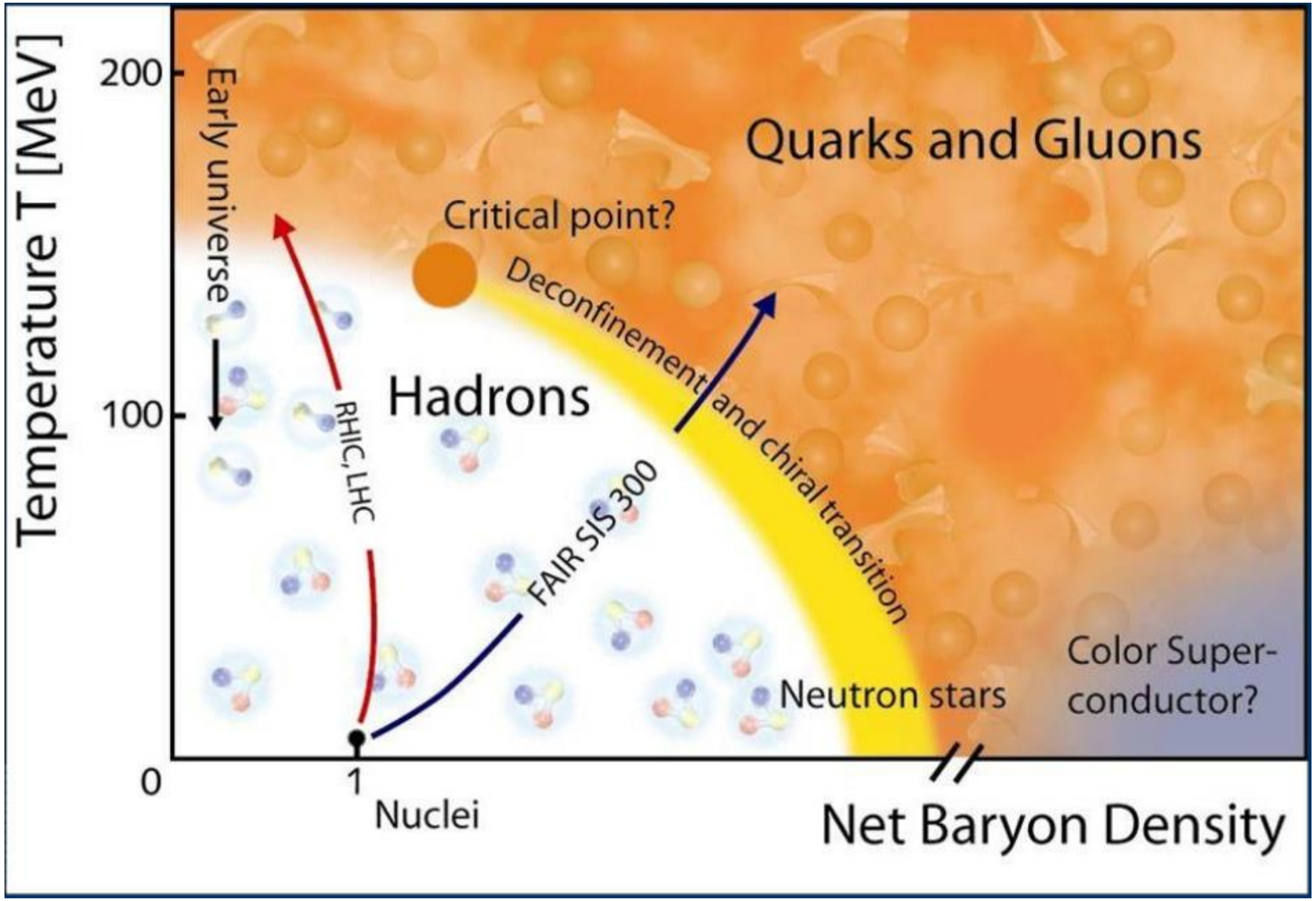
- Motivation
- What is the CL-approach
- Intro to boundary-terms
- Results
- Dynamical stabilization

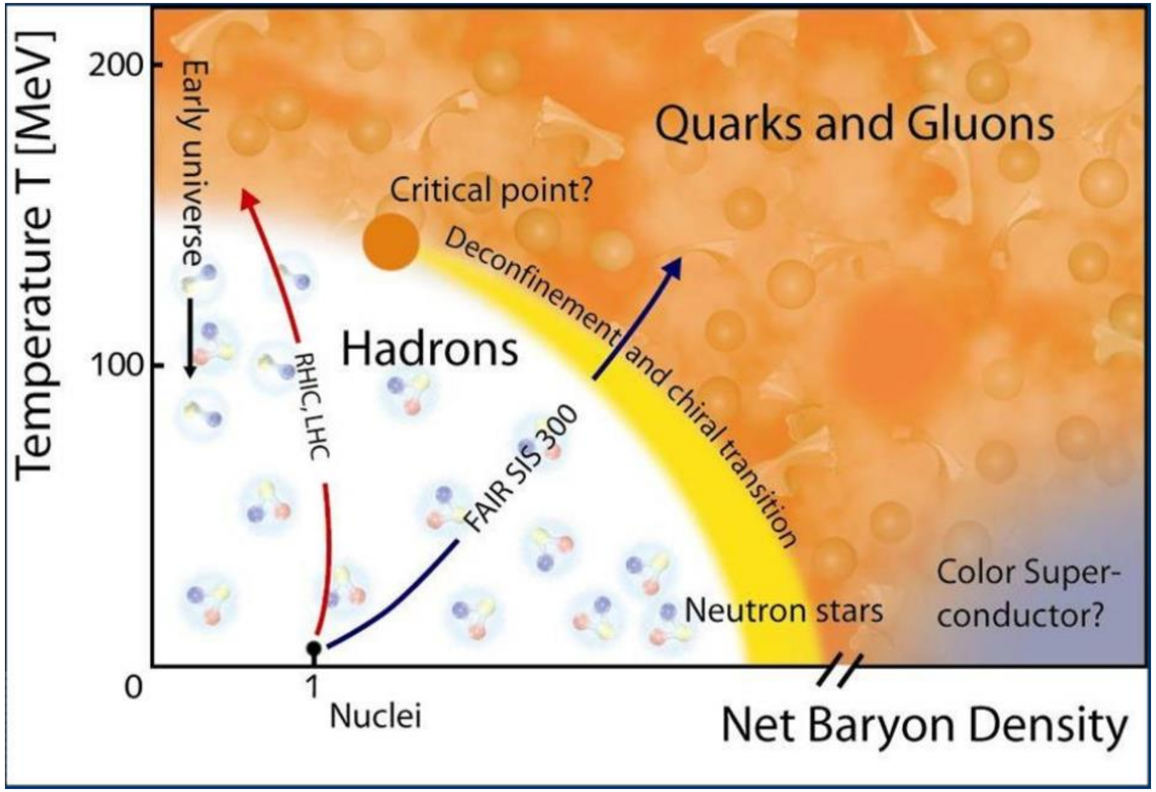
Motivation

- Investigate compressibility of nuclear matter, and existence of critical point
- Sign-problem
- Difficult HMC calculations for large chemical potential
- Reweighting (Determinant costs $O(N_s^9)$)
- Taylor expansion (Limited convergence radius)



Sign-problem

- Non-zero chemical potential
- Complex action => complex valued probability densities?
- This breaks importance sampling
  - Metropolis-Hastings, Heatbath….
- Models with these problems
  - XY-model, SU(3) spin model, QCD



Complex Langevin

- Complex action => Sign problem
- Using stochastic equation
  instead of importance sampling.
- With the correct configuration space

$$\rho(\phi) \propto \exp(-S[\phi])$$

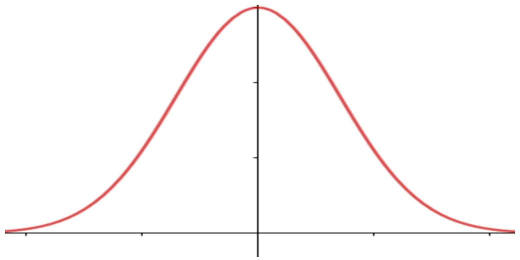
- Complex Langevin equation (CLE)

$$\Re(d\phi) = \Re(K)dt + d\omega, \qquad \Im(d\phi) = \Im(K)dt$$

$$K = -\frac{d}{d\phi}S[\phi]$$

$$L_c = (\partial_z + K_z)\partial_z$$

$$\partial_t P(\phi, t) = L_c P(\phi, t)$$



Complex Langevin Operator

- $L_C^T = \partial_x(\partial_x - K_x)$
- Solves the Fokker-Planck Equation (FPE)

$$L_C \rho(x) = \partial_t \rho(x)$$

Since $\rho(x) \propto \exp(-S[x])$, makes both sides of FPE equal to 0, due to the choice of the drift

Boundary terms

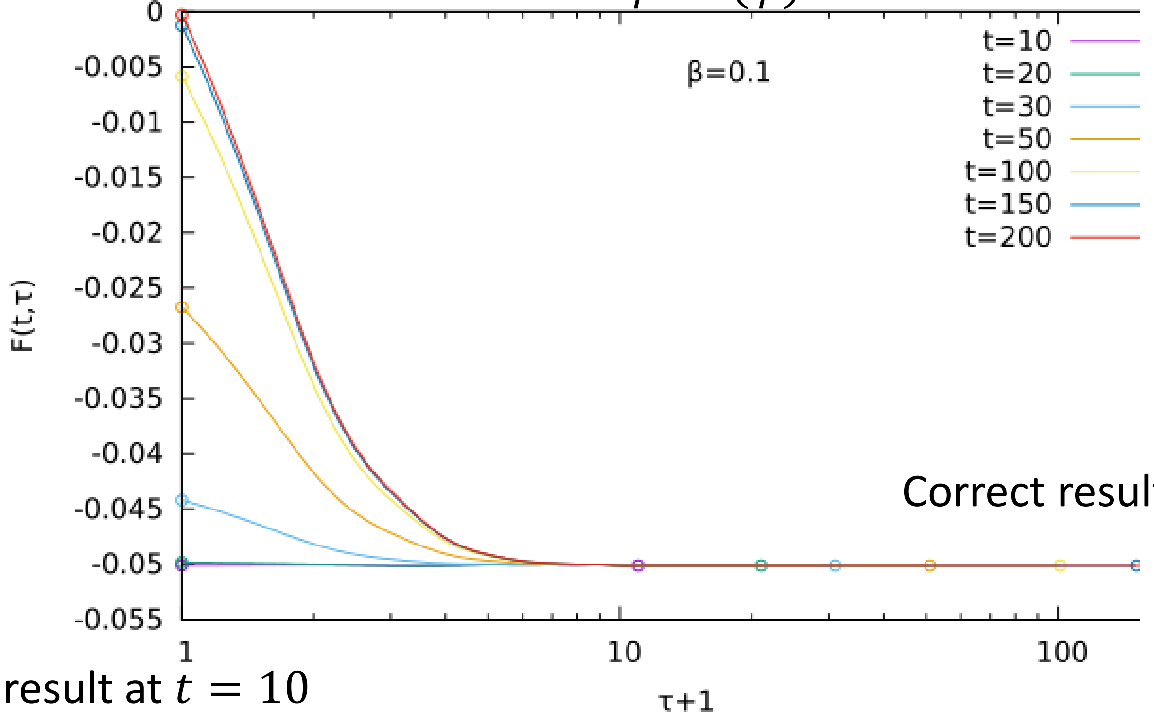
- Interpolation function between $P(t)$ and $\rho(t)$
- $F_O(t,\tau) = \int P(x,y,t-\tau)\exp(\tau L_c)\, O(x+iy)\, dxdy$

$$F_O(t,0) = \langle O \rangle_{P(t)}, \qquad F_O(t,t) = \langle O \rangle_{\rho(t)}$$

