Theory Aspects of Nuclear Reactions in Astrophysics

n_TOF Winter School
2024

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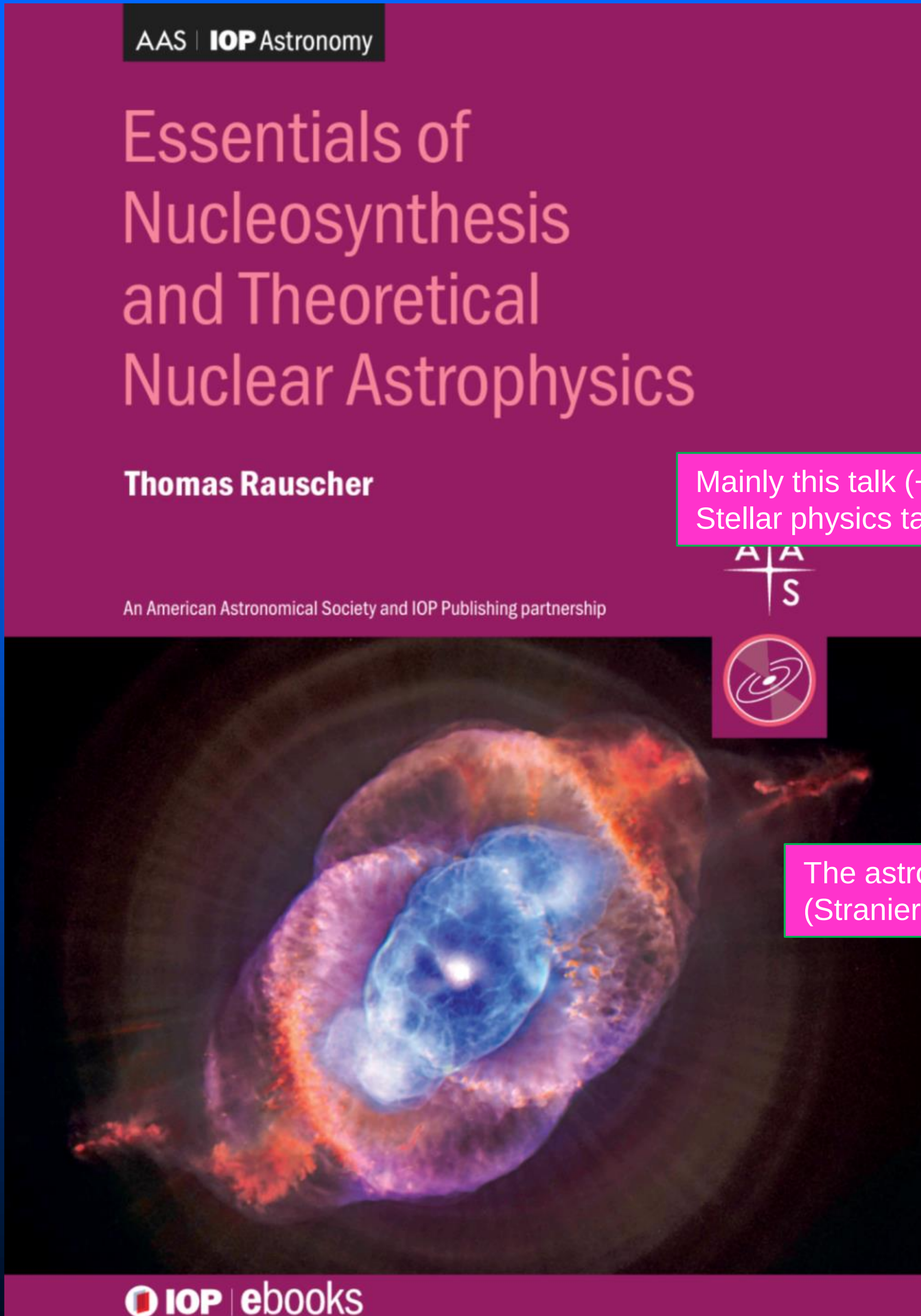

Mainly this talk (+Gunsing);
Stellar physics talk (Straniero)

The astrophysics talks
(Straniero, Pignatari)

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Contents (summary), 2 parts in 1 volume:

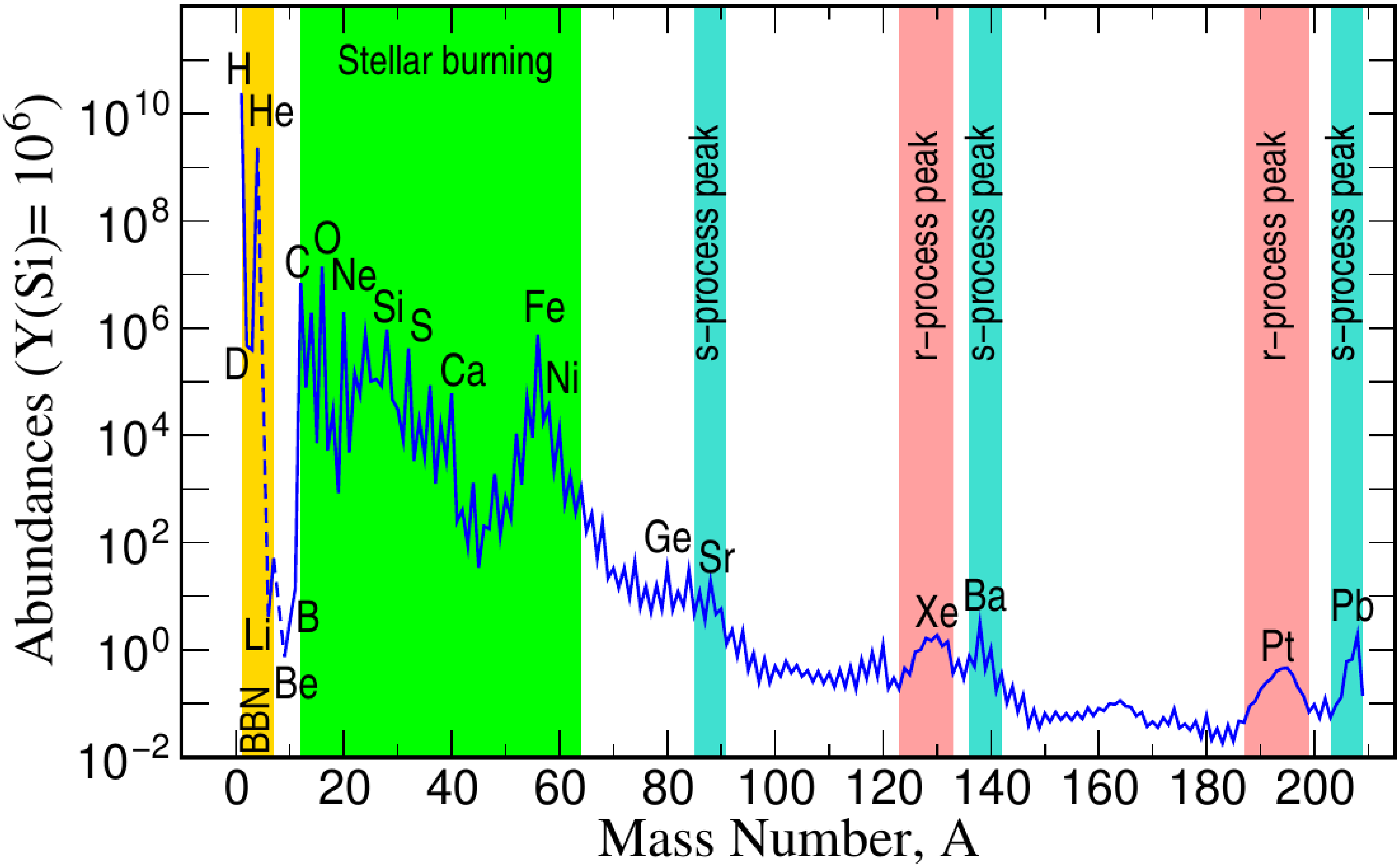
- Part 1: Essentials
  - Basic definitions, equations of state, stellar structure, nuclear physics and reactions, stellar effects on cross sections, astrophysical reaction rates, reaction networks and reaction equilibria
- Part 2: Nucleosynthesis
  - Stellar evolution, hydrostatic and explosive burning, origin of the elements beyond Fe, Big Bang nucleosynthesis, Galactic Chemical Evolution

https://iopscience.iop.org/book/978-0-7503-1149-6

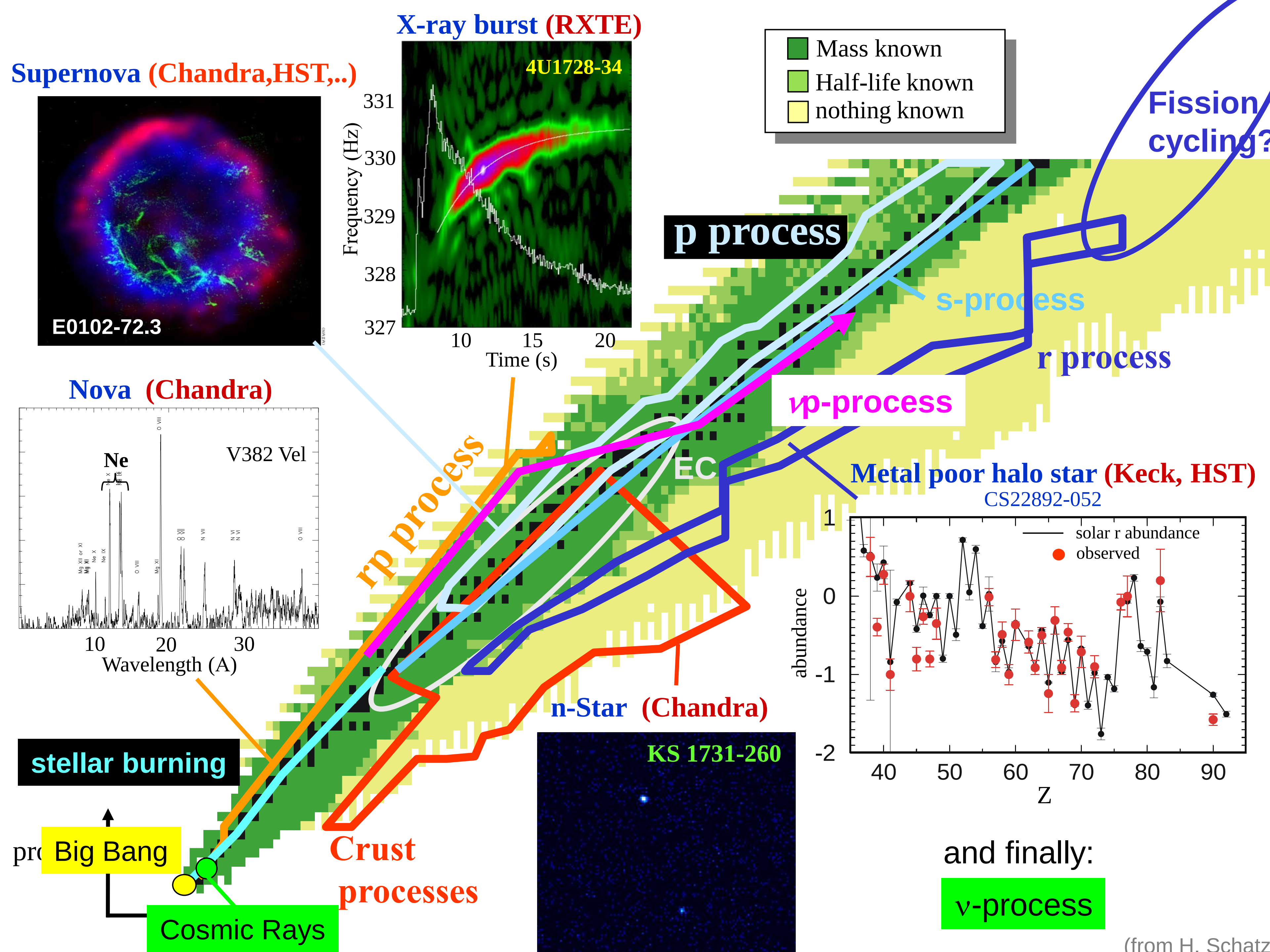
https://store.ioppublishing.org/page/detail/Essentials-of-Nucleosynthesis-and-Theoretical-Nuclear-Astrophysics/?K=9780750311496

Outline

- Nuclear reactions
  - Optical model
  - Direct reactions
  - Compound reactions
    - Resonant, statistical (Hauser-Feshbach)
- Astrophysical reaction rates
  - Reaction network
  - Definition of reaction rates
  - Relevant energy window (Gamow window)
  - Stellar Modification of reaction cross sections
    - contribution of excited target states to the stellar rate
  - Reciprocity of stellar rates
- Considerations regarding rate determinations for light and heavy nuclides

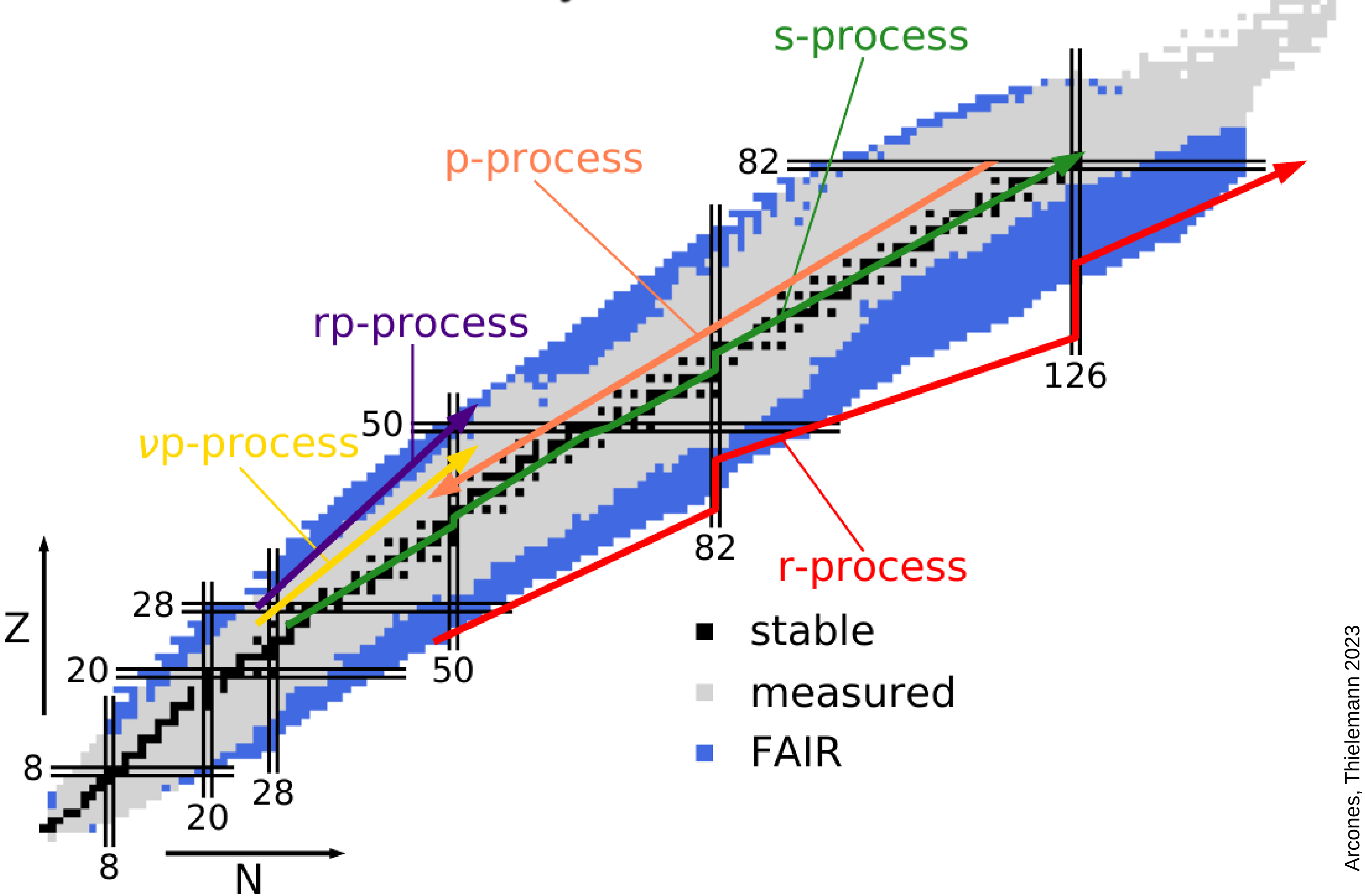

Stellar burning
s–process peak
r–process peak
s–process peak
r–process peak
s–process peak
H
He
D
Li
Be
B
C
O
Ne
Si
S
Ca
Fe
Ni
Ge
Sr
Xe
Ba
Pt
Pb
BBN
10^{10}
10^{8}
10^{6}
10^{4}
10^{2}
10^{0}
10^{-2}
0
20
40
60
80
100
120
140
160
180
200
Mass Number, A
Abundances (Y(Si)= 10^6)

Cowan+ 2022



X-ray burst (RXTE)
4U1728-34
Frequency (Hz)
Time (s)
Supernova (Chandra,HST,..)
E0102-72.3
Mass known
Half-life known
nothing known
Fission cycling?
p process
s-process
r process
νp-process
rp process
EC
Metal poor halo star (Keck, HST)
CS22892-052
solar r abundance
observed
abundance
Z
Nova (Chandra)
V382 Vel
Ne
Wavelength (A)
stellar burning
Big Bang
Cosmic Rays
Crust processes
n-Star (Chandra)
KS 1731-260
and finally:
ν-process
(from H. Schatz

Nucleosynthesis Processes

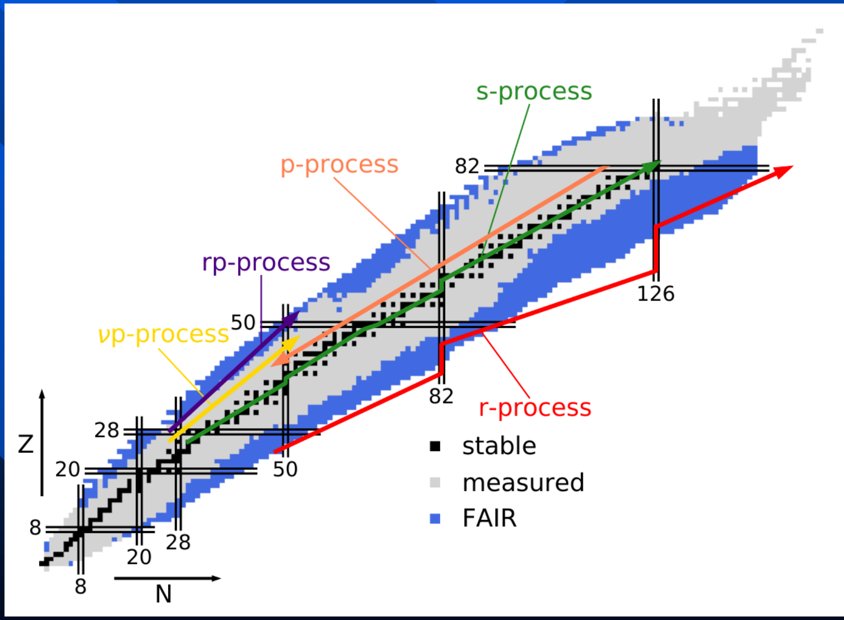


Typical plasma temperatures and nuclear energies

- Hydrostatic burning:
  - H-, He- (s-process), C-burning: 0.01 – 0.3 GK, neutrons 1 – 90 keV, protons few 100 keV, alphas few MeV
  - Late burning stage (Ne-, O-, Si-): up to few GK

- Explosive:
  - p/γ-process: 2 – 4 GK, neutrons 200 – 400 keV, protons < 6 MeV, alphas < 14 MeV
  - rp-, νp-process: 1 – 2 GK, protons < 4 MeV, alphas < 10 MeV
  - r-process: 1 GK, neutrons 100 keV
  - Others see talk by Pignatari

Note: r-process, rp- , and νp-process involve reaction equilibria, individual cross sections are not important! (see later)



Have to consider:

- What are the astrophysically relevant interaction energies (given plasma temperature)?
- What type of reaction mechanism dominates for given nucleus at energies corresponding to the stellar plasma temperature?
- Astrophysical modifications of the usual (nuclear physics) cross section?

Nuclear Reaction Basics

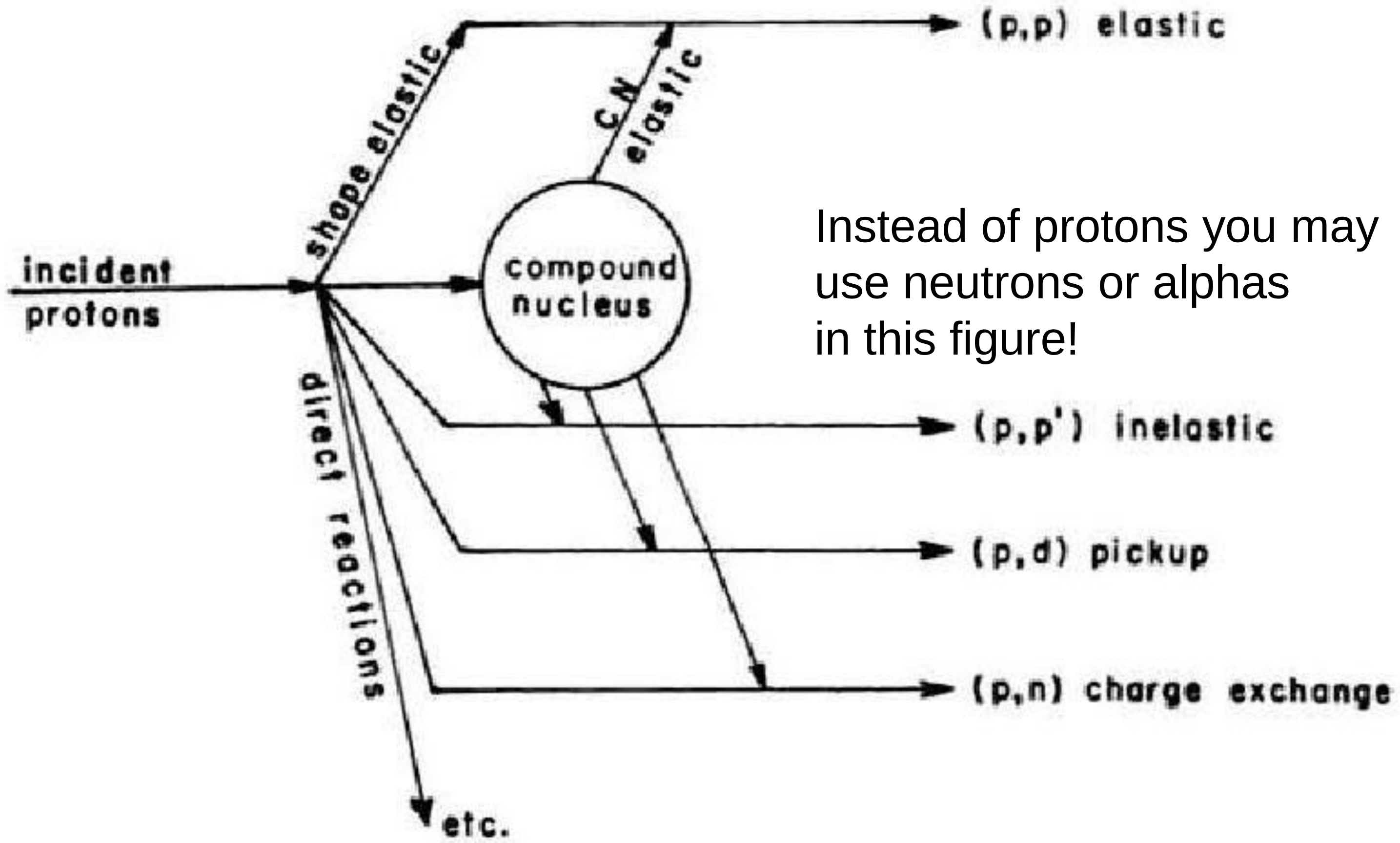
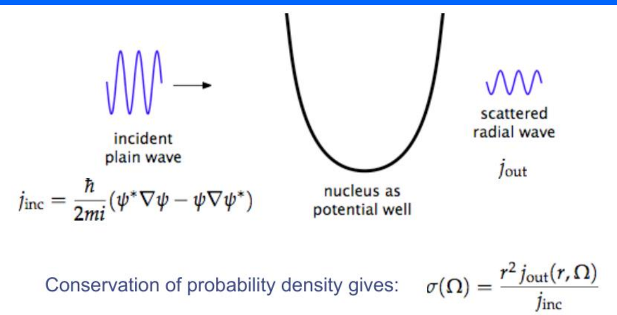


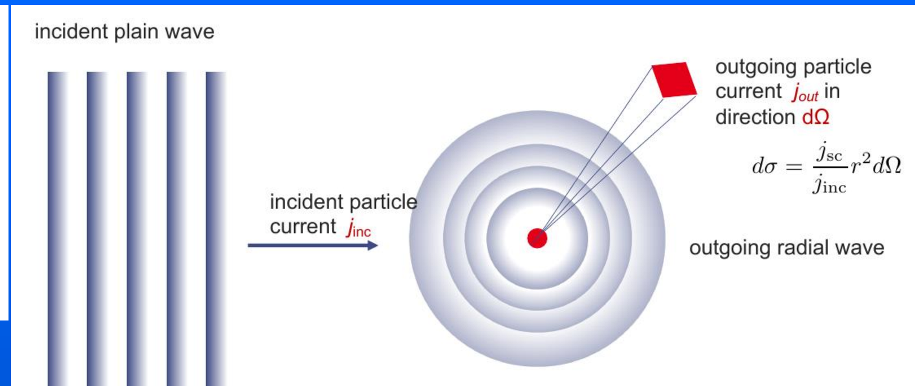
Fig. 1.2. Depiction of the processes that are typical of proton–nucleus interactions. (Adapted from P. E. Hodgson, 1971.)

Instead of protons you may use neutrons or alphas in this figure!

Optical model potential

(talk by F. Gunsing)





$$\frac{d^2 u_\ell}{dr^2} + \frac{2m}{\hbar^2}\left[E - V_{nuc}(r) - \frac{\ell(\ell+1)\hbar^2}{2\,mr^2}\right]u_\ell = 0,$$

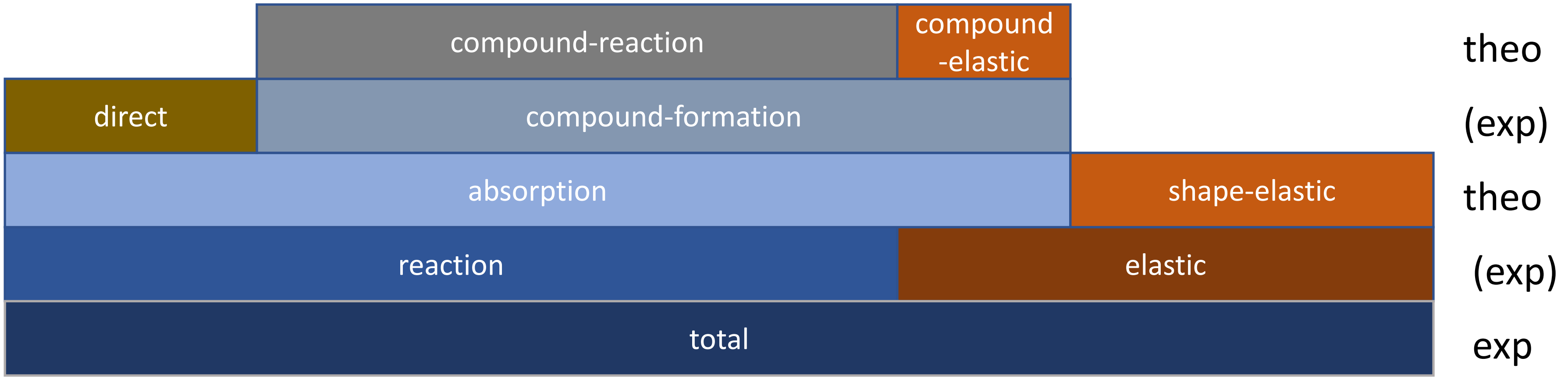
Radial time-independent Schrödinger equation for partial wave

$V_{nuc}(r)=V_{coul}(r)+Vf_{Re}(r)+Wf_{Imag}(r)$

Effective interaction potential for scattering with real and imaginary part (optical potential)

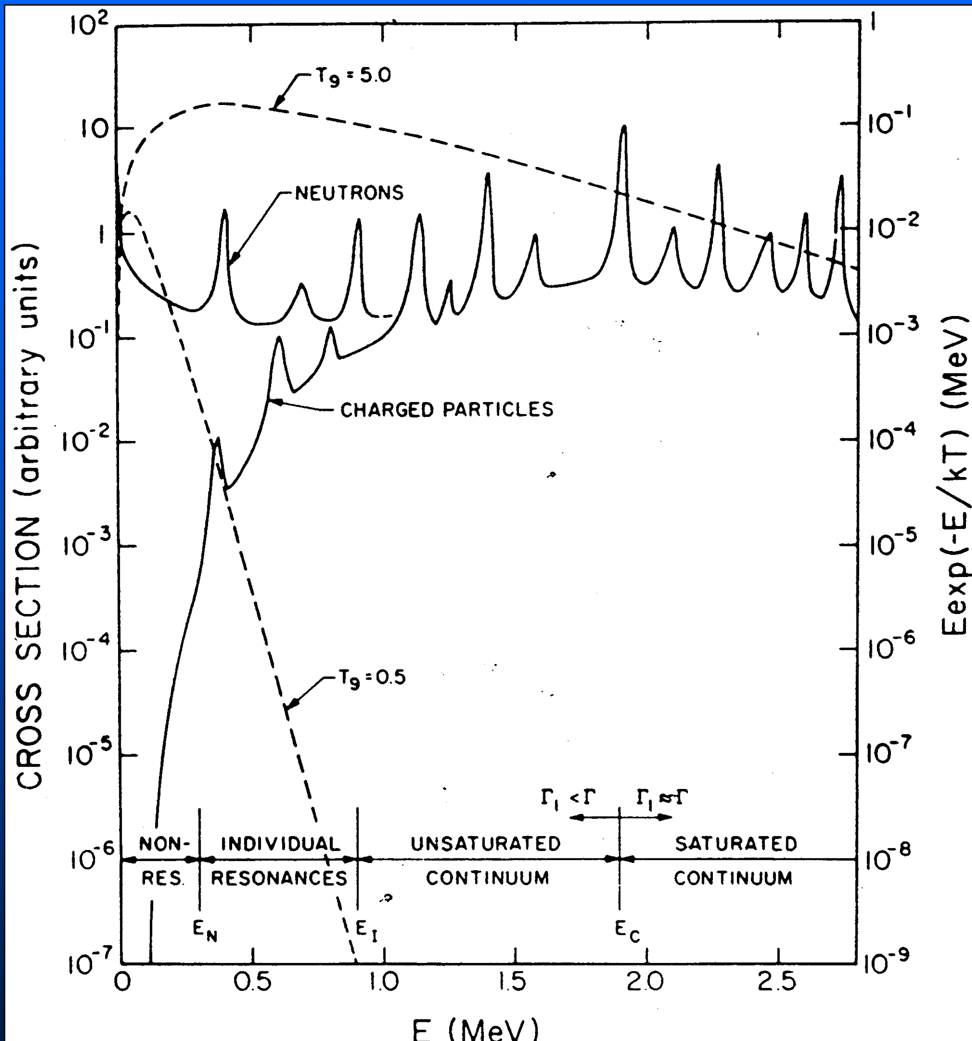
Interaction potential can be identified with a refraction index.
With real potential: flux conservation in elastic scattering; with imaginary potential: flux is lost from elastic scattering channel (absorption factor $e^{-k_2 r}$ in solution).
This is comparable to loss when shining light on opaque “crystal ball”, therefore “optical model”

Decomposition of cross sections



- Not all of these can be directly measured.
- Absorption by imaginary potential (optical model) gives split between reaction and elastic (scattering) c.s.; scattering is used to determine optical potential, but:
  - also “elastic” may include “compound-elastic”
  - does not necessarily define reaction mechanism
- Direct and compound reactions can be distinguished experimentally by angular distribution of reaction products.
  - Compound can be isolated resonances or many unresolved, “statistical” res.
- Depending on projectile energy one reaction mechanism may dominate.

