Entropy constraints on effective field theory

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Based on **arXiv:2201.00931** with **Qing-Hong Cao (Peking University)**

arXiv:2211.08065 with **Qing-Hong Cao (Peking University)**, and **Naoto Kan (Osaka University)**

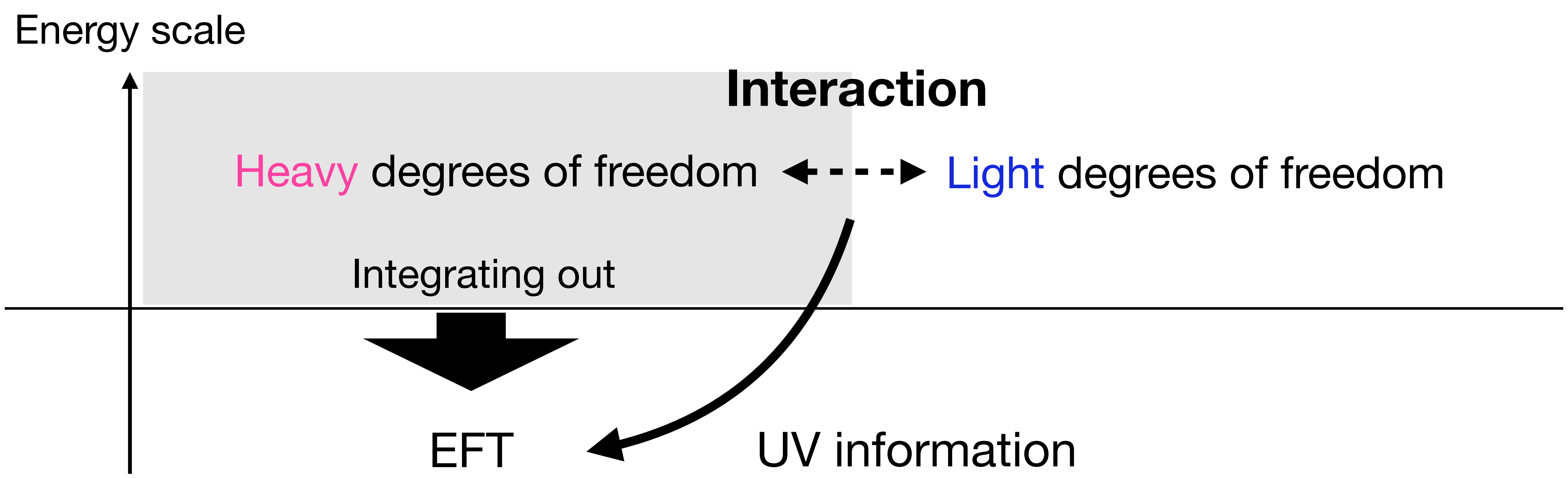
Introduction

Introduction

- Effective Field Theory (EFT):

  - EFT is generated by integrating out dynamical degrees of freedom

  - Information on UV theory is transferred through interaction b/w heavy and light degrees of freedom



Differences between theories with and without interaction characterize UV information

⇒ Relative entropy characterizes their difference

Relative entropy and our idea

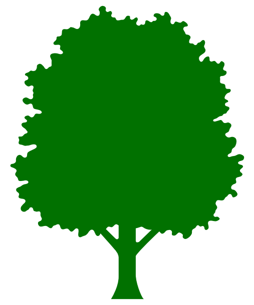
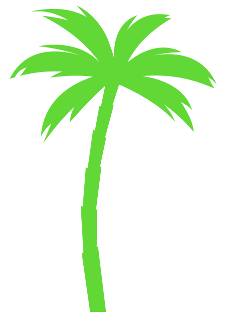
- Definition of relative entropy b/w two probability distribution functions ρ_A and ρ_B

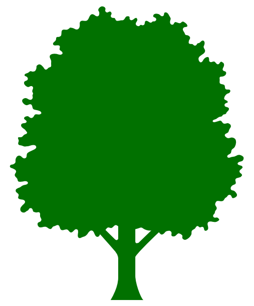
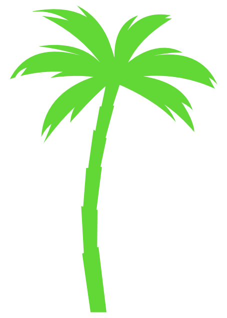
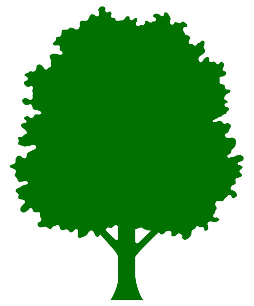