- If $F_O(t,\tau)$ is constant in tau, then the observables are correct

CLE result at $t = 200$

$S = i\beta\cos(\phi)$



Correct result

CLE result at $t = 10$

Cut-off effect

- Big error at run-aways
- Limit the imaginary part, to “cut-off” run-aways

$$B_n(Y,t) = \partial_\tau^n F_O(t,\tau)|_{\tau=0}$$
$$= -\int_{|y|<Y} \partial_t^n P(x,y,t)O(x+iy)dxdy + \int_{|y|<Y} P(x,y,t)L_c^n O(x+iy)dxdy$$

- First integral vanishes as $t \to \infty$
- Second is easy to calculate on the lattice
- Higher order boundary terms

$$B_n(Y,t) = \int_{|y|<Y} P(x,y,t)\, L_c^n\, O(x+iy)dxdy$$

- Use unitarity norm, for gauge fields UN= $Tr[U_\mu(x)U_\mu^\dagger(x) - 1]$

Toy model

- Action:

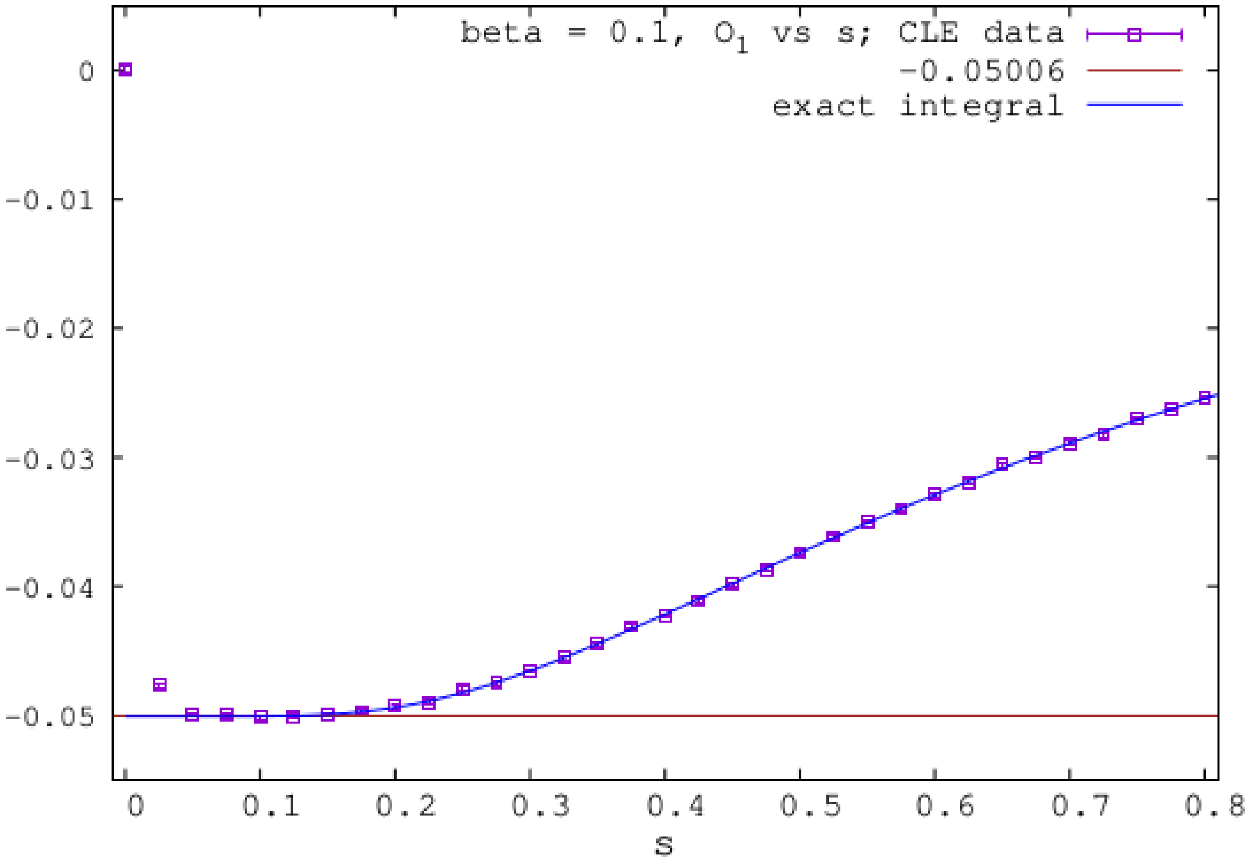
$$S[\phi] = i\beta\cos(\phi) + \frac{1}{2}s\phi^2$$

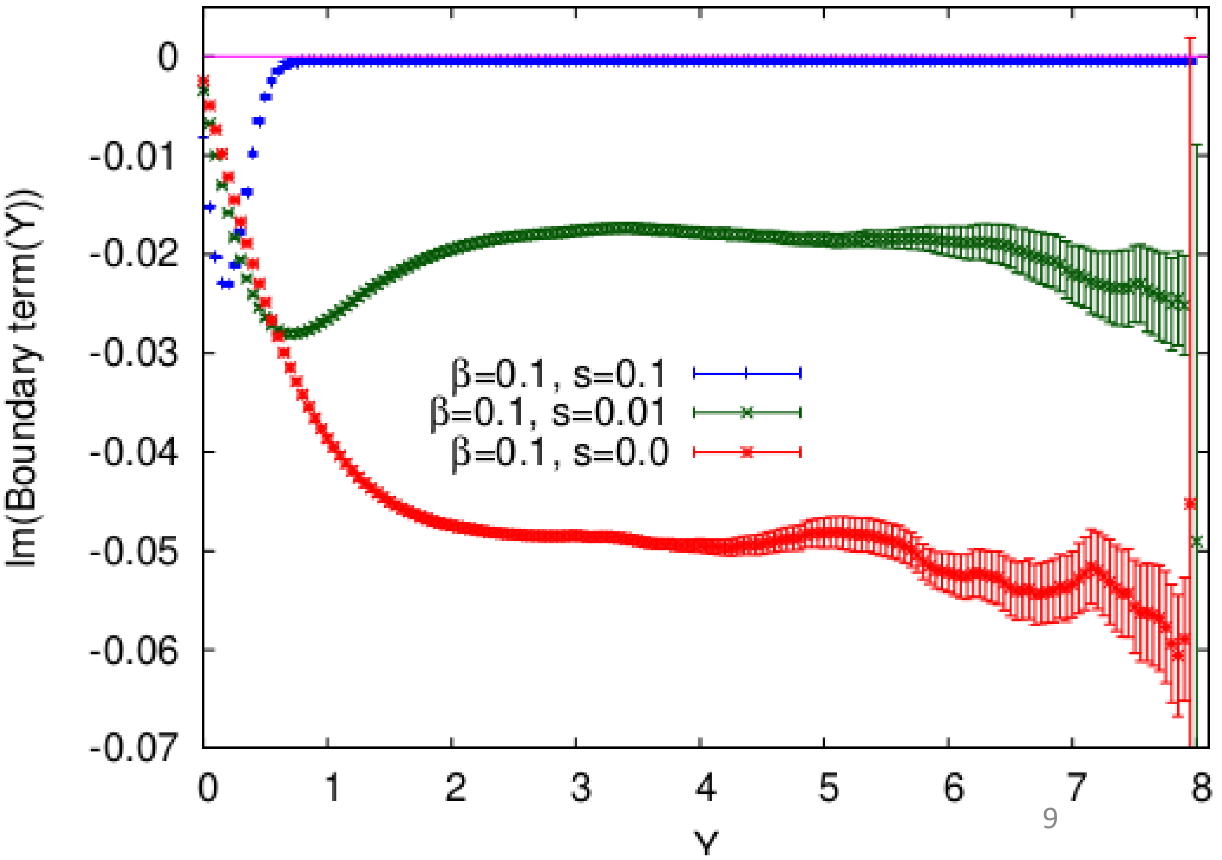
- Observable:

$$O[\phi] = \exp(i\phi)$$

- Bounary term:

