Progress on the next-to-next-to-leading formulation of the BFKL approach

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based on
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Outline

Outline

Introduction

BFKL approach

- Reggeization
- BFKL in the LLA
- BFKL in the NLLA
- Beyond the NLLA

Introduction

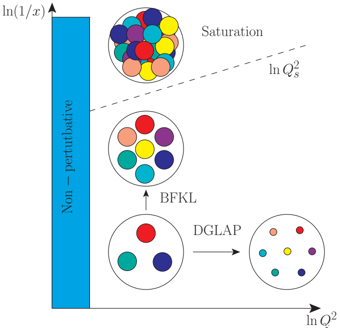
- **Semi-hard** collision process

$$s \gg Q^2 \gg \Lambda^2_{\rm QCD}$$

$\nearrow$

Regge kinematic region

$$\alpha_s(Q^2)\ \ln\left(\frac{s}{Q^2}\right) \sim 1$$



- **Linear regime** of high-energy QCD

  The **BFKL** (Balitsky-Fadin-Kuraev-Lipatov) approach

  i. Leading-Logarithmic-Approximation (**LLA**): $(\alpha_s \ln s)^n$

  ii. Next-to-Leading-Logarithmic-Approximation (**NLLA**): $\alpha_s(\alpha_s \ln s)^n$

- **Non-linear** (saturation) **regime**

  B-JIMWLK (Balitsky — Jalilian-Marian, Iancu, McLerran, Weigert, Kovner, Leonidov) evolution equations

Outline

Before QCD

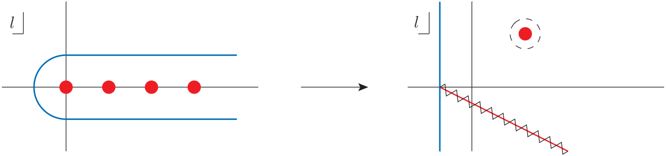
- **Partial wave expansion** ($2 \to 2$ amplitude)

$$A_{ab\to cd}(s,t) = \sum_{l=0}^{\infty}(2l+1)a_l(t)P_l\left(1+\frac{2s}{t}\right)$$

- Complex angular momenta l-plane [A. Sommerfeld (1949)]

$$A(s,t) = \frac{1}{2i}\oint_C dl(2l+1)\frac{a(l,t)}{\sin(\pi l)}P\left(l,1+\frac{2s}{t}\right)$$

- Sommerfeld-Watson transformation



- Only poles

$$A(s,t) = \frac{1}{2i}\int_{-1/2-i\infty}^{-1/2+i\infty} dl\frac{(2l+1)}{\sin(\pi l)}\sum_{\eta=\pm 1}\frac{\eta+e^{-i\pi l}}{2}a^{(\eta)}(l,t)P\left(l,1+\frac{2s}{t}\right)$$

$$+\sum_{\eta=\pm 1}\sum_{n_\eta}\frac{\eta+e^{-i\pi\alpha_{n_\eta}(t)}}{2}\frac{\bar{\beta}_{n_\eta}(t)}{\sin\pi\alpha_{n_\eta}(t)}P\left(\alpha_{n_\eta}(t),1+\frac{2s}{t}\right)$$

Reggeization

- Asymptotic behavior of Legendre Polynomial

$$P_l\left(1+\frac{2s}{t}\right)\xrightarrow[s\gg|t|]{}\frac{\Gamma(2l+1)}{\Gamma^2(l+1)}\left(\frac{s}{2t}\right)^l$$

- Asymptotic behavior of amplitudes in the Regge region

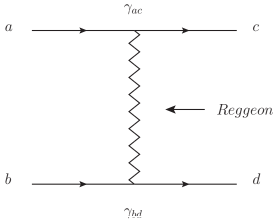
$$\mathcal{A}(s,t)\xrightarrow[s\gg|t|]{}\frac{\eta+e^{-i\pi\alpha(t)}}{2}\beta(t)s^{\alpha(t)}$$

- Definition of **Reggeization**

A particle of mass M and spin J is said to Reggeize if the amplitude, $\mathcal{A}$, for a process involving the exchange in the t-channel of the quantum numbers of that particle behaves asymptotically in s as

$$\mathcal{A}\propto s^{\alpha(t)}$$

where $\alpha(t)$ is the trajectory and $\alpha(M^2)=J$, so that the particle itself lies on the trajectory



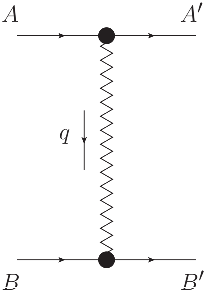
The Reggeized gluon in pQCD

- Elastic scattering process $A + B \longrightarrow A' + B'$

  i. ***Gluon quantum numbers*** in the t-channel

  ii. ***Regge limit*** $\longrightarrow s \simeq -u \to \infty$, $t = q^2$ fixed (i.e not growing with s)

  iii Valid in LLA ($\alpha_s^n \ln^n s$ resummed) and NLLA ($\alpha_s^{n+1} \ln^n s$ resummed)



$$(\mathcal{A})_{AB}^{A'B'} = \Gamma_{A'A}^{c} \left[\left(\frac{-s}{-t} \right)^{j(t)} - \left(\frac{s}{-t} \right)^{j(t)} \right] \Gamma_{B'B}^{c}$$

$$j(t) = 1 + \omega(t), \qquad j(0) = 1$$

$j(t)$-***Reggeized gluon trajectory***

$$\Gamma_{A'A}^{c} = g \langle A' | T^c | A \rangle \, \Gamma_{A'A}$$

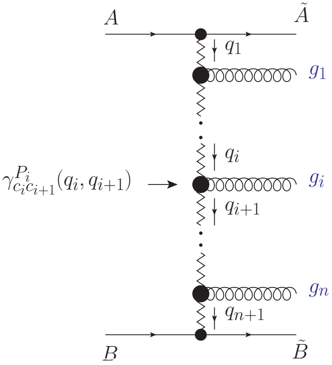
T^c- fundamental(quarks) or adjoint(gluons)

- LLA [L. N. Lipatov (1976)]

$$\Gamma_{A'A}^{(0)} = \delta_{\lambda_{A'} \lambda_A}, \quad \omega^{(1)}(t) = \frac{g^2 t}{(2\pi)^{(D-1)}} \frac{N}{2} \int \frac{d^{D-2} k_\perp}{k_\perp^2 (q-k)_\perp^2} = -g^2 \frac{N \Gamma(1-\epsilon)}{(4\pi)^{2+\epsilon}} \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)} (\vec{q}^{\,2})^{\epsilon}$$

BFKL in LLA

- Inelastic scattering process $A + B \longrightarrow \tilde{A} + \tilde{B} + n$ in the LLA



i. ***Leading-logarithm resummation***

$\downarrow$

Multi-Regge kinematics (MRK)

ii. Exchange of fermions suppressed in LLA

iii. Vertical gluons become Reggeized due to loop radiative corrections

iv. $\gamma^{P_i}_{c_i c_{i+1}}(q_i, q_{i+1}) \rightarrow$ ***Lipatov vertex***

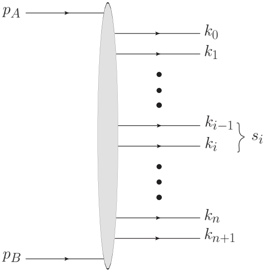
- ***Multi-Regge form of inelastic amplitudes***

$$\Re\mathcal{A}^{\tilde{A}\tilde{B}+n}_{AB} = 2s\Gamma^{c_1}_{\tilde{A}A}\left(\prod_{i=1}^{n}\gamma^{P_i}_{c_i c_{i+1}}(q_i, q_{i+1})\left(\frac{s_i}{s_0}\right)^{\omega(t_i)}\frac{1}{t_i}\right)\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{s_0}\right)^{\omega(t_{n+1})}\Gamma^{c_{n+1}}_{\tilde{B}B}$$