Reaction Mechanisms

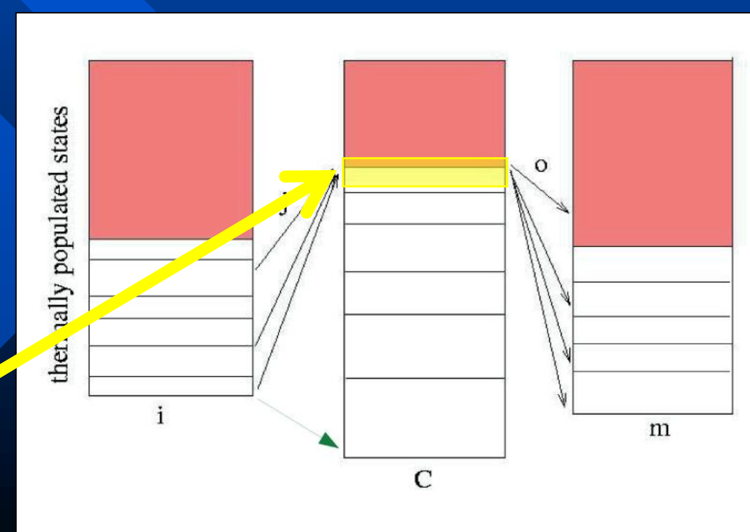


Wagoner 1959

Determined by nucl. level density

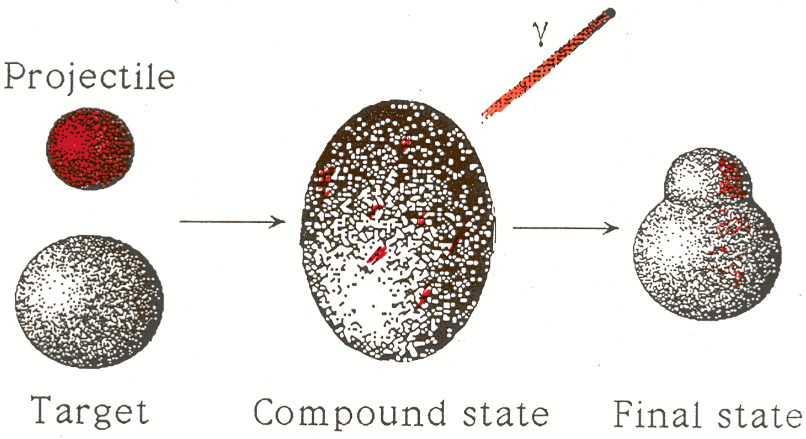
Regimes:

1. Overlapping resonances: statistical model (Hauser-Feshbach)
2. Single resonances: Breit-Wigner, R-matrix (RGM, GCM in light nuclides)
3. Without or in between resonances or at high energy: Direct reactions (DWBA, potential model)

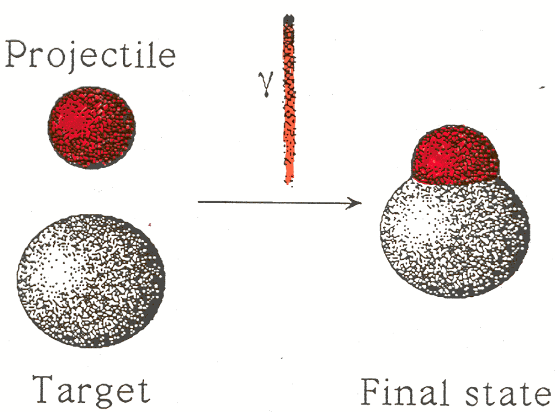


Reaction Mechanisms II

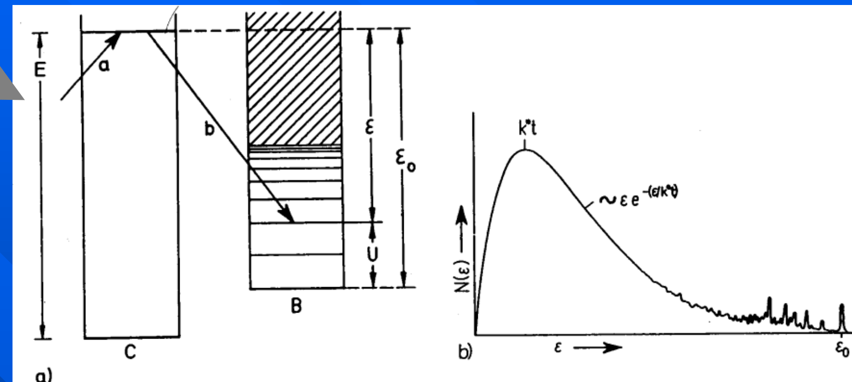
Capture via Formation of Compound State



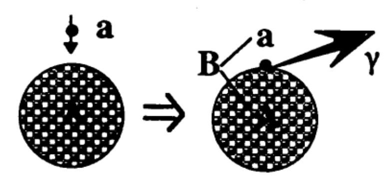
Direct Capture



$$\sigma^{\mathrm{CN}}_{\alpha\rightarrow\beta} = \sigma^{\mathrm{form}}_{\alpha} b_{\beta} = \sigma^{\mathrm{form}}_{\alpha} \frac{\langle\Gamma_{\beta}\rangle}{\langle\Gamma_{\mathrm{tot}}\rangle} \propto \frac{\langle\Gamma_{\alpha}\rangle\langle\Gamma_{\beta}\rangle}{\langle\Gamma_{\mathrm{tot}}\rangle}$$



Direct Capture (DC)



$$A + a \rightarrow B + \gamma$$

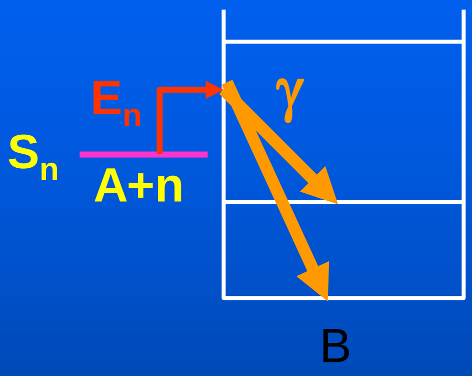
A	… target nucleus
a	… projectile
B = A ⊕ a	… residual nucleus

$$\frac{d\sigma}{d\Omega} = \left|\langle\phi_{\beta}|O_{EM}|\chi_{\alpha}\phi_{\alpha}\rangle\right|^{2} \propto S\left|\int d\vec{R}\,\phi_{Aa} O_{EM}\chi_{\alpha}\right|^{2}$$

Basic reaction mechanisms involving strong or electromagnetic interaction:

Example: neutron capture A + n -> B + γ

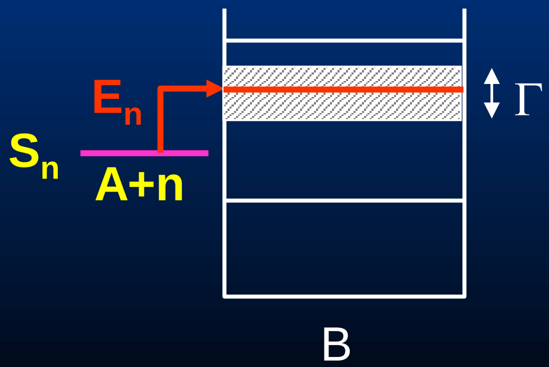
I. Direct reactions (for example, direct capture)



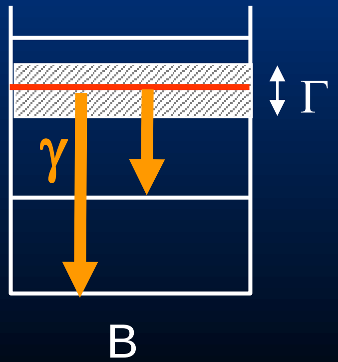
direct transition into bound states

II. Resonant reactions (for example, resonant capture)

Step 1: Compound nucleus formation
(in an unbound state)

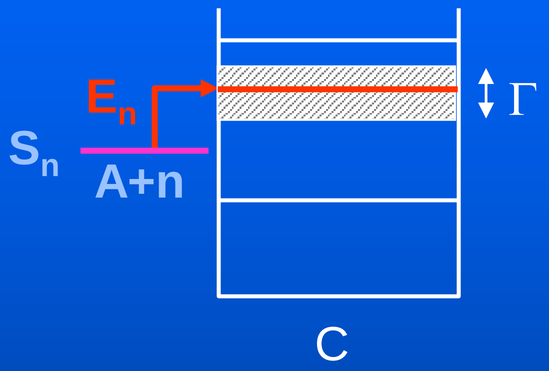


Step 2: Compound nucleus decay

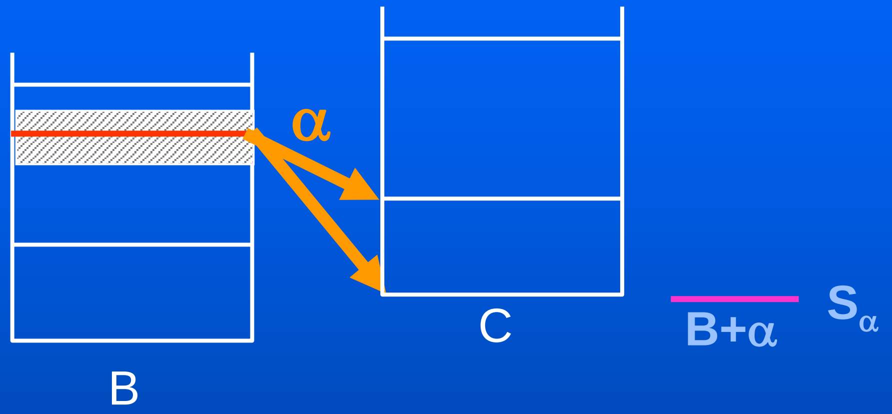


For example, a resonant A(n,α)B reaction:

Step 1: Compound nucleus formation (in an unbound state)



Step 2: Compound nucleus decay



For resonant reactions, E_n has to "match" an excited state (but all excited states have a width and there is always some cross section through tails)

But enhanced cross section for $E_n \sim E_x - S_n$

Direct reactions - for example direct capture:

a + A -> B + γ

Direct transition from initial state |a+A> to final state <f| (some state in B)

$$\sigma \propto \pi \lambda_a^2 \cdot \left| \langle f | H | a + A \rangle \right|^2 \cdot P_l(E)$$

geometrical factor (deBroglie wave length of projectile - "size" of projectile)

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

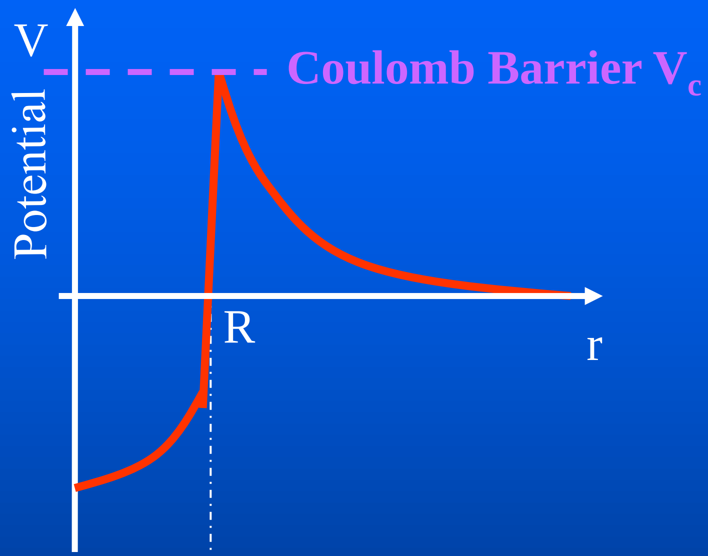
Interaction matrix element (nuclear structure, overlap between initial and final state, independent of E)

Penetrability: probability for projectile to reach the target nucleus for interaction.
Depends on projectile Angular momentum l and Energy E

$$\sigma \propto \frac{1}{E} \cdot \left| \langle f | H | a + A \rangle \right|^2 \cdot P_l(E)$$

Penetrability: 2 effects that can strongly reduce penetrability:

1. Coulomb barrier



for a projectile with Z_2 and a nucleus with Z_1

$$V_c = \frac{Z_1 Z_2 e^2}{R}$$

or

$$V_c\,[\mathrm{MeV}] = 1.44\,\frac{Z_1 Z_2}{R\,[\mathrm{fm}]} \approx 1.2\,\frac{Z_1 Z_2}{(A_1^{1/3} + A_2^{1/3})}$$

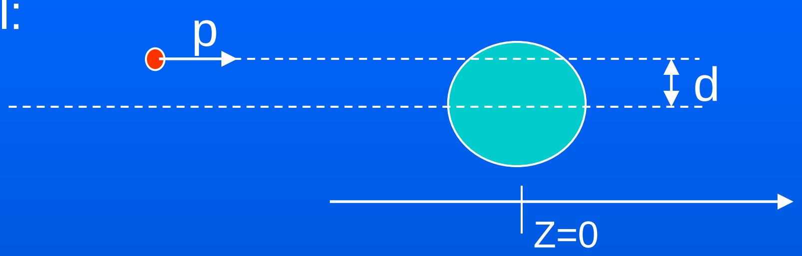
Example: ^{12}C(p,γ) V_C= 3 MeV

Typical particle energies for light nuclei in astrophysics are kT=1-100 keV ! Therefore, all charged particle reaction rates in nuclear astrophysics occur way below the Coulomb barrier – fusion is only possible through tunneling

2. Angular momentum barrier

Incident particles can have orbital angular momentum L

Classical:



Momentum p
Impact parameter d

$$L = pd$$

In quantum mechanics the angular momentum of an incident particle can have discrete values:

$$L = \sqrt{l(l+1)}\,\hbar$$

With

$l = 0$ s-wave
$l = 1$ p-wave
$l = 2$ d-wave
...

And parity of the wave function: $(-1)^l$

For radial motion (with respect to the center of the nucleus), angular momentum conservation (central potential !) leads to an energy barrier for non zero angular momentum.

Classically, one needs the radial kinetic energy to overcome the central potential, but if d != 0 then there is an increasing amount of "non radial kinetic energy", which one needs to supply as well (at z=0 for example, K_r=0, but of course K != 0)

Energy E of a particle with angular momentum L (still classical)

$$E = \frac{L^2}{2mr^2}$$

Similar here in quantum mechanics:

$$V_l = \frac{l(l+1)\hbar^2}{2\mu r^2}$$

μ : reduced mass of projectile-target system

Peaks again at nuclear radius (like Coulomb barrier) when combined with nuclear potential

Or in MeV using the nuclear radius and mass numbers of projectile A_1 and target A_2:

$$V_l\,[\mathrm{MeV}] = 12\,\frac{l(l+1)}{\left(\frac{A_1 A_2}{A_1 + A_2}\right)(A_1^{1/3} + A_2^{1/3})}$$

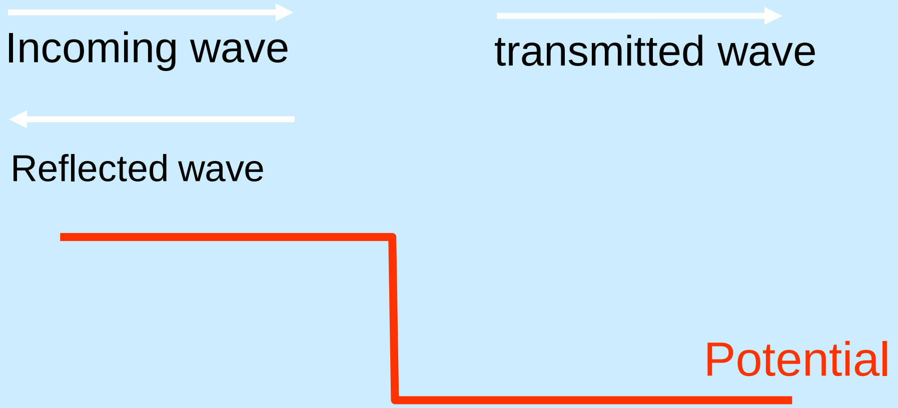
Direct reactions – the simplest case: s-wave neutron capture

No Coulomb or angular momentum barriers: $V_l = 0$

$V_C = 0$

s-wave capture therefore always dominates at low energies

But, change in potential still causes reflection – even without a barrier
Recall basic quantum mechanics:



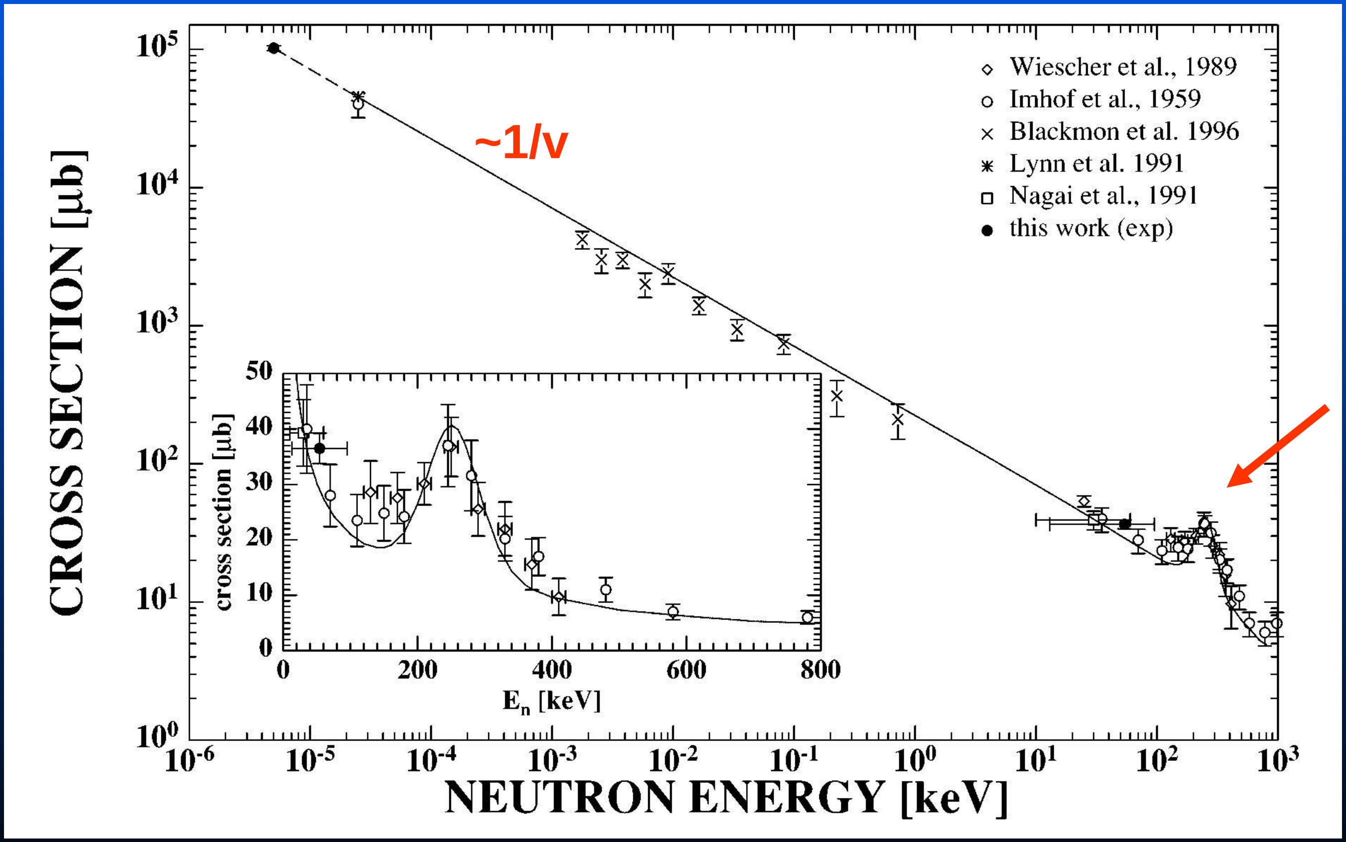
Transmission proportional to $\sqrt{E}$

Therefore, for direct s-wave neutron capture:

Penetrability $P_l(E) \propto \sqrt{E}$

Cross section: $\sigma \propto \frac{1}{\sqrt{E}}$ Or $\sigma \propto \frac{1}{v}$

Example: ^{7}Li(n,γ)



Deviation from 1/v due to resonant contribution

Direct reactions – neutron captures with higher orbital angular momentum

For neutron capture, the only barrier is the angular momentum barrier

The penetrability scales with

$$P_l(E) \propto E^{1/2+l}$$

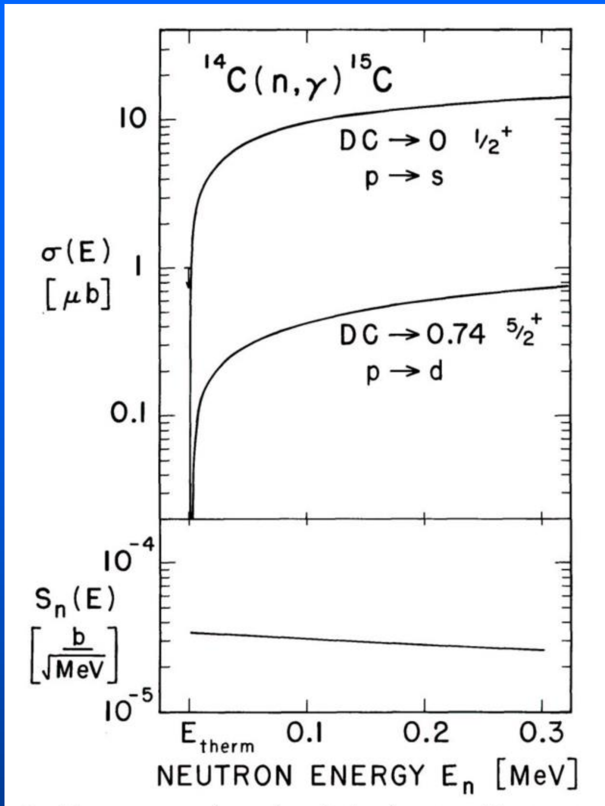
and therefore the cross section is

$$\sigma \propto E^{l-1/2}$$

for **l**>**0** cross section decreases with decreasing energy (as there is a barrier present)

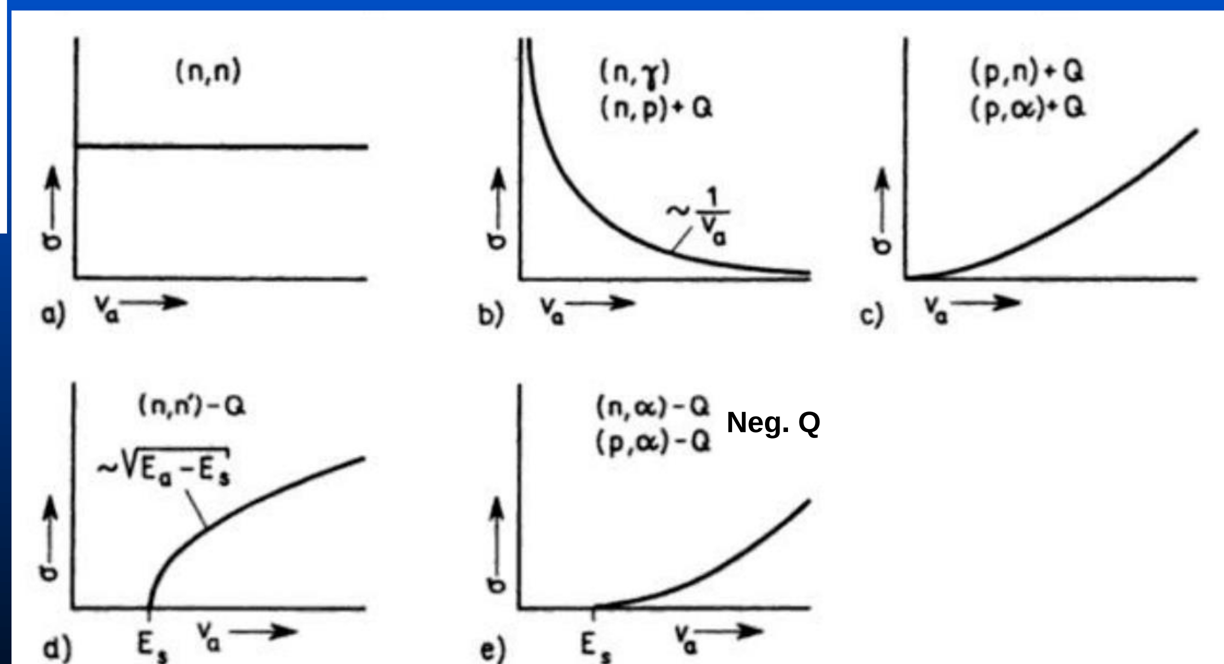
Therefore, s-wave capture in general dominates at low energies, in particular at thermal energies. Higher l-capture usually plays only a role at higher energies. What “higher” energies means depends on case to case - sometimes s-wave is strongly suppressed because of angular momentum selection rules (as it would then require higher gamma-ray multipolarities)

Example: p-wave capture in $^{14}C(n,\gamma)^{15}C$



$$\sigma \propto \sqrt{E}$$

Depending on barrier penetration, cross section can have different energy dependence:



(always s-waves in incident channel)

Resonant Reactions

If in the energy range reachable by the incoming projectile there is an excited state (or part of it, as states have a width) in the Compound nucleus then the cross section will have a resonant contribution.

If the center of the state is located in this energy range, then:

• The resonant contribution to the cross section tends to dominate by far

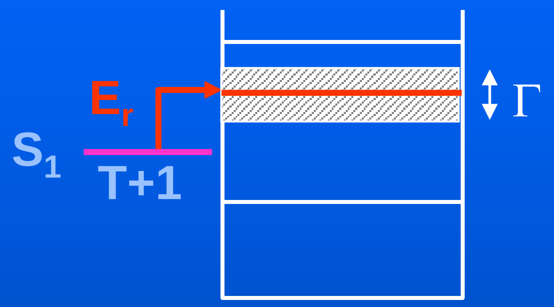
• The cross section becomes extremely sensitive to the properties of the resonant state

Reaction: 1 + T ⟶ C ⟶ F+2

With: Projectile 1
Target nucleus T
Compound nucleus C
Final nucleus F
Outgoing particle 2

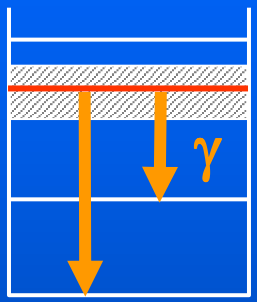
For capture 2 is a γ ray and F=C

Step 1



C

Step 2



C = F

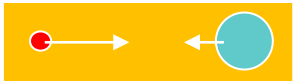
S_1: Particle 1 separation energy in C.