$$L_c O[\phi] = i(i - S'[\phi])\exp(i\phi)$$



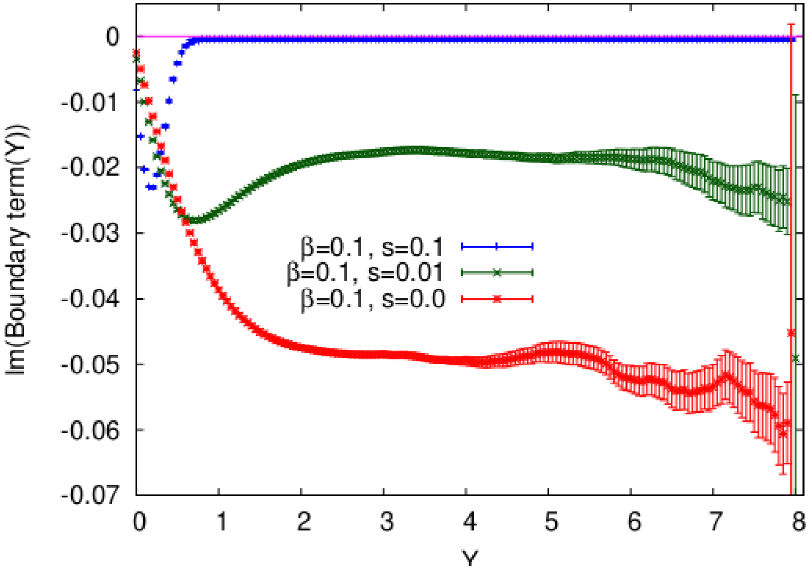


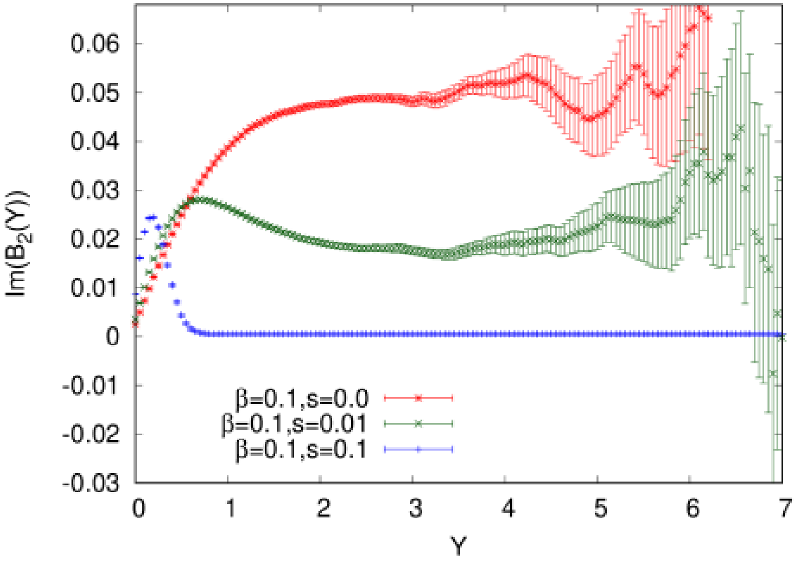
Boundary terms correction

- Correcting using boundary terms

$$F(t,0) - F(t,t) = \frac{B_1^2}{B_2}$$

- B_2 might be difficult to get





β, s	B_1	B_2	B_1^2/B_2	CL error	CL	correct	corrected CL
0.1, 0	-0.04859(45)	0.0493(11)	0.04786(79)	0.04891(45)	-0.00115(45)	-0.05006	-0.04901(62)
0.1, 0.01	-0.01795(49)	0.01801(80)	0.01789(60)	0.01689(50)	-0.03318(50)	-0.05006	-0.05106(40)
0.1, 0.1	-0.00048(30)	0.00057(35)	0.00039(28)	0.00049(31)	-0.04957(31)	-0.05006	-0.04997(6)
0.5, 0	-0.2474(11)	0.237(11)	0.258(11)	0.25818(23)	0.00003(23)	−0.25815	-0.258(11)
0.5, 0.3	−0.05309(86)	0.0552(51)	0.0507(41)	0.04183(70)	-0.19658(70)	−0.23841	−0.2473(37)

Updating the lattice using CLE

- Update:

$$U_\mu^{n+1}(x) = \exp\left[i\lambda_a\left(\epsilon K_{\mu a}(x) + \sqrt{\epsilon}\,\eta_{\mu a}(x)\right)\right] U_\mu^n(x)$$

- Using the left derivative

$$K_{\mu a}(x) = -D_{\mu a}S(x),$$

$$D_{\mu a}f(U) = \partial_\alpha f\left(\exp(i\alpha\lambda_a)\; U_\mu(x)\right)\Big|_{\alpha=0}$$

- If the drift is complex ⇒ $U \in SL(N)$
- Needs gauge cooling after each step

Reweighting

- Change the weights

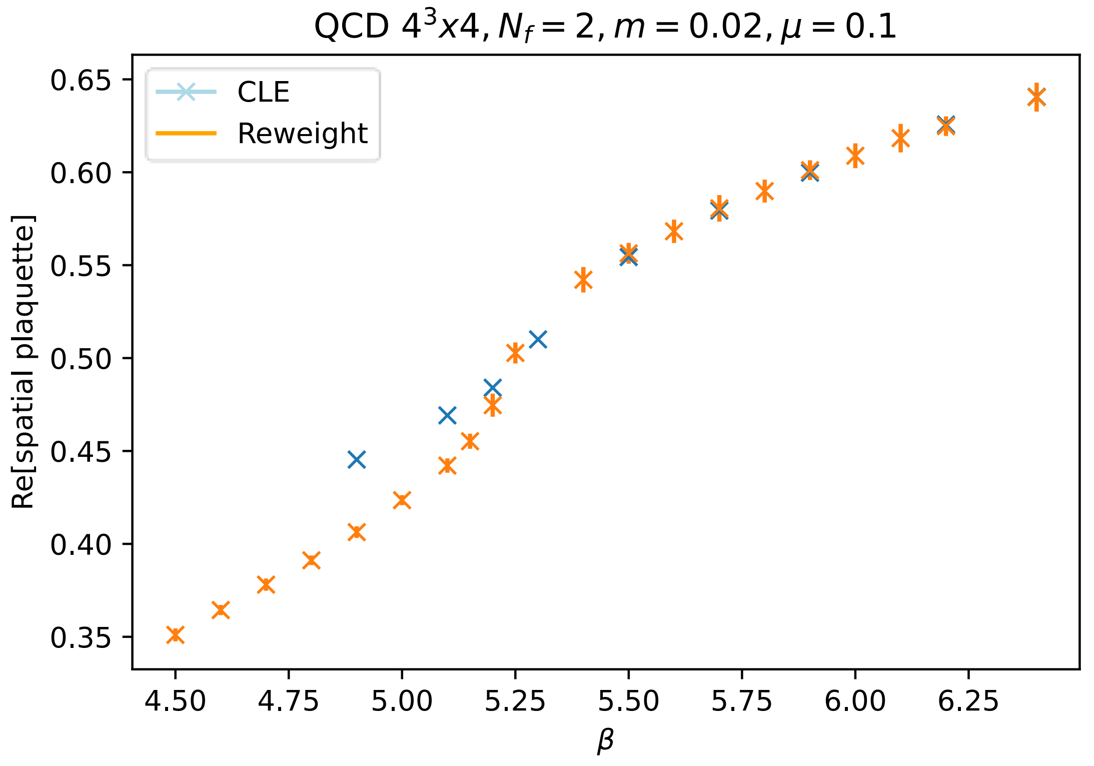
$$\langle x\rangle_w = \frac{\sum w_i x_i}{\sum w_j} = \frac{\sum w_i x_i \frac{w_i'}{w_i'}}{\sum w_j \frac{w_j'}{w_j'}} = \frac{\sum w_i' x_i \frac{w_i}{w_i'}}{\sum w_j' \frac{w_j}{w_j'}} = \frac{\left\langle x \frac{w}{w'}\right\rangle_{w'}}{\left\langle \frac{w}{w'}\right\rangle_{w'}}$$

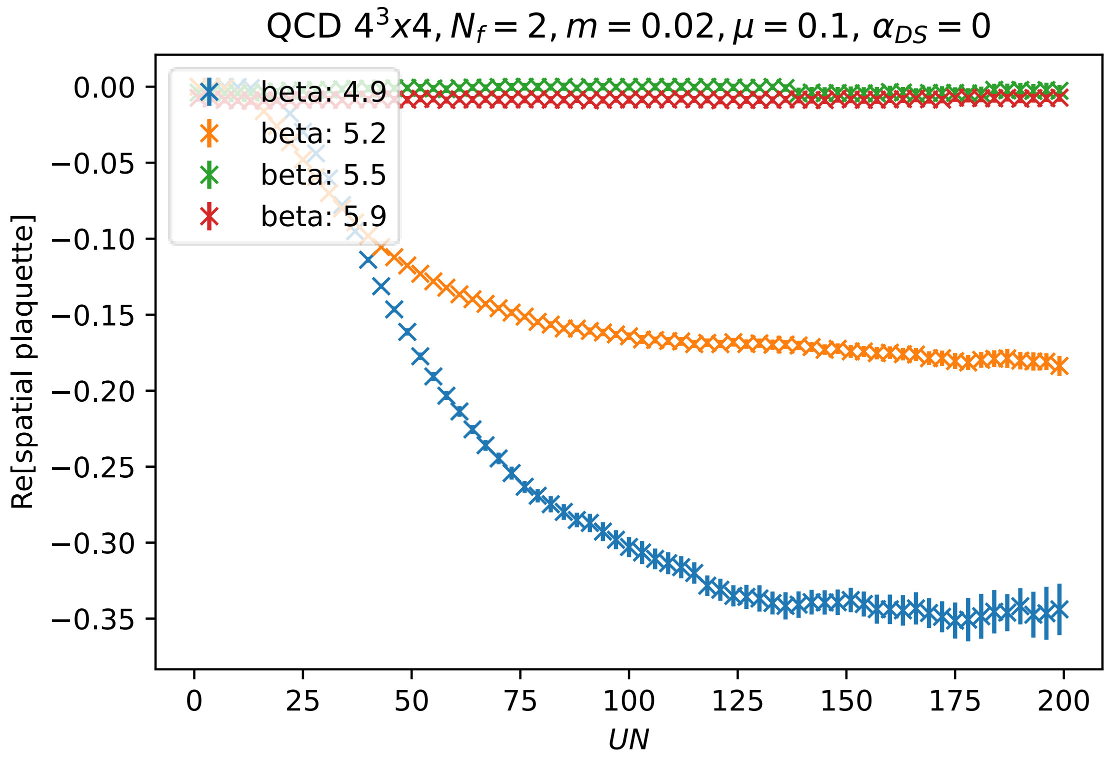
- Used in HMC, to simulate non-zero chemical potential

$$\left\langle \frac{w}{w'}\right\rangle = \left\langle \frac{\det M(\mu)}{\det M(\mu=0)}\right\rangle = \exp\left(-\frac{V}{T}\Delta F(\mu,t)\right)$$