Multi-Regge kinematics

- ***Sudakov decomposition***

$$k_i = z_i p_A + \lambda_i p_B + k_{i\perp} \qquad p_A^2 = p_B^2 = 0$$



- ***Multi-Regge kinematics (MRK)***

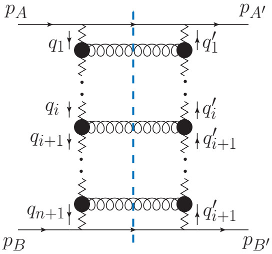
$$z_0 \gg z_1 \gg \ldots \gg z_n \gg z_{n+1}$$

$$\lambda_{n+1} \gg \lambda_n \gg \ldots \gg \lambda_1 \gg \lambda_0$$

$$k_{0\perp} \sim k_{1\perp} \sim \ldots \sim k_{n\perp} \sim k_{n+1\perp}$$

- Cutkosky rules

$$\Im \mathcal{A}_{AB}^{A'B'} = \frac{1}{2} \sum_n \int d\Phi_{\tilde{A}\tilde{B}+n} \mathcal{A}_{AB}^{\tilde{A}\tilde{B}+n} \left(\mathcal{A}_{A'B'}^{\tilde{A}\tilde{B}+n} \right)^*$$



- Integration over phase space

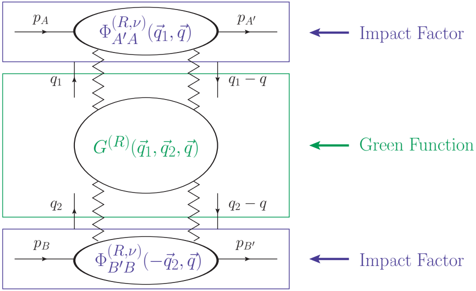
Each integration over s_i (or z_i)

$\downarrow$

One energy logarithm

BFKL resummation

- Diffusion $A + B \longrightarrow A' + B'$ in the ***Regge kinematical region***
- BFKL factorization for $\Im \mathcal{A}_{AB}^{A'B'} \to$ convolution of a ***Green function*** (process independent) with the ***Impact factors*** of the colliding particles (process dependent)



$$\Im \mathcal{A}_{AB}^{A'B'} = \frac{s}{(2\pi)^{D-2}} \int \frac{d^{D-2} q_1}{\vec{q}_1^{\,2} (\vec{q}_1 - \vec{q}\,)^{\,2}} \frac{d^{D-2} q_2}{\vec{q}_2^{\,2} (\vec{q}_2 - \vec{q}\,)^{\,2}}$$

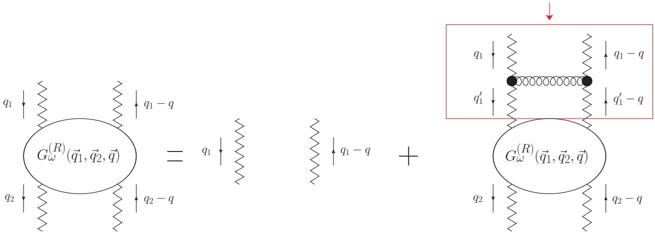
$$\times \sum_{\nu} \Phi_{A'A}^{(R,\nu)}(\vec{q}_1, \vec{q}, s_0) \int \frac{d\omega}{2\pi i} \left[\left(\frac{s}{s_0} \right)^{\omega} G_{\omega}^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}\,) \right] \Phi_{B'B}^{(R,\nu)}(-\vec{q}_2, \vec{q}, s_0)$$

- $\mathcal{R} = 1(\text{singlet}), 8^-(\text{octect}), \ldots$

BFKL resummation

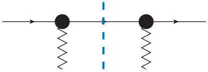
- $G_\omega^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}\,)$-Mellin transform of the Green function for the Reggeon-Reggeon scattering

$$\omega G_\omega^{(R)}(\vec{q}_1, \vec{q}_2; \vec{q}\,) = \vec{q}_1^{\,2}(\vec{q}_1 - \vec{q}\,)^2 \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + \int \frac{d^{D-2}q_1'}{\vec{q}_1^{\,\prime 2}(\vec{q}_1^{\,\prime} - \vec{q}\,)^2} \mathcal{K}^{(R)}(\vec{q}_1, \vec{q}_1^{\,\prime}; \vec{q}\,) G_\omega^{(R)}(\vec{q}_1^{\,\prime}, \vec{q}_2; \vec{q}\,)$$



- $\Phi_{P'P}^{(R,\nu)}$- LO impact factor in the t-channel color state (R, ν)

$$\Phi_{PP'}^{(R,\nu)} = \langle cc'|\hat{\mathcal{P}}|\nu\rangle \sum_{\{f\}} \int \frac{ds_{PR}}{2\pi} d\rho_f \Gamma^c_{\{f\}P} (\Gamma^{c'}_{\{f\}P'})^*$$



BFKL resummation

- ***BFKL equation*** ($\vec{q}^{\,2} = 0$ and singlet color state representation)
  [I. I. Balitsky, V. S. Fadin, E. A. Kuraev, Lipatov (1975)]

Redefinition :
$$G_\omega(\vec{q}_1,\vec{q}_2) \equiv \frac{G_\omega^{(0)}(\vec{q}_1,\vec{q}_2,0)}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}\ , \quad \mathcal{K}(\vec{q}_1,\vec{q}_2\ ) \equiv \frac{\mathcal{K}^{(0)}(\vec{q}_1,\vec{q}_2\ ,0)}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}}$$

$$\downarrow$$

$$\boxed{\omega G_\omega(\vec{q}_1,\vec{q}_2) = \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + \int d^{D-2}q_r\ \mathcal{K}(\vec{q}_1,\vec{q}_r\ )G(\vec{q}_r,\vec{q}_2)}$$

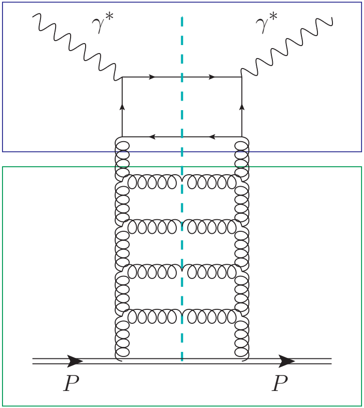
- Elastic amplitude factorization

$$\Im\mathcal{A}_{AB}^{AB} = \frac{s}{(2\pi)^{D-2}}\int d^{D-2}q_1 d^{D-2}q_2$$
$$\times\ \frac{\Phi_{AA}^{(0)}(\vec{q}_1,s_0)}{\vec{q}_1^{\,2}}\int\frac{d\omega}{2\pi i}\left[\left(\frac{s}{s_0}\right)^\omega G_\omega(\vec{q}_1,\vec{q}_2)\right]\frac{\Phi_{BB}^{(0)}(-\vec{q}_2,s_0)}{\vec{q}_2^{\,2}}$$

- ***Optical Theorem***

$$\sigma_{AB} = \frac{\Im\mathcal{A}_{AB}^{AB}}{s}$$

Description of Deep-Inelastic-Scattering



- k_T-factorization for DIS

$$\sigma_{\gamma^* P}(x) = \int \frac{d^2\vec{k}}{\vec{k}^{\,2}} \, \Phi_{\gamma^*\gamma^*}(\vec{k}\,) \mathcal{F}(x,\vec{k})$$

- Unintegrated gluon density (UGD)

$$\mathcal{F}(x,\vec{k}) = \Phi_{PP}(\vec{k}') \ \otimes_{\vec{k}'} \ G_{\rm BFKL}(x,\vec{k}',\vec{k})$$

$$\downarrow$$

$$\sigma_{\gamma^* P}(x) \sim \left(\frac{s}{Q^2}\right)^{\omega_0} = \left(\frac{1}{x}\right)^{\omega_0}$$

- Evolution equation

$$\frac{\partial \mathcal{F}}{\partial \ln(1/x)} = \mathcal{K} \otimes_{\vec{k}} \mathcal{F}$$

- Relevant BFKL kinematics

$$1 \gg x_1 \gg ... \gg x_i \gg ... \gg x_n = x \qquad \vec{k}_i^2 \sim Q^2$$

- Collinear kinematics

$$1 \geq x_1 \geq x_2 \geq ... \geq x_{n-1} \geq x_n = x \qquad 0 \ll \vec{k}_1^2 \ll \vec{k}_2^2 ... \ll \vec{k}_{n-1}^2 \ll \vec{k}_n^2 \ll Q^2$$

BFKL in the NLLA

- Simple factorized form of inelastic amplitudes

$$\downarrow$$

[V. S. Fadin, L. N. Lipatov (1989)]