Excited states above S_1 are unbound and can decay by emission of particle 1 (in addition to other decay modes). Such states can serve as resonances

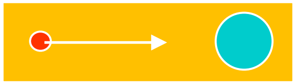
For capture, S_1 = Q-value

E_r: **Resonance energy**. Energy needed to populate the center of a resonance state

Reminder: Center of mass system



Laboratory system

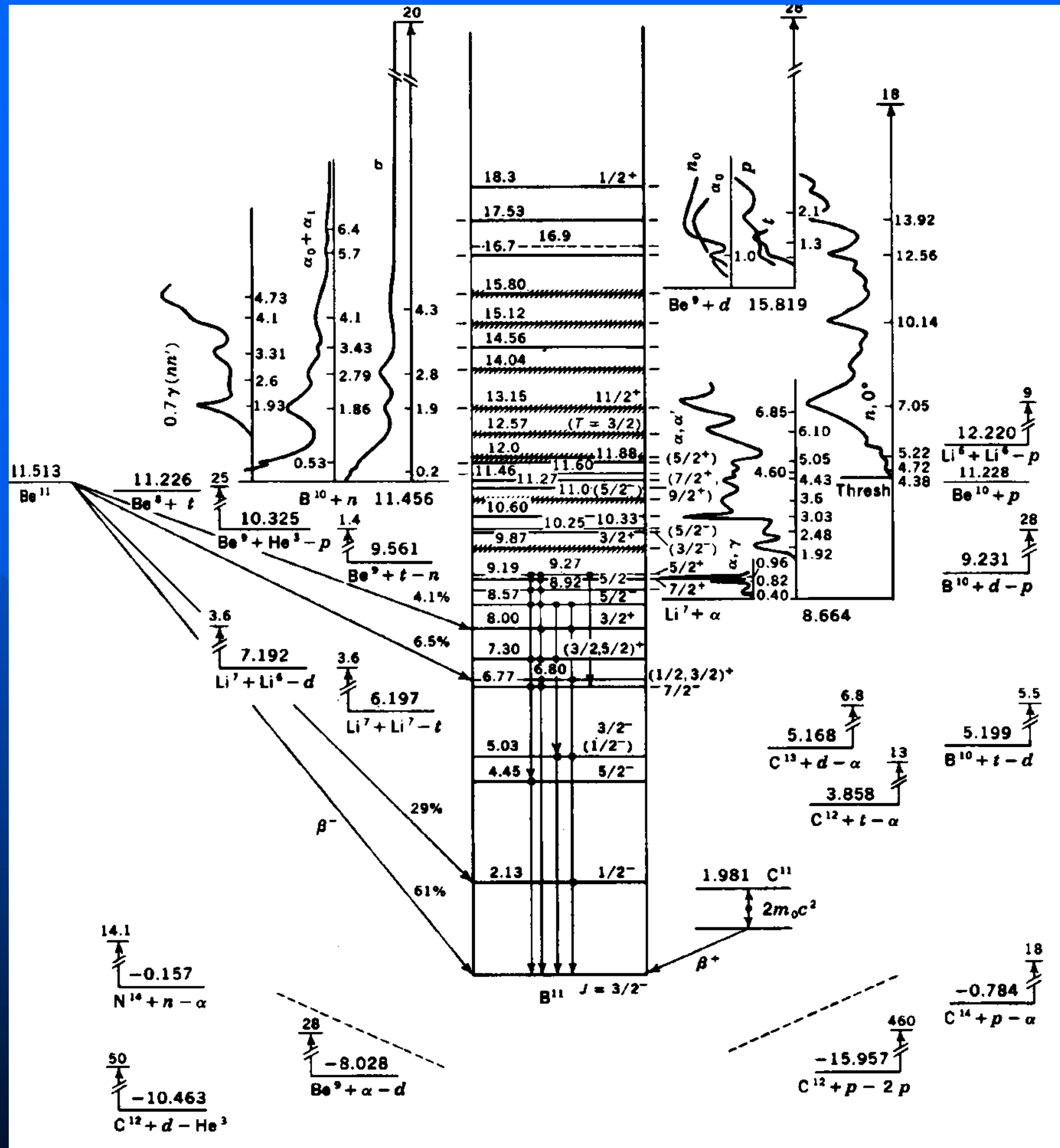


$$E_{CM} = \frac{1}{2}\mu v^2 \qquad \mu = \frac{m_p m_T}{m_p + m_T}$$

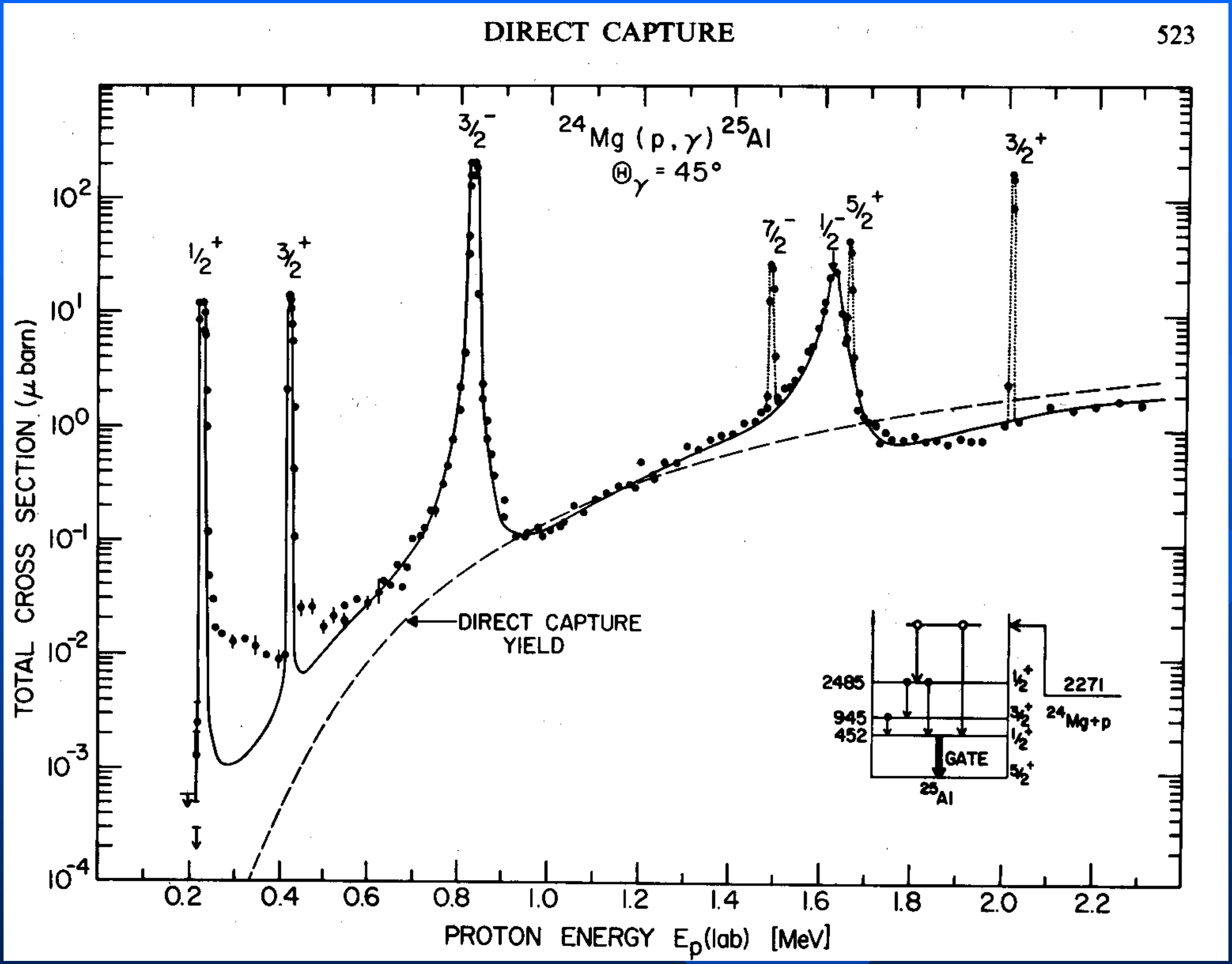
$$E_{Lab} = \frac{1}{2} m_p v^2$$

A Real Example

These are not just single-particle states but also configurations from excitations of one or more nucleons within the nucleus!



Example:



Resonance contributions are on top of direct capture cross sections

… and the corresponding S-factor

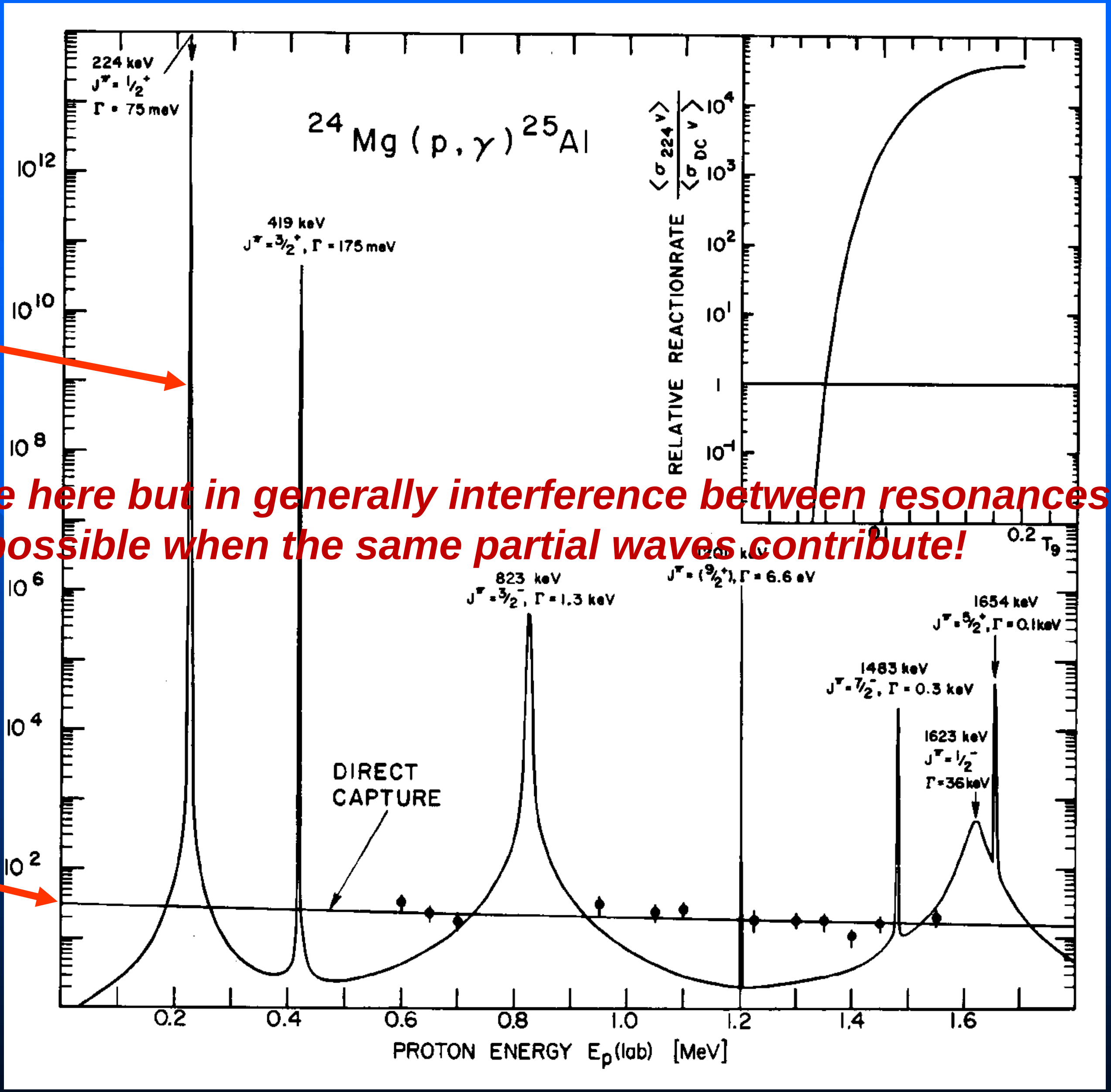
Note varying widths !

S-Factor:

$S(E)=\sigma E e^{2\pi\eta}$,

$\eta \propto (1/E) Z_1 Z_2$

Not constant S-factor for resonances (log scale !!!!)



Note: no interference here but in generally interference between resonances and direct reaction possible when the same partial waves contribute!

~ constant S-factor for direct capture

The cross section contribution due to a single resonance is given by the Breit-Wigner formula:

$$\sigma(E) = \pi\lambda\!\!\!^{-}{}^2 \cdot \omega \cdot \frac{\Gamma_1\Gamma_2}{(E-E_r)^2 + (\Gamma/2)^2}$$

Usual geometric factor

$$= \frac{656.6}{A}\frac{1}{E}\,\text{barn}$$

Spin factor:

$$\omega = \frac{2J_r + 1}{(2J_1+1)(2J_2+1)}$$

$\propto \Gamma_1$ Partial width for decay of resonance by emission of particle 1
= Rate for formation of Compound nucleus state

$\propto \Gamma_2$ Partial width for decay of resonance by emission of particle 2
= Rate for decay of Compound nucleus into the right exit channel

$\Gamma = \Gamma_1 + \Gamma_2 + \ldots$ Total width (including all energetically possible channels) is in the denominator as a large total width reduces the relative probabilities for formation and decay into specific channels.

Energy dependence of widths

Partial and total widths depend sensitively on the decay energy. Therefore:

- widths depend sensitively on the excitation energy of the state
- widths for a given state are a function of energy !

  (they are NOT constants in the Breit Wigner Formula)

Particle widths:

$$\Gamma_1 = \frac{2\hbar v}{R} P_l'(E_1)\,\Theta_l^2$$

* - see note below

Main energy dependence (can be calculated)

"reduced width" contains the nuclear structure

Photon widths:

$$\Gamma_\gamma = B(l)\, E_\gamma^{2l+1}$$

Reduced matrix element

* Our previously defined $P_l(E)=vP'_l(E)$ – usually width Γ is used instead of $P_l(E)$

For particle capture:

$$E_1 = E_r$$
$$E_\gamma = Q + E_r$$

For other cases:

$$E_1 = E_r$$
$$E_2 = S_2 + E_r$$

Typically E_r << Q and mostly also E_r << S_2 and therefore in many cases:

- $\Gamma_{\text{incoming particle}}$ has ***strong dependence*** on E_r (especially if it is a charged particle !)
- $\Gamma_{\text{outgoing particle}}$ has only ***weak dependence*** on E_r

So, for capture of particle 1, the main energy dependence of the cross section comes from λ^2 and Γ_1

Particle partial widths have the same (approximate) energy dependence than the "Penetrability" factor that we discussed in terms of the direct reaction mechanism.

Note:

$$\sigma(E) = \pi\lambdabar \Gamma_1(E) \frac{\Gamma_2}{(E-E_r)^2 + (\Gamma/2)^2}$$

Same energy dependence than direct reaction

For E << E_r very weak energy dependence

Far from the resonance the contribution from wings has a similar energy dependence than the direct reaction mechanism.

In particular, for s-wave neutron capture there is often a 1/v contribution at thermal energies through the tails of higher/lower lying s-wave resonances.

Therefore, resonant tail contributions and direct contributions to the reaction rate can be parametrized in the same way (for example S-factor)
Tails and DC are often mixed up in the literature.

Though they look the same, direct and resonant tail contributions are different things:

- in direct reactions, no compound nucleus forms
- resonance contributions can be determined from resonance properties measured at the resonance, far away from the relevant energy range (but need to consider interference !)

Breit-Wigner Formula

Isolated, non-interfering resonances are described through (partial) widths of states for absorption and emission of particles and photons:

$$\sigma(j,k) = \frac{\pi^2}{k_j^2}\frac{(1+\delta_{ij})}{(2I_i+1)(I_j+1)}\sum_n (2J_n+1)\frac{\Gamma_{j,n}\Gamma_{o,n}}{(E-E_n)^2+(\Gamma_n/2)^2}.$$

Here, we sum over *n* resonances in the reaction *i*(*j*,*k*)*o* , each with a total width Γ_n:

$$\Gamma_n = \Gamma_{j,n} + \Gamma_{o,n} + \ldots$$

From Breit-Wigner to Hauser-Feshbach

When having many overlapping, indistinguishable resonances we can make an average:

$$\left\langle \sigma(j,k) \right\rangle = \frac{\pi^2}{k_j^2}\frac{(1+\delta_{ij})}{(2I_i+1)(I_j+1)}\left\langle \sum_n (2J_n+1)\frac{\Gamma_{j,n}\Gamma_{o,n}}{(E-E_n)^2+(\Gamma_n/2)^2} \right\rangle$$

Using the mathematical relation

$$\int_{-\infty}^{+\infty} \frac{\Gamma_{j,n}\Gamma_{o,n}}{(E-E_n)^2+(\Gamma_n/2)^2} = 2\pi\frac{\Gamma_{j,n}\Gamma_{o,n}}{\Gamma_n}$$

$$\left\langle \frac{\Gamma_{j,n}\Gamma_{o,n}}{(E-E_n)^2+(\Gamma_n/2)^2} \right\rangle = \frac{1}{\Delta E}\int \ldots dE \approx \frac{2\pi}{\Delta E}\frac{\Gamma_{j,n}\Gamma_{k,n}}{\Gamma_n}$$

$$\left\langle \sum_n (2J_n+1)\frac{\Gamma_{j,n}\Gamma_{o,n}}{(EE_n)^2+(\Gamma_n/2)^2} \right\rangle$$

$$= \sum_{J,\pi}(2J+1)2\pi\frac{\Delta n(J,\pi)}{\Delta E}\left\langle \frac{\Gamma_{j,J,\pi}\Gamma_{o,J,\pi}}{\Gamma_{J,\pi}} \right\rangle$$

$$= \sum_{J,\pi}(2J+1)\frac{2\pi}{D_{J,\pi}}\frac{\langle\Gamma_{j,J,\pi}\rangle\langle\Gamma_{o,J,\pi}\rangle}{\langle\Gamma_{J,\pi}\rangle}W(j,o,J,\pi)$$

D is level spacing

Hauser-Feshbach Averaged Cross Section (Statistical Model)

$$\sigma_i(j,o)_{HF}$$

$$= \frac{\pi}{k_j^2}\sum_J (2J+1)\frac{(1+\delta_{ij})}{(2I_i+1)(2I_j+1)} W(j,o,J,\pi)\frac{T_j(E,J,\pi)T_o(E,J,\pi)}{T_{tot}(E,J,\pi)}$$

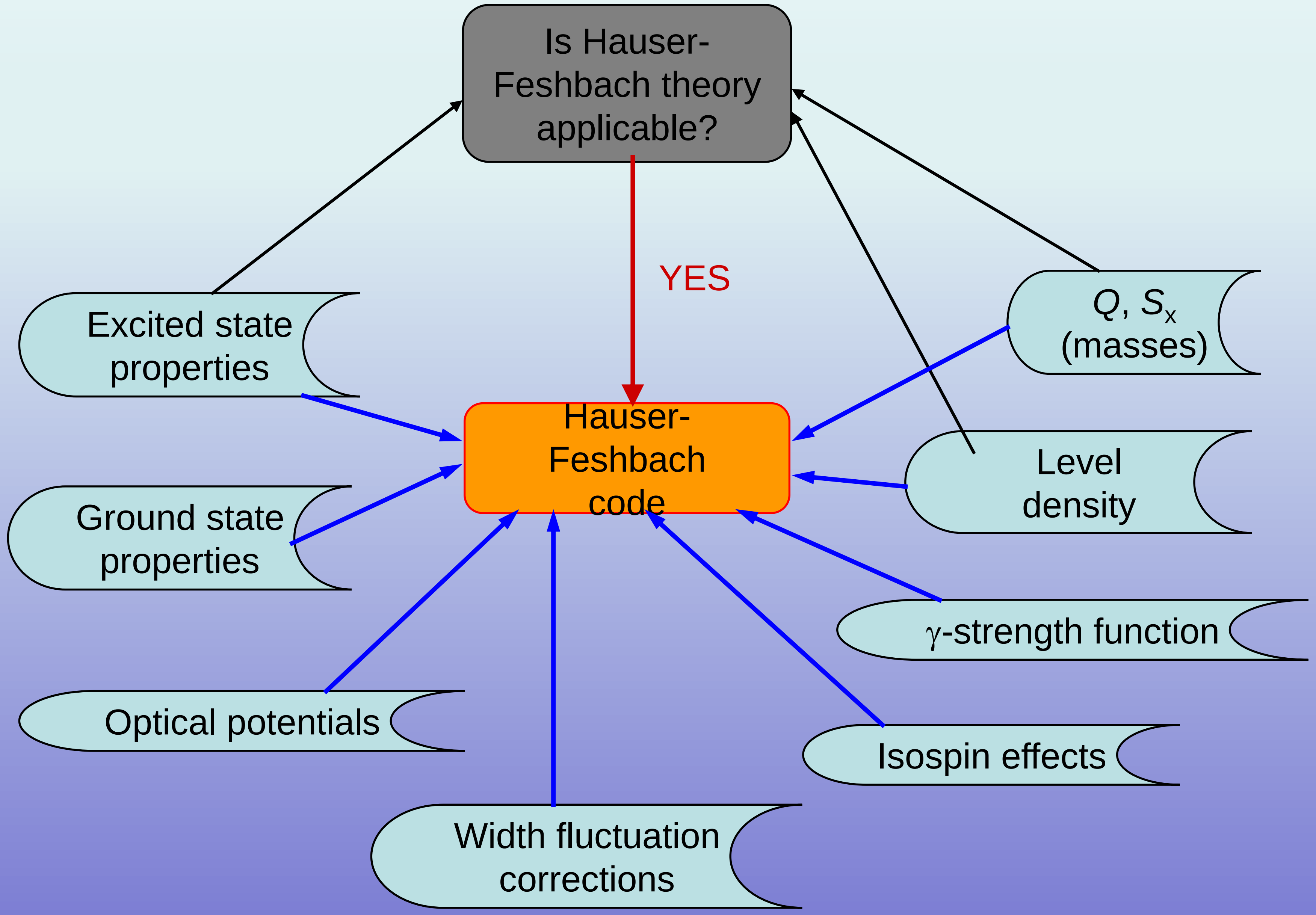
Transmission coefficients are solutions of Schrödinger equation:

$$T_{J,\pi} = \frac{2\pi}{D_{J,\pi}}\langle\Gamma_{J,\pi}\rangle = 2\pi\rho_{J,\pi}\langle\Gamma_{J,\pi}\rangle$$

ρ is level density

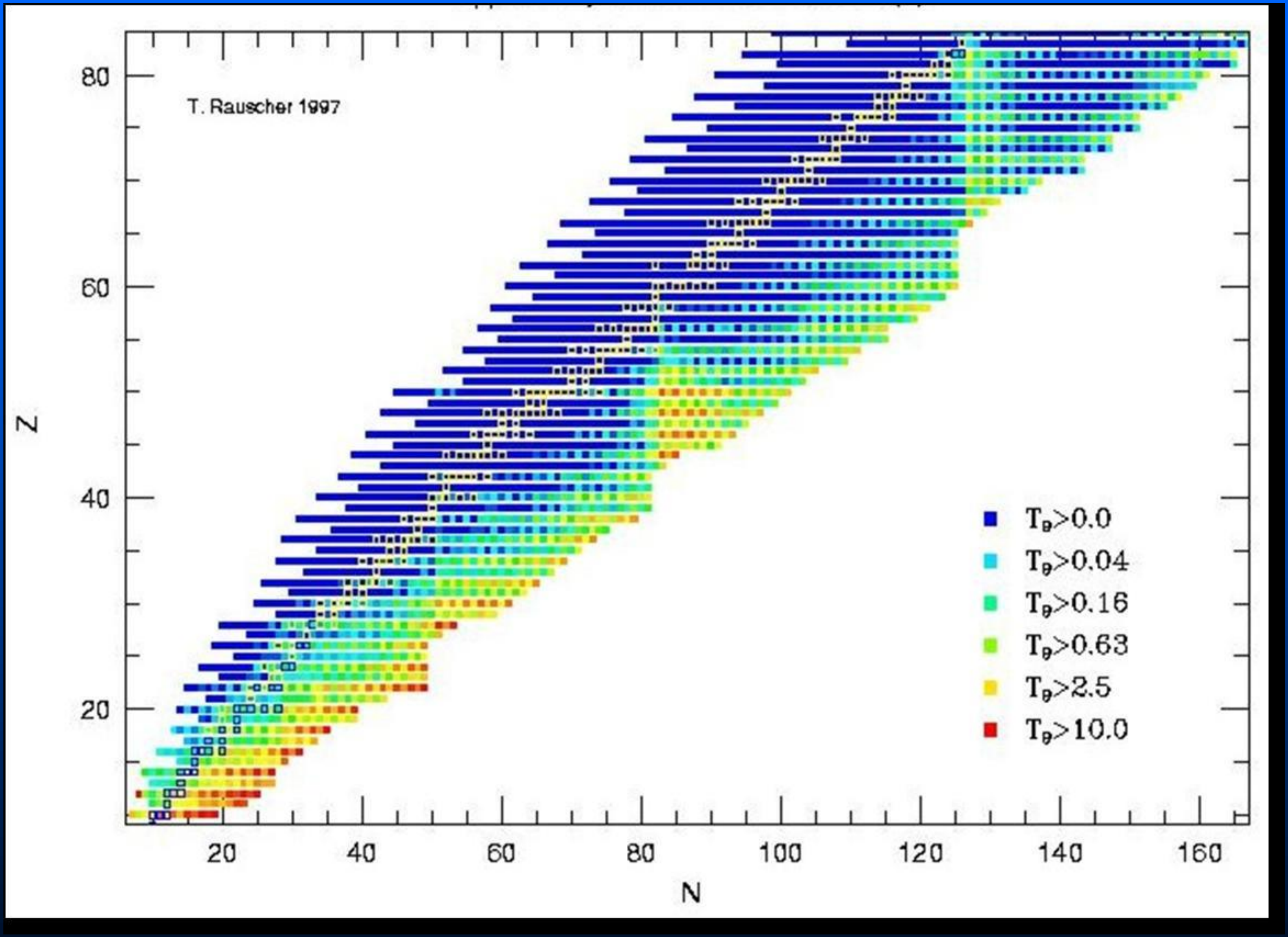
Width fluctuation corrections account for non-statistical correlations between entrance and exit channels; formally:

$$W(j,o,E,J,\pi) = \left\langle \frac{\Gamma_j(E,J,\pi)\Gamma_o(E,J,\pi)}{\Gamma_n(E,J,\pi)} \right\rangle \cdot \frac{\langle\Gamma(E,J,\pi)\rangle}{\langle\Gamma_j(E,J,\pi)\rangle\langle\Gamma_o(E,J,\pi)\rangle}$$


Is Hauser-Feshbach theory applicable?
YES
Hauser-Feshbach code
Excited state properties
Ground state properties
Optical potentials
Width fluctuation corrections
Q, S_x (masses)
Level density
γ-strength function
Isospin effects

Astrophysical applicability of the Statistical Model (Hauser-Feshbach)