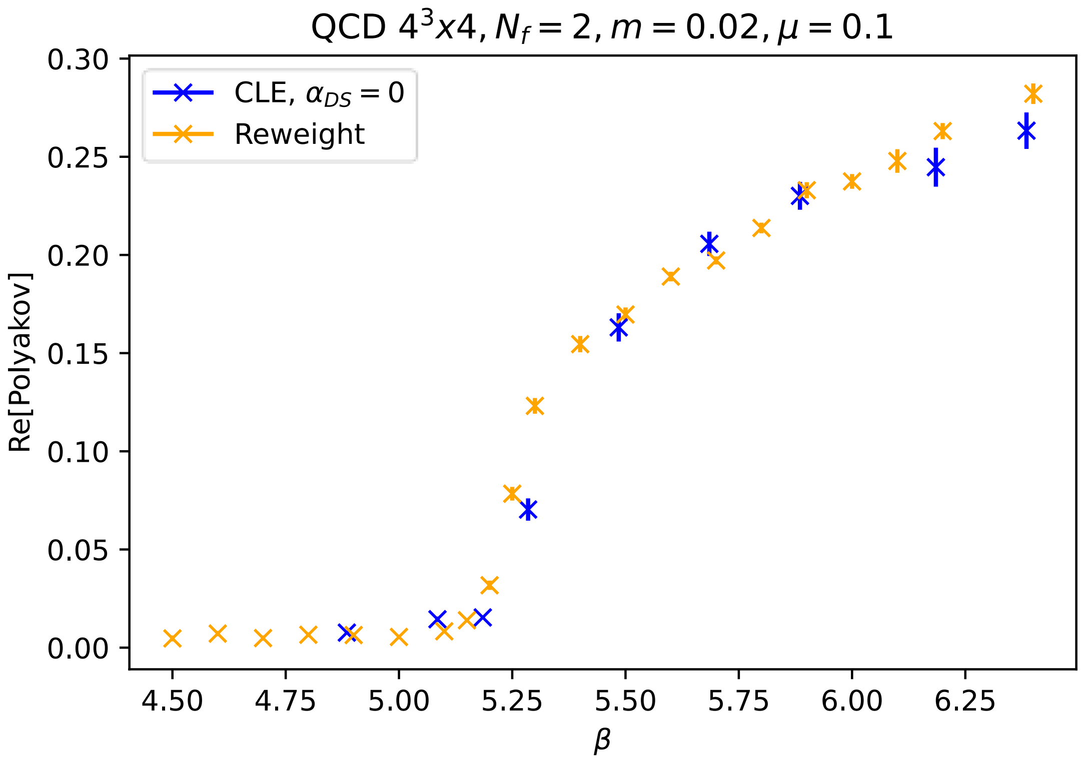
- Large $\mu \Rightarrow \left\langle \frac{w}{w'}\right\rangle$ goes towards zero

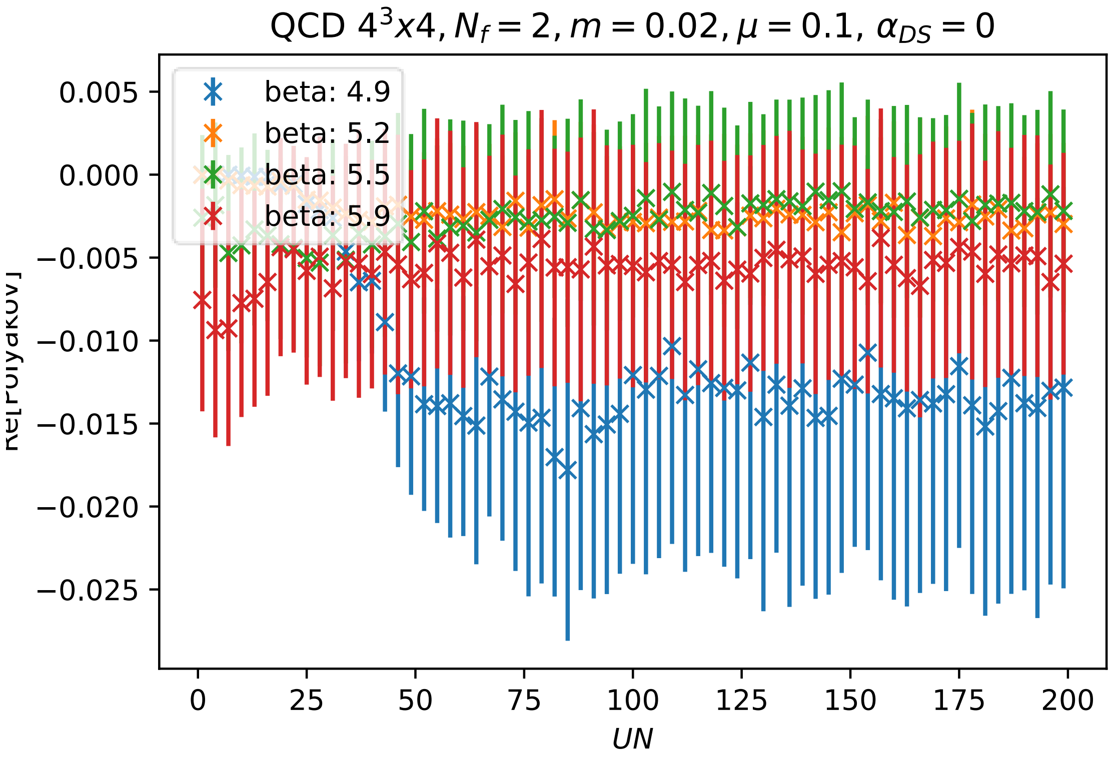
Results – Plaquettes





Results – Polyakov loops





Dynamical stabilization

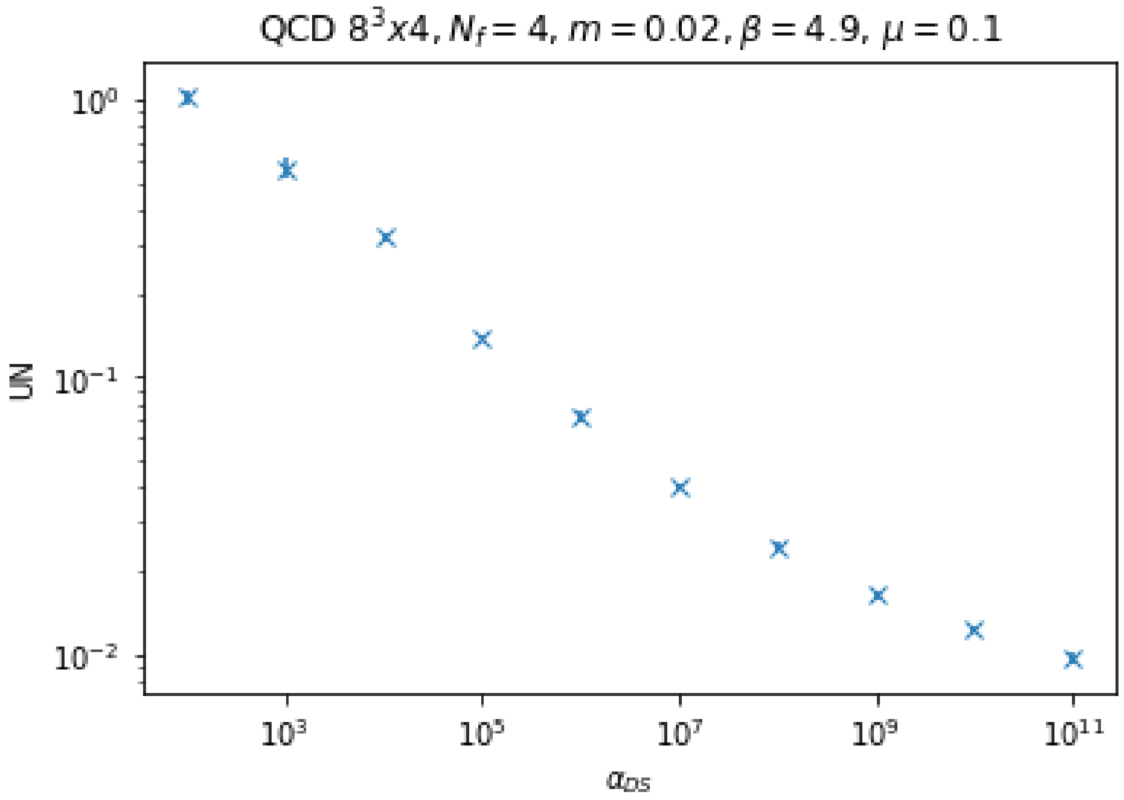
(Attanasio, Jäger, arxiv: 1808.04400)

- Introducing a Gauge invariant force, to the drift
- Designed to grow rapidly with the unitarity norm

$$K_{\mu a}(x) \to K_{\mu a}(x) + i\alpha_{DS} M_a(x)$$

$$M_a(x) = i b_a \left(\sum_c b_c(x) b_c(x) \right)^3$$

$$b_a(x) = Tr\left[\lambda_a \sum_\mu U_\mu(x) U_\mu^\dagger(x) \right]$$



Dynamical stabilization - methods

- $K_{\mu a}(x) \rightarrow K_{\mu a}(x) + i\alpha_{DS} M_{\mu a}(x)$
- Where $M_{\mu a}$ now depends on the direction
- $M_{\mu a}(x) = iTr\left[\lambda_a U_\mu(x) U_\mu^\dagger(x)\right]\left(2Tr\left[\left(U_\mu(x) U_\mu^\dagger(x)\right)^2\right] - \frac{2}{3} Tr\left[U_\mu(x) U_\mu^\dagger(x)\right]^2\right)^3$
- Note that unitary links => $M_{\mu a} = 0$