Straightforward program of computations

- Resummation of subleading logarithms means a ***new kinematics***

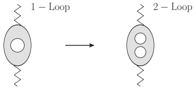
  i. Multi-Regge kinematics (MRK)

  ii. Quasi multi-Regge kinematics (QMRK)

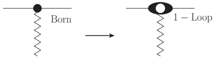
- ***Multi-Regge kinematics***

  Previous quantity must be calculated at higher loops (one α_s more)

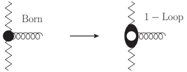
  i. $\omega^{(1)}(t) \longrightarrow \omega^{(2)}(t)$



  ii. $\Gamma^{c(0)}_{P'P} \longrightarrow \Gamma^{c(1)}_{P'P}$



  iii. $\gamma^{G_i(0)}_{c_i c_{i+1}} \longrightarrow \gamma^{G_i(1)}_{c_i c_{i+1}}$

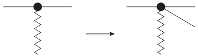


BFKL in the NLLA

- ***Quasi Multi-Regge kinematics***

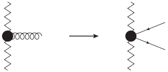
  A pair of particles (but only one!) may have longitudinal Sudakov variables of the same order (one logarithm less)

  i. $\Gamma^{c(0)}_{P'P} \longrightarrow \Gamma^{c(0)}_{\{f\}P}$



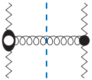
  ii. $\gamma^{G(0)}_{c_i c_{i+1}} \longrightarrow \gamma^{GG(0)}_{c_i c_{i+1}}$

  iii. $\gamma^{G(0)}_{c_i c_{i+1}} \longrightarrow \gamma^{QQ(0)}_{c_i c_{i+1}}$

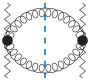


- 3 ***new contributions to the real kernel***

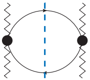
$$\mathcal{K}_r\left(\vec{q}_1, \vec{q}_2\right) = \mathcal{K}^{(1)}_{RRG}\left(\vec{q}_1, \vec{q}_2\right) + \mathcal{K}^{(0)}_{RRGG}\left(\vec{q}_1, \vec{q}_2\right) + \mathcal{K}^{(0)}_{RRQ\bar{Q}}\left(\vec{q}_1, \vec{q}_2\right).$$



1−Loop



Born

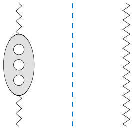


Born

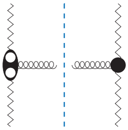
BFKL in NNLLA: Naive program

- ***Multi-Regge kinematics*** (4(5)-point amplitudes at 3(2,1)-loop)

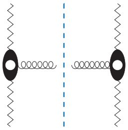
[G. Falcioni, E. Gardi, N. Maher, C. Milloy, L. Vernazza (2022)]
[F. Caola, A. Chakraborty, G. Gambuti, A. von Manteuffel, L. Tancredi (2022)]
[V. S. Fadin, M. F., A. Papa (2023)]



3-loop Regge trajectory

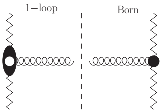


2-loop Lipatov vertex



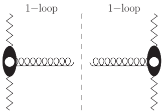
1-loop Lipatov vertex

- $\mathcal{K}_{RRG}^{1-\mathrm{loop}\times\mathrm{Born}}$ (NLO kernel)



$$\gamma = \tfrac{1}{\epsilon^2}A + \tfrac{1}{\epsilon}B + C$$

- $\mathcal{K}_{RRG}^{1-\mathrm{loop}\times 1-\mathrm{loop}}$ (NNLO kernel)



$$\boxed{\gamma = \frac{1}{\epsilon^2}A + \frac{1}{\epsilon}B + C + D\epsilon + E\epsilon^2}$$

Lipatov vertex at LO

- ***Lipatov effective vertex***

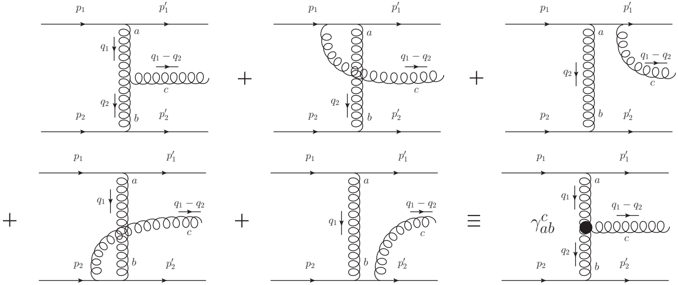
Coupling constant ↓ ; Lorentz structure ↙

$$\gamma^{G_i}_{c_i,c_{i+1}}(q_i,q_{i+1}) = gT^{d_i}_{c_i,c_{i+1}}\, e^*_\mu(p_i)\, C^\mu(q_i,q_{i+1})$$

↗ Matrix elements of $SU(N)$ generators (adjoint representation)

↖ Polarization 4-vector of the outgoing gluon

- *Gauge invariance:* $p_{i,\mu}C^\mu(q_i,q_{i+1}) = 0$



- The amplitude must be computed at NLO to extract the one-loop correction to the Lipatov vertex

Lipatov vertex at NLO

- Gauge invariant structure of the Lipatov vertex at one-loop accuracy

$$gC^{\mu,1-\text{loop}} = R^\mu + I^\mu \frac{\omega_1 - \omega_2}{4} \ln\left(\frac{s_1(-s_1)s_2(-s_2)}{s(-s)(\vec{p}^{\,2})^2}\right)$$

- Real part and imaginary of the vertex

$$R^\mu = 2g\left\{C^\mu(q_2,q_1) + \bar{g}^2\left(C^\mu(q_2,q_1)R_1(\mathcal{I}_{4A},\mathcal{I}_{4B},\mathcal{I}_5,\mathcal{L}_3) + \mathcal{P}^\mu R_2(\mathcal{I}_{4A},\mathcal{I}_{4B},\mathcal{I}_5,\mathcal{L}_3)\right)\right\}$$

$$I^\mu = 2g\bar{g}^2\left(C^\mu(q_2,q_1)I_1(\mathcal{I}_3) + \mathcal{P}^\mu I_2(\mathcal{I}_3)\right) \qquad \mathcal{P}^\mu = \frac{p_A^\mu}{s_1} - \frac{p_B^\mu}{s_2}$$

$p_\mu \mathcal{P}^\mu = 0 \rightarrow$ Gauge invariant structure

- $\mathcal{I}_3, \mathcal{I}_{4A}, \mathcal{I}_{4B}, \mathcal{I}_5, \mathcal{L}_3$ are integrals in $d = 2 + 2\epsilon$
- *Soft limit* of the emitted particle

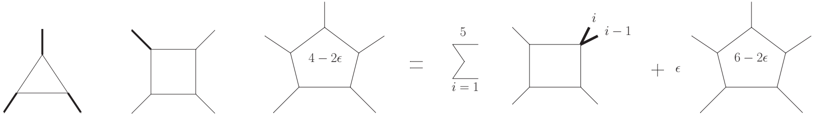
[V. S. Fadin, R. Fiore, M. Kotsky (1996)]

$$R^\mu = 2gC^\mu(q_2,q_1)\left(1 + \bar{g}^2\frac{\Gamma^2(\epsilon)}{2\Gamma(2\epsilon)}(\vec{p}^{\,2})^\epsilon\left[\psi(\epsilon) - \psi(1-\epsilon)\right]\right)$$

$$I^\mu = -\frac{2g\bar{g}^2}{\omega_1 - \omega_2}C^\mu(q_2,q_1)\frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)}(\vec{p}^{\,2})^\epsilon$$

Integrals

- ***Triangle, Box and Pentagonal integrals***



$$\hat{I}_5 = \frac{1}{2}\left[\sum_{i=1}^{5} \gamma_i \hat{I}_4^{(i)} - 2\epsilon \Delta_5 \hat{I}_5^{(D=6+2\epsilon)}\right]$$

[Z. Bern, L. Dixon, D. Kosower (1993)]