Neutron induced reactions

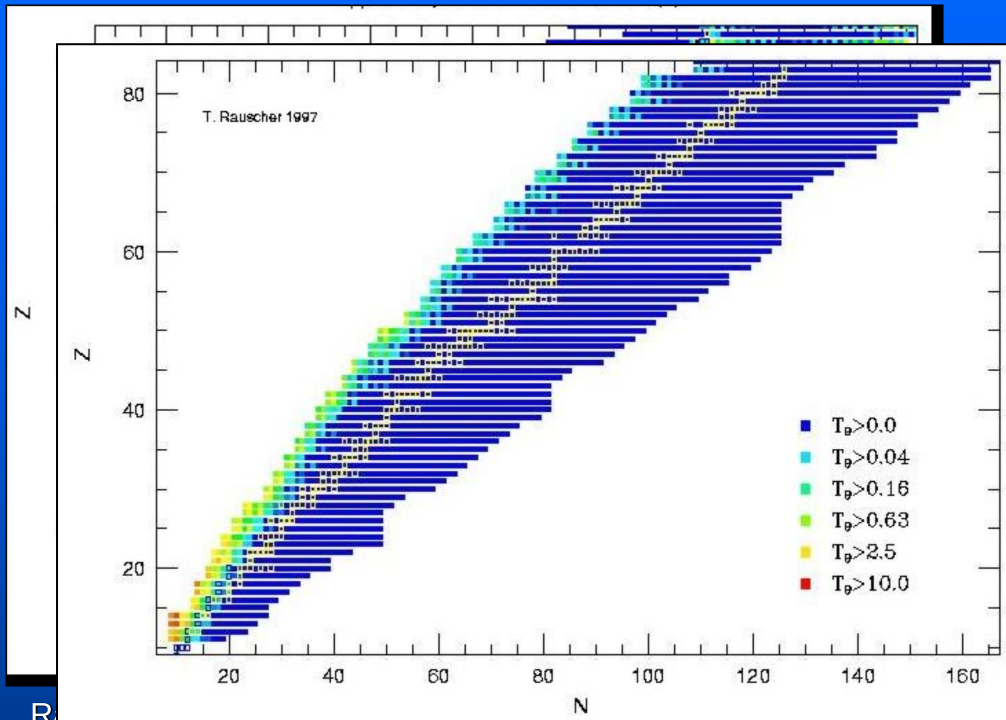


Rauscher et al. 1997

Reactions Far Off Stability

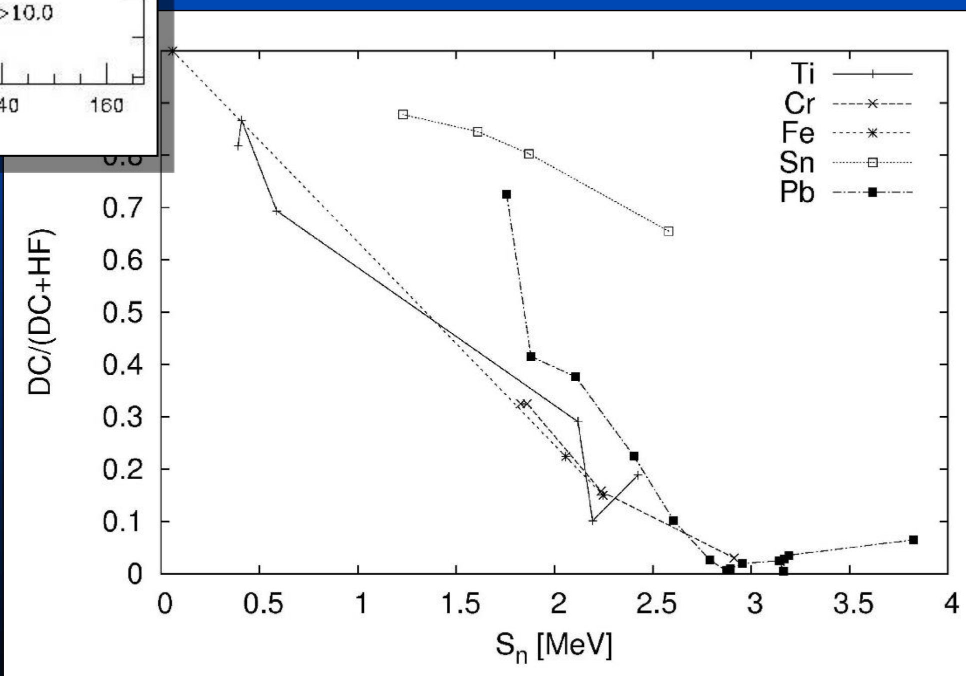
Neutron induced reactions

Applicability of statistical model



Proton-induced reactions

Comparison DC and Hauser-Feshbach

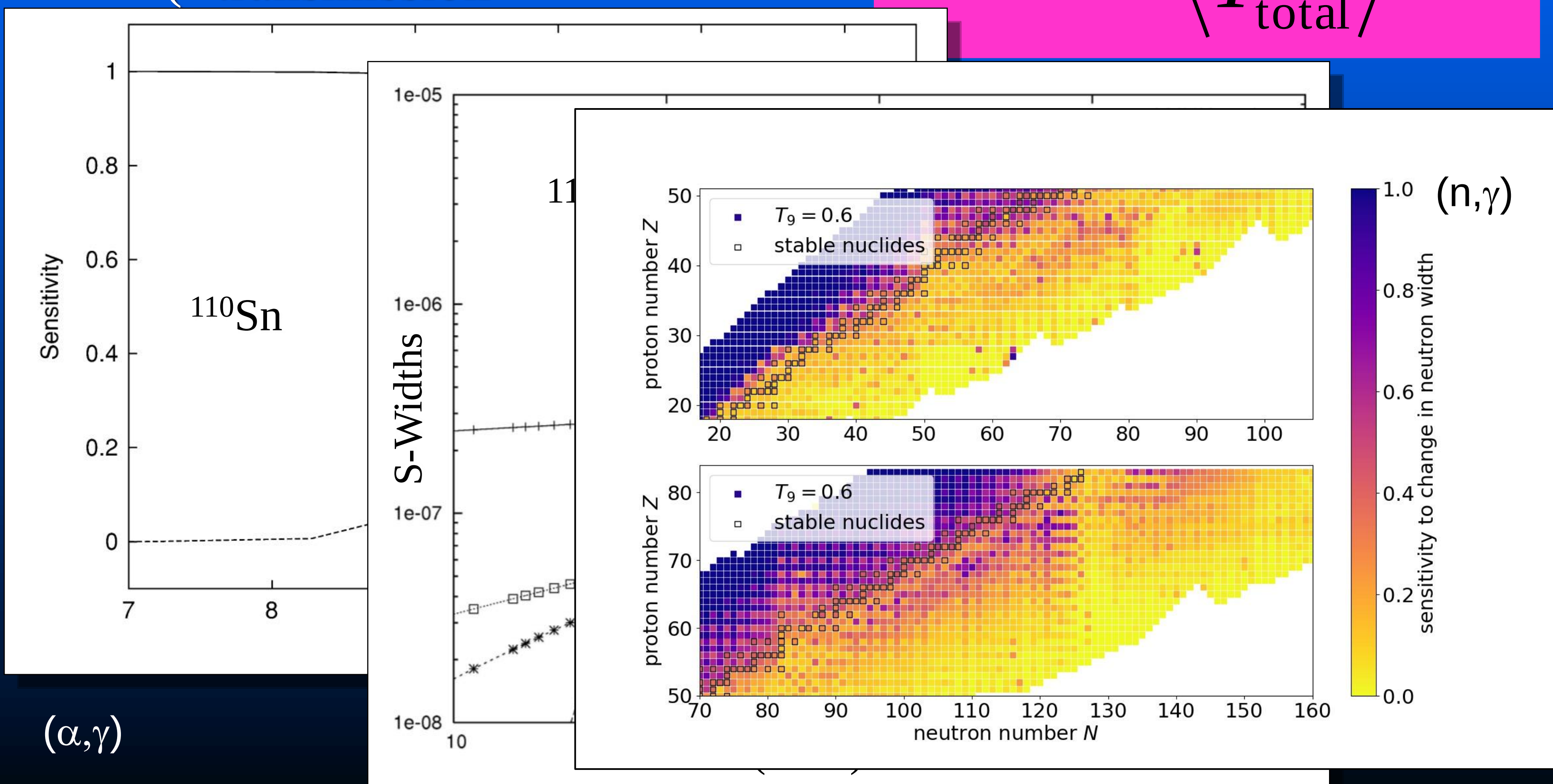


T. Rauscher; J. Phys. G 35 (2008) 014026

Relative importance of widths

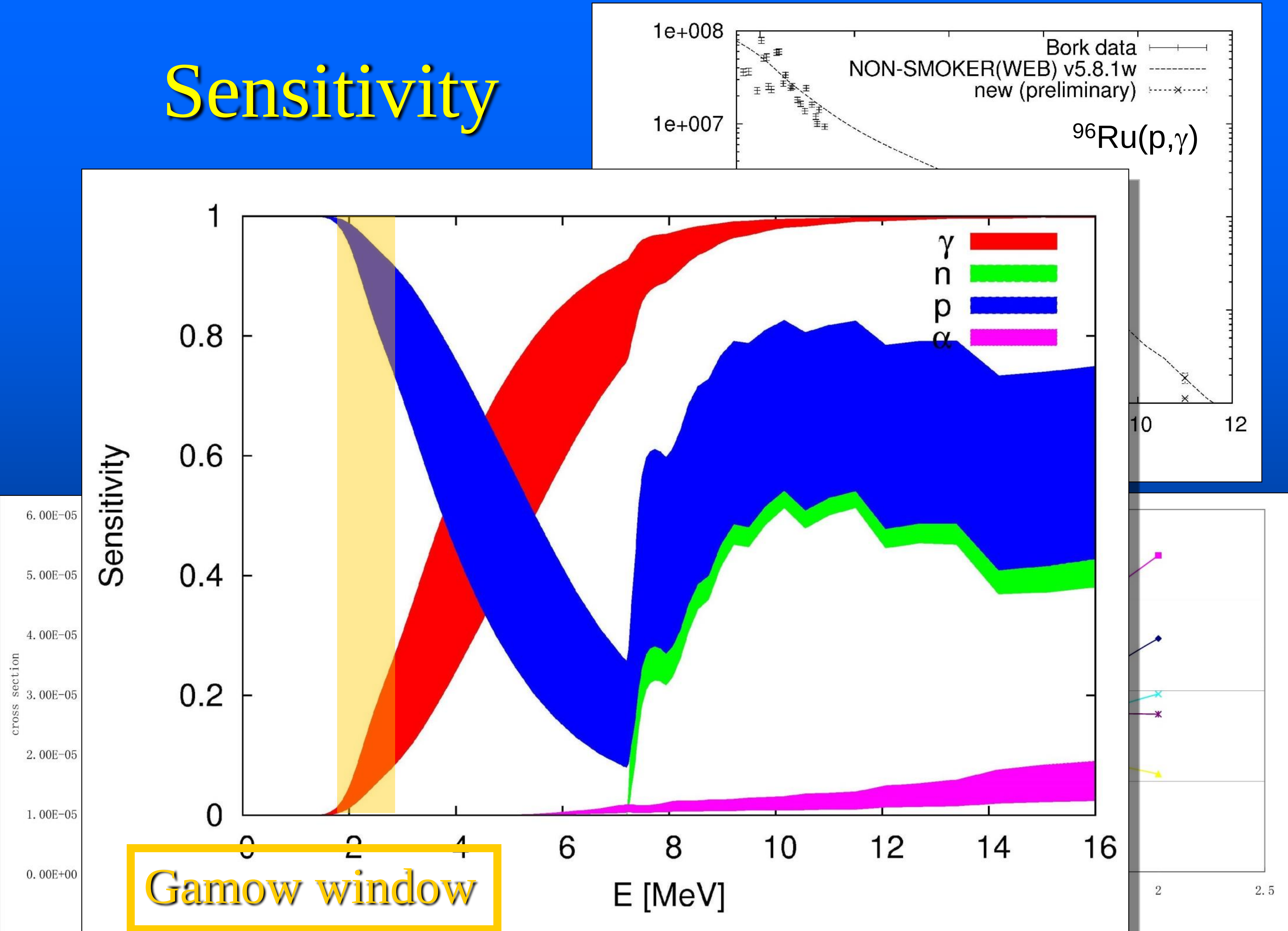
- Average widths (=transmission

$$\sigma \propto \frac{\langle T_{\mathrm{entrance}} \rangle \langle T_{\mathrm{exit}} \rangle}{\langle T_{\mathrm{total}} \rangle}$$



(α,γ)

Sensitivity



Reaction networks
and
Astrophysical Reaction Rates

("Stellar" rates)

Astrophysical Definitions

- (Mass)Density [g cm^{-3}]: ρ_i, $\Sigma\,\rho_i=\rho$
- Number density [cm^{-3}]: $n_i=N_i/V$, $\Sigma\,n_i=n$
- Mass fraction: $X_i=\rho_i/\rho$, $\Sigma\,X_i=1$
- Abundance: $Y_i=n_i/(\rho N_A)$, $Y_i=X_i/A_i$, $Y_{\mathrm{e}}=\Sigma\,Z_i/A_i$
- Plasma temperature: T_6 [10^6 K], T_9 [10^9 K]
- Typical Energy (MB distribution): $E=kT=T_9/11.6045$ MeV
- S-Factor [MeV b]: $S(E)=\sigma E e^{2\pi\eta}$, $\eta\propto(1/E)Z_1Z_2$

Reaction Networks I

Reactions *i(j,k)m* lead to change in plasma composition:

- NN reactions:

$$\left(\frac{\partial n_i}{\partial t}\right)_\rho = \left(\frac{\partial n_j}{\partial t}\right)_\rho = -r_{ij} = -\frac{1}{1+\delta_{ij}} n_i n_j \left\langle \sigma^* v \right\rangle_{ij}$$

$$\left(\frac{\partial n_k}{\partial t}\right)_\rho = \left(\frac{\partial n_m}{\partial t}\right)_\rho = +r_{ij} = \frac{1}{1+\delta_{ij}} n_i n_j \left\langle \sigma^* v \right\rangle_{ij}$$

- Nγ, NL reactions, decays:

$$\left(\frac{\partial n_i}{\partial t}\right)_\rho = -r_i = -n_i \lambda_i \quad ; \quad \left(\frac{\partial n_m}{\partial t}\right)_\rho = +r_i = n_i \lambda_i$$

Reaction Networks II

Want density independent measure, interested in changes caused by reactions, not density fluctuations

⇒ use abundances $Y_k(n_k(t),\rho(t))=n_k(t)/(\rho(t)N_A)$:

$$\dot{Y}_k = \frac{dY_k}{dt} = \frac{1}{\rho N_A}\left(\frac{\partial n_k}{\partial t}\right)_\rho$$

Network equations:

$$\dot{Y}_k = \sum_i N_i^k \lambda_i Y_i + \sum_{ij} \frac{N_{ij}^k}{1+\delta_{ij}} \rho N_A \left\langle \sigma^* v \right\rangle_{ij} Y_i Y_j$$

(with *M* species in the plasma we obtain <u>*M* equations</u>)

Mass fraction and abundance:

Mass fraction X_i is fraction of total mass of sample that is made up by nucleus of species i

$$n_i = \frac{X_i \rho}{m_i}$$

ρ : mass density (g/cm[3])

m_i mass of nucleus of species i

(CGS only !!!)

with $m_i \approx A_i \cdot m_u$ and $m_u = m_{12C}/12 = 1/\mathrm{N_A}$ as atomic mass unit (AMU)

$$n_i = \frac{X_i}{A_i} \rho \, \mathrm{N_A}$$

call this abundance Y_i

note: we neglect **here** nuclear binding energy and electrons (mixing atomic and nuclear masses) - therefore strictly speaking our ρ is slightly different from the real ρ, but differences are negligible in terms of the accuracy needed for densities in astrophysics

so $n_i = Y_i \rho N_A$ with $Y_i = \frac{X_i}{A_i}$

note: Abundance has no units only valid in CGS

The abundance Y is proportional to number density but changes only if the nuclear species gets destroyed or produced. Changes in density are factored out.

Thermonuclear Reaction Rates

Definition: Number of reactions per volume and time between two components of the stellar plasma:

$$r_{ij} = \int \sigma^*(v) v \, dn_i \, dn_j \quad , \text{with } v = \left| \vec{v}_i - \vec{v}_j \right| .$$

$$dn_i = n_i \phi(\vec{v}_i) d^3 v_i$$

The velocity distribution

$$\int_0^\infty \phi(\vec{v}) d^3 v = 1$$

depends on the particle statistics and can be derived from thermodynamics.

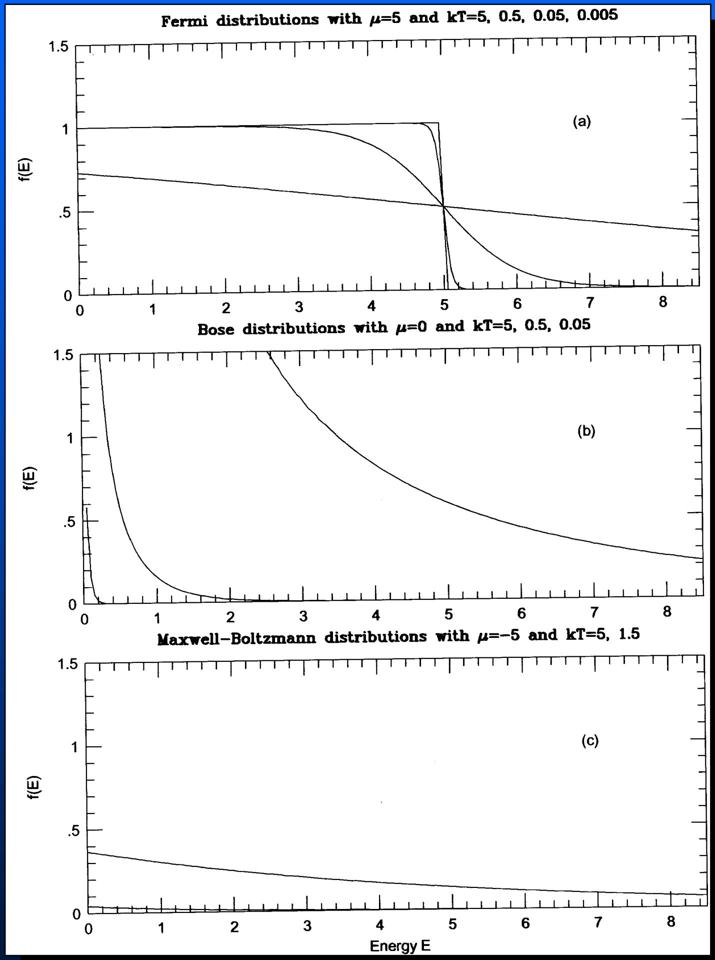
Particle Statistics

Occupation probabilities of states with energy E and chemical potential μ:

$$f(E) = \left[e^{\frac{E-\mu}{kT}} + 1 \right]^{-1} \quad \text{Fermions}$$

$$f(E) = \left[e^{\frac{E-\mu}{kT}} - 1 \right]^{-1} \quad \text{Bosons}$$

$$f(E) = e^{-\frac{E-\mu}{kT}} \quad \text{Maxwell - Boltzmann}$$



Low ρ + high T, $-\mu/kT \rightarrow -\infty$, then MB applies (H-, He-burning).

Reaction Rate (MB)

$$r_{12} = \frac{1}{1+\delta_{12}} n_1 n_2 \left\langle \sigma^* v \right\rangle_{12} = \frac{1}{1+\delta_{12}} \rho^2 Y_1 Y_2 N_A^2 \left\langle \sigma^* v \right\rangle_{12}$$

Number of reactions per time and volume

σ^* … Stellar cross section, see later.

Angle brackets denote reactivity (rate per particle pair): cross section times relative velocity averaged over velocity distribution.

Often, kinetic energy is used instead of velocity (same result).

Relevant Energies – Gamow Window

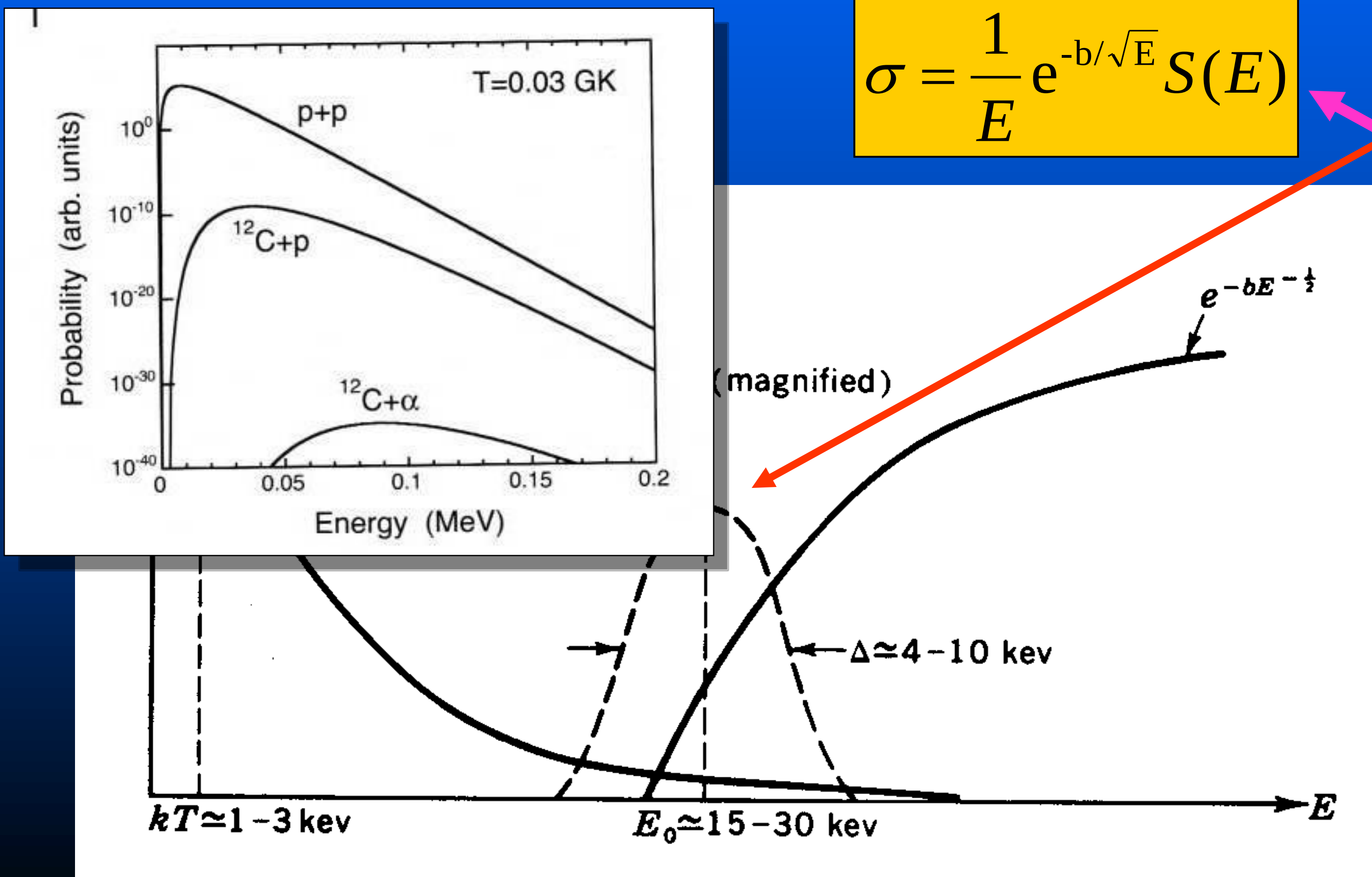
for charged particle reactions, this is the reactivity (rate per particle pair):

$$<\sigma v> = \sqrt{\frac{8}{\pi\mu}}(kT)^{-3/2}\int \sigma(E)E\,e^{-\frac{E}{kT}}\,dE = \sqrt{\frac{8}{\pi\mu}}(kT)^{-3/2}\int S(E)e^{-\left(\frac{b}{\sqrt{E}}+\frac{E}{kT}\right)}dE$$

Gamow Peak

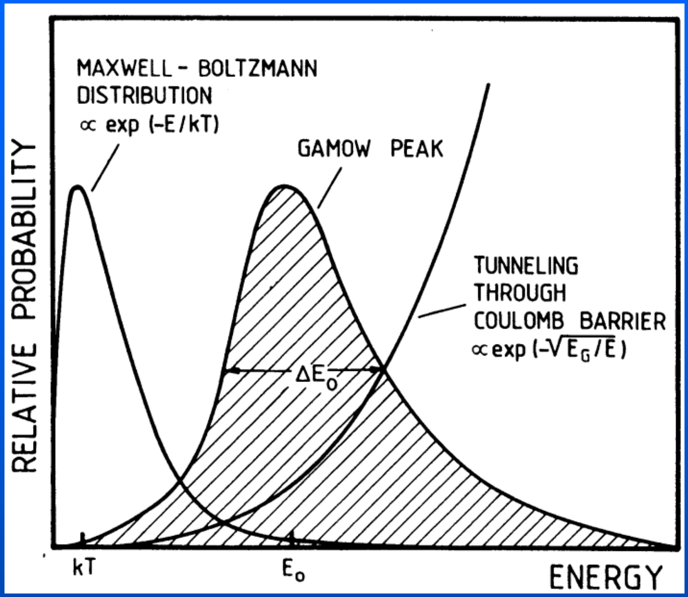
$$\sigma = \frac{1}{E}e^{-b/\sqrt{E}}S(E)$$

Astrophysical S-factor

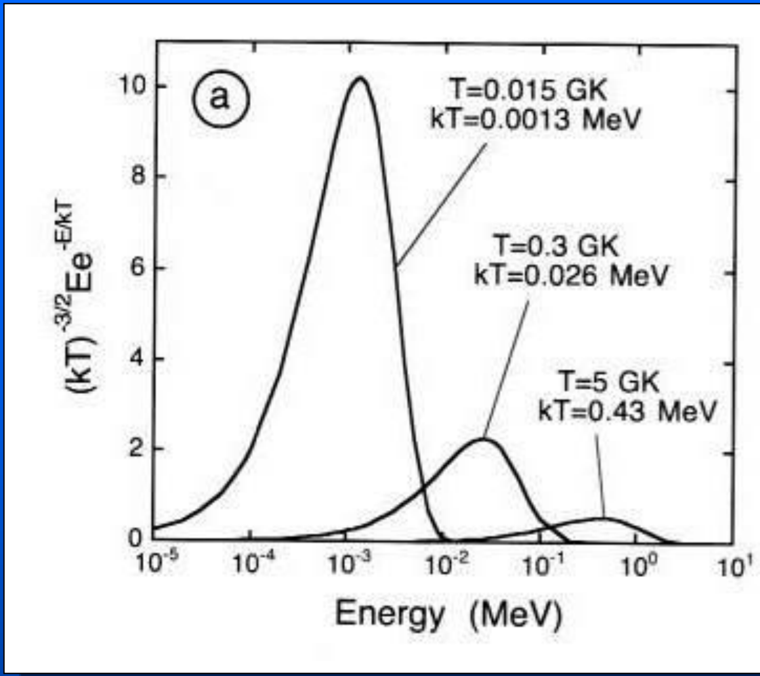


Note: relevant cross section in tail of M.B. distribution, much larger than kT (very different from n-capture !)

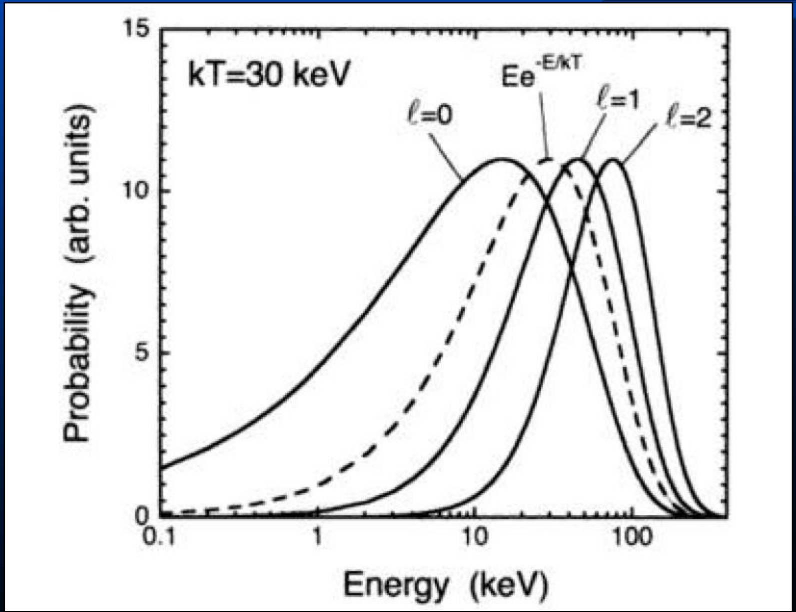
„Gamow peak“ for neutrons



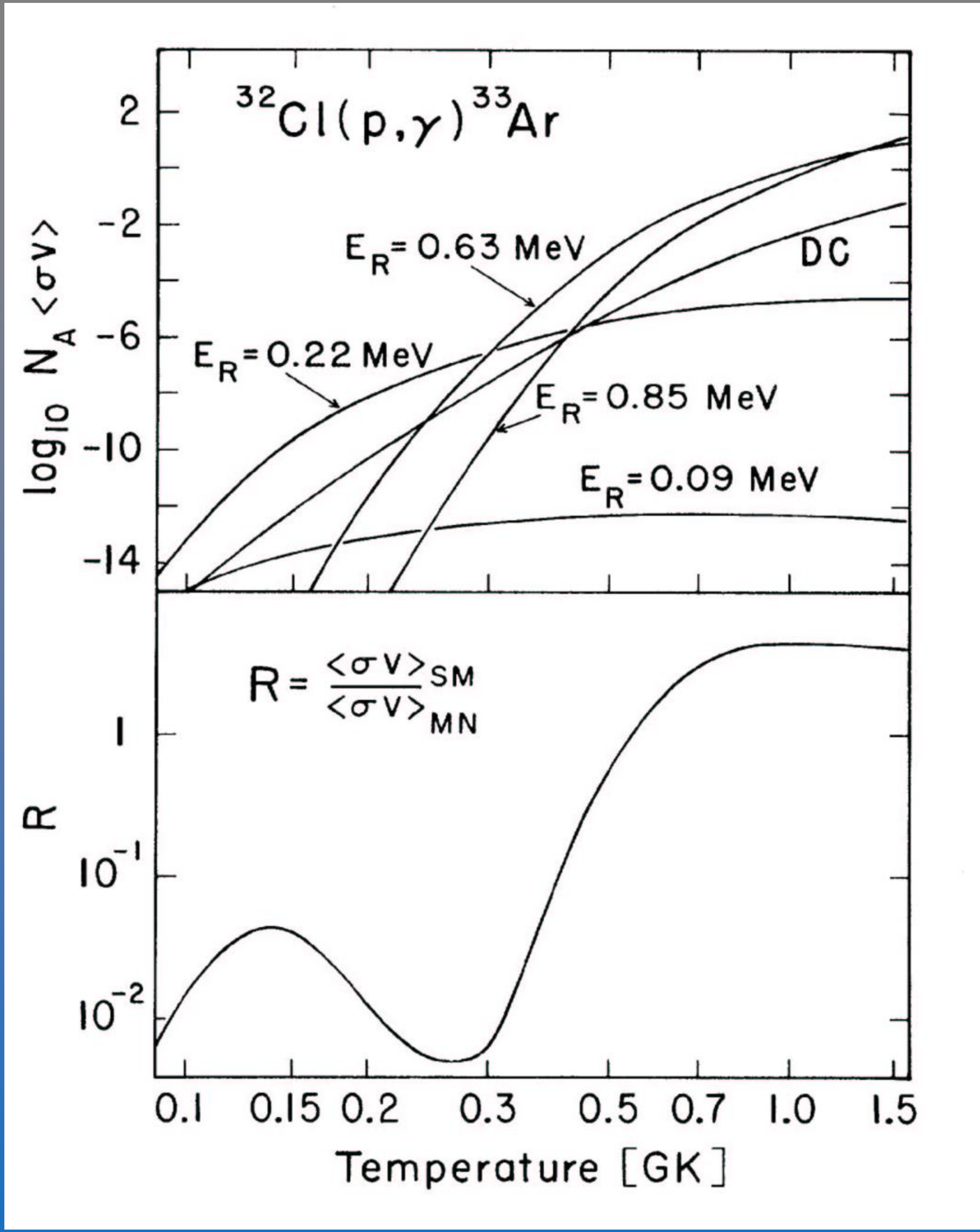
Rolfs & Rodney 1985



Neutrons have typical energy $kT=T_9/11.605$ MeV.



Iliadis 2006



Gamow Window:

0.1 GK: 130-220 keV

0.5 GK: 330-670 keV

1 GK: 500-1100 keV

But note: Gamow window has been defined for direct reaction energy dependence !

For heavier nuclides, the Gamow window can be located at several MeV, close to 10 MeV for alpha-particles. This is still below the Coulomb barrier!

The Gamow window moves to higher energies with increasing temperature – therefore different resonances play a role at different temperatures.

Some other remarks:

- If a resonance is in or near the Gamow window it tends to dominate the reaction rate by orders of magnitude

- As the level density increases with excitation energy in nuclei, higher temperature rates tend to be dominated by resonances, lower temperature rates by direct reactions.

- As can be seen from the equations for resonant rates, **the reaction rate is extremely sensitive to the resonance energy**.

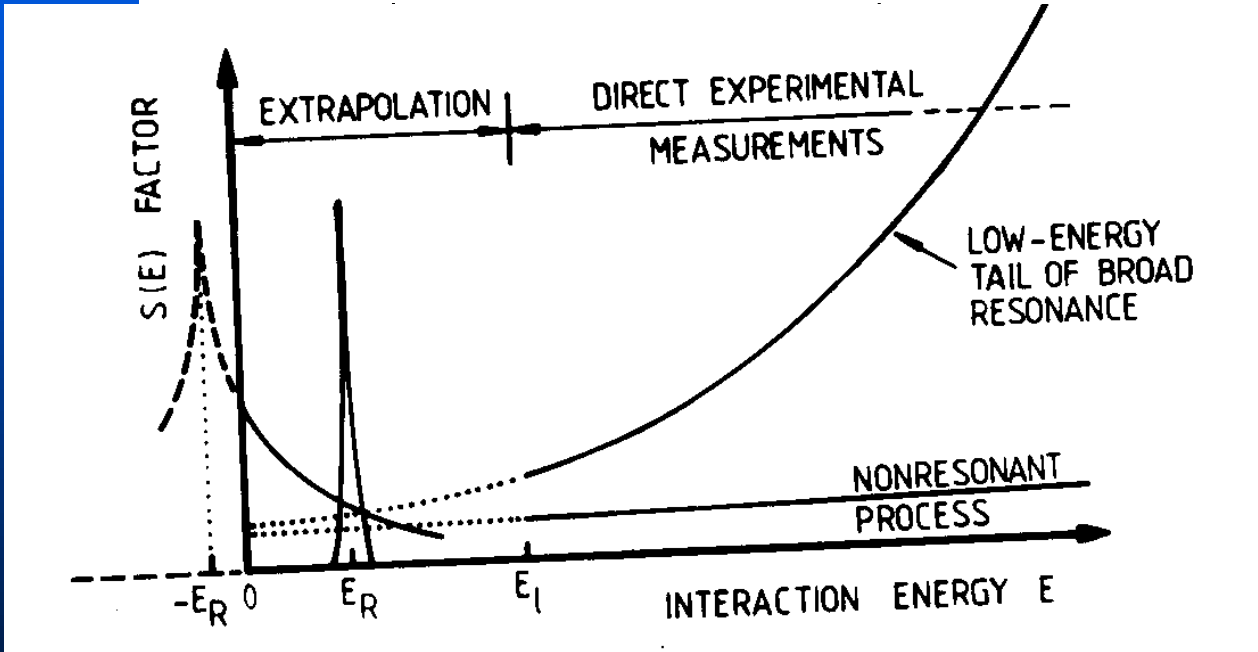
For p-capture this is due to the $\exp(E_r/kT)$ term **AND** $\Gamma_p(E)$ (Penetrability) !

As $E_r=E_x-Q$ one needs accurate excitation energies **and** masses !