Simple test model

- Action:

$$-S = \beta_1 Tr\, U + \beta_2 Tr\, U^{-1}$$

$$\beta_1 = \beta + \kappa e^{\mu}, \qquad \beta_2 = \beta + \kappa e^{-\mu}$$

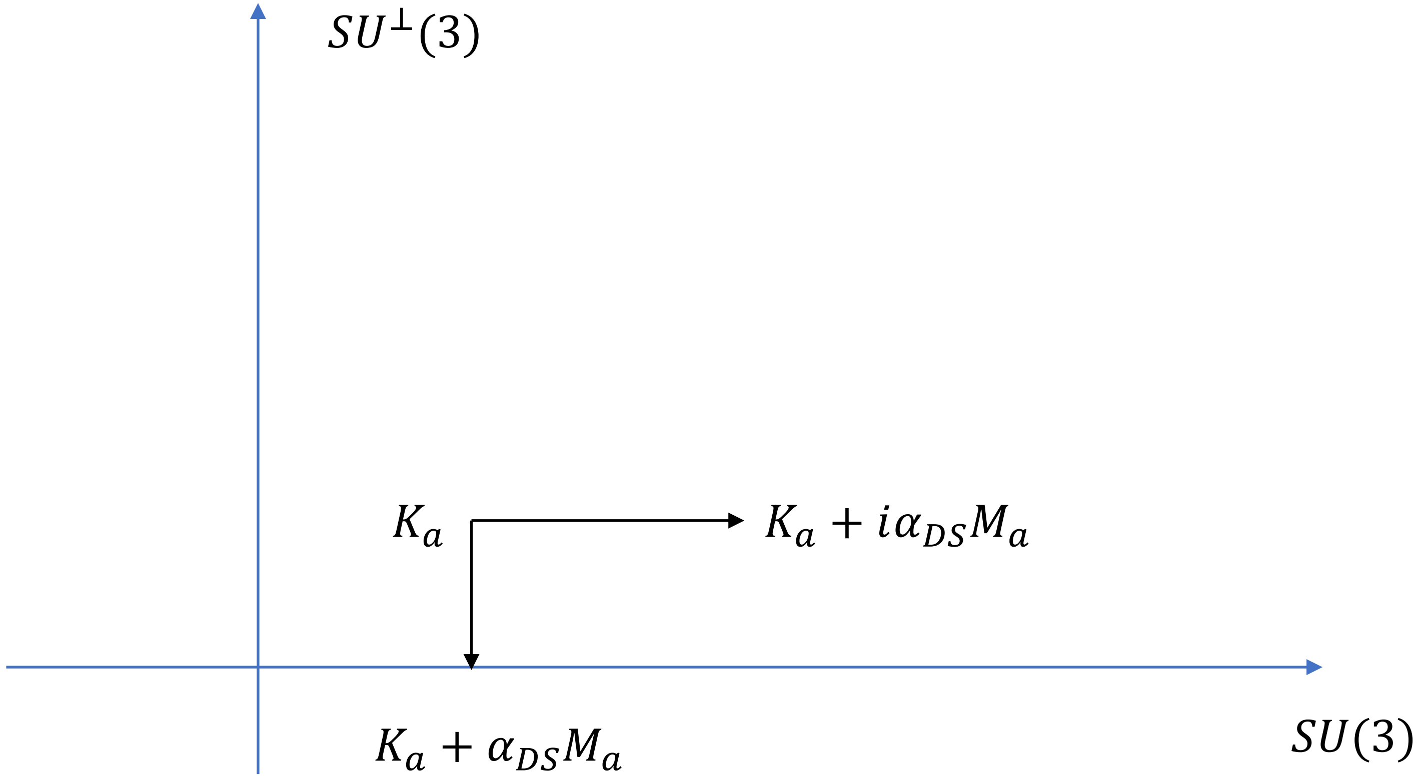
- For $\mu \neq 0$, a sign problem appears, since S is complex
- Drift:

$$K_a = i\beta_1 Tr\, \lambda_a U - i\beta_1 Tr\, \lambda_a U^{-1}$$

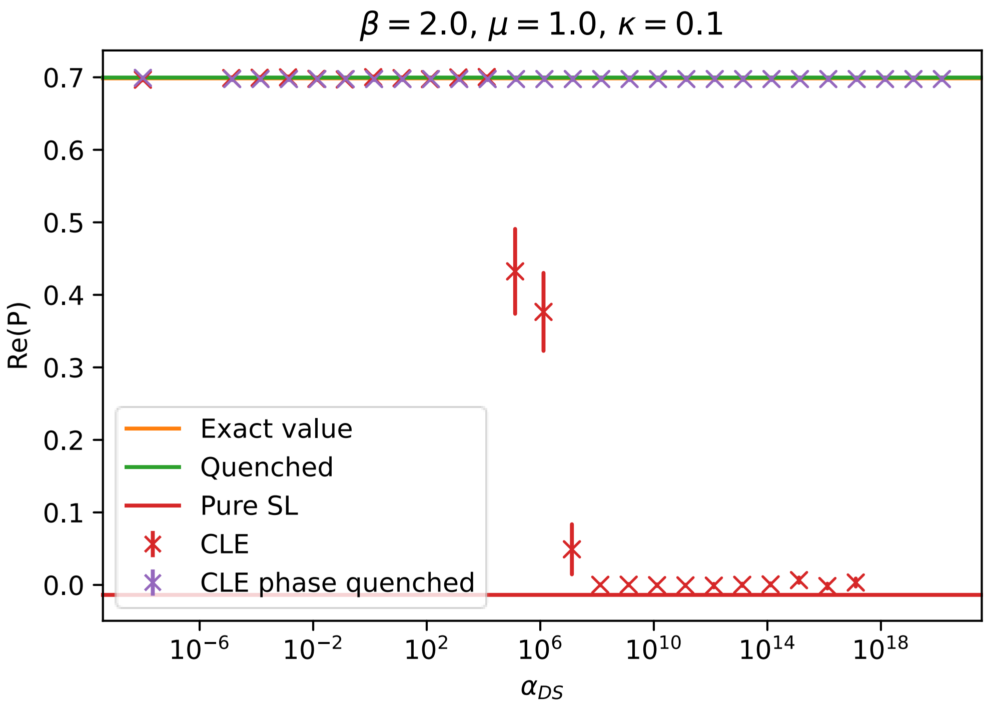
- And finally

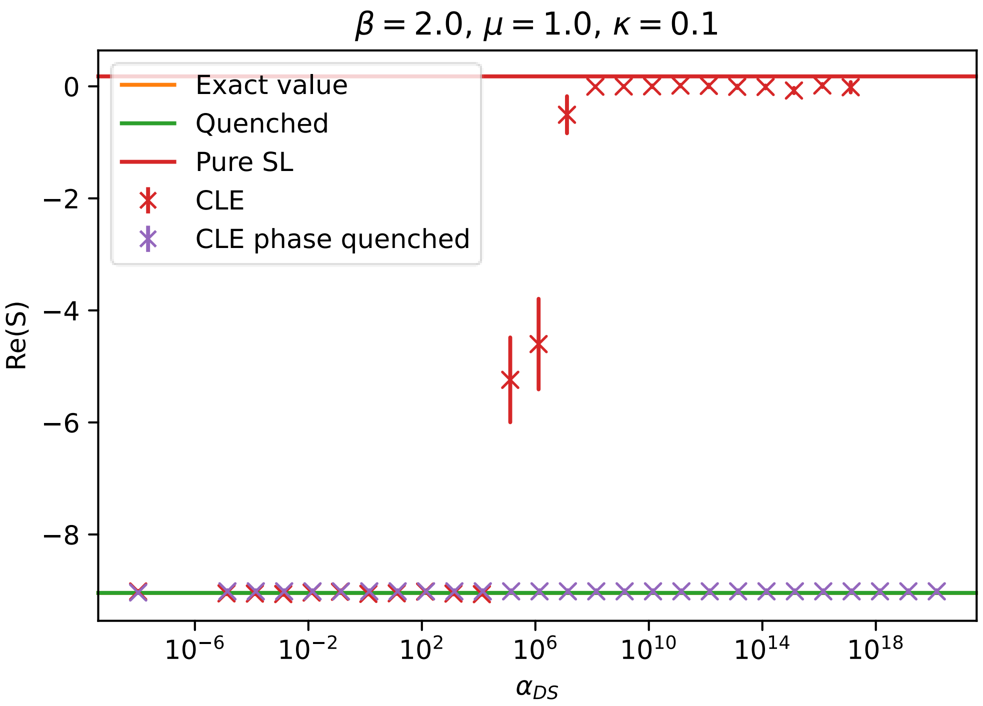
$$\lambda_a K_a = 2i\left(M - \frac{1}{3} Tr\, M\right), M = \beta_1 U - \beta_2 U^{-1}$$