- ***Direct Feynman integration*** and ***partial differential equation*** techniques

$$\mathcal{I}_{4B} = \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)}(-t_2)^{\epsilon-1}\left[\psi(\epsilon) - \psi(2\epsilon) + \sum_{n=1}^{\infty} \epsilon^{n-1}\left(-\frac{\ln^n (t_1/t_2)}{n!} + \epsilon\, (-1)^n\, \mathrm{S}_{1,n}\left(1 - \frac{t_1}{t_2}\right)\right)\right]$$

$$\mathcal{I}_3 \sim \frac{1}{\epsilon}\hat{\mathcal{S}}\left\{(\vec{q}_2^{\,2})^{\epsilon}\left(\frac{\vec{p}^{\,2} + \vec{q}_1^{\,2} - \vec{q}_2^{\,2}}{\vec{q}_2^{\,2}\vec{q}_1^{\,2}\vec{p}^{\,2}}\right) + \frac{\epsilon^2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}\vec{p}^{\,2}}((\vec{q}_2^{\,2})^2 - \vec{q}_1^{\,2}\vec{q}_2^{\,2} - \vec{q}_2^{\,2}\vec{p}^{\,2})I_{\vec{q}_1^{\,2},\vec{q}_2^{\,2},\phi}\right\} + \ldots$$

$$I_{\vec{q}_1^{\,2},\vec{q}_2^{\,2},\phi} = -\frac{2}{|\vec{q}_1||\vec{q}_2|\sin\phi}\left[\ln\rho \arctan\left(\frac{\rho\sin\phi}{1-\rho\cos\phi}\right) + \Im\left(-\mathrm{Li}_2(\rho e^{i\phi})\right)\right] \qquad \rho = \min\left(\frac{|\vec{q}_1|}{|\vec{q}_2|}, \frac{|\vec{q}_2|}{|\vec{q}_1|}\right)$$

Pentagonal integral

- $\mathcal{I}_5 - \mathcal{L}_3 \longrightarrow$ ***Double nested harmonic sums***

  [V. Del Duca, C. Duhr, N. Glover, V. A. Smirnov (2010)]

$$S_{i\vec{j}}(n) = \sum_{k=1}^{n} \frac{S_{\vec{j}}(k)}{k^i} \qquad \mathcal{M}(\vec{i},\vec{j},\vec{k};x_1,x_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \binom{n_1+n_2}{n_1}^2 S_{\vec{i}}(n_1) S_{\vec{j}}(n_2) S_{\vec{k}}(n_1+n_2) x_1^{n_1} x_2^{n_2}$$

- A simpler form fo the ϵ-term of $\mathcal{I}_5 - \mathcal{L}_3$ is extremely desirable

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- A simpler form fo the ϵ-term of $\mathcal{I}_5 - \mathcal{L}_3$ is extremely desirable

Outlook

- In QCD higher-ϵ orders of the Lipatov vertex can be used to check the NLL and future NNLL *bootstrapt conditions*

[J. Bartels, V. Fadin, and R. Fiore (2003)]

[V.S. Fadin, R. Fiore, M.G. Kozlov, A.V. Reznichenko (2006)]

- Lipatov vertex in $\mathcal{N} = 4$ SYM is the only non-trivial ingredients to extract the high-energy behaviour ***Remainder functions*** of the ABDK-BDS ansatz

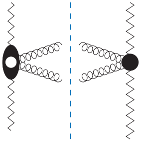
[C. Anastasiou, Z. Bern, L. Dixon, D. Kosower, V. A. Smirnov (2003-2005)]

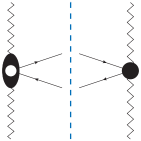
[V. S. Fadin, L. N. Lipatov (2012)]

BFKL in the NNLLA: Naive program

- ***Quasi-multi-Regge kinematics*** (1-loop 6-point amplitudes)

[E. P. Byrne, V. Del Duca, L. J. Dixon, E. Gardi, J. M. Smillie (2022)] ($\mathcal{N} = 4$)

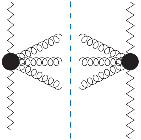


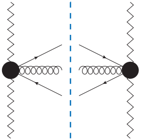


1-loop vertices for two partons not strongly ordered in rapidity

- ***Next-to-quasi-multi-Regge kinematics*** (Born 7-point amplitudes)

[V. Del Duca, A. Frizzo, F. Maltoni (2000)] [D. de Florian and J. Zurita (2006)]





Born vertices for three partons not strongly ordered in rapidity

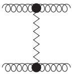
- Main challenge → carry out the phase space integration

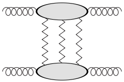
Violation of the pole Regge form

- ***Violation of the Pole Regge form in NNLA***

  [V. Del Duca, N. Glover (2001)]

$$(\mathcal{A})_{AB}^{A'B'} = \Gamma^c_{A'A}\left[\left(\frac{-s}{-t}\right)^{j(t)} - \left(\frac{s}{-t}\right)^{j(t)}\right]\Gamma^c_{B'B} + \text{multiple Reggeon exchange}$$





- Violation → ***Regge cuts*** in the complex angular momenta plane
- Cuts breaks Regge factorization and universality in QCD amplitudes
- Effective theory based on Wilson lines [S. Caron-Huot (2013)]
  i. Three reggeon and three Reggeon/single Reggeon mixing
  ii. Up to four loop computations
  iii. Separation of Regge cuts and Regge poles to all orders in pQCD based on a planar ansatz [S. Mandelstam (1963)]

  [G. Falcioni, E. Gardi, N. Maher, C. Milloy, L. Vernazza (2022)]
- QCD feynman diagrams approach [Fadin (2017-2018), Fadin-Lipatov (2018)]
  i. Three reggeon only
  ii. Up to four loop computations
  iii. Non-applicability of planar ansatz in QCD [V. S. Fadin (2023)]

Combining BFKL and DGLAP

- High-energy factorization for F_2

  [S. Catani, M. Ciafaloni, F. Hautmann (1990-1994)]

$$F_2\left(x,Q^2\right) = \int_x^1 \frac{dz}{z} \int d^2\vec{k}\, C\left(\frac{x}{z},\vec{k},Q^2\right) \mathcal{F}(z,\vec{k})$$

  i. C is an off-shell continuation of the collinear coefficient function

  ii. (Leading twist) k_T and collinear factorizations should be recovered in the respective limits

- **Altarelli-Ball-Forte (ABF)**

  i. NLLA resummation using **duality relations**

  ii. Only LO coefficient functions

- **Catani-Ciafaloni-Fiorani-Marchesini (CCFM)**

  i. Interpolation at level of diagrams using the **angular ordering** of the emissions

  ii. Only LLA evolution

- More recent approach based on Shockwave approach

  [R. Boussarie, Y. Mehtar-Tani (2021)]

  i. Operator level definition of UGD

  ii. Possibility of keeping track of multiple Reggeon exchange

Summary

- The BFKL approach gives the description of QCD-scattering amplitudes in the region $s \gg |t|$ (Regge region), with various colour states in the t channel
- The evolution equation for the UGD appears in this approach as a particular result for the imaginary part of the forward-scattering amplitude
- In the LLA and NLLA the approach is well established and some impact factors have been calculated up to NLO

  [Samuel's talk]
- The NNLL formulation requires two main ingredients:
  1. Amplitude calculations at the frontier (up to 7-point functions or up to 3-loop computations)
  2. Understanding of multi-Reggeon cuts in QCD
- There are two main approaches to treat Regge cuts
  1. Effective Wilson line approach [S. Caron-Huot (2013)]
  2. QCD Feynman diagrams approach [V. S. Fadin (2017)]
- The idead of a fully consistent interpolation between BFKL and DGLAP is fascinating, but still in early steps...