The stellar reaction rate of a nuclear reaction is determined by the sum of

- sum of direct transitions to the various bound states
- sum of all narrow resonances in the relevant energy window
- tail contribution from higher lying resonances

Or as equation: $$<\sigma v> = \sum <\sigma v>_{DC->state i} + \sum <\sigma v>_{Res;i} + <\sigma v>_{tails}$$



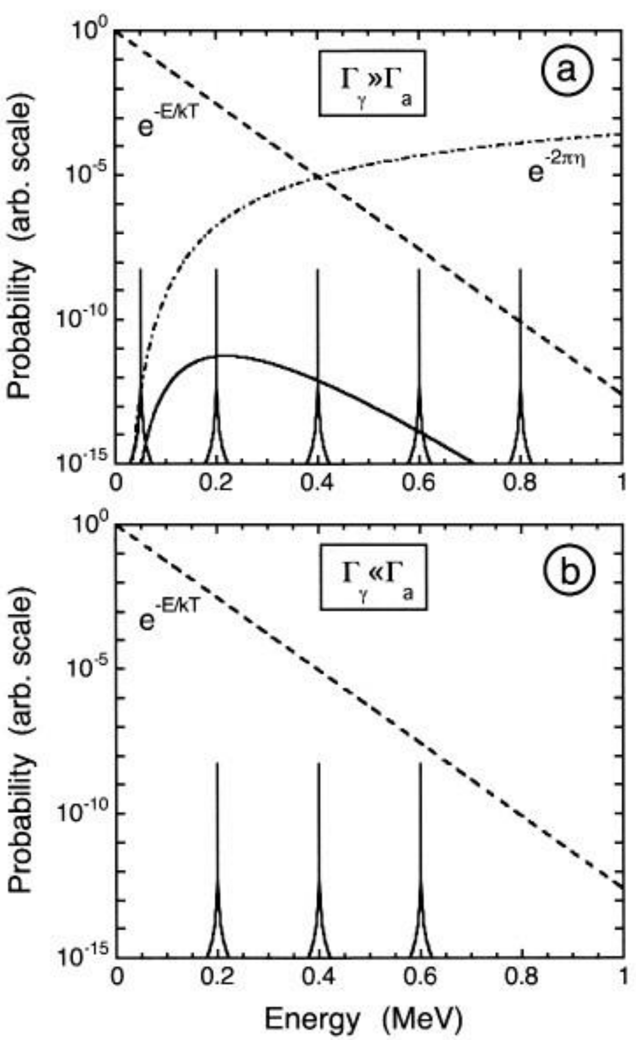
(Rolfs & Rodney)

Caution: Interference effects are possible (constructive or destructive addition) among

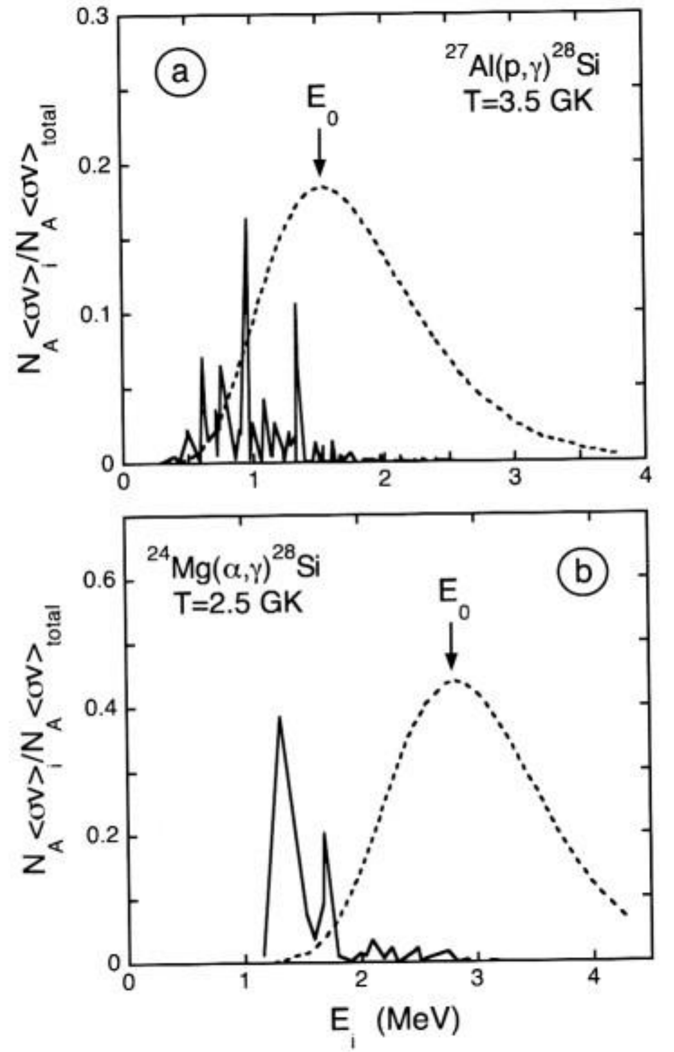
- Overlapping resonances with same quantum numbers
- Same wave direct capture and resonances

Limitation of Gamow peak concept

Narrow resonances can also be important below the Gamow window when width of exit channel smaller than width of entrance channel!



No barrier penetration factor for gammas!



Iliadis 2006

Revised Gamow peaks for intermediate and heavy target nuclides

$$\sigma \propto \frac{\langle T_{\mathrm{entrance}} \rangle \langle T_{\mathrm{exit}} \rangle}{\langle T_{\mathrm{total}} \rangle}$$

$\langle T \rangle$...(averaged) width

only valid when entrance channel determines energy dependence of cross section!

The peak is not symmetrical around E_0 but, nevertheless, is often approximated by a Gaussian function,

$$\mathcal{T}(E) = \mathcal{T}_{\max} e^{-[4(E-E_0)^2]/\Delta^2}, \quad (7)$$

widely used textbook formula!

where $\mathcal{T}_{\max} = \exp[-3E_0/(kT)]$ is the maximal value of the product of the two exponentials in Eq. (3) and $\Delta = 4\sqrt{E_0 kT/3}$ is the $1/e$ width of the peak. Inserting the proper numerical factors and units in Eqs. (6) and (7) leads to the more practical form [1,2,4]

$$E_0 = 0.12204\left(\mu_A Z_1^2 Z_2^2 T_9^2\right)^{\frac{1}{3}}, \quad (8)$$

$$\Delta = 0.23682\left(\mu_A Z_1^2 Z_2^2 T_9^5\right)^{\frac{1}{6}}. \quad (9)$$

Easy to see, for example, with (n,p) or (n,α) reactions…

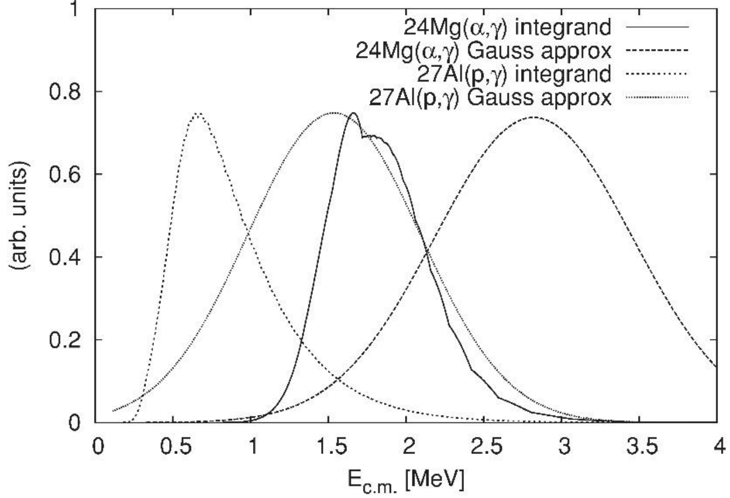


FIG. 5. Comparison of actual reaction rate integrand $\mathcal{F}$ and Gaussian approximation of the Gamow window for the reactions ^{24}Mg$(\alpha,\gamma)^{28}$Si at $T = 2.5$ GK and ^{27}Al$(p,\gamma)^{28}$Si at $T = 3.5$ GK. The integrands and Gaussians have been arbitrarily scaled to yield similar maximal values.

Examples of revised energy windows

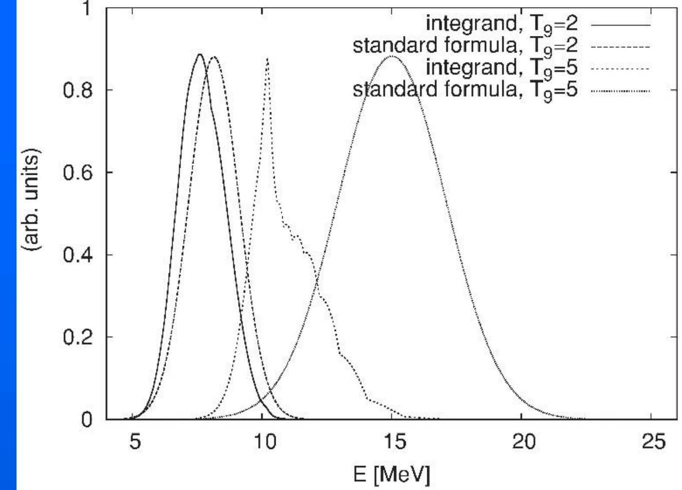


FIG. 6. Comparison of actual reaction rate integrands $\mathcal{F}$ and Gaussian approximations of the Gamow window for the reaction ^{169}Tm$(\alpha,\gamma)^{173}$Lu at $T = 2$ and 5 GK. The integrands and Gaussians have been arbitrarily scaled to yield similar maximal values. While the shift is small for $T_9 = 2$, it is about 5 MeV at $T_9 = 5$. Also, the asymmetry of the integrand can be clearly seen at $T_9 = 5$.

- revised energy windows can be shifted by several MeV
- important to know because experiments measure at the detection limit
  - can relevant energy window be reached?

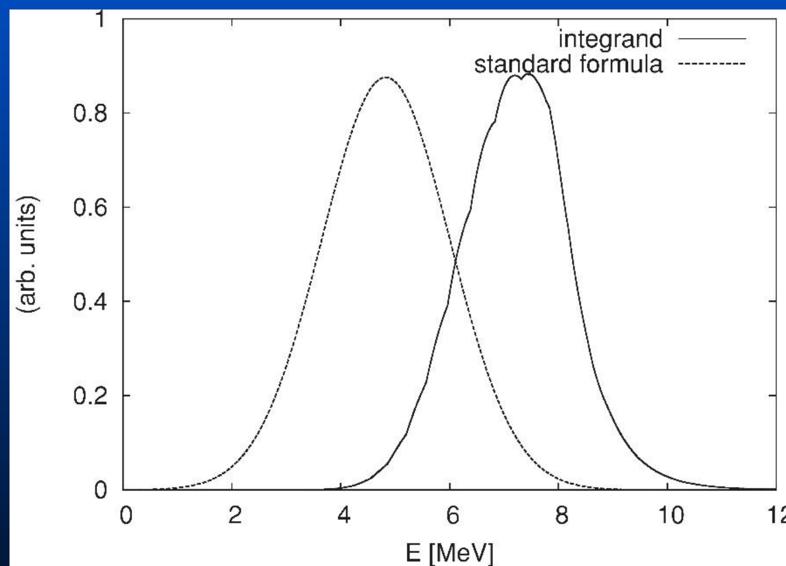


FIG. 9. Comparison of the actual reaction rate integrand $\mathcal{F}$ and the Gaussian approximation of the Gamow window for the reaction ^{112}Sn$(p,\alpha)^{109}$In at $T = 5$ GK. The two curves have been arbitrarily

TABLE I. Effective energy windows $\widetilde{E}_{hi} - \widetilde{\Delta} \leqslant E \leqslant \widetilde{E}_{hi}$ for a given plasma temperature T. Also listed is the energy $\widetilde{E}_0$ of the maximum in the reaction rate integrand and its shift δ relative to the standard formula. The latter is $\delta = \widetilde{E}_0 - E_0$ relative to the location of the Gamow peak E_0 for charged-particle-induced reactions and $\delta = \widetilde{E}_0 - E_{MB}$ relative to the maximum of the MB distribution at E_{MB} for neutron-induced reactions. This table lists only a few examples. The full table is available from Ref. [7].

Target	Reaction	T (GK)	$\widetilde{E}_{hi}$ (MeV)	$\widetilde{\Delta}$ (MeV)	$\widetilde{E}_0$ (MeV)	δ (MeV)
^{24}Mg	(α,γ)	2.5	2.36	1.05	1.66	−1.16
^{27}Al	(p,γ)	3.5	1.47	1.12	0.65	−0.89
^{40}Ca	(α,γ)	2.0	3.62	1.39	2.85	−0.63
		4.0	4.66	1.97	3.56	−1.97
^{60}Fe	(n,γ)	5.0	1.20	1.20	0.13	−0.30
^{62}Ni	(n,γ)	3.5	1.00	1.00	0.15	−0.15
^{106}Cd	(α,γ)	3.5	10.07	3.44	8.08	−1.17
^{120}Sn	(n,α)	5.0	9.54	4.16	6.92	+6.49
^{144}Sm	(α,γ)	3.5	11.97	3.99	9.90	−1.10
^{169}Tm	(α,γ)	2.0	9.20	2.94	7.61	−0.54
		5.0	13.20	4.27	10.22	−4.79

$$\widetilde{E}_{hi} - \widetilde{\Delta} \leqslant E \leqslant \widetilde{E}_{hi}.$$

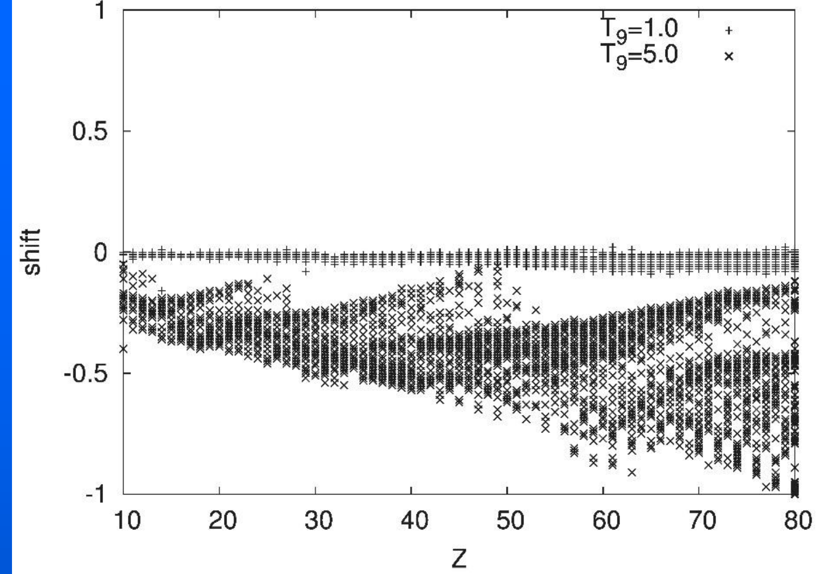


FIG. 1. Shifts δ (MeV) of the maximum of the integrand relative to E_0 of the Gaussian approximation as a function of the target charge Z for (p,n) reactions at two temperatures. Almost no shift is observed a

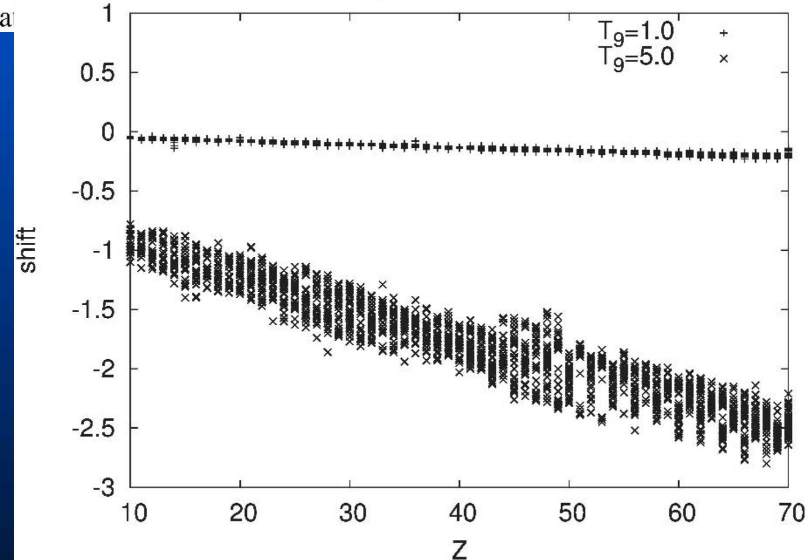


FIG. 2. Shifts δ (MeV) of the maximum of the integrand relative to E_0 of the Gaussian approximation as a function of the target charge Z for (α, n) reactions at two temperatures. Almost no shift is observed at $T_9 = 1.0$ and shifts reach a few mega–electron volts for $T_9 = 5.0$.

Rauscher, PRC 81, 045807

Rate of reaction through a narrow resonance

Narrow means: $\Gamma << \Delta E$

In this case, the resonance energy must be "near" the relevant energy range ΔE to contribute to the stellar reaction rate.

Recall:

$$<\sigma v>=\sqrt{\frac{8}{\pi\mu}}\frac{1}{(kT)^{3/2}}\int_0^\infty \sigma(E)E\,e^{-\frac{E}{kT}}dE$$

and

$$\sigma(E)=\pi\lambda\!\!\!^{-}\,\omega\,\frac{\Gamma_1(E)\Gamma_2(E)}{(E-E_r)^2+(\Gamma(E)/2)^2}$$

For a **narrow** resonance assume:

M.B. distribution $\Phi(E)\propto E\,e^{-\frac{E}{kT}}$ constant over resonance $\Phi(E)\approx\Phi(E_r)$

All widths $\Gamma(E)$ constant over resonance $\Gamma_i(E)\approx\Gamma_i(E_r)$

$\lambda\!\!\!^{-}\,^2$ constant over resonance

Then one can carry out the integration analytically and finds:

For the contribution of a single narrow resonance to the stellar reaction rate:

$$N_A < \sigma v > = 1.54 \cdot 10^{11} (A T_9)^{-3/2} \, \omega\gamma \, [\mathrm{MeV}] \, e^{\frac{-11.605 \, E_r [\mathrm{MeV}]}{T_9}} \quad \frac{\mathrm{cm}^3}{\mathrm{s\ mole}}$$

The rate is entirely determined by the "resonance strength" $\omega\gamma$

$$\omega\gamma = \frac{2J_r + 1}{(2J_1 + 1)(2J_T + 1)} \frac{\Gamma_1 \Gamma_2}{\Gamma}$$

Which in turn depends mainly on the total and partial widths of the resonance at resonance energies.

Often $\Gamma = \Gamma_1 + \Gamma_2$, then for $\Gamma_1 << \Gamma_2 \longrightarrow \Gamma \approx \Gamma_2 \longrightarrow \frac{\Gamma_1 \Gamma_2}{\Gamma} \approx \Gamma_1$

$\Gamma_2 << \Gamma_1 \longrightarrow \Gamma \approx \Gamma_1 \longrightarrow \frac{\Gamma_1 \Gamma_2}{\Gamma} \approx \Gamma_2$

And reaction rate is determined by the smaller width !

Rate for broad resonances or non-resonant reactions

Often (for example with theoretical reaction rates) one approximates the rate calculation by assuming the S-factor is constant over the Gamow Window:

$S(E)=S(E_0)$

$$\sigma = \frac{1}{E} e^{-b/\sqrt{E}} S(E)$$

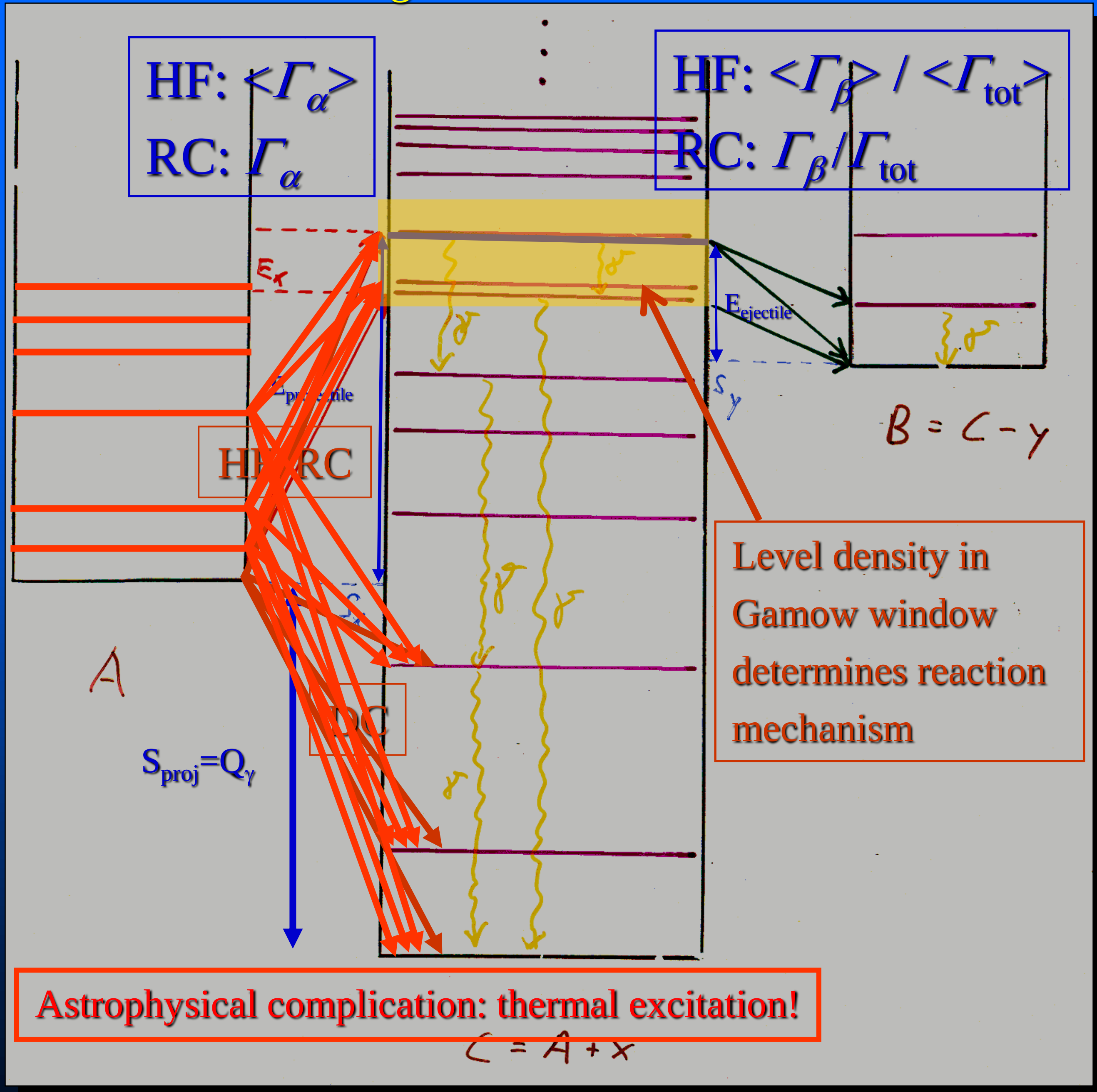
Then one finds the useful equation:

$$N_A < \sigma v > = 7.83 \cdot 10^9 \left(\frac{Z_1 Z_2}{A_R T_9^{\ 2}} \right)^{1/3} S(E_0)[\mathrm{MeV\ barn}]\ e^{-4.2487 \left(\frac{Z_1^2 Z_2^2 A_R}{T_9} \right)^{1/3}}$$

(A_R reduced mass number $A_1 A_2/(A_1+A_2)$)

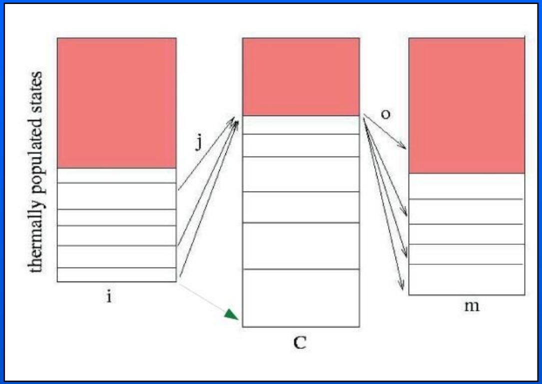
“Stellar” cross sections

Energetics in Nuclear Reactions



Thermally excited target nuclei in the stellar plasma

Ratio of nuclei in a thermally populated excited state to nuclei in the ground state is given by the Saha Equation:



$$\frac{n_{\mathrm{ex}}}{n_{\mathrm{gs}}} = \frac{g_{\mathrm{ex}}}{g_{\mathrm{gs}}} e^{-\frac{E_x}{kT}}$$

$$g = (2J + 1)$$

Ratios of order 1 for E_x~kT

- Only small correction for:
  - light nuclei (level spacing several MeV)
  - Gamow window at low energy: at low T
- **LARGE correction**, when
  - low lying (~100 keV) excited state(s) exist(s) in the target nucleus (heavy nuclei)
  - temperatures are high (explosive nucleosynthesis)
  - the populated state has a very different rate

The correction for this effect has to be calculated. Importance often underestimated…

Stellar rate and stellar cross section

$$r^* = \frac{n_a n_A}{1+\delta_{aA}} \int_0^\infty \sigma^*(E)\Phi(E,T)\, dE = \frac{n_a n_A}{1+\delta_{aA}} R^*$$

Stellar rate

$$R^*(T) = w_0 R_0 + w_1 R_1 + w_2 R_2 + \ldots$$

Stellar reactivity

$$R_i(T) = \int \sigma_i(E_i)\, \Phi(E_i,T)\, dE_i$$

$$w_i = (2J_i + 1) e^{-E_i/(kT)}$$

Boltzmann weights

The measured cross section σ_0 determines R_0

$$\sigma^*(E,T) = \frac{\sigma^{\mathrm{eff}}(E)}{G_0(T)} = \frac{1}{\sum_i P_i} \sum_i \sum_j \frac{2J_i+1}{2J_0+1} \frac{E-E_i}{E} \sigma^{i \to j}(E-E_i)$$

$$= \frac{1}{\sum_i P_i} \sum_i \sum_j \frac{2J_i+1}{2J_0+1} W_i \sigma^{i \to j}(E-E_i) \quad ,$$

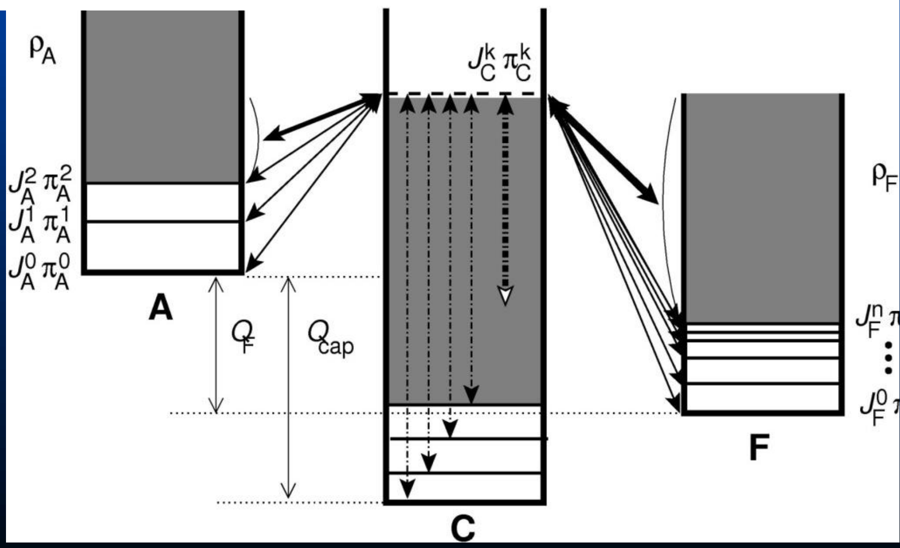
Stellar cross section

$$P_i = \frac{2J_i+1}{2J_0+1} \exp\left(-\frac{E_i}{kT}\right)$$

Population factor

$$W_i = \frac{E-E_i}{E} = 1 - \frac{E_i}{E}$$

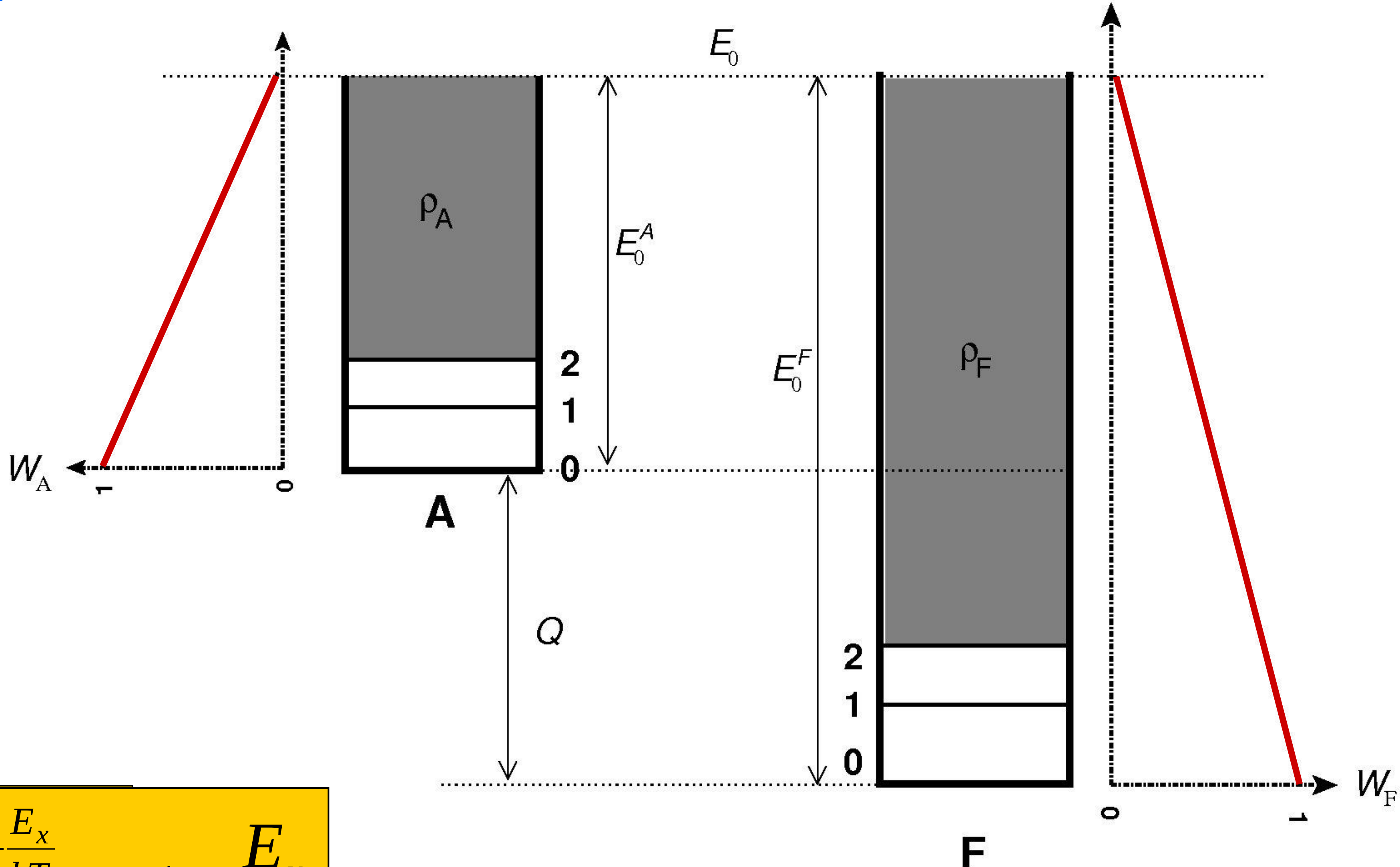
Weight of excited state



Using pop. fact. as measure of importance underestimates impact!

