Three-loop mixing matrix for flavor-singlet operators

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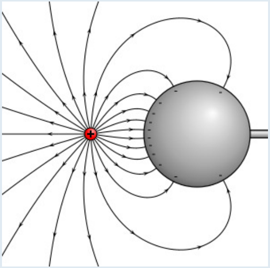
based on: V. Braun, K. Chetyrkin, A. Manashov, Phys.Lett. B834 (2022) 137409

QCD Evolution, May 2023

L.I. Magnus (1831):
Inversion transformation

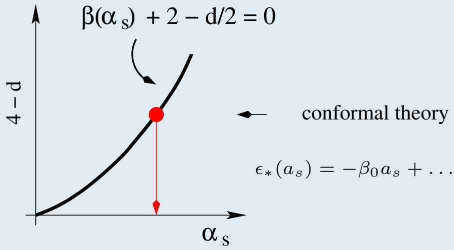
$$x^\mu \to \frac{x^\mu}{x^2}$$



- Small distances $\leftrightarrow$ large distances ??
- Asymptotic freedom $\leftrightarrow$ confinement ??

QCD?

QCD is not a conformal theory, but



$$\epsilon_*(a_s) = -\beta_0 a_s + \ldots\,, \qquad a_s = \frac{\alpha_s}{4\pi}$$

$$\boxed{\mathcal{A}_{\rm QCD} = \mathcal{A}^{\rm conf}_{\rm QCD} + \frac{\beta(g)}{g}\Delta\mathcal{Q}}$$

"Conformal QCD": QCD in $d-2\epsilon$ at Wilson-Fischer critical point $\beta(\alpha_S)=0$

V.B., A. Manashov, Eur.Phys.J.C 73 (2013) 2544

- No corrections $\sim \beta(\alpha_s)$ for the counterterms
  $\Rightarrow$ exact conformal symmetry of the QCD RG equations in MS-schemes

Mixing with total derivatives

Renormalization of off-forward matrix elements

$$\langle p'|\mathcal{O}_{Nk}|p\rangle = \langle p'|\bar{q}(0)\gamma_{\mu_1} \overleftarrow{D}_{\mu_2} \ldots \overleftarrow{D}_{\mu_k} \overrightarrow{D}_{\mu_{k+1}} \ldots \overrightarrow{D}_{\mu_N} q(0)|p\rangle$$

Mixing matrix with operators containing total derivatives with anomalous dimensions on the diagonal

$$\begin{pmatrix} \gamma_1 & 0 & 0 & 0 \\ * & \gamma_2 & 0 & 0 \\ * & * & \gamma_3 & 0 \\ * & * & * & \gamma_4 \end{pmatrix}$$

need off-diagonal elements

[evolution kernels in x-space are functions of two variables]

- In conformal theories, operators with different scaling dimensions are orthogonal

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \frac{\text{const}}{|x_1 - x_2|^{2\Delta_1}}\, \delta_{\Delta_1\Delta_2}$$

- n-loop ADs from n-loop two-particle functions

VB, K.Chetyrkin, A.Manashov, PLB 834 (2022) 137409

- Gauge-invariant two-point functions
- Simple algorithmic implementation

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simple example:

$$O_2(x) = \partial_+^2\, \bar q_1(x)\, C_2^{(3/2)}\left(\frac{\overleftarrow{D}_+ - \overrightarrow{D}_+}{\overleftarrow{D}_+ + \overrightarrow{D}_+}\right)\gamma_+ q_2(x) \qquad O_1(x) = \partial_+^2 \bar q_1(x)\gamma_+ q_2(x)\,,$$

$$\gamma = \begin{pmatrix} \gamma_{11} & 0 \\ \gamma_{21} & \gamma_{22}\end{pmatrix}$$

$$\gamma_{11} = 0$$

$$\gamma_{22} = \frac{100}{9} a_s + \left[\frac{34450}{243} - \frac{830}{81} n_f\right] a_s^2 + \dots$$

$$\gamma_{21} = \gamma_{21}^{(2)} a_s^2 + \dots = ?$$

Let

$$\begin{pmatrix} [\mathbb{O}_1] \\ [\mathbb{O}_2]\end{pmatrix} = \begin{pmatrix} [O_1] \\ [O_2] + \lambda_{21}[O_1]\end{pmatrix}\,, \qquad \lambda_{21} = \frac{\gamma_{21}}{\gamma_{22} - \gamma_{11}}$$

At the critical point

$$\Big(\mu\partial_\mu + \gamma_{11}\Big)\mathbb{O}_1 = 0\,, \qquad \Big(\mu\partial_\mu + \gamma_{22}\Big)\mathbb{O}_2 = 0\,,$$

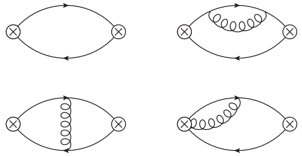
and conformal symmetry requires that to all orders of perturbation theory

$$\langle \mathbb{O}_2(x)\mathbb{O}_1(0)\rangle = \langle [O_2](x)[O_1](0)\rangle + \lambda_{21}\langle [O_1](x)[O_1](0)\rangle = 0$$