Thanks for your attention

Backup

Reggeization

Before QCD

- Assumptions on S-matrix ($S_{ab} = \langle b_{out} | a_{in} \rangle$):
  - Lorentz invariance:
    It can be expressed as a function of Lorentz invariant scalar product, e.g (s, t) for $2 \to 2$ particle scattering.
  - Analiticity
    Causality $\to$ Analytic function with only those singularity required by unitarity.
  - Unitarity

Cutkosky rules

$$2\Im\mathcal{A}_{ab} = (2\pi)^4 \delta^4 (\sum_a p_a - \sum_b p_b) \sum_c \mathcal{A}_{ac}\mathcal{A}^\dagger_{cb}$$

Optical theorem

$$2\Im\mathcal{A}_{aa}(s, 0) = F\sigma_{tot}$$

- Unitarity $\to$ relates the imaginary parts of amplitudes to sum of products of other amplitudes, **dispersion relations** $\to$ reconstruct the corresponding real parts
- More generally subtracted **dispersion relation** $\to$ we must know the asymptotic behavior of amplitudes $\to$ **Regge theory**

Positive and negative signature

- Partial wave expansion:

$$A_{ab\to cd}(s,t) = \sum_{l=0}^{\infty}(2l+1)a_l(t)P_l\left(1+\frac{2s}{t}\right)$$

- Complex angular momenta l-plane (Sommerfeld(1949))

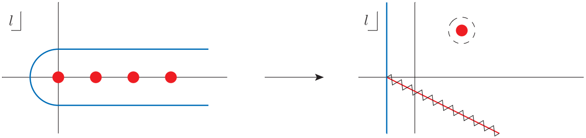
$$A(s,t) = \frac{1}{2i}\oint_C dl(2l+1)\frac{a(l,t)}{\sin(\pi l)}P\left(l,1+\frac{2s}{t}\right)$$

- $a(l,t)$ unique? $\to$ Carlson (1914)

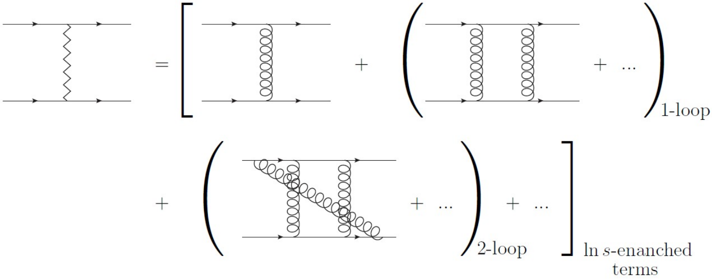
  $\to$ Contributions to partial wave amplitudes $\propto (-1)^l$

  $\to$ Two analytic functions $a^{(+)}(l,t)$ and $a^{(-)}(l,t)$ which are the analytic continuation of of even and odd partial wave amplitudes.

$$A(s,t) = \frac{1}{2i}\oint_C dl(2l+1)\sum_{\eta=\pm 1}\frac{\eta+e^{i\pi l}}{2}\frac{a^{(\eta)}(l,t)}{\sin(\pi l)}P\left(l,1+\frac{2s}{t}\right)$$



Meaning of Reggeization in pQCD



$$\Gamma^a_{1'1}\frac{s}{t}\left[\left(\frac{s}{-t}\right)^{\omega(t)}+\left(\frac{-s}{-t}\right)^{\omega(t)}\right]\Gamma^a_{2'2}\simeq\left\{\Gamma^{a(0)}_{1'1}\frac{2s}{t}\Gamma^{a(0)}_{2'2}\right\}_{LLA}$$

$$+\left\{\Gamma^{a(0)}_{1'1}\frac{s}{t}\left[\omega^{(1)}(t)\ln\left(\frac{s}{-t}\right)+\omega^{(1)}(t)\ln\left(\frac{-s}{-t}\right)\right]\Gamma^{a(0)}_{2'2}\right\}_{LLA}+\left\{\Gamma^{a(1)}_{1'1}\frac{2s}{t}\Gamma^{a(0)}_{2'2}+\Gamma^{a(0)}_{1'1}\frac{2s}{t}\Gamma^{a(1)}_{2'2}\right\}_{NLLA}$$

$$+\left\{\Gamma^{a(0)}_{1'1}\frac{s}{t}\left[\frac{(\omega^{(1)}(t))^2}{2}\ln^2\left(\frac{s}{-t}\right)+\frac{(\omega^{(1)}(t))^2}{2}\ln^2\left(\frac{-s}{-t}\right)\right]\Gamma^{a(0)}_{2'2}\right\}_{LLA}$$

$$+\left\{\Gamma^{a(1)}_{1'1}\frac{s}{t}\omega^{(1)}(t)\left[\ln\left(\frac{s}{-t}\right)+\ln\left(\frac{-s}{-t}\right)\right]\Gamma^{a(0)}_{2'2}+\Gamma^{a(0)}_{1'1}\frac{s}{t}\omega^{(1)}(t)\left[\ln\left(\frac{s}{-t}\right)+\ln\left(\frac{-s}{-t}\right)\right]\Gamma^{a(1)}_{2'2}\right.$$

$$\left.+\Gamma^{a(0)}_{1'1}\frac{s}{t}\left[\omega^{(2)}(t)\ln\left(\frac{s}{-t}\right)+\omega^{(2)}(t)\ln\left(\frac{-s}{-t}\right)\right]\Gamma^{a(0)}_{2'2}\right\}_{NLLA}+\left\{\Gamma^{a(2)}_{1'1}\frac{2s}{t}\Gamma^{a(0)}_{2'2}+\dots\right\}_{NNLLA}$$

BFKL approach

Solution of the BFKL equation

- Let's solve the equation

$$\omega G_\omega(\vec{q}_1, \vec{q}_2) = \delta^{(D-2)}(\vec{q}_1 - \vec{q}_2) + \int d^{D-2} q_r \, \mathcal{K}(\vec{q}_1, \vec{q}_r \,) G(\vec{q}_r, \vec{q}_2)$$

$$\mathcal{K}(\vec{q}_1, \vec{q}_r) = \mathcal{K}^{(R)}(\vec{q}_1, \vec{q}_r) + 2\omega(\vec{q}_1^{\,2})\delta^{(2)}(\vec{q}_1 - \vec{q}_r)$$

- We can see $\mathcal{K}(\vec{k}, \vec{k}')$ as the integral kernel of an operator acting on a space of complex functions (defined on a bi-dimensional vector space)

$$\hat{\mathcal{K}} \left[f(\vec{k}) \right] = \int d^2 \vec{k}' \mathcal{K}(\vec{k}, \vec{k}') \, f(\vec{k}')$$

- We solve the eigenvalue problem for the Kernel

  Eigenvalues $\longrightarrow \omega_n(\nu) = \bar{\alpha}_s \chi_n(\nu) \;, \quad \bar{\alpha}_s = \frac{\alpha_s N}{\pi}$

  Eigenfunctions $\longrightarrow \phi_\nu^n(\vec{q}\,) = \frac{1}{\pi\sqrt{2}} \left(\vec{q}^{\,2}\right)^{-\frac{1}{2}+i\nu} e^{in\theta}$

- Then we are able to reconstruct the $G_\omega(\vec{q}_1, \vec{q}_2)$

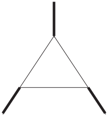
$$G_\omega(\vec{q}_1, \vec{q}_2) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\nu \left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}} \right)^{i\nu} \frac{e^{in(\theta_1 - \theta_2)}}{2\pi^2 q_1 q_2} \frac{1}{\omega - \bar{\alpha}_s \chi(n, \nu)} \longrightarrow G_s(\vec{q}_1, \vec{q}_2) \sim s^{\omega_0}$$

$\omega_0 = 4\bar{\alpha}_s \ln 2 \simeq 0.40$ for $\alpha_s = 0.15$

Lipatov vertex

Triangle integral with three external scales

- ***Massless triangle with three external scales***



$$\mathcal{I}_3 = \int \frac{d^{2+2\epsilon} k}{\pi^{1+\epsilon}\Gamma(1-\epsilon)} \frac{1}{\vec{k}^{\,2}(\vec{k}-\vec{q}_1)^2(\vec{k}-\vec{q}_2)^2}$$

- Solution by **direct Feynman integration (dFi)** at order ϵ

$$\mathcal{I}_3 \sim \frac{1}{\epsilon}\hat{\mathcal{S}}\left\{(\vec{q}_2^{\,2})^\epsilon\left(\frac{\vec{p}^{\,2}+\vec{q}_1^{\,2}-\vec{q}_2^{\,2}}{\vec{q}_2^{\,2}\vec{q}_1^{\,2}\vec{p}^{\,2}}\right)+\frac{\epsilon^2}{\vec{q}_1^{\,2}\vec{q}_2^{\,2}\vec{p}^{\,2}}((\vec{q}_2^{\,2})^2-\vec{q}_1^{\,2}\vec{q}_2^{\,2}-\vec{q}_2^{\,2}\vec{p}^{\,2})I_{\vec{q}_1^{\,2},\vec{q}_2^{\,2},\phi}\right\}+\dots$$

$$I_{\vec{q}_1^{\,2},\vec{q}_2^{\,2},\phi} = -\frac{2}{|\vec{q}_1||\vec{q}_2|\sin\phi}\left[\ln\rho\arctan\left(\frac{\rho\sin\phi}{1-\rho\cos\phi}\right)+\Im\left(-\mathrm{Li}_2(\rho e^{i\phi})\right)\right] \qquad \rho = \min\left(\frac{|\vec{q}_1|}{|\vec{q}_2|},\frac{|\vec{q}_2|}{|\vec{q}_1|}\right)$$