T. Rauscher, Int. J. Mod. Phys. E 20, 1071 (2011)

Effective weights [Gamow energy] states



$$e^{-\frac{E_x}{kT}} \rightarrow 1 - \frac{E_x}{E_0}$$

By shifting energy scales of MB distributions to g.s. energy, weights are modified (Fowler 1975).

Reaction Rate (MB)

$$r_{12} = \frac{1}{1+\delta_{12}} n_1 n_2 \left\langle \sigma^* v \right\rangle_{12} = \frac{1}{1+\delta_{12}} \rho^2 Y_1 Y_2 N_A^2 \left\langle \sigma^* v \right\rangle_{12}$$

Number of reactions per time and volume

$$\left\langle \sigma v \right\rangle^*_{Aa} \propto \frac{1}{G_A^{\mathrm{norm}}} \sum_\mu \left(\int \left\{ \frac{g_A^\mu}{g_A^0} \sigma_{Aa}^\mu E_A^\mu e^{-\left(E_A^\mu + \varepsilon_A^\mu\right)/(kT)} \right\} dE_A^\mu \right)$$

$$= \ldots = \frac{1}{G_A^{\mathrm{norm}}} \int \left\{ \sum_\mu \frac{g_A^\mu}{g_A^0} E_A^\mu \sigma_{Aa}^\mu e^{-E_A^0/(kT)} \right\} dE_A^0 = \frac{1}{G_A^{\mathrm{norm}}} \int \sigma_A^{\mathrm{eff}} E_A^0 e^{-E_A^0/(kT)} dE_A^0$$

stellar reactivity

Simplification of Stellar Rate

MB distributed projectiles act on every excited state, have to do a weighted sum:

$$\langle \sigma v \rangle^{*}_{Aa} \propto \frac{1}{G_A^{\text{norm}}} \sum_{\mu} \left(\int \left\{ \frac{g_A^{\mu}}{g_A^{0}} \sigma_{Aa}^{\mu} E_A^{\mu} e^{-\left(E_A^{\mu}+\varepsilon_A^{\mu}\right)/(kT)} \right\} dE_A^{\mu} \right)$$

$$= \ldots = \frac{1}{G_A^{\text{norm}}} \int \left\{ \sum_{\mu} \frac{g_A^{\mu}}{g_A^{0}} E_A^{\mu} \sigma_{Aa}^{\mu} e^{-E_A^{0}/(kT)} \right\} dE_A^{0} = \frac{1}{G_A^{\text{norm}}} \int \sigma_A^{\text{eff}} E_A^{0} e^{-E_A^{0}/(kT)} dE_A^{0}$$

with effective cross section

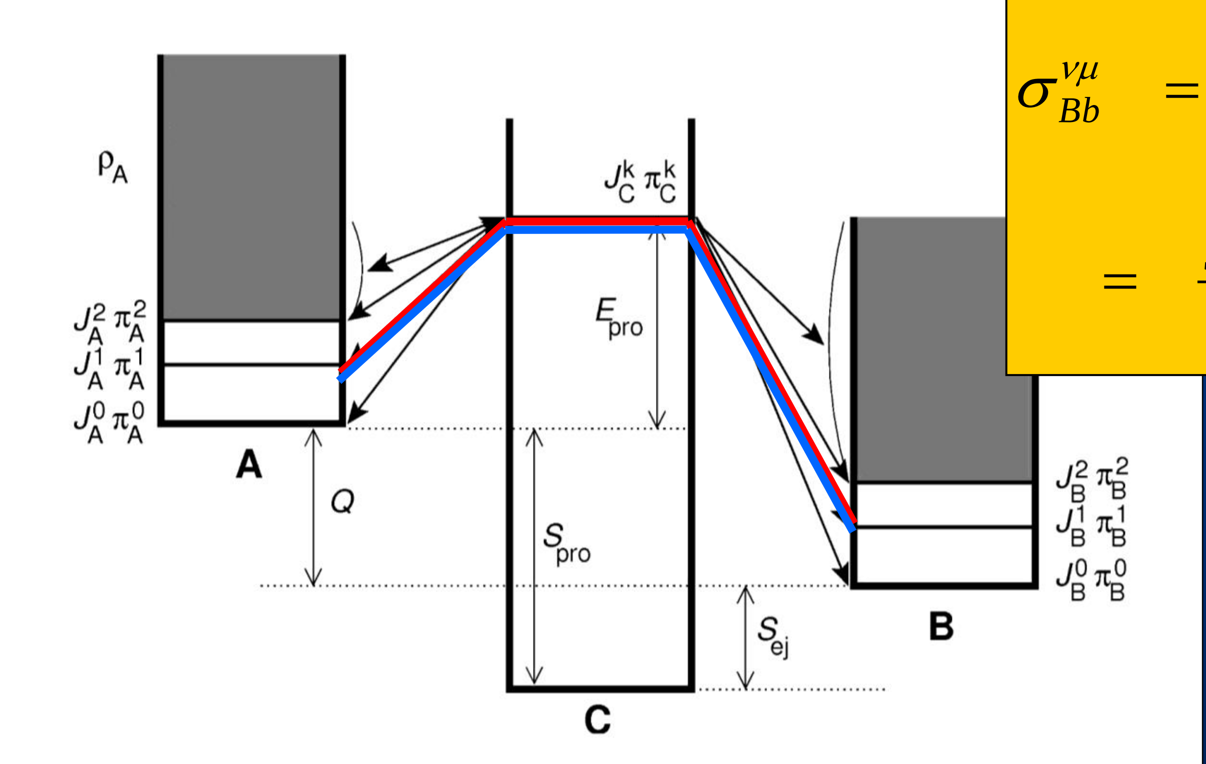
$$\sigma_{Aa}^{\text{eff}} = \sum_{\mu} \sum_{\nu} \frac{g_A^{\mu}}{g_A^{0}} \frac{E_A^{\mu}}{E_A^{0}} \sigma_{Aa}^{\mu\nu}$$

$g = 2J + 1$

G^{norm} … normalized partition function

Effective cross section sums over all accessible excited states μ, ν in initial and final nucleus!

Reciprocity of stellar rates



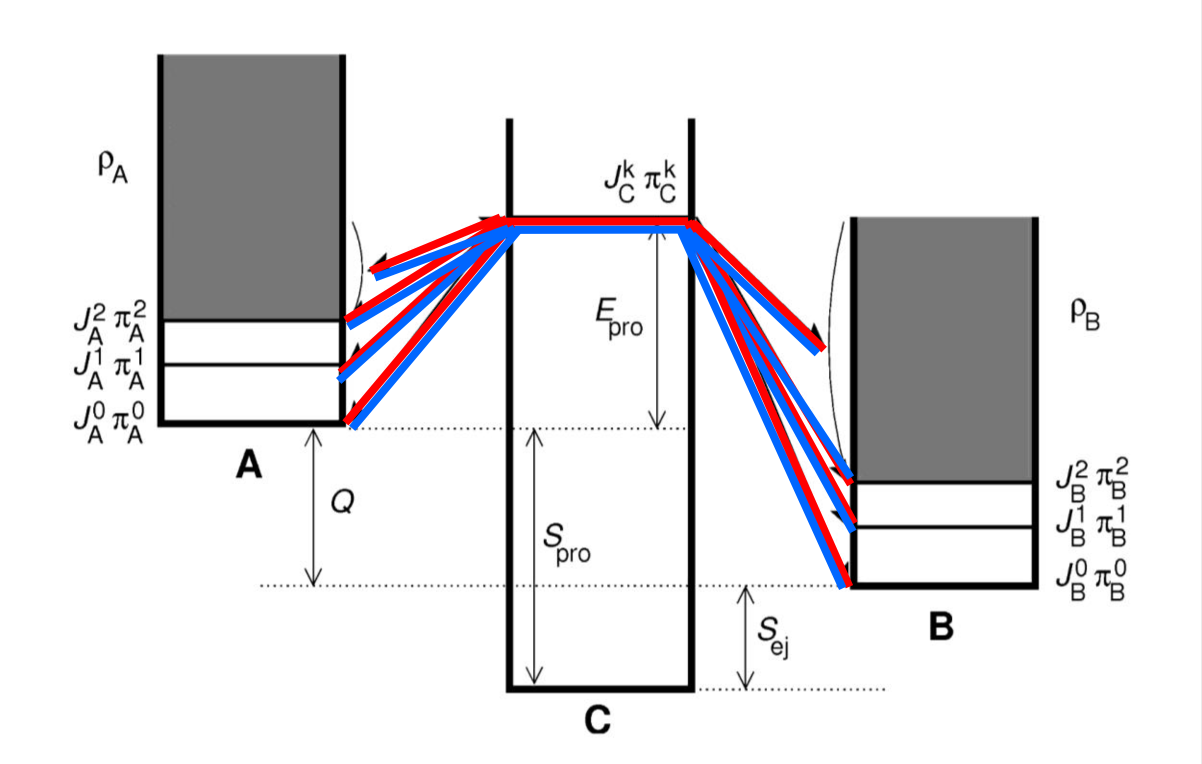
$$\sigma_{Bb}^{\nu\mu} = \frac{(2J_A^{\mu}+1)(2J_a+1)}{(2J_B^{\nu}+1)(2J_b+1)}\frac{k_{Aa}^2}{k_{Bb}^2}\sigma_{Aa}^{\mu\nu} = \frac{g_A^{\mu}g_a}{g_B^{\nu}g_b}\frac{m_A}{m_B}\frac{E_A}{E_B}\sigma_{Aa}^{\mu\nu}$$

Reciprocity relation

$$\sigma_{\text{lab}} = \sigma_{Aa}^{0} = \sum_{\nu}\sigma_{Aa}^{0\nu}$$

Lab cross section; no reciprocity with σ_{Bb}^{0}

$$\left(\text{in general}: \sigma_{Aa}^{\mu} = \sum_{\nu}\sigma_{Aa}^{\mu\nu}\right)$$



$$\sigma_{Bb}^{\nu\mu} = \frac{g_A^{\mu} g_a}{g_B^{\nu} g_b} \frac{m_A}{m_B} \frac{E_A^{\mu}}{E_B^{\nu}} \sigma_{Aa}^{\mu\nu}$$

Reciprocity relation

For fun, let's postulate "effective" cross section:

$$\sigma_{Aa}^{\text{eff}} = \sum_{\mu}\sum_{\nu} \frac{g_A^{\mu}}{g_A^{0}} \frac{E_A^{\mu}}{E_A^{0}} \sigma_{Aa}^{\mu\nu} = \sum_{\mu} \frac{g_A^{\mu}}{g_A^{0}} \frac{E_A^{\mu}}{E_A^{0}} \sigma_{Aa}^{\mu}$$

$$\sigma_{Bb}^{\text{eff}} = \sum_{\nu} \frac{g_B^{\nu}}{g_B^{0}} \frac{E_B^{\nu}}{E_B^{0}} \sigma_{Bb}^{\nu}$$

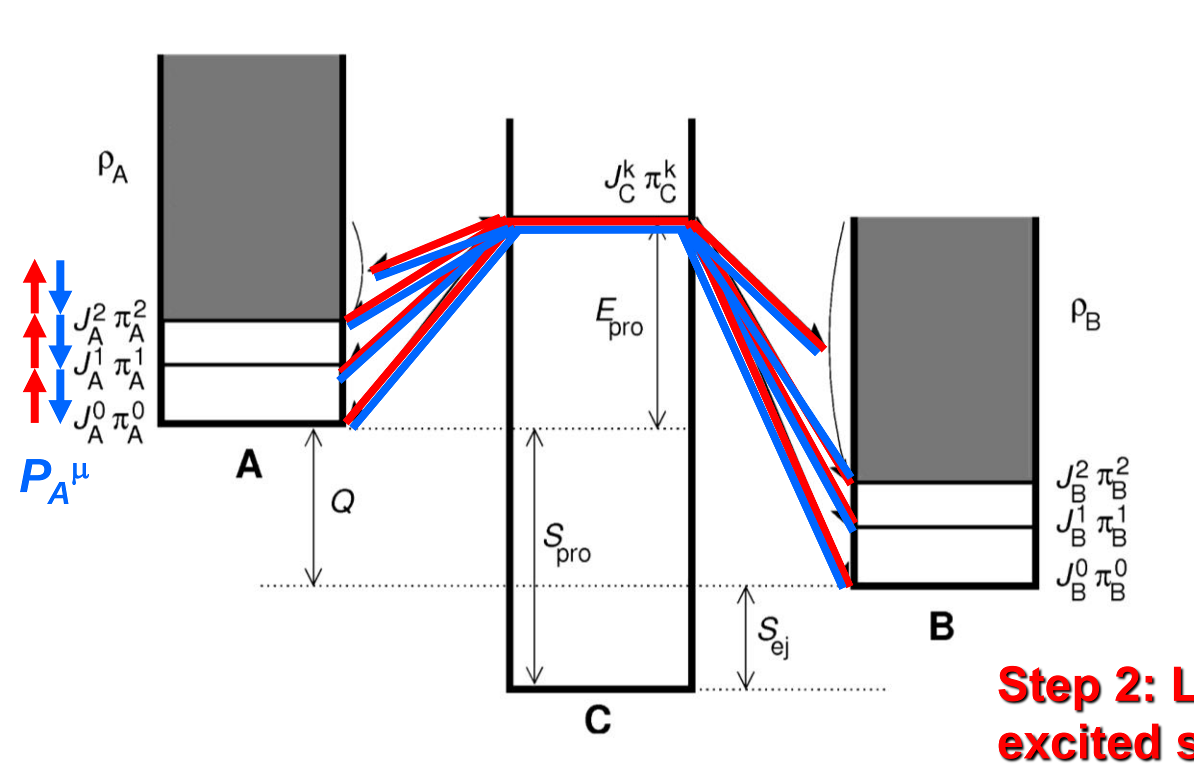
Reciprocity again!!

$$\sigma_{Bb}^{\text{eff}} = \frac{g_A^{0} g_a}{g_B^{0} g_b} \frac{m_A}{m_B} \frac{E_A^{0}}{E_B^{0}} \sigma_{Aa}^{\text{eff}}$$

or

$$\sigma_{Bb}^{\text{eff}} E_B^{0} = \frac{g_A^{0} g_a}{g_B^{0} g_b} \frac{m_A}{m_B} E_A^{0} \sigma_{Aa}^{\text{eff}}$$

But: unmeasureable!



$$\sigma_{Aa}^{\text{eff}} = \sum_{\mu} \frac{g_A^{\mu}}{g_A^0} \frac{E_A^{\mu}}{E_A^0} \sigma_{Aa}^{\mu}$$

Effective c.s.

$$\sigma_{Bb}^{\text{eff}} E_B^0 = \frac{g_A^0 g_a}{g_B^0 g_b} \frac{m_A}{m_B} E_A^0 \sigma_{Aa}^{\text{eff}}$$

Reciprocity relation

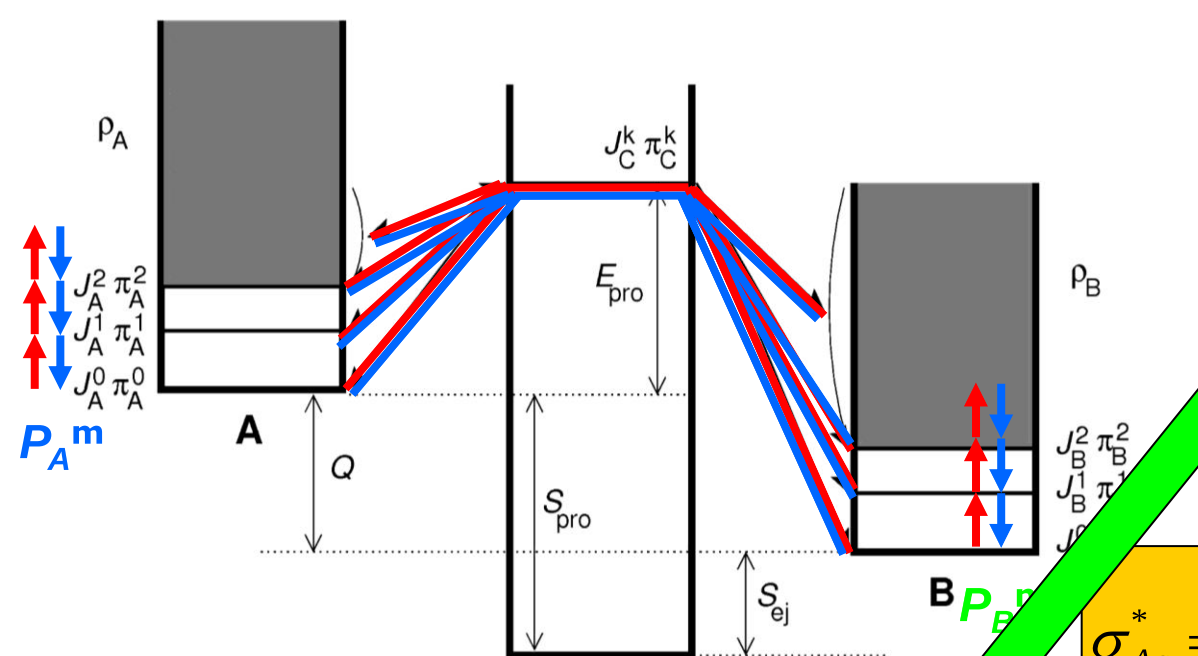
Step 2: Let's add thermal population of excited states → Detailed Balance

$$P_A^{\mu} = \frac{N_A^{\mu}}{N_A^{\text{tot}}} = \frac{g_A^{\mu} e^{-\varepsilon_A^{\mu}/(kT)}}{\sum_{\mu} g_A^{\mu} e^{-\varepsilon_A^{\mu}/(kT)}} = \frac{g_A^{\mu} e^{-\varepsilon_A^{\mu}/(kT)}}{G_A(T)} = \frac{g_A^{\mu} e^{-\varepsilon_A^{\mu}/(kT)}}{g_A^0 G_A^{\text{norm}}}$$

From Saha equation; $G(T)$ is partition function

$$\sigma_{Aa}^{*} = \sum_{\mu} P_A^{\mu} \sigma_{Aa}^{\mu}$$

Stellar cross section



Step 3: Insert in stellar rate

$$\sigma^{\rm eff}_{Aa} = \sum_\mu \frac{g^\mu_A}{g^0_A}\frac{E^\mu_A}{E^0_A}\sigma^\mu_{Aa}$$

Effective c.s.

$$\sigma^0_{Bb} E^0_B = \frac{g^0_A g_a}{g^0_B g_b}\frac{m_A}{m_B} E^0_A \sigma^{\rm eff}_{Aa}$$

Reciprocity relation

$$\sigma^*_{Aa} = \sum_\mu P^\mu_A \sigma^\mu_{Aa} = \sum_\mu \frac{g^\mu_A e^{-\varepsilon^\mu_A/(kT)}}{g^0_A G^{\rm norm}_A}\sigma^\mu_{Aa}$$

Stellar cross section

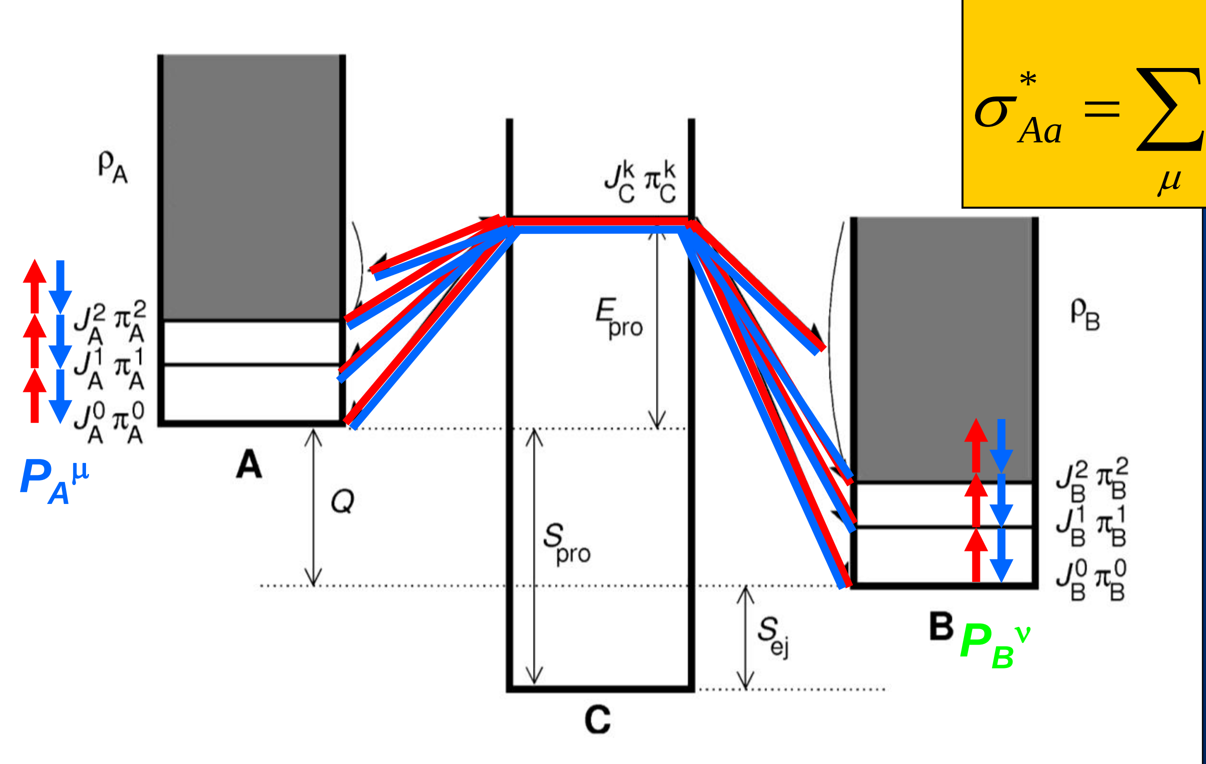
One MB distribution instead of many!

$$\langle \sigma v\rangle^*_{Aa} \propto \int \sigma^*_{Aa} E_A e^{-E_A/(kT)} dE_A = \frac{1}{G^{\rm norm}_A}\int \left\{ \sum_\mu \frac{g^\mu_A}{g^0_A}\sigma^\mu_{Aa} E_A e^{-\left(E_A+\varepsilon^\mu_A\right)/(kT)} \right\} dE_A$$

$$= \ldots = \frac{1}{G^{\rm norm}_A}\int \left\{ \sum_\mu \frac{g^\mu_A}{g^0_A} E^\mu_A \sigma^\mu_{Aa} e^{-E^0_A/(kT)} \right\} dE^0_A = \frac{1}{G^{\rm norm}_A}\int \sigma^{\rm eff}_A E^0_A e^{-E^0_A/(kT)} dE^0_A$$

Stellar rates obey reciprocity! This implies thermal equilibrium in BOTH nuclei A, B!

$$\sigma^{*}_{Aa} = \sum_{\mu} P^{\mu}_{A} \sigma^{\mu}_{Aa} = \sum_{\mu} \frac{g^{\mu}_{A} e^{-\varepsilon^{\mu}_{A}/(kT)}}{g^{0}_{A} G^{\text{norm}}_{A}} \sigma^{\mu}_{Aa}$$



Stellar cross section

Reciprocity relation for stellar rates:

$$\langle \sigma v \rangle^{*}_{Bb} = \left(\frac{m_{Aa}}{m_{Bb}} \right)^{3/2} \frac{g_A g_a}{g_B g_b} \frac{G^{\text{norm}}_{A}}{G^{\text{norm}}_{B}} e^{-\frac{Q_{Aa}}{kT}} \langle \sigma v \rangle^{*}_{Aa}$$

(Similar for photodisintegration)

<u>Prerequisite:</u>
Fast thermal equilibration in all channels!
Fulfilled in most cases, unless there are isomeric states.

Always determine rate in direction of positive Q_{Aa}, to maximize g.s. contribution and numerical errors. For numerical stability in reaction networks, forward and backward rates have to be computed from ONE source!

Nucleus-Photon Rate

With Planck distribution of photons:

$$r_{m\gamma} = n_m \lambda_{m\gamma}(T)$$

$$\lambda_{m\gamma}(T) = \frac{1}{\pi^2 c^2 \hbar^3} \int_0^\infty \frac{\sigma^*_{m\gamma}(E_\gamma) E_\gamma^2}{\mathrm{e}^{E_\gamma/kT} - 1} dE_\gamma$$

Connection to capture rate by <u>detailed balance</u>:

$$\lambda_{m\gamma} = \left(\frac{A_i A_j}{A_m} \right)^{3/2} \frac{(2J_i+1)(2J_j+1)}{2J_m+1} \frac{G_i(T)}{G_m(T)} \left(\frac{\mu kT}{2\pi\hbar^2} \right)^{3/2} \mathrm{e}^{-Q_{ij}/kT} \left\langle \sigma^* v \right\rangle_{ij}$$

Nuclear Partition Functions

$$G_0(T) = \frac{1}{2J_0+1}\left[\sum_{i=0}^{k}(2J_i+1)e^{-E_i/kT} + \int_{E_k}^{E_{max}}\sum_{J,\pi}(2J+1)e^{-\varepsilon_i/kT}\rho(\varepsilon,J,\pi)d\varepsilon\right]$$

G_0 (or G^{norm}) is normalized to the g.s. $(2J_0+1)$. PF is proportional to number of different configurations at given temperature *T*. Corrections due to loss of nucleons to the continuum may apply at *T* > 10.

Reciprocity in Stellar Rates

Some considerations:

- Detailed balance: thermalization required
  - Problematic for nuclei with isomeric states
  - e.g., ^{26}Al, ^{180}Ta
  - Use "internal" network to follow all particle and photon transitions between states in a nucleus
- ONE source for forward and reverse reaction in network for numerical stability and proper equilibria
  - Usually direction of positive Q value ("Q-value rule")
- Photodisintegration in lab tests only few transitions, better use capture and compute reverse rate

Ground state contribution to stellar rate

$$X = \frac{R_0}{R^* G_0} = \frac{\int \sigma^{\mathrm{lab}}(E)\Phi_{\mathrm{MB}}(E,T)dE}{\int \sigma^{\mathrm{eff}}(E)\Phi_{\mathrm{MB}}(E,T)dE}$$

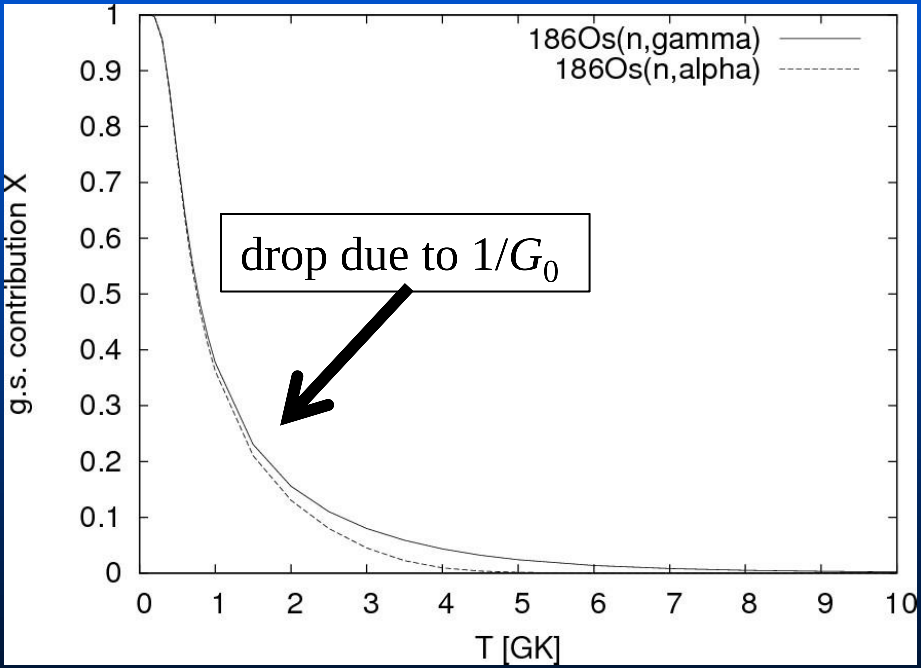
wrong

$$f_{\mathrm{SEF}} = \frac{R^*}{R_0}$$

stellar enhancement factor

neglecting partition function

- <u>g.s. contribution (X)</u>
  - gives g.s. contribution to stellar rate
  - =1 at T=0
  - confined to 0<=X<=1
  - monotonically decreasing to 0
  - Uncertainty scales with G_0 and is related to X:
    - $u=(1-X)u'$



Partition function G_0 related to g.s. population

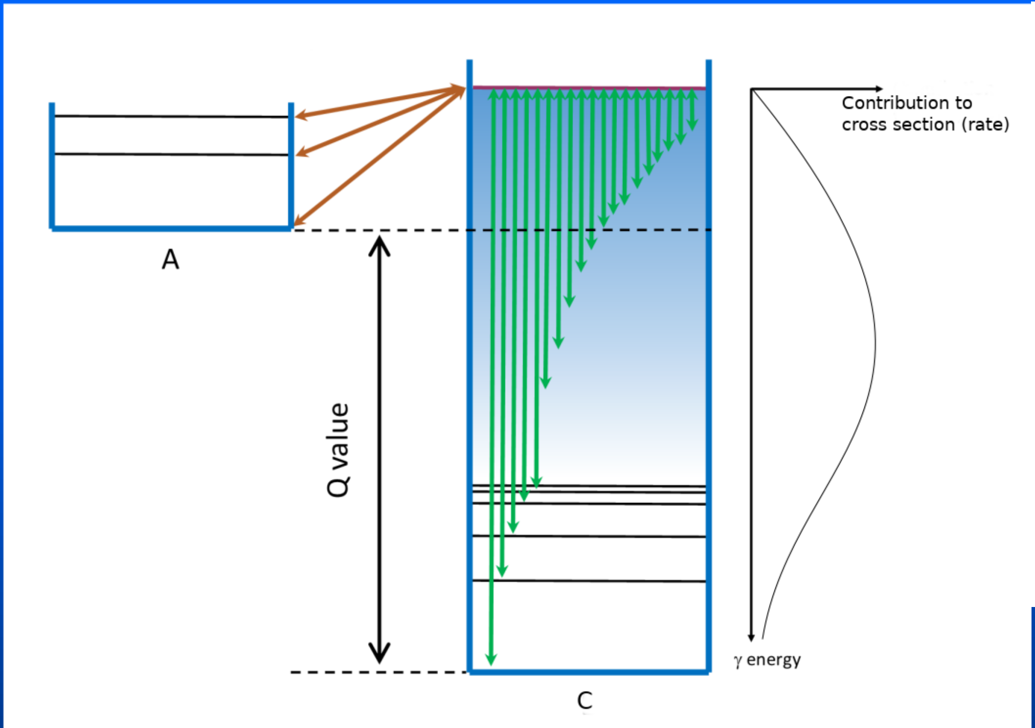
T. Rauscher, ApJLett 755, L10 (2012)   Rauscher et al, Ap. J. 738, 143 (2011)

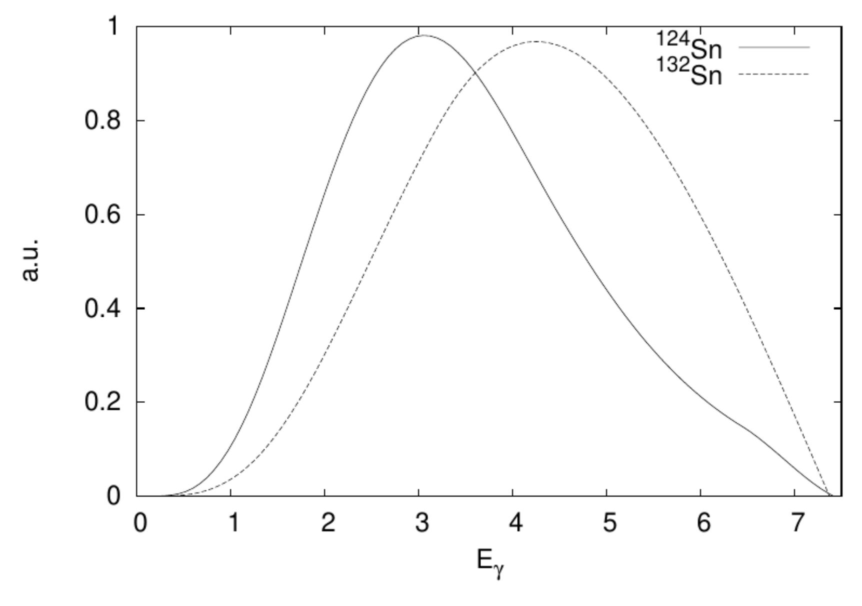
Table 4.1: Comparison of ground-state contributions ${}^*X^0$ for selected A + neutron $\leftrightarrow$ C + γ reactions at $T = 2.5$ GK. Reactions are identified by the target nucleus A of the neutron capture reaction.