Dynamical Stabilization

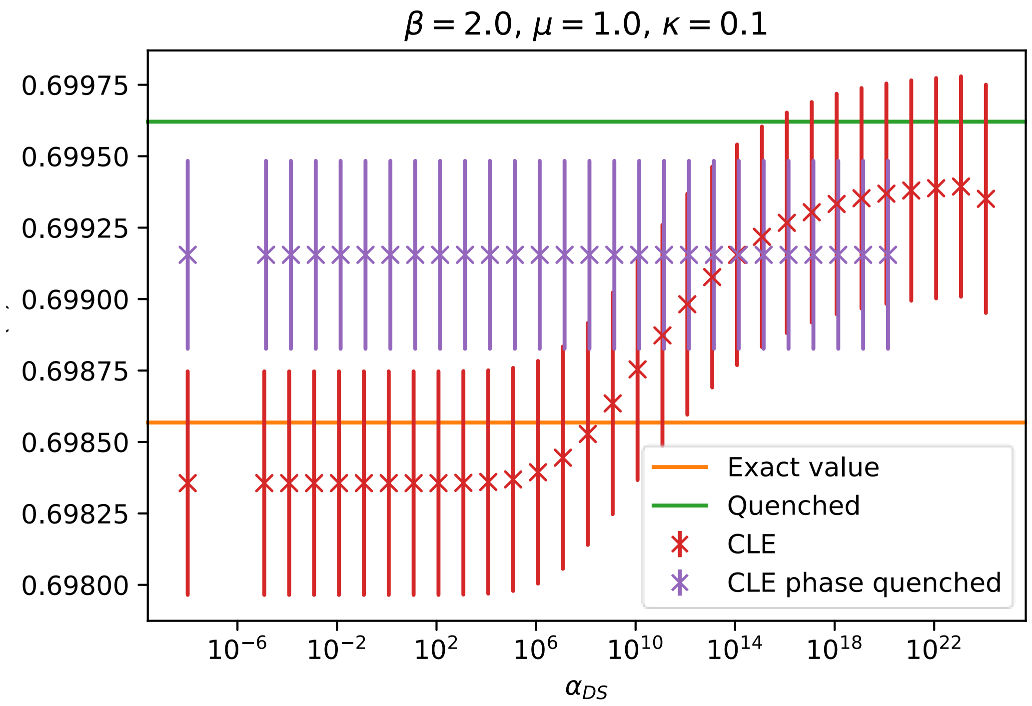


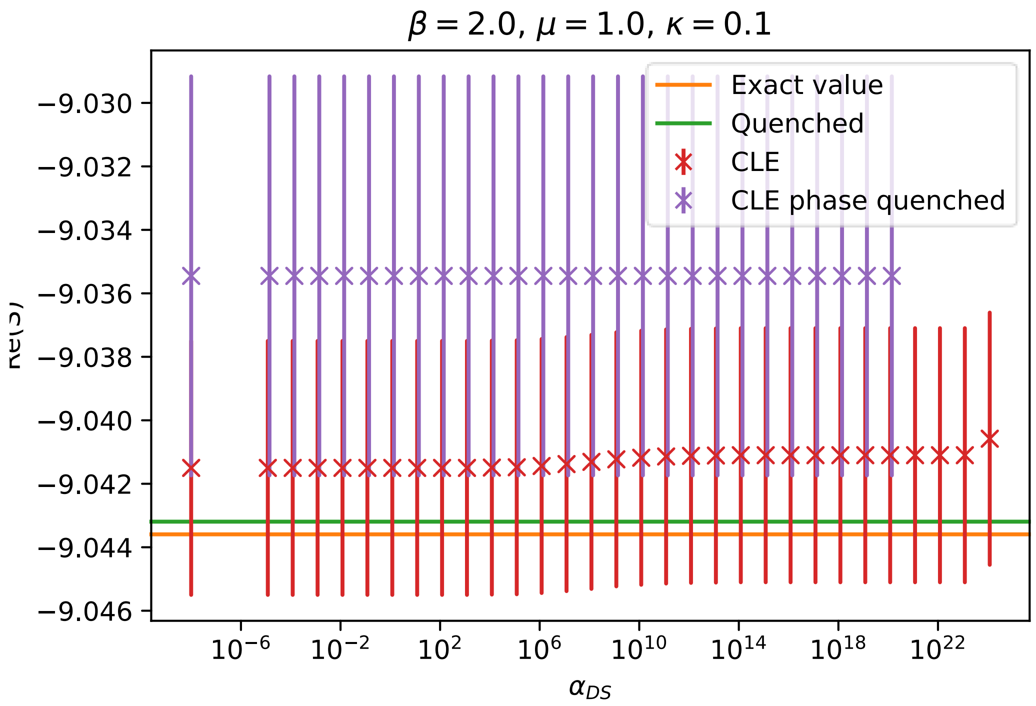
Increasing along $i\alpha_{DS}M_a$



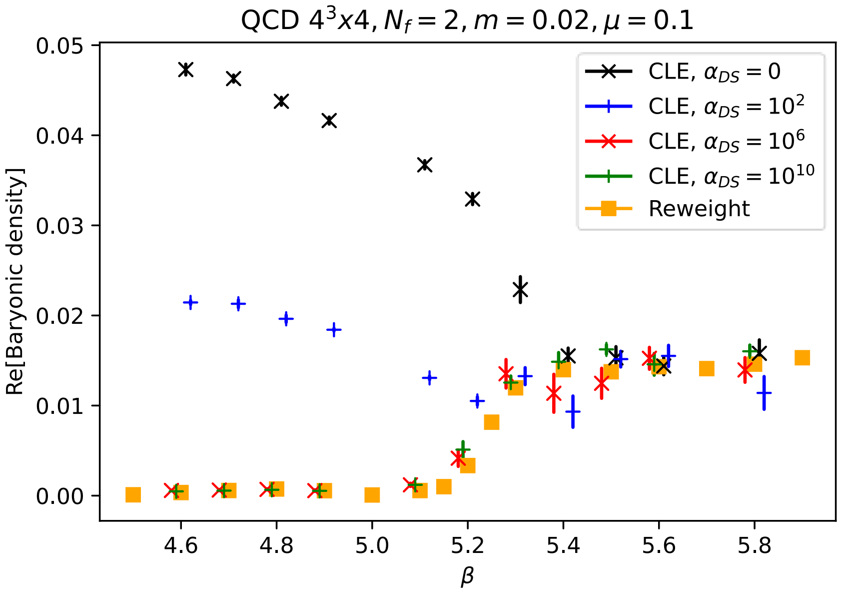


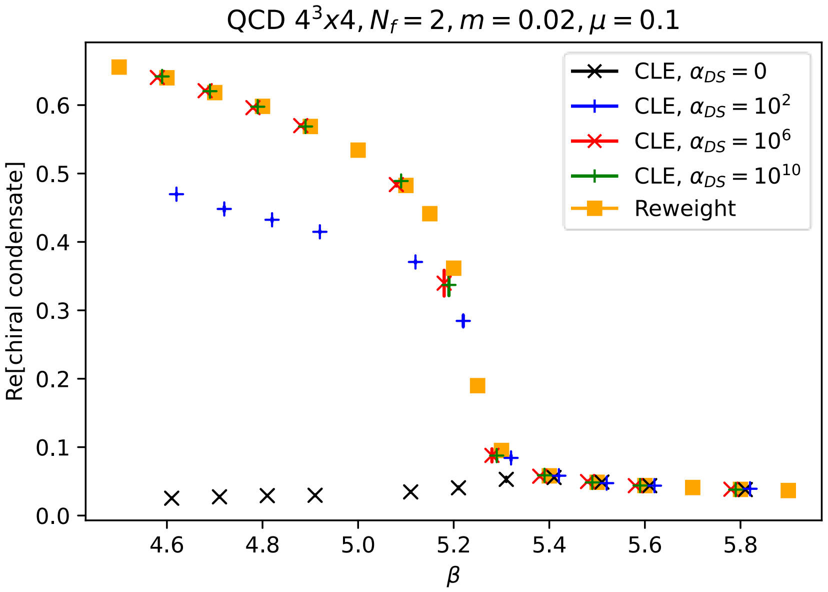
Increasing along $\alpha_{DS} M_a$

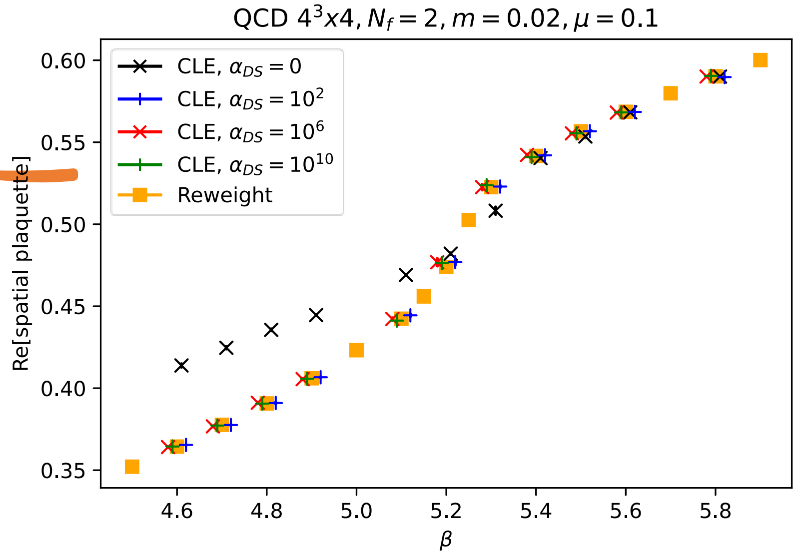


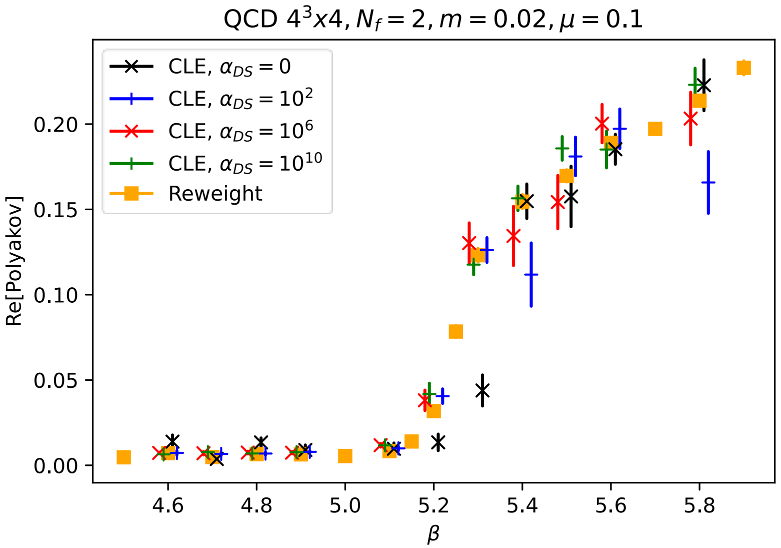


Small lattice with DS-corrections







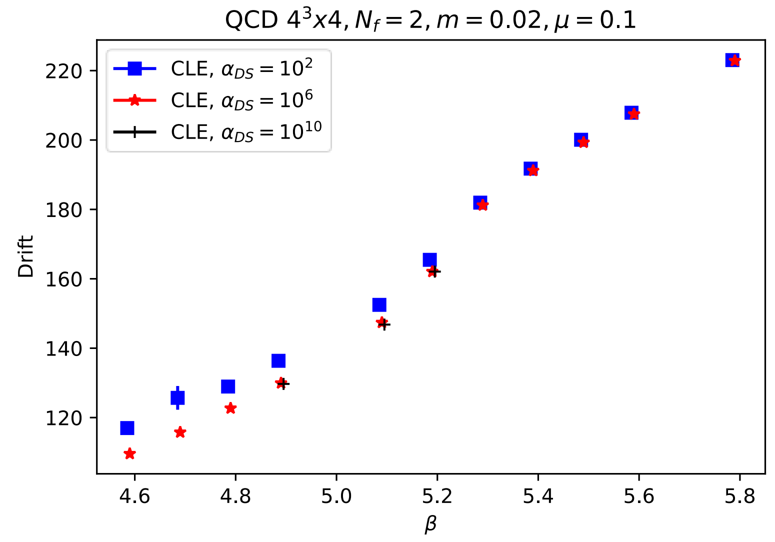


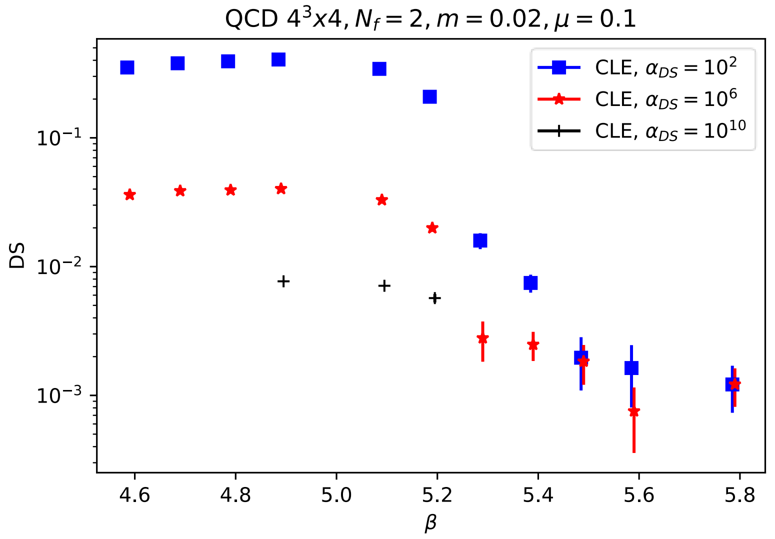
DS and Drift terms

Considering the drift term and the DS, for various values of β.