- All order solution by the **Bern-Dixon-Kosower (BDK)** method

[Bern, Dixon, Kosower (1993)]

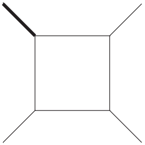
$$\mathcal{I}_3 = \alpha_1\alpha_2\alpha_3\left(-\frac{1}{2}\frac{\Gamma(2-\epsilon)\Gamma^2(\epsilon)}{\Gamma(1-\epsilon)\Gamma(2\epsilon-1)}\right)\frac{\hat{\Delta}_3^{1/2-\epsilon}}{(1-\epsilon)^2}\left[f(\delta_1)+f(\delta_2)+f(\delta_3)+c\right]$$

$$f(\delta) = \frac{1}{i}\left[\left(\frac{1+i\delta}{1-i\delta}\right)^{1-\epsilon}{}_2F_1\left(2-2\epsilon,1-\epsilon,2-\epsilon;-\left(\frac{1+i\delta}{1-i\delta}\right)\right)+(\delta\to-\delta)\right]$$

i. $\hat{\Delta}_3, \alpha_i's, \delta_i's$ are functions of $\vec{q}_1^{\,2}, \vec{q}_2^{\,2}, \vec{p}^{\,2}$

ii. The solution can be expanded at ϵ^2 and beyond

Box integrals with one external mass



- ***Box related integral*** $\mathcal{I}_{4B}(\mathcal{I}_{4A})$

$$\mathcal{I}_{4B} = \int_0^1 \frac{dx}{x} \int \frac{d^{D-2}k}{\pi^{1+\epsilon}\Gamma(1-\epsilon)} [f(x) - f(0)]$$

$$f(x) = \frac{1-x}{\left(x\vec{k}^2 + (1-x)(\vec{k}-\vec{q}_1)^2\right)(\vec{k}-(1-x)(\vec{q}_1-\vec{q}_2))^2}$$

- Exact solution by dFi

$$\mathcal{I}_{4B} = \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)}(-t_2)^{\epsilon-1}\left[\psi(\epsilon) - \psi(2\epsilon) + \sum_{n=1}^{\infty} \epsilon^{n-1}\left(-\frac{\ln^n(t_1/t_2)}{n!} + \epsilon\,(-1)^n\,\mathrm{S}_{1,n}\left(1-\frac{t_1}{t_2}\right)\right)\right]$$

- Relation between ***massless box with one external scale*** (MRK) and $\mathcal{I}_{4B}$

$$I_{4B} = -\frac{\pi^{2+\epsilon}\Gamma(1-\epsilon)}{s_2}\left[\frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)}(-t_2)^{\epsilon-1}\left(\ln\left(\frac{-s_2}{-t_2}\right) + \psi(1-\epsilon) - 2\psi(\epsilon) + \psi(2\epsilon)\right) + \mathcal{I}_{4B}\right]$$

- I_{4B} can be solved exactly (in ϵ and kinematics) by dFi or BDK method

$$\mathcal{I}_{4B} = \frac{\Gamma^2(1+\epsilon)}{\Gamma(1+2\epsilon)}\frac{2}{\epsilon^2}\frac{(-t_1)^{\epsilon}}{-t_2}\left[\left(\frac{t_2}{t_1}\right)^{\epsilon}\left(-\frac{1}{2} + \frac{\pi\epsilon}{\sin(\pi\epsilon)}\cos(\pi\epsilon) - \epsilon\ln\left(1-\frac{t_2}{t_1}\right)\right.\right.$$
$$\left.\left. + \sum_{n=2}^{\infty}\epsilon^n\zeta(n)\left(1-(-1)^n\left(2^{n-1}-1\right)\right)\right) - 1 + \epsilon\ln\left(1-\frac{t_1}{t_2}\right) + \sum_{n=2}^{\infty}(-\epsilon)^n\mathrm{Li}_n\left(\frac{t_1}{t_2}\right)\right]$$

Pentagonal integral

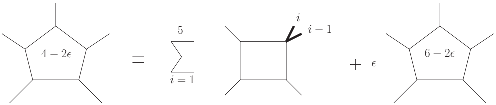
- $\mathcal{I}_5, \mathcal{L}_3$ are quite complicated but they appears in combination in the vertex

$$I_5 = \frac{\pi^{2+\epsilon}\Gamma(1-\epsilon)}{s}\left[\ln\left(\frac{(-s)(\vec{q}_1-\vec{q}_2)^2}{(-s_1)(-s_2)}\right)\mathcal{I}_3 + \mathcal{L}_3 - \mathcal{I}_5\right]$$

- The ***massless pentagonal*** integral satisfy the iterative relation

[Bern, Dixon, Kosower (1993)]

$$\hat{I}_5 = \frac{1}{2}\left[\sum_{i=1}^{5}\gamma_i \hat{I}_4^{(i)} - 2\epsilon\Delta_5 \hat{I}_5^{(D=6+2\epsilon)}\right]$$



- All five divergent boxes can be computed exactly, e.g.

$$\hat{I}_4^{(1)}(s_1, s_2, s) \simeq \frac{\Gamma(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1+2\epsilon)}\frac{2}{\epsilon^2}\left(\frac{(-s_1)(-s_2)}{(-s)}\right)^{\epsilon}\left\{1 + \sum_{n=1}^{\infty}\epsilon^{2n}\, 2\left(1 - \frac{1}{2^{2n-1}}\right)\zeta(2n)\right\}$$

- Finite pentagon in $6-2\epsilon \longrightarrow$ ***Double nested harmonic sums***

[Del Duca, Duhr, Glover, Smirnov (2010)]

$$S_{i\vec{j}}(n) = \sum_{k=1}^{n}\frac{S_{\vec{j}}(k)}{k^i} \qquad \mathcal{M}(\vec{i}, \vec{j}, \vec{k}; x_1, x_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\binom{n_1+n_2}{n_1}^2 S_{\vec{i}}(n_1)S_{\vec{j}}(n_2)S_{\vec{k}}(n_1+n_2)x_1^{n_1}x_2^{n_2}$$

Partial differential equation technique

- Massless box

$$I_4(s,t) = \Gamma(2+\epsilon)\int_0^1 d^4 a_i \delta\left(1-\sum_i a_i\right)\frac{1}{[-s a_1 a_3 - t a_2 a_4]^{2+\epsilon}}$$

- Definition of an ausiliar integral

$$J_{4;1} \equiv \Gamma(1+\epsilon)\int_0^1 da_3 \int_0^{1-a_3} da_2 \int_0^{1-a_2-a_3} da_1 \frac{\partial}{\partial a_1}\frac{1}{[-s a_1 a_3 - t a_2 (1-a_1-a_2-a_3))]^{1+\epsilon}}$$

- Evaluation the integrand at the boundaries $a_1 = 1 - a_2 - a_3$ and $a_1 = 0$

$$J_{4;1} \equiv \Gamma(1+\epsilon)\int_0^1 da_3 \int_0^{1-a_3} da_2 \int_0^{1-a_2-a_3} da_1 \frac{\partial}{\partial a_1}\frac{1}{[-s a_1 a_3 - t a_2 (1-a_1-a_2-a_3))]^{1+\epsilon}}$$

$$= \Gamma(1+\epsilon)\int_0^1 da_1 da_2 da_3 \frac{\delta\left(1-\sum_{i=1}^3 a_i\right)}{[-s a_1 a_3]^{1+\epsilon}} - \Gamma(1+\epsilon)\int_0^1 da_2 da_3 da_4 \frac{\delta\left(1-\sum_{i=2}^4 a_i\right)}{[-t a_2 a_4]^{1+\epsilon}}$$

- Triangular integrals lead to

$$J_{4;1} = \frac{r_\Gamma}{\epsilon^2}\left((-s)^{-1-\epsilon} - (-t)^{-1-\epsilon}\right)$$