A	${}^*X^0_A$	${}^*X^0_{C\gamma}$	A	${}^*X^0_A$	${}^*X^0_{C\gamma}$	A	${}^*X^0_A$	${}^*X^0_{C\gamma}$
^{85}Sr	0.771	0.00059	^{185}W	0.0788	0.00049	^{197}Pt	0.0396	0.0018
^{89}Zr	0.98	0.00034	^{184}Re	0.0148	0.00021	^{196}Au	0.0815	0.00035
^{95}Zr	0.875	0.0061	^{186}Re	0.0356	0.00024	^{195}Hg	0.0433	0.00043
^{93}Mo	0.992	0.0043	^{185}Os	0.0318	0.00016	^{197}Hg	0.066	0.00084
^{141}Nd	0.737	0.0028	^{189}Pt	0.0537	0.000069	^{203}Hg	0.551	0.0088
^{154}Gd	0.0914	0.0012	^{191}Pt	0.0541	0.00011	^{203}Pb	0.719	0.0059

Note: all these captures have positive Q-values (Q-value rule!)

Importance of γ-energies

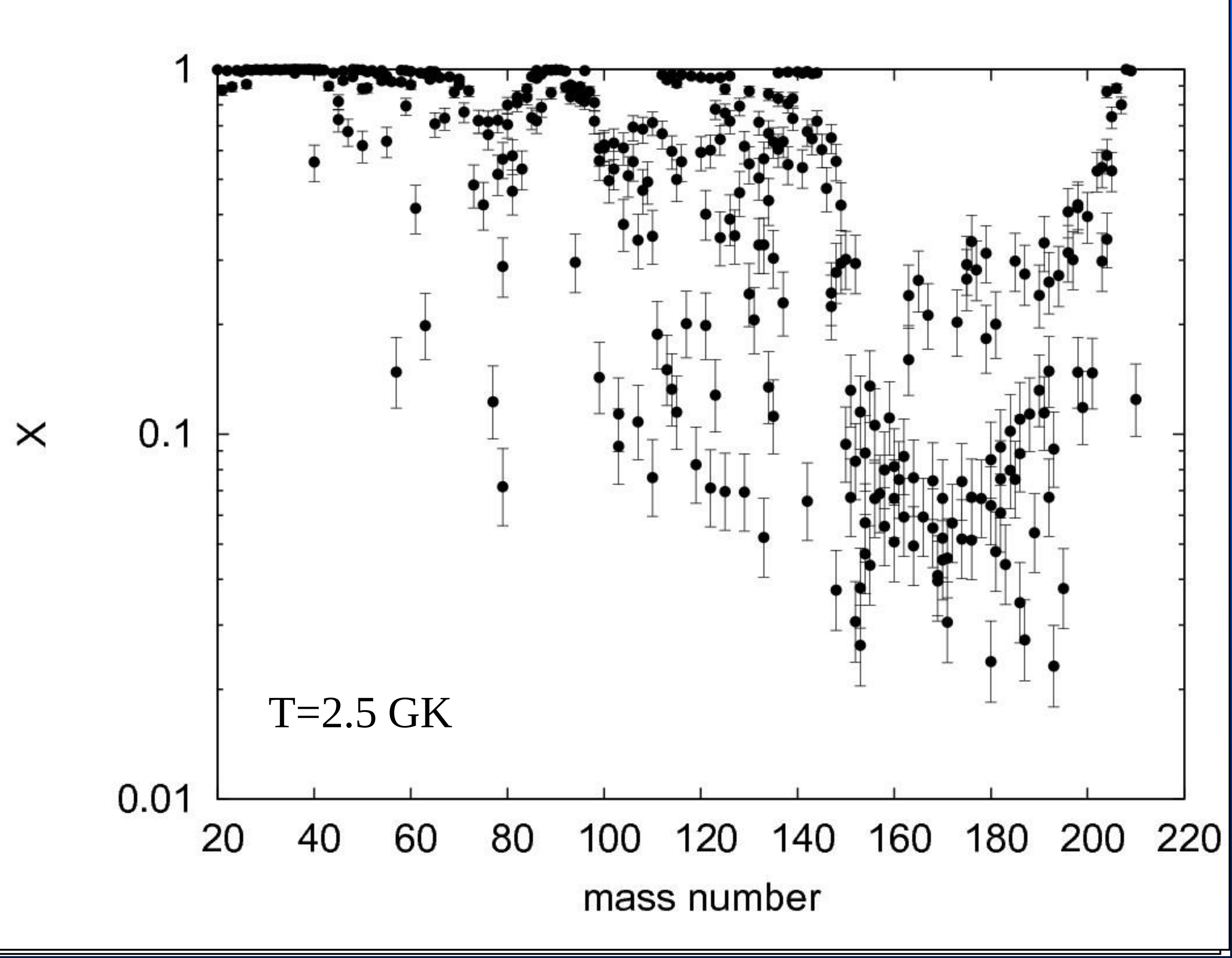




Rel. contribution to rate

A photodisintegration experiment would only measure transition from/to g.s.!
Not suited to directly constrain the reaction rate!
g.s. contribution much larger in capture direction.

Ground-state contributions to s-process neutron capture?



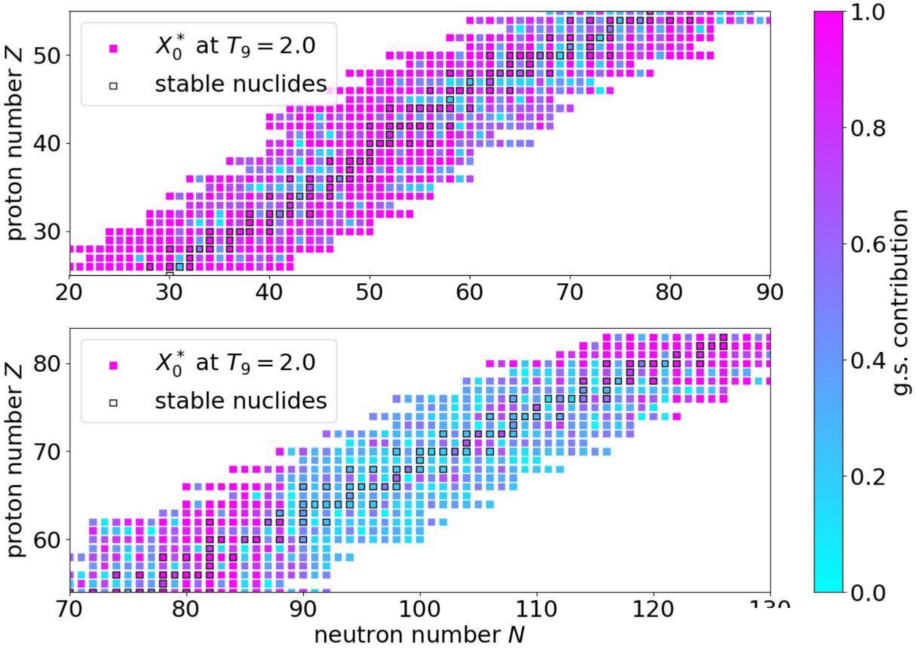
X directly also gives the maximally possible reduction in (theory) uncertainty by experiments!

- Nuclides from KADoNiS
- (n,γ) at kT=30 keV

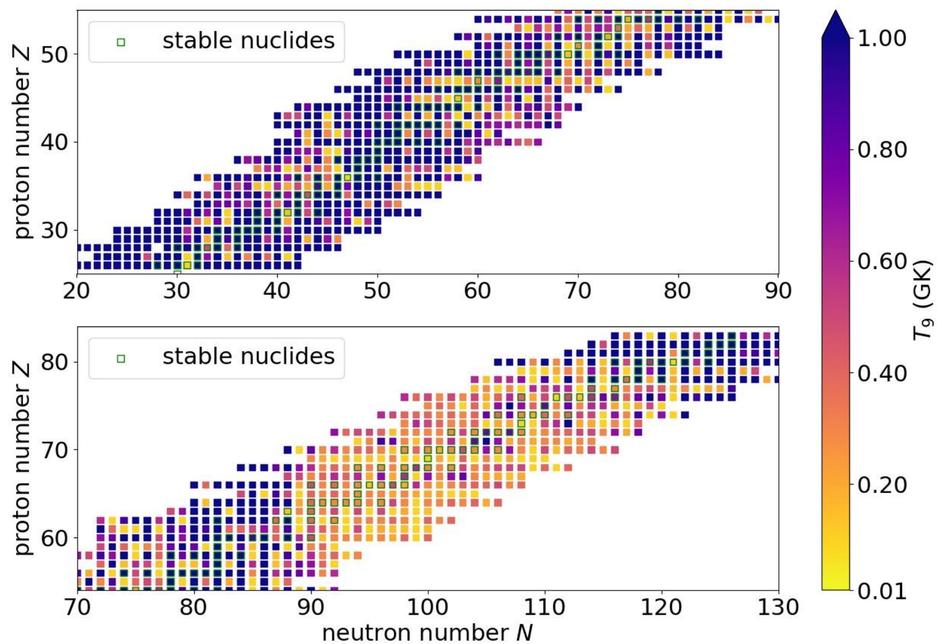
Black squares are nuclei for which error cannot be reduced by more than 80%

Rauscher P. Mohr, I. Dillmann, R. Plag; Ap. J. 738 (2011) 143.

G_0 known for s-process conditions!

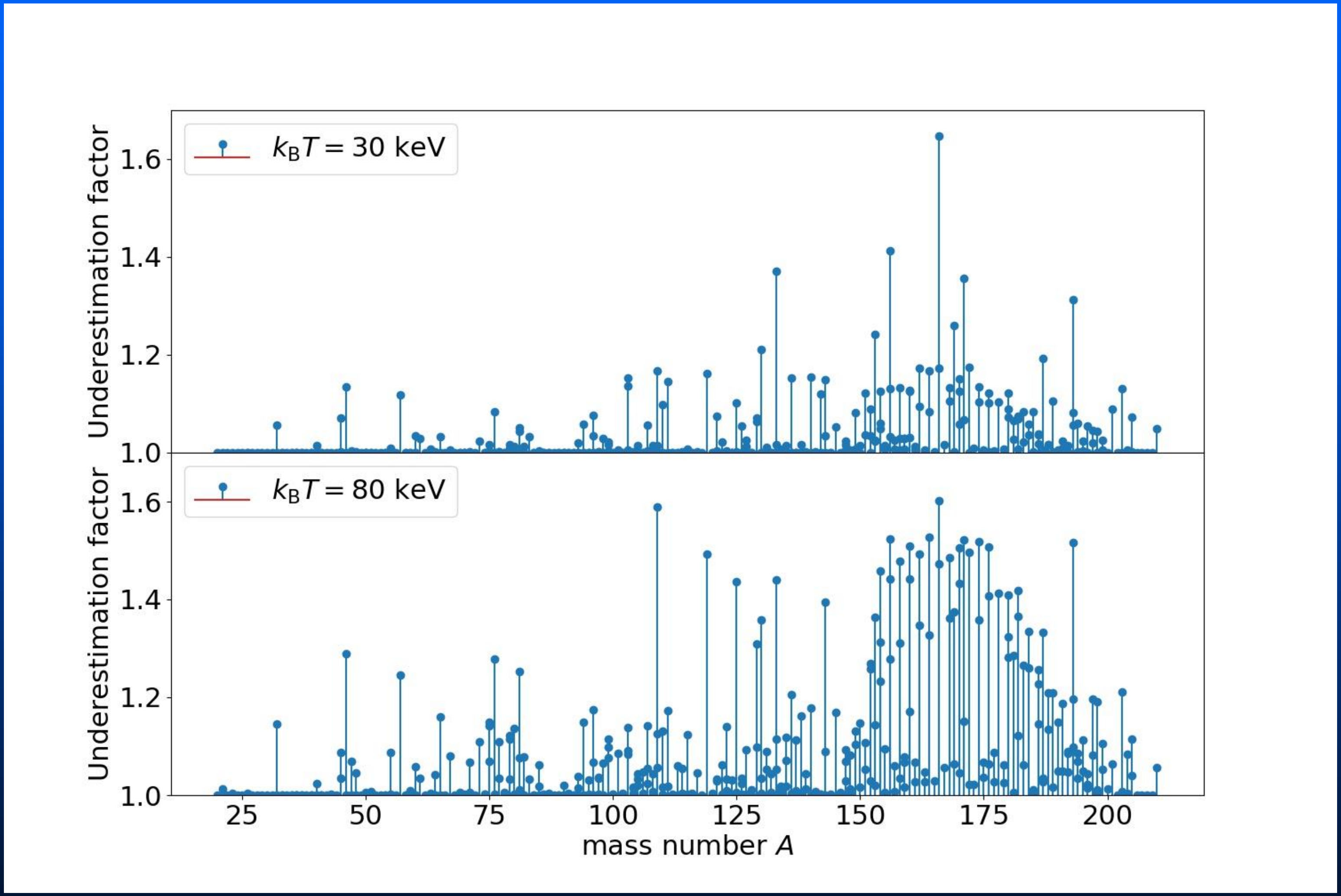


g.s. contribution X_0* for (n,γ)
(2 GK is much higher than s-process temperature



T_9 at which X_0* < 0.8 for (n,γ)

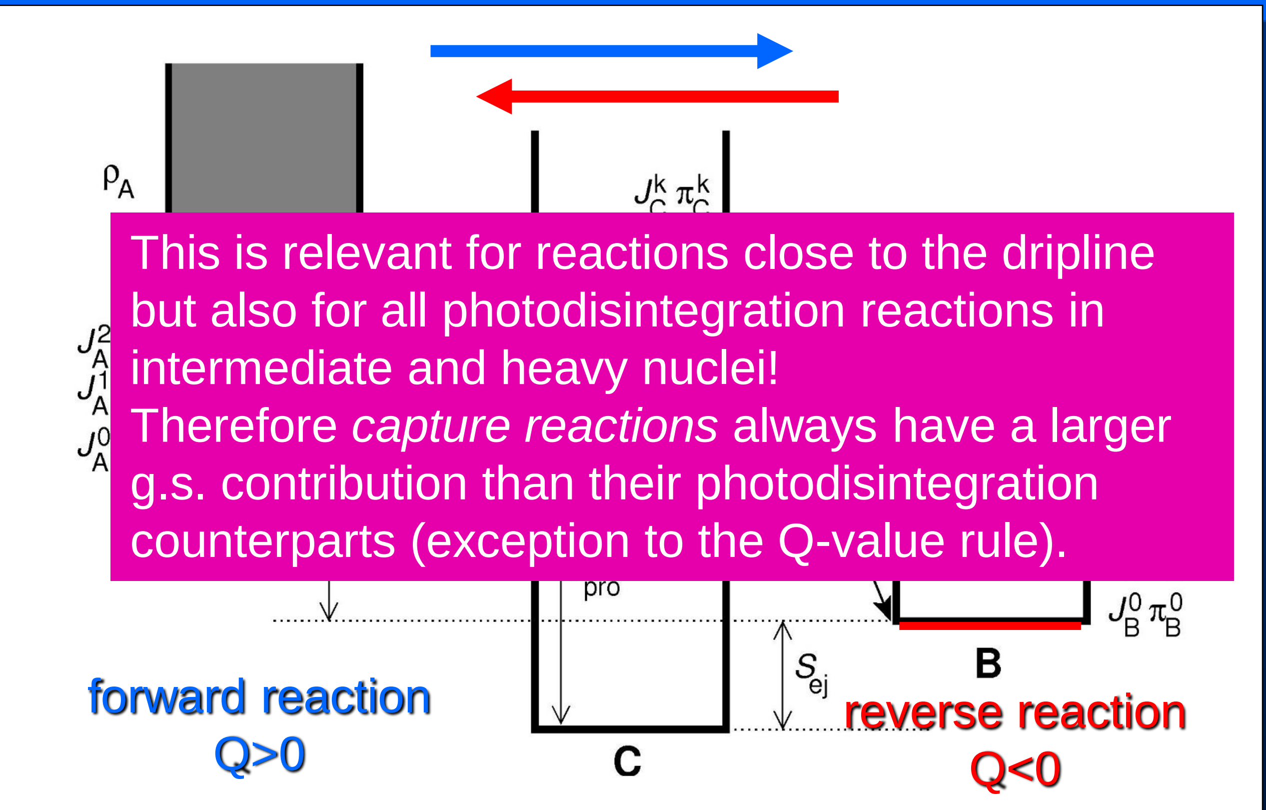
Underestimation of excited state contribution to neutron capture rate when using SEF



(target nuclei along stability)

Rauscher, EPJA 58, 214

Coulomb enhancement of g.s. contribution



This is relevant for reactions close to the dripline but also for all photodisintegration reactions in intermediate and heavy nuclei!
Therefore *capture reactions* always have a larger g.s. contribution than their photodisintegration counterparts (exception to the Q-value rule).

It is usually assumed that $X_{forw} > X_{rev}$ and therefore a measurement of the forward reaction will be closer to stellar cross section.

However, low energy transitions of charged particles will be suppressed even when they are favored by spin selection. Thus, for reactions with different Coulomb barriers in the channels, an inversion is possible!

g.s. contribution: $X = \frac{r^{\mathrm{lab}}}{G_0 r^*}$

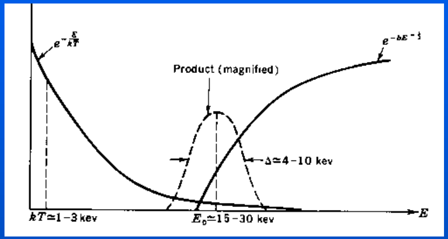
MB population: $P_i = (2J_i + 1)\, e^{-\frac{E_i}{kT}}$

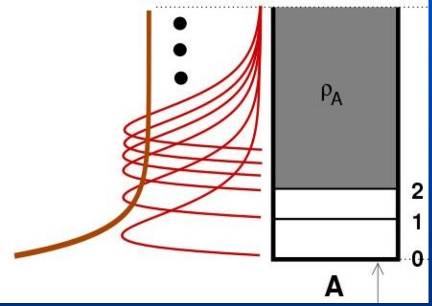
transition probability: T_i

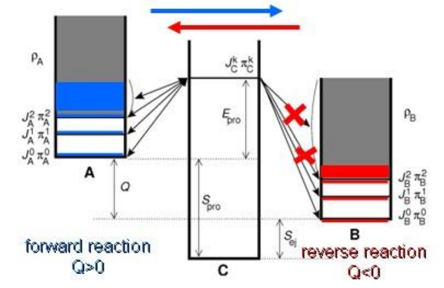
$$X_{\mathrm{forw}} < X_{\mathrm{rev}}$$

Stellar Reaction Rates

When assessing impact of nuclear physics or planning experiments, pay attention to:







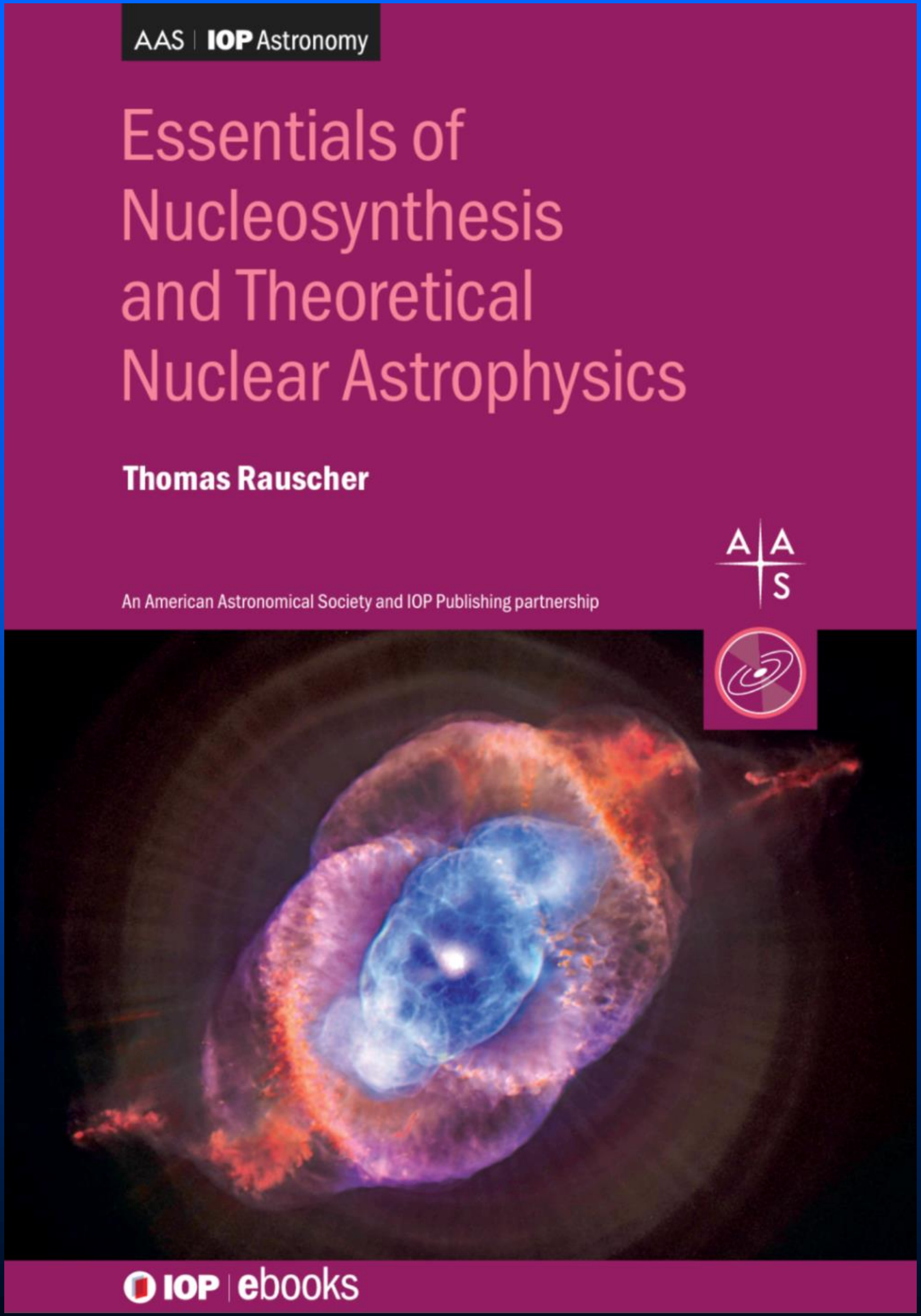
- Relevant energy range!
  - simple Gamow peak formula NOT correct!
  - *incorrect in some text books*
- Stellar modification of the rates
  - Many additional transitions from excited states!
  - NOT simple Boltzmann factor!
  - *incorrect in some text books*
- Ground state contribution of the measured reaction
  - Photodisintegration rate never good for direct measurement

Review: Rauscher, Int. J. Mod. Phys. E 20, 1071 (2011)

Keep in mind

There is a fundamental difference in reactions acting in nucleosynthesis of light nuclei and heavier nuclei (A>30) stemming from differences in Coulomb barrier, level density and stellar plasma temperatures:

- Light nuclides: Low level density, large level spacings, low Coulomb barriers, low synthesis temperature.
  - Few transitions contributing, g.s. contribution large
  - Experiments may be able to probe all contributing transitions and constrain stellar rates
- Heavier nuclides: Large level density, small level spacings, high Coulomb barriers, high temperatures in nucleosynthesis site (perhaps except for s-process), large contributions of the excited target states to the stellar rate, *unstable nuclei*.
  - Lab measurement can only constrain a fraction of the stellar rate by c.s. measurement and only a fraction of the relevant transitions (if c.s. measurement not feasible)
  - Experiment can be used to test and improve certain features of reaction models or predictions of nuclear properties
  - The majority of reaction rates has to come from theory (prediction of resonance properties problematic if individual resonances are important).



AAS-IOP ebook series
IOP Publishing, July 2020

ISBN: 978-0-7503-1149-6 (e-book)
ISBN: 978-0-7503-1150-2 (print)

Contents (summary), 2 parts in 1 volume:

- Part 1: Essentials
  - Basic definitions, equations of state, stellar structure, nuclear physics and reactions, stellar effects on cross sections, astrophysical reaction rates, reaction networks and reaction equilibria
- Part 2: Nucleosynthesis
  - Stellar evolution, hydrostatic and explosive burning, origin of the elements beyond Fe, Big Bang nucleosynthesis, Galactic Chemical Evolution

https://iopscience.iop.org/book/978-0-7503-1149-6

https://store.ioppublishing.org/page/detail/Essentials-of-Nucleosynthesis-and-Theoretical-Nuclear-Astrophysics/?K=9780750311496

Additional links/references

- The text book shown on the previous slide and references therein:
  - Details on nuclear physics as well as astrophysics; covers all theory aspects of nuclear astrophysics topics of the school.

- See also the references given on the slides.

- Further textbook references:
  - Iliadis, Nuclear Physics of Stars, 2nd edition, Wiley 2015
  - Krane, Introductory Nuclear Physics, Wiley & Sons 1988
  - Blatt & Weisskopf, Theoretical Nuclear Physics, Springer 1988
  - Hodgson, Gadioli & Gadioli-Erba, Introductory Nuclear Physics, Clarendon 1997
  - Satchler, Direct Nuclear Reactions, Clarendon 1983
  - Glendenning, Direct Nuclear Reactions, World Scientific 2004
  - Fröbrich & Lipperheide, Theory of Nuclear Reactions, Clarendon 1996
  - + those given in other talks.

- Sensitivity plots (for cross sections + reactivities) and g.s. contributions (for reactivities) can be found by selecting a reaction at https://nucastro.org/reacs .

- Publication list: Many publications for specific reactions and applications to astrophysics can be found at https://thomasrauscher.ch/pubs.html .

What I have no time to talk about *in detail*

- Other types of reactions: decay, fission, …
- Screening of reactions in the plasma
- Simplifications of reaction networks

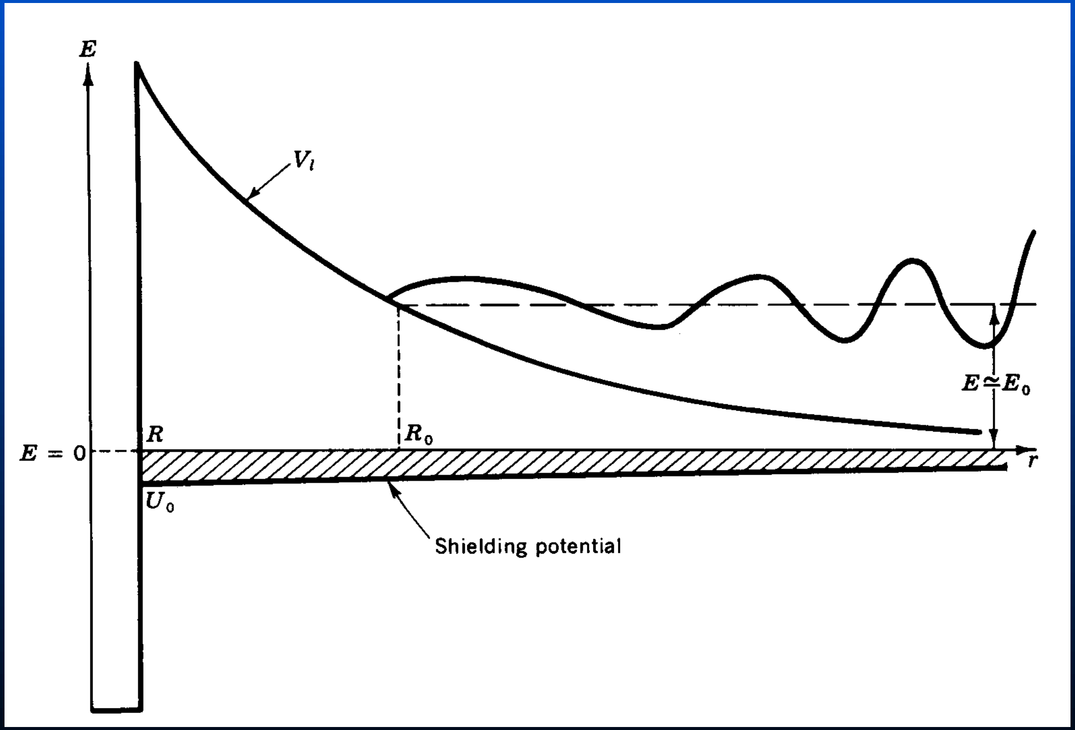
Electron screening

The nuclei in an astrophysical plasma undergoing nuclear reactions are fully ionized.