— an equation for γ_{21}

NB: this equation holds at $d = 4 - 2\epsilon_*$, but the result for γ_{21} is valid for $d = 4$

to leading order $\mathcal{O}(a_s^2)$

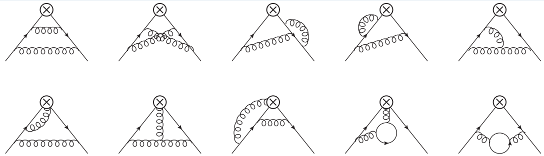


$$\langle \mathsf{T}\{O_1(x)O_1(0)\}\rangle = \mathcal{N}\Big[-105 + \mathcal{O}(a_s,\epsilon)\Big],$$
$$\langle \mathsf{T}\{O_2(x)O_1(0)\}\rangle = \mathcal{N}\Big[63\epsilon + 70 a_s + \mathcal{O}(a_s^2, a_s\epsilon, \epsilon^2)\Big],$$
$$\mathcal{N} = \frac{(n\cdot x)^6}{(4\pi)^d}\left(\frac{4}{-x^2+i0}\right)^{d+4}.$$

Using these expressions and the one-loop γ_{22}, expanding to $\mathcal{O}(a_s)$, and replacing $\epsilon \mapsto -\beta_0 a_s$, one obtains

$$\gamma_{21}^{(2)} = \frac{260}{9} - \frac{40}{9} n_F$$

compare:



- Straightforward to generalize to higher orders, more derivatives and flavor-singlet

Three-loop flavor singlet anomalous dimensions

$$\mathcal{O}^q_{nk} = i\,\partial_+^n \sum_{f=1}^{n_f} \bar{q}^f\, C_k^{(3/2)}\left(\frac{\overleftarrow{D}_+ - \overrightarrow{D}_+}{\overleftarrow{D}_+ + \overrightarrow{D}_+}\right)\gamma_+ q^f,$$

$$\mathcal{O}^g_{nk} = 6\,\partial_+^{n-1}\, F^{\mu,+}\, C_{k-1}^{(5/2)}\left(\frac{\overleftarrow{D}_+ - \overrightarrow{D}_+}{\overleftarrow{D}_+ + \overrightarrow{D}_+}\right)F_{\mu,+}.$$

$$\boldsymbol{\gamma}_n(a) = \begin{pmatrix} \gamma_{11} & 0 & \cdots & 0 \\ \gamma_{31} & \gamma_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n1} & \gamma_{n2} & \cdots & \gamma_{nn} \end{pmatrix}.$$

$$\gamma_{nk} = \begin{pmatrix} \gamma^{qq}_{nk} & \gamma^{qg}_{nk} \\ \gamma^{gq}_{nk} & \gamma^{gg}_{nk} \end{pmatrix} = a_s\gamma^{(1)}_{nk} + a_s^2\gamma^{(2)}_{nk} + a_s^3\gamma^{(3)}_{nk} + \ldots$$

- Three-loop off-forward anomalous dimensions

$$\gamma^{(3)} = \gamma^{(3,0)} + n_f \gamma^{(3,1)} + n_f^2 \gamma^{(3,2)}$$

$$\gamma_{31}^{(3,0)} = \begin{pmatrix} \frac{36623912}{54675} & 0 \\ -\frac{2430374}{3645} & \frac{261063}{50} \end{pmatrix} \qquad \gamma_{51}^{(3,0)} = \begin{pmatrix} \frac{8049304723}{31255875} & 0 \\ -\frac{26632998209}{112521150} & \frac{2829671009}{329280} \end{pmatrix},$$

$$\gamma_{53}^{(3,0)} = \begin{pmatrix} \frac{320657981731}{520931250} & 0 \\ -\frac{29333397389}{20837250} & \frac{14378664569}{6860000} \end{pmatrix} \qquad \gamma_{71}^{(3,0)} = \begin{pmatrix} \frac{7192640196053}{56710659600} & 0 \\ \frac{52031947546}{506345175} & \frac{49155659027}{3969000} \end{pmatrix},$$

$$\gamma_{73}^{(3,0)} = \begin{pmatrix} \frac{159898280729473}{525098700000} & 0 \\ -\frac{5108698450661}{3750705000} & \frac{832037077}{441000} \end{pmatrix} \qquad \gamma_{75}^{(3,0)} = \begin{pmatrix} \frac{220023775251709}{396974617200} & 0 \\ -\frac{10780083012803}{7088832450} & \frac{16149051685793}{9724050000} \end{pmatrix}$$

etc.

Summary:

> three-loop off-forward vector ADs obtained using three-loop two-particle correlators

- Gauge-invariant two-point functions
- Simple algorithmic implementation
- Axial-vector flavor singlets — work in progress
- Straightforward to extend to four loops (MINCER $\to$ FORCER)

- $2N - 2$ open indices, progressing to high N calculationally expensive
- Restore evolution kernels from AD matrices?