Partial differential equation technique

- Evaluation by explicit differentiation

$$J_{4;1} = -\frac{1}{2st}\left(s^2\frac{\partial \hat{I}_4}{\partial s} - t^2\frac{\partial \hat{I}_4}{\partial t}\right) \qquad \hat{I}_4 = \frac{I_4}{st}$$

- First partial differential equation

$$\boxed{s^2\frac{\partial \hat{I}_4}{\partial s} - t^2\frac{\partial \hat{I}_4}{\partial t} = -\frac{2r_\Gamma}{\epsilon}st\left((-s)^{-1-\epsilon} - (-t)^{-1-\epsilon}\right)}$$

- Second differential equation

$$\boxed{s\frac{\partial \hat{I}_4}{\partial s} + t\frac{\partial \hat{I}_4}{\partial t} = -\epsilon\hat{I}_4}$$

- One can solve the complete system of differential equations to get

$$\hat{I}_4 = \frac{2r_\Gamma}{\epsilon^2}\left[t^{-\epsilon}{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;1+\frac{t}{s}\right) + s^{-\epsilon}{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;1+\frac{s}{t}\right)\right]$$

BFKL and ADBK-BDS

- **ABDK-BDS ansatz**: Iterative structure of higher loop amplitudes with maximal helicity violation (MHV) in Yang-Mills theories with maximal supersymmetry ($\mathcal{N} = 4$ SYM) in the planar limit

  [Anastasiou, Bern, Dixon, Kosower, Smirnov (2003-2005)]
- The ansatz is violated $\longrightarrow$ ***Remainder functions***
- *Hypothesis of dual conformal invariance*: MHV amplitudes are given by products of the BDS amplitudes and the remainder functions depend only on anharmonic ratios of kinematic invariants

$$M_{\rm MHV} = R\, M_{\rm BDS}$$

  M_{BDS} contains all the infrared singularities
- *Hypothesis of scattering amplitude/Wilson loop correspondence*: Remainder functions are given by the expectation values of Wilson loops
- **Different hypothesis can be tested by the BFKL approach**

$$\downarrow$$

[Fadin, Lipatov (2012)]

$$Re^{i\pi\delta} = \cos\pi\omega_{ab} + i\frac{a}{2}\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{w}{w^*}\right)^{\frac{n}{2}}\int_{-\infty}^{\infty}\frac{|w|^{2i\nu}d\nu}{\nu^2+\frac{n^2}{4}}\Phi_{Reg}(\nu,n)\left(-\frac{1}{\sqrt{u_2u_3}}\right)^{\omega(\nu,n)}$$

- Product of Reggeon-gluon transition impact factors
- One-loop Lipatov vertex in $\mathcal{N} = 4$ SYM is the only non-trivial ingredients

Wilson lines approach to high-energy amplitudes

- Eikonal nature of the interaction → ***Path-ordered Wilson lines***

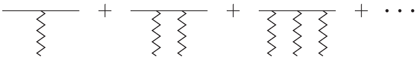
$$U^{\eta}_{\vec{z}_i} = \mathcal{P} \exp \left[ig \mathbf{T}^a \int_{-\infty}^{+\infty} dz_i^+ A_\eta^{-a} \left(z_i^+, z_i^- = 0, \vec{z}_i \right) \right]$$

- Construction of Reggeized gluon field W^a

$$\frac{d}{d\eta} W^a(p) = \omega(p) W^a(p) + O\left(g^4 W^3\right)$$

- Expansion of the Wilson line in terms of this latter operator

$$U = \exp(ig\mathbf{T}^a W^a) \sim$$



- Amplitude written as

$$\mathcal{M}_{ij\to ij} = \langle \psi_j | e^{-H\eta} | \psi_i \rangle \qquad\qquad |\psi_i\rangle = \sum_{n=1}^{\infty} (r_\Gamma \alpha_s)^{(n-1)/2} \, |i_n\rangle$$

$|\psi_i\rangle$ $(|\psi_j\rangle)$ → projectile (target) $\quad$ $|i_n\rangle$ → state with n number of Reggeons

- Evolution determined by the Balitsky-JIMWLK Hamiltonian re-written in terms of functional derivatives of W^a

Shockwave approach

Saturation physics

- ***Martin-Froissart bound***

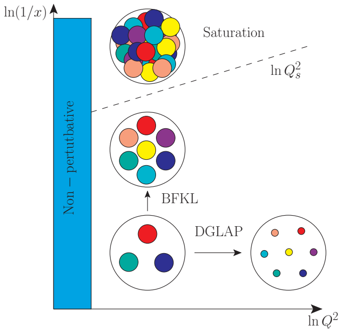
$$\sigma_{tot} \lesssim c \ln^2 s$$

- DIS total cross-section

$$\sigma_{\gamma^* P}(x) = \Phi_{\gamma^*\gamma^*}(\vec{k}\,) \otimes_{\vec{k}} \mathcal{F}(x, \vec{k})$$

$$\downarrow$$

$$\sigma_{\gamma^* P}(x) \sim \left(\frac{s}{Q^2}\right)^{\omega_0} = \left(\frac{1}{x}\right)^{\omega_0}$$



- ***Saturation effects***
  - *i.* Very dense system $\Longrightarrow$ Recombination effects
  - *ii.* In large nuclei $\Longrightarrow$ Multiple re-scattering ($\alpha_s^2 A^{1/3}$ resummation)

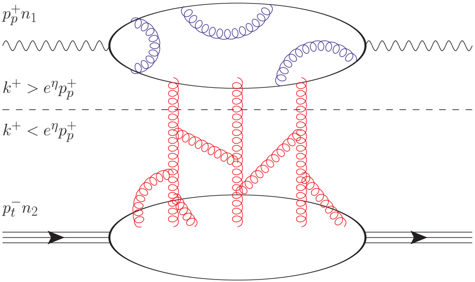
- Characteristic ***Saturation scale***

$$Q_s^2 \sim \left(\frac{A}{x}\right)^{1/3} \Lambda^2_{\text{QCD}} \qquad \alpha_s(Q_s^2) \ll 1 \implies \textit{Weakly coupled QCD}$$

Saturation window: $Q^2 < Q_s^2$

Shockwave approach

- High-energy approximation $s = (p_p + p_t)^2 \gg \{Q^2\}$
- n_1^μ, n_2^μ are light-cone vectors (+/- directions)



- Separation of the gluonic field into "fast" (quantum) part and "slow" (classical) part through a rapidity parameter $\eta < 0$

[Balitsky (2001)]

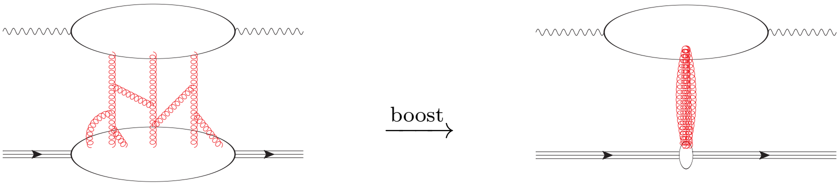
$$\mathcal{A}^\mu(k^+, k^-, \vec{k}) = A^\mu(k^+ > e^\eta p_p^+, k^-, \vec{k}) + b^\mu(k^+ < e^\eta p_p^+, k^-, \vec{k})$$

$e^\eta \ll 1$

Shockwave approach

- Large longitudinal Boost: $\Lambda = \sqrt{\frac{1+\beta}{1-\beta}} \sim \frac{\sqrt{s}}{m_t}$

$$\begin{cases} b^+(x^+, x^-, \vec{x}) & = \Lambda^{-1} b_0^+(\Lambda x^+, \Lambda^{-1} x^-, \vec{x}) \\ b^-(x^+, x^-, \vec{x}) & = \Lambda b_0^-(\Lambda x^+, \Lambda^{-1} x^-, \vec{x}) \\ b^i(x^+, x^-, \vec{x}) & = b_0^i(\Lambda x^+, \Lambda^{-1} x^-, \vec{x}) \end{cases}$$



$b_0^\mu(x)$

$b^\mu(x^+, x^-, \vec{x}) = \delta(x^+)\mathbf{B}(\vec{x}) n_2^\mu + \mathcal{O}\left(\Lambda^{-1}\right)$