- constant for large beta (correct convergence)

- decreasing for low beta (incorrect convergence)

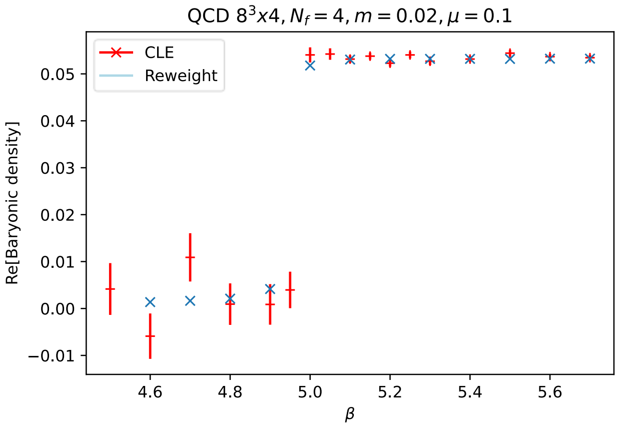


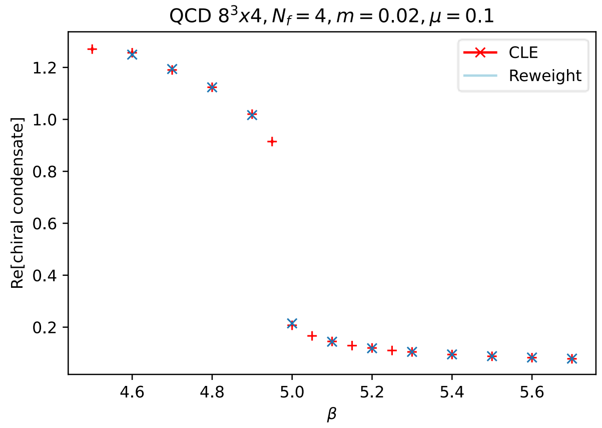


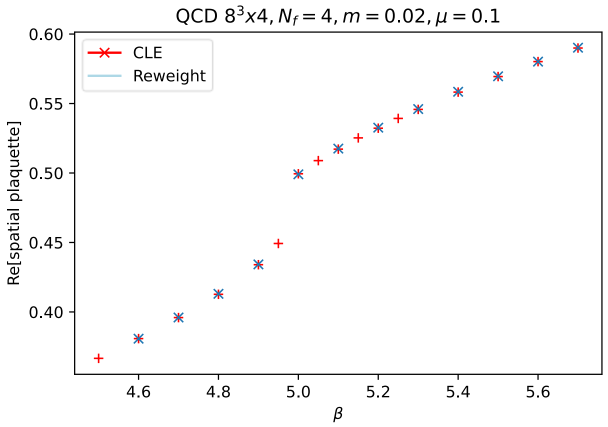
The observables

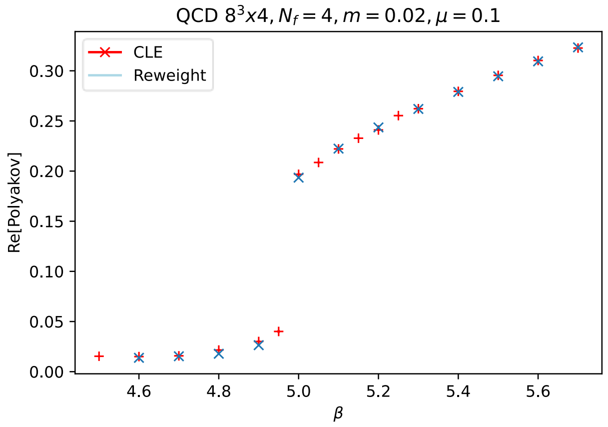
Tests with $\alpha_{DS} = 10^6$

- Plaquettes (spatial/temporal)
- Poliakov Loop (normal/inverse)
- Baryonic density
- (Chiral Condensate)