However, they are immersed in a dense electron gas, which leads to some shielding of the Coulomb repulsion between projectile and target for charged particle reactions.

Charged particle reaction rates are therefore enhanced in a stellar plasma, compared to reaction rates for bare nuclei.

The Enhancement depends on the stellar conditions



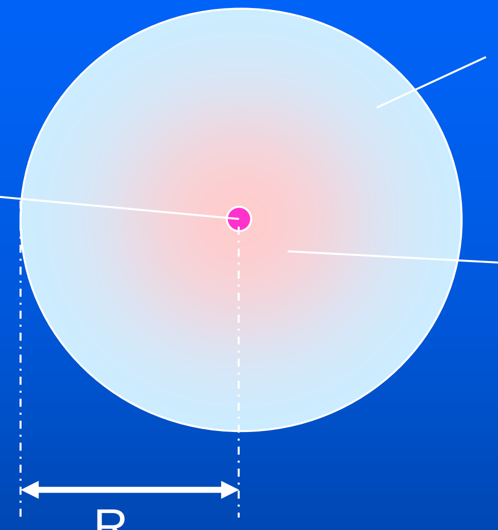
$$V(r) = \frac{Z_1 Z_2 e^2}{r} + U(r)$$

Bare nucleus Coulomb

Extra Screening potential

(attractive, so <0)

For weak screening, each ion is surrounded by a sphere of ions and electrons that are somewhat polarized by the charge of the ion (Debeye Huckel treatment)



Exp: Quicker drop off due to screening

Then potential around ion

$$V_1(r) = \frac{eZ}{r} e^{-r/R_D}$$

With

$$R_D = \sqrt{\frac{kT}{4\pi e^2 \rho N_A \xi}}$$

$$\xi = \sum_i (Z_i^2 + Z_i)^2 Y_i$$

Thus, complete screening for $r \gg R_D$.

Reaction Equilibria

At high temperatures and/or densities, reaction equilibria can be attained. They simplify the network equations and can be used to speed up the calculations.

- NSE: Nuclear statistical equilibrium
  - all reactions are fast and equilibrated; individual rates need not to be known, abundances determined by nuclear mass differences
  - Si-burning, ejecta from the innermost parts of a core-collapse supernova or neutron star mergers
- QSE: Quasi-statistical equilibrium
  - groups of equilibrated nuclei, slow connecting reactions have to be known
  - O-, Si-burning in massive stars
- Waiting Point Approximation, (n,γ)-(γ,n) equilibrium, (p,γ)-(γ,p) equilibrium, β-flow equilibrium
  - QSE-type equilibria where isotopic, isotonic or isobaric chains of nuclides are equilibrated
  - r-process, rp-process, νp-process
- Steady flow
  - Reaction chain operating for extended times; s-process, r-process (between peaks), hydrostatic burning phases in stars

Reaction rates cancel between forward and reverse reaction → no cross section needed!

Reaction Equilibria

At high temperatures and/or densities, reaction equilibria can be attained. They simplify the network equations and can be used to speed up the calculations.

- NSE
- QSE
- Waiting Point Approximation

Nuclear Statistical Equilibrium I

$T_9 > 4\text{-}5$: Strong, el.-magn. interactions in equilibrium
$\Rightarrow$ individual reactions not important for abundances:

$$Y(Z,N) = G_{Z,N}\left(\rho N_A\right)^{A-1} \frac{A^{3/2}}{2^A}\left(\frac{2\pi\hbar^2}{m_\mathrm{u} kT}\right)^{\frac{3}{2}(A-1)} \mathrm{e}^{B_{Z,N}/kT}\, Y_n^N Y_p^Z$$

$$\sum_\mathrm{i} A_i Y_i = 1$$

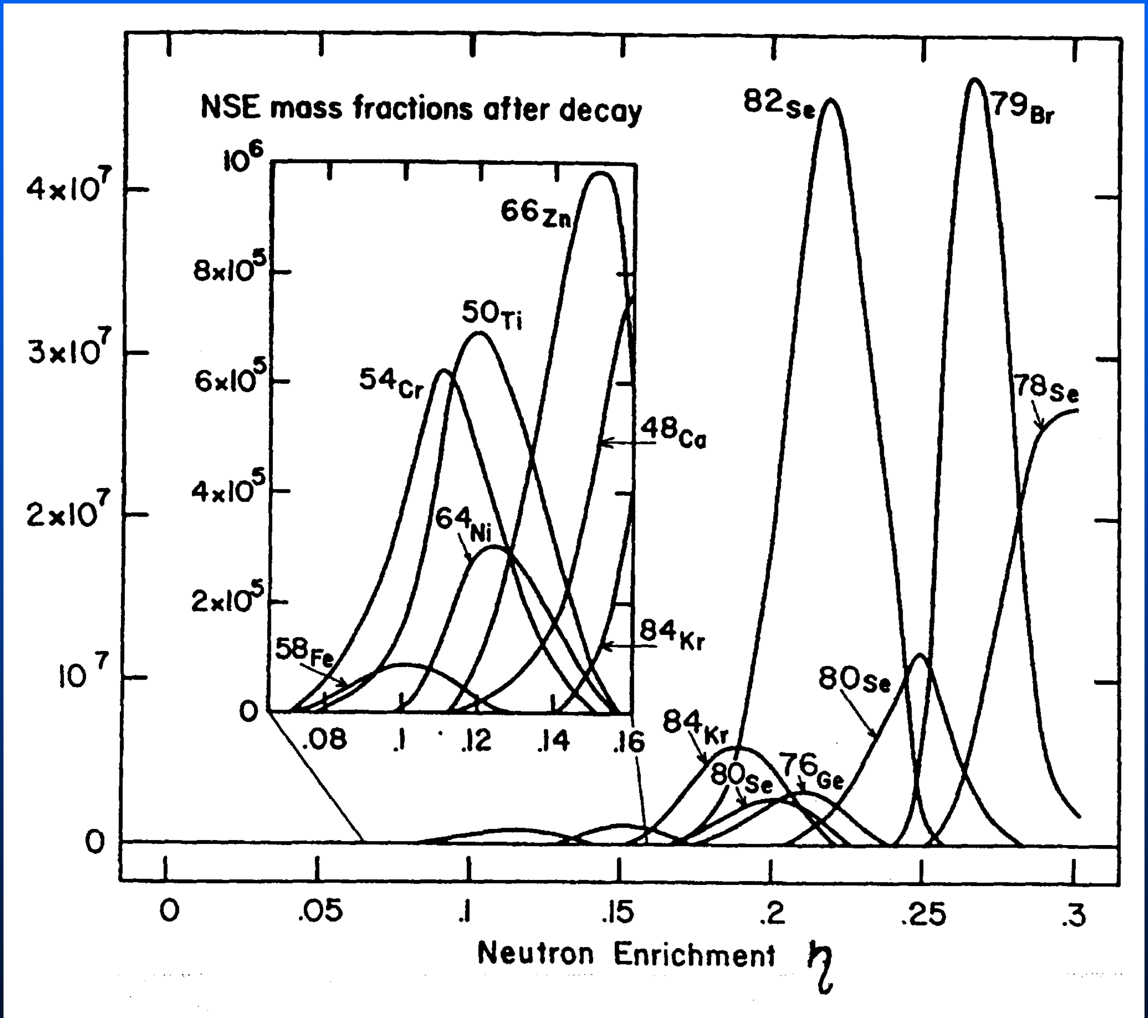
$$\sum_i Z_i Y_i = Y_\mathrm{e}$$

$Y_{\mathrm{n,p}}$...free neutrons, protons; Y_e electron abundance (weak interaction)

Nuclear Statistical Equilibrium II

1. Term ρ^{A-1}: High densities yield heavy nuclei.
2. Term $(1/kT)^{3/2(A-1)}$: High temperatures yield light nuclei.
3. Term $e^{B/kT}$: Always nuclei with high binding energy B are favored.

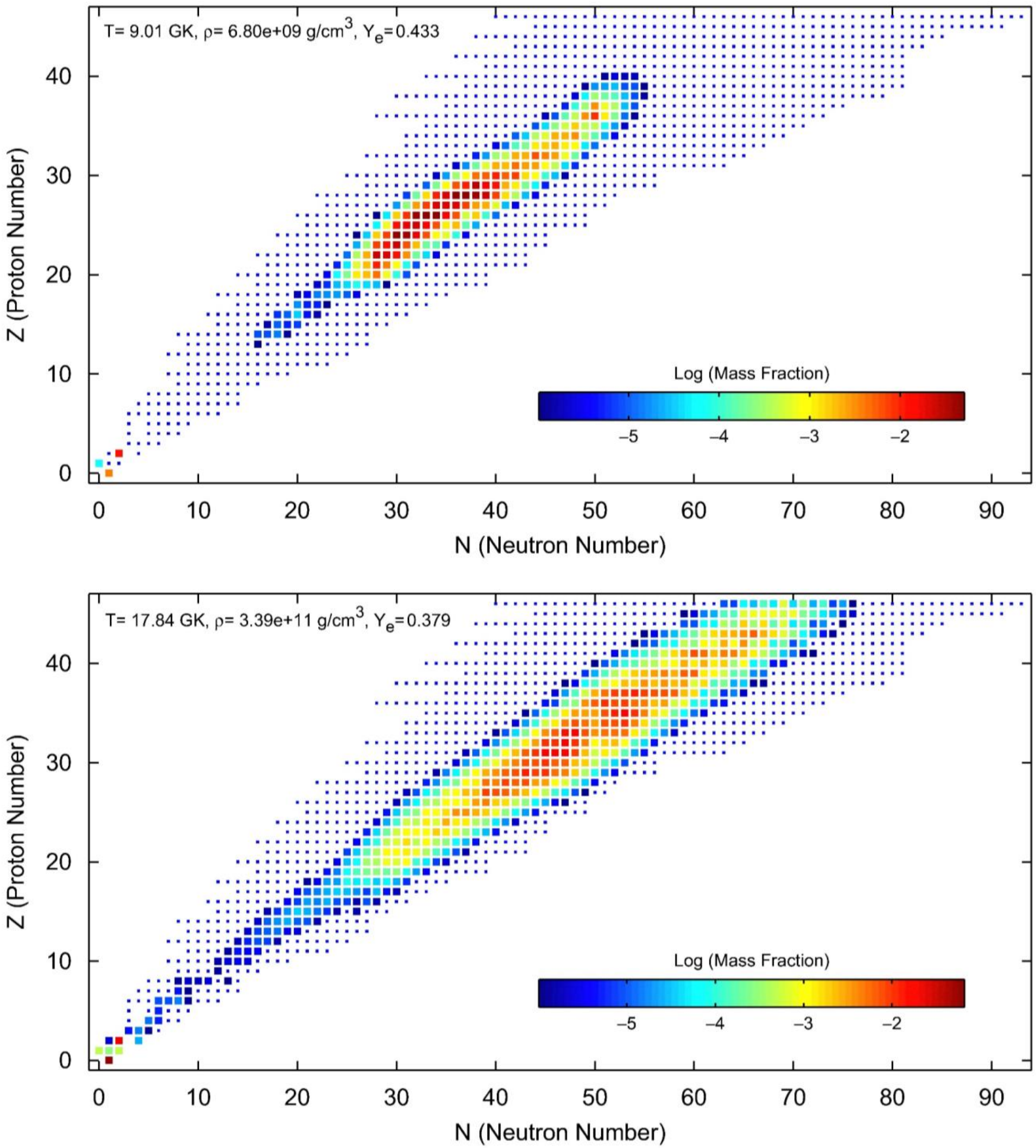
Nuclear Statistical Equilibrium III



Abundances peak at nucleus with $Z/A = Y_e$

Neutron enrichment: $\eta = 1\text{-}2Y_e$

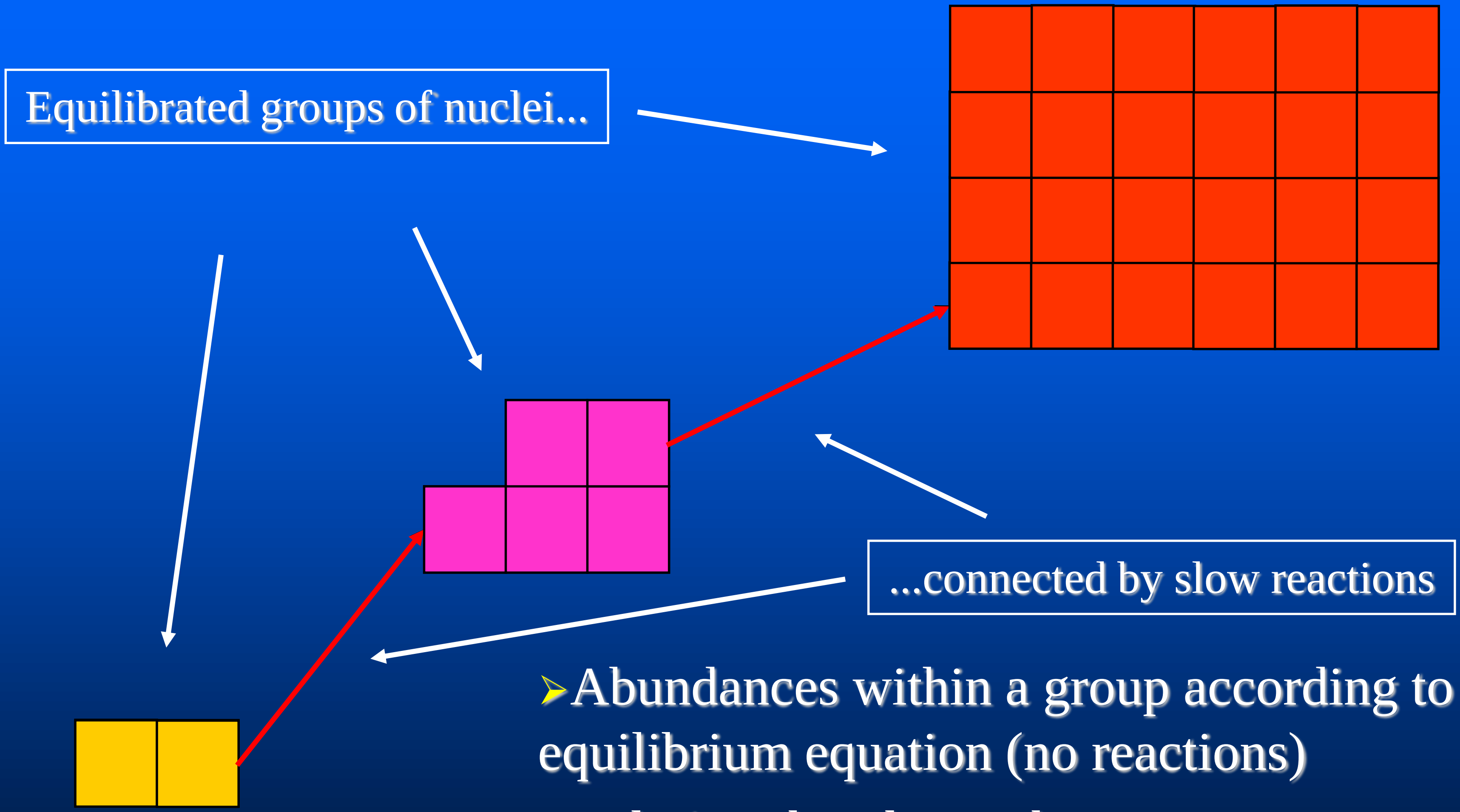
Examples of NSE distributions from interior region of ccSN



- Slope determined by Y_e
- Extension in mass number is given by ρ, T
- Within region most tightly bound nuclei are most abundant

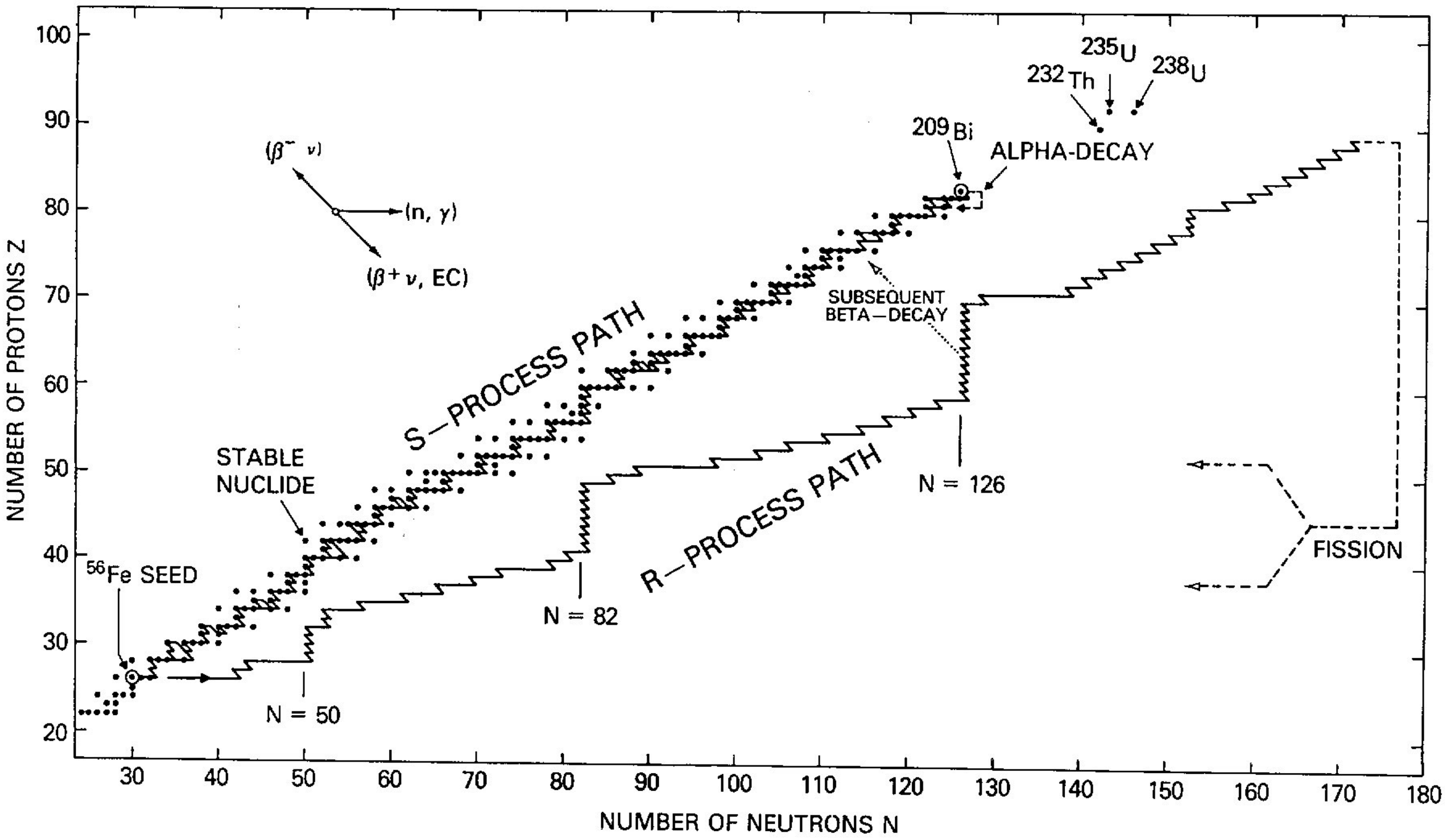
Courtesy of W. R. Hix

Quasi-Statistical Nuclear Equilibrium

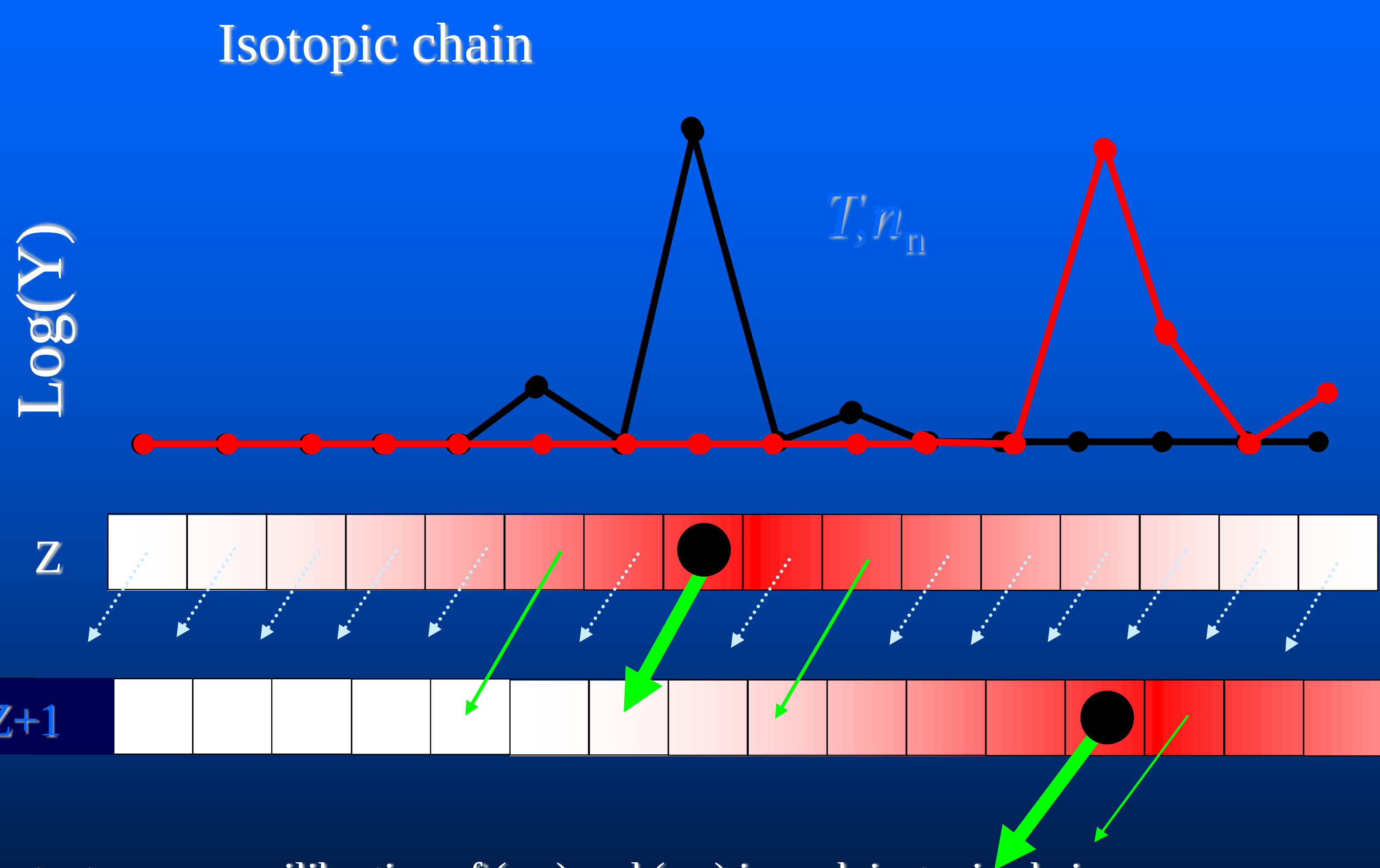


- Abundances within a group according to equilibrium equation (no reactions)
- Relative abundances between groups determined by connecting reactions (model of connected pools)

r-Process Path and Waiting Point Approximation



Definition of „Waiting Points“



- Instantaneous equilibration of (n,γ) and (γ,n) in each isotopic chain
- *β*-decay of isotope(s) with highest abundance (i.e. waiting point) populates next chain
- „Path“ defined by connecting maximum abundances in each chain

Waiting Point Approximation

Assuming $\lambda_\beta << \lambda_n$ (r-process):

$$dY_{(Z,A)}/dt = \lambda_{\gamma(A+1)} Y_{(Z,A+1)} - n_n <\sigma v>_{n\gamma(A)} Y_{(Z,A)}$$

In (n,γ)↔(γ,n) equilibrium $dY/dt = 0$ and

$$Y_{(Z,A+1)}/Y_{(Z,A)} = n_n <\sigma v>_{n\gamma(A)} / \lambda_{\gamma(A+1)}$$

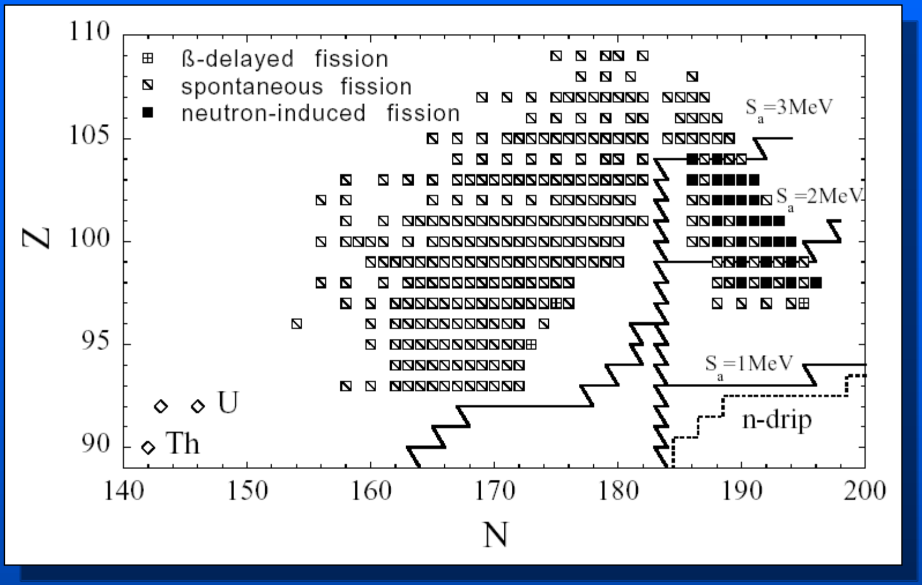
Applying detailed balance yields:

$$\frac{Y_{(Z,A+1)}}{Y_{(Z,A)}} = n_n \frac{G_{A+1}}{2G_A} \left(\frac{A+1}{A}\right)^{3/2} \left(\frac{2\pi\hbar^2}{\mu kT}\right)^{3/2} e^{(A+1)S_n/kT}$$

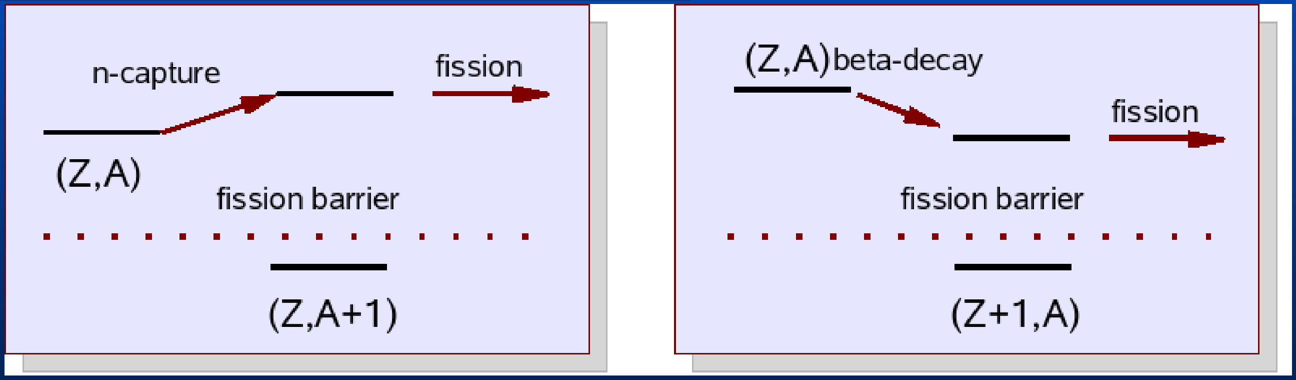
Parameters n_n, T; r-process path located around S_n=2-3 MeV.

Fission in Astrophysics

Fission: Endpoint of the r-Process



Goriely & Clerbaux 1999



Important to know: fission barriers, fission fragment distribution!
Impact on: fission cycling in r-process, production of rare-earth peak, maximal extension of r-process production (endpoint)

Decays

- Modification of half-lives in stellar environment:
  - Nuclei are thermally excited
  - For electron captures:
    - Electrons not from atomic K-shell but from free electrons in the plasma
  - For β-:
    - At extremely high densities, blocking of exit channel energies (Pauli exclusion)

Decay Rate

$$r_i = \lambda_i n_i = -\dot{N}_i$$

$$N(t) = N_0 \, \mathrm{e}^{-\lambda_i t}$$

Lifetime $\tau = 1/\lambda$, half-life $T_{1/2}$:

$$N(T_{1/2}) = \frac{1}{2} N_0 = N_0 \, \mathrm{e}^{-\lambda_i T_{1/2}} \Rightarrow \lambda_i = \frac{\ln 2}{T_{1/2}}$$

Nucleus-Lepton Rate

In reactions with leptons (electrons, positrons, neutrinos) their masses are negligible:

$$r_{i\mathrm{L}} = n_i \int \sigma_{\mathrm{L}}(v_{\mathrm{L}}) v_{\mathrm{L}} \, dn_{\mathrm{L}} = n_i \lambda_{i\mathrm{L}}(\rho, T)$$

Use FD or MB distributions according to density and temperature conditions.

Importance of nuclear input

- Energy generation
  - Evolution and lifetime of stars (+GCE)
  - Timescale and time structure of explosive events (eg. Novae, X-ray bursts, r-process)
- Nucleosynthesis
  - Products of stars, explosive events $\Rightarrow$ galactic chemical evolution
  - Explain observed stellar and galactic abundances
- Equation of state
  - Collapse of massive stellar cores
  - Neutron star properties
  - Black hole formation
- Strong sensitivity of astrophysics to nuclear properties!!
  - Can rule out astrophysical scenario
  - (or point to need for improved nuclear physics)
  - Different sensitivities of different scenarios/processes