Shockwave approximation

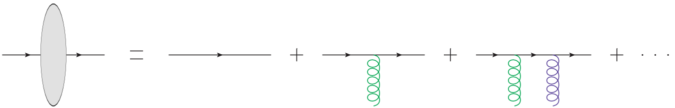
- Light-cone gauge $A \cdot n_2 = 0$

$$A \cdot b = 0 \implies \textit{Simple effective Lagrangian}$$

Shockwave approach

- Multiple interactions with the target → ***path-ordered Wilson lines***

$$U^{\eta}_{\vec{z}_i} = \mathcal{P}\exp\left[ig\int_{-\infty}^{+\infty} dz_i^+ b_\eta^-\left(z_i^+, \vec{z}_i\right)\right]$$



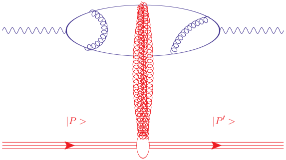
$$U_{\vec{z}_i} = 1 + ig\int_{-\infty}^{+\infty} dz_i^+ b_\eta^-\left(z_i^+, \vec{z}_i\right) + (ig)^2\int_{-\infty}^{+\infty} dz_i^+ dz_j^+ b_\eta^-\left(z_i^+, \vec{z}_i\right) b_\eta^-\left(z_j^+, \vec{z}_i\right)\theta\left(z_{ij}^+\right) + \cdots$$

- Factorization in the Shockwave approximation

$$\mathcal{M}^\eta = N_c\int d^d z_{1\perp} d^d z_{2\perp}\,\Phi^\eta(z_{1\perp}, z_{2\perp})\left\langle P'\left|\left[\frac{1}{N_c}\mathrm{Tr}\left(U^\eta_{\vec{z}_1} U^{\eta\dagger}_{\vec{z}_2}\right) - 1\right](\vec{z}_1, \vec{z}_2)\right|P\right\rangle$$

- ***Dipole operator***

$$\mathcal{U}^\eta_{ij} = \frac{1}{N_c}\mathrm{Tr}\left(U^\eta_{\vec{z}_i} U^{\eta\dagger}_{\vec{z}_j}\right) - 1$$



Balitsky-JIMWLK evolution equations

- ***Balitsky-JIMWLK evolution equations*** for the dipole
  [Balitsky, Jalilian-Marian, Iancu, McLerran, Weigert, Kovner, Leonidov]

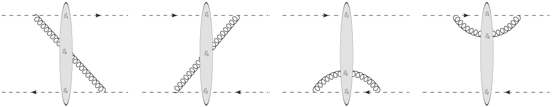
$$\frac{\partial \mathcal{U}_{12}^{\eta}}{\partial \eta} = \frac{\alpha_s N_c}{2\pi^2} \int d^2 \vec{z}_3 \left(\frac{\vec{z}_{12}^{\,2}}{\vec{z}_{23}^{\,2} \vec{z}_{31}^{\,2}} \right) \Big[\underbrace{\mathcal{U}_{13}^{\eta} + \mathcal{U}_{32}^{\eta} - \mathcal{U}_{12}^{\eta}}_{\text{BFKL}} - \mathcal{U}_{13}^{\eta} \mathcal{U}_{32}^{\eta} \Big]$$

$$\frac{\partial \mathcal{U}_{13}^{\eta} \mathcal{U}_{32}^{\eta}}{\partial \eta} = \cdots$$

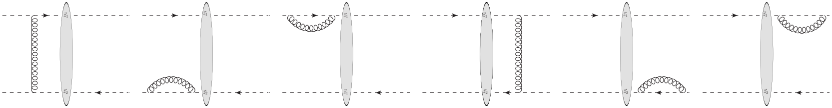
$\longleftarrow$ *Balitsky hierarchy*

$$\vdots$$

- Double dipole contribution and Dipole contribution

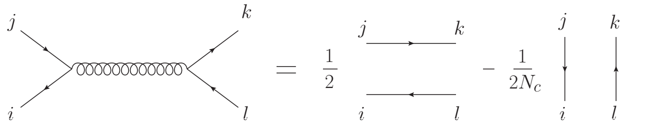


- Dipole contribution



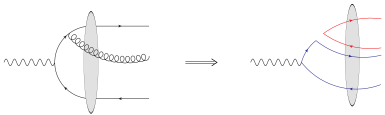
Balitsky-Kovchegov evolution equation

- Large-N_c limit ['t Hooft (1974)]



$$\boxed{t^a_{ij}t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)}$$

- Double dipole $\rightarrow$ Dipole $\times$ dipole



$$\langle \mathcal{U}^\eta_{13}\mathcal{U}^\eta_{32}\rangle \rightarrow \langle \mathcal{U}^\eta_{13}\rangle\,\langle \mathcal{U}^\eta_{32}\rangle$$

- Hierarchy of equations broken $\rightarrow$ closed non-linear BK equation
[Balitsky (1995)] [Kovchegov (1999)]

$$\frac{\partial\langle \mathcal{U}^\eta_{12}\rangle}{\partial\eta} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_3\left(\frac{\vec{z}^{\,2}_{12}}{\vec{z}^{\,2}_{23}\vec{z}^{\,2}_{31}}\right)\left[\langle \mathcal{U}^\eta_{13}\rangle + \langle \mathcal{U}^\eta_{32}\rangle - \langle \mathcal{U}^\eta_{12}\rangle - \langle \mathcal{U}^\eta_{13}\rangle\langle \mathcal{U}^\eta_{32}\rangle\right]$$

with $\langle \mathcal{U}^\eta_{12}\rangle \equiv \langle P'|\mathcal{U}^\eta_{12}|P\rangle$

Combining BFKL and DGLAP

Altarelli-Ball-Forte (ABF)

- Gluon parton distribution function expressed as

$$f(\xi, t) \qquad\qquad \xi = \ln\left(\frac{1}{x}\right) \qquad \ln\left(\frac{Q^2}{\mu}\right)$$

- BFKL equation

$$\frac{d}{d\xi} f\left(\xi, q^2\right) = \int_0^\infty \frac{dk^2}{k^2} K\left(\frac{q^2}{k^2}\right) f\left(\xi, k^2\right) \qquad\qquad \frac{d}{d\xi} f(\xi, M) = \chi\left(\alpha_s, M\right) f(\xi, M)$$

- DGLAP equation

$$\frac{d}{dt} f\left(N, q^2\right) = \gamma\left(N\right) f\left(N, q^2\right)$$

- Double Mellin transform and double logarithmic approximation (DLA)

$$\left[M - \gamma\left(\alpha_s, N\right)\right] f(N, M) = F_0(N)$$
$$\left[N - \chi\left(\alpha_s, M\right)\right] f(N, M) = \tilde{F}_0(M)$$

$$F_0(N) = \left[e^{-Mt} f(N, t)\right]_{t=-\infty}^{t=+\infty} \qquad\qquad \tilde{F}_0(M) = \left[e^{-N\xi} f(\xi, M)\right]_{\xi=0}^{\xi=+\infty}$$

Altarelli-Ball-Forte(ABF)

- Solution

$$f(N,t) = \int_{c-i\infty}^{c+i\infty} \frac{dM}{2\pi i} e^{Mt} \frac{F_0(N)}{M - \gamma(\alpha_s, N)} = F_0(N) e^{\gamma(\alpha_s, N)t}$$

$$f(N,t) = \int_{c-i\infty}^{c+i\infty} \frac{dM}{2\pi i} e^{Mt} \frac{\tilde{F}_0(M)}{N - \chi(\alpha_s, M)} = \frac{\tilde{F}_0(\bar{M})}{-\chi'(\alpha_s, \bar{M})} e^{\overline{M}t} + \text{ higher twist}$$

- Duality relations

$$\chi(\alpha_s, \gamma(\alpha_s, N)) = N \quad \leftrightarrow \quad \gamma(\alpha_s, \chi(\alpha_s, M)) = M$$

- This relation cannot be satisfied if the two anomalous dimensions are not resummed

$$\gamma(N=1) = 0 \implies \chi(M=0) = 1$$

- Reversing the argument and imposing the duality one can resum collinear terms in the BFKL kernel or viceversa

Angular ordering

- Sequential gluon emissions in a cascade satisfy the angular constraint

$$\theta_i < \theta_{i-1}$$

- The angular ordering comes from **QCD color coherence**