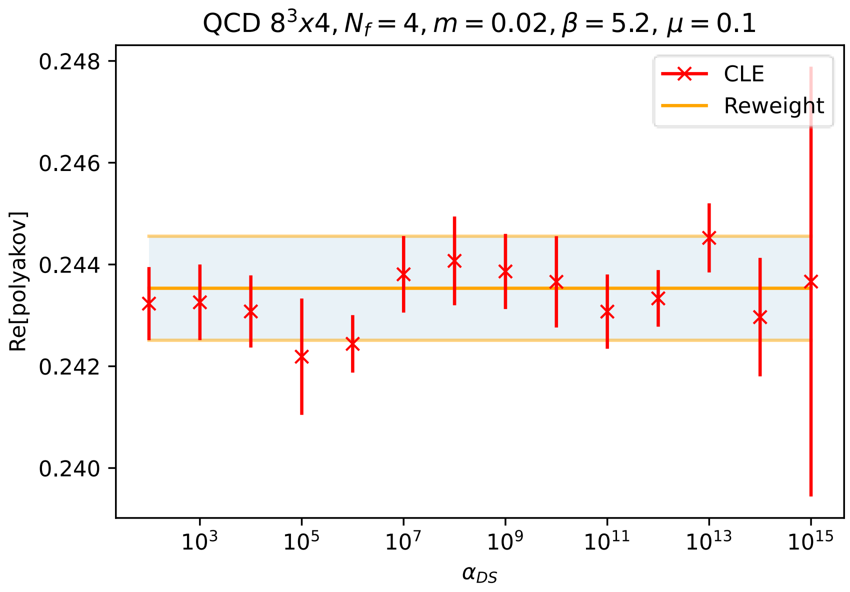


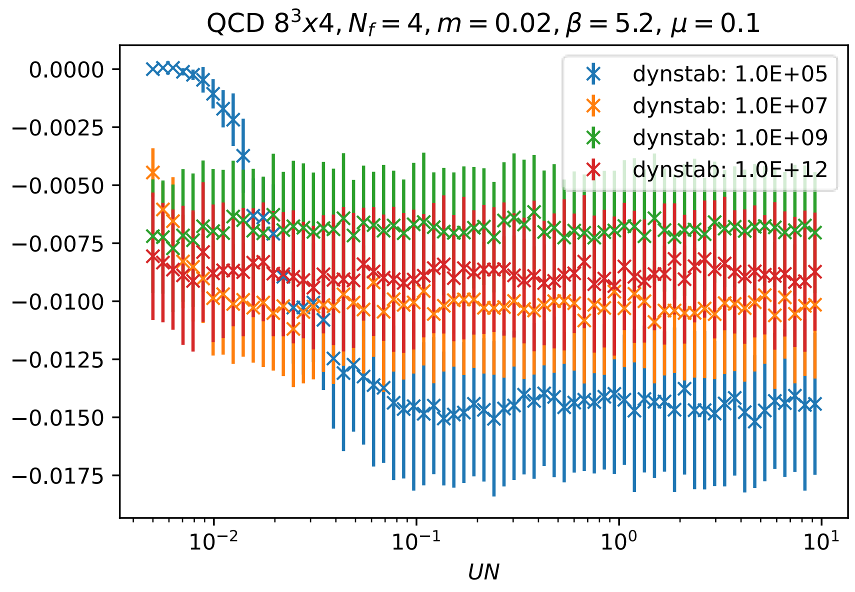




DynStab – High temperature

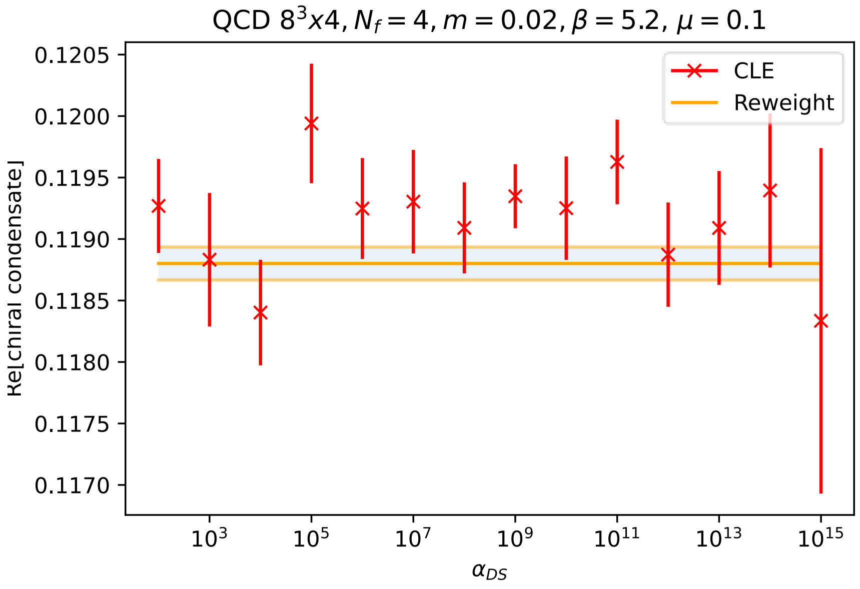
- Polyakov loop

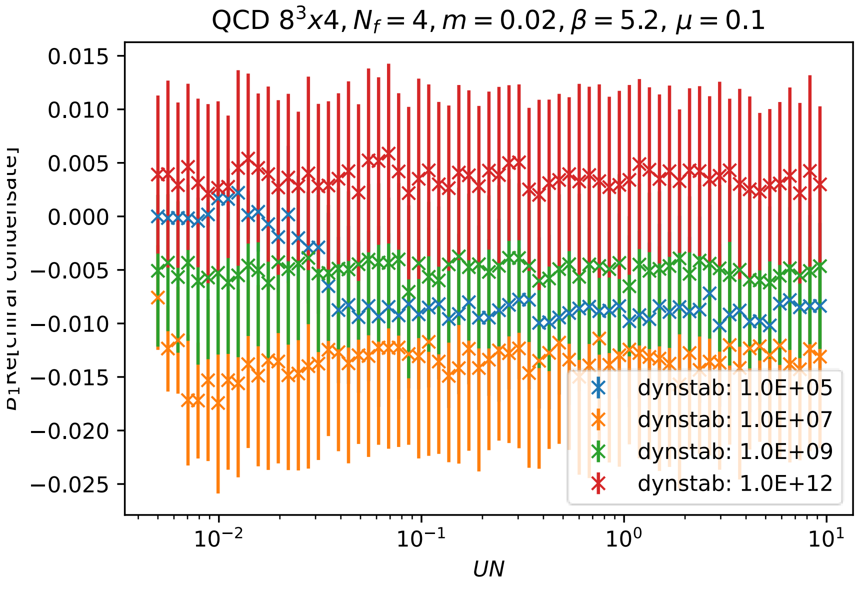




DynStab – High temperature

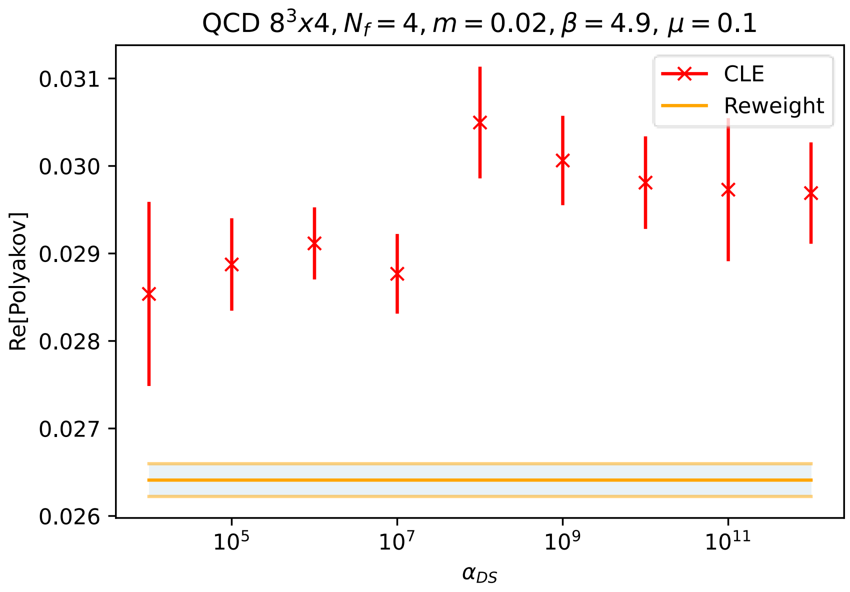
Chiral condensate

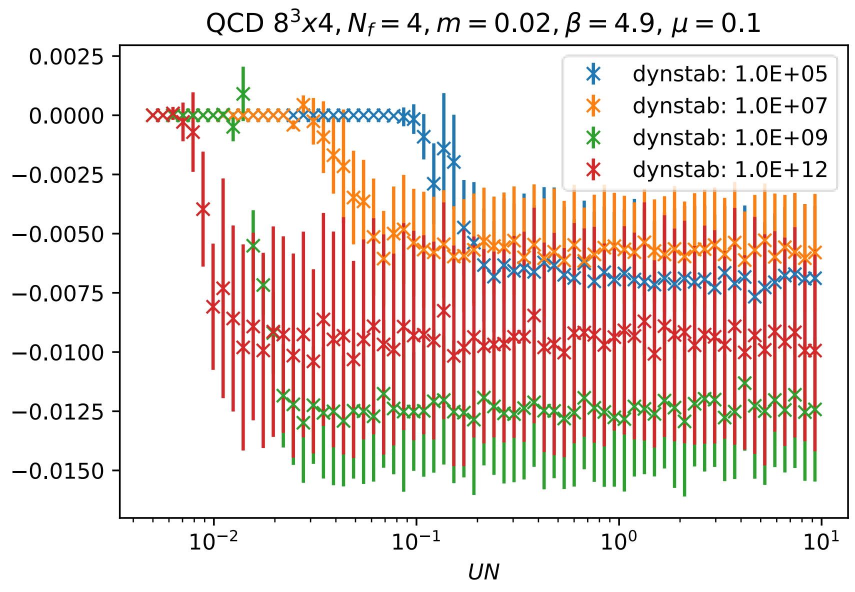




Dynstab - Low Temperature

Polyakov loop





Dynstab - Low Temperature

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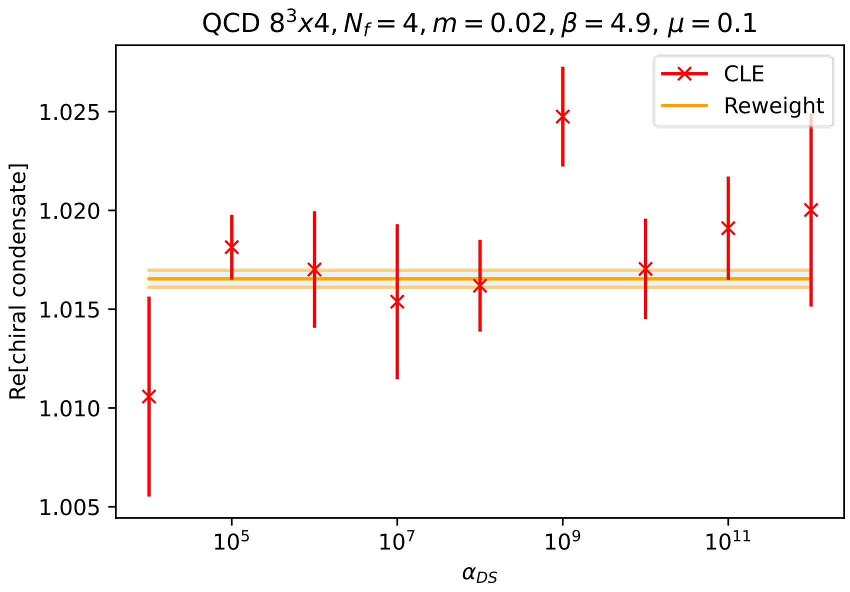

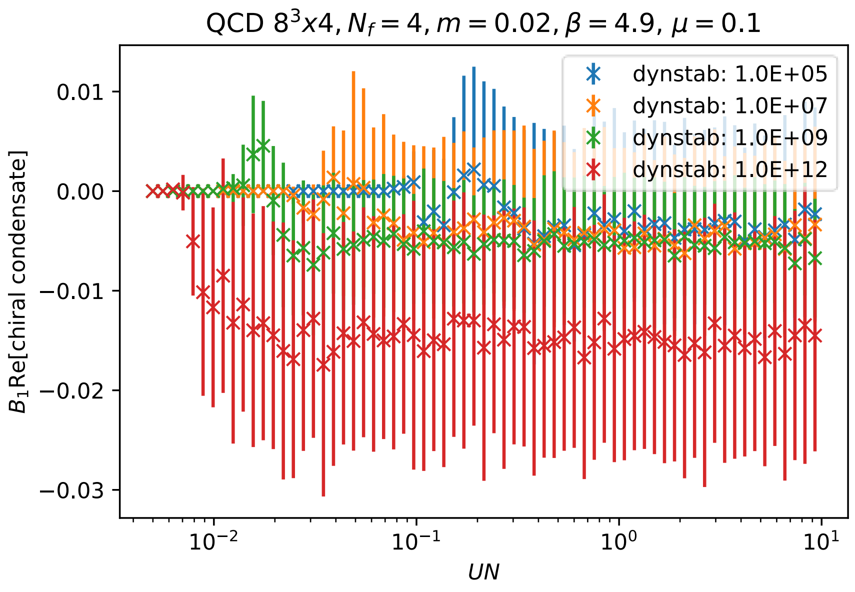

Conclusion

- CL works very well for non-zero density (high temperature)
- Low temperature deviations can be estimated using boundary terms
- Dynamical Stabilization can slow the drifts from SU(3) to SL(3)
- Dynamical Stabilization can help correct most observables in low temperature

Calculating the boundary

- Long and difficult calculations
- $\Sigma = \frac{1}{|\Omega|} Tr(M^{-1})$
- $L_c\Sigma = \frac{2}{|\Omega|} \frac{N^2-1}{N} \left(Tr(M^{-1}) - mTr(M^{-2})\right)$

$$+ \frac{1}{|\Omega|} \sum_{j \in \Omega} 2Tr\left(M^{-1}\left(D_a^j M\right) M^{-1} \left(D_a^j M\right) M^{-1}\right)$$

$$+ \frac{1}{|\Omega|} \sum_{j \in \Omega} K_a^j D_a^j \, Tr(M^{-1})$$

$$\begin{aligned}
\frac{1}{2} \sum_{j \in \Omega} \mathrm{Tr}(M^{-1} D_i^j M M^{-1} D_i^j M M^{-1}) = \\
& - \mathrm{Tr}\left[\left(M^{-2}_{a,a-\mu}\right)\left(M_{a-\mu,a}\right)\right] \mathrm{Tr}\left[\left(M^{-1}_{a,a-\mu}\right)\left(M_{a-\mu,a}\right)\right] \\
& - \mathrm{Tr}\left[\left(M^{-2}_{a,a}\right)\right] \mathrm{Tr}\left[\left(M^{-1}_{a+\mu,a+\mu}\right)\right] \\
& - \mathrm{Tr}\left[\left(M^{-2}_{a,a}\right)\right] \mathrm{Tr}\left[\left(M^{-1}_{a-\mu,a-\mu}\right)\right] \\
& - \mathrm{Tr}\left[\left(M^{-2}_{a,a+\mu}\right)\left(M_{a+\mu,a}\right)\right] \mathrm{Tr}\left[\left(M^{-1}_{a,a+\mu}\right)\left(M_{a+\mu,a}\right)\right] \\
& + \frac{1}{N} \mathrm{Tr}\left[\left(M^{-2}_{a,a-\mu}\right)\left(M_{a-\mu,a}\right)\left(M^{-1}_{a,a-\mu}\right)\left(M_{a-\mu,a}\right)\right] \\
& + \frac{1}{N} \mathrm{Tr}\left[\left(M^{-2}_{a,a}\right)\left(M_{a,a+\mu}\right)\left(M^{-1}_{a+\mu,a+\mu}\right)\left(M_{a+\mu,a}\right)\right] \\
& + \frac{1}{N} \mathrm{Tr}\left[\left(M^{-2}_{a,a}\right)\left(M_{a,a-\mu}\right)\left(M^{-1}_{a-\mu,a-\mu}\right)\left(M_{a-\mu,a}\right)\right] \\
& + \frac{1}{N} \mathrm{Tr}\left[\left(M^{-2}_{a,a+\mu}\right)\left(M_{a+\mu,a}\right)\left(M^{-1}_{a,a+\mu}\right)\left(M_{a+\mu,a}\right)\right]
\end{aligned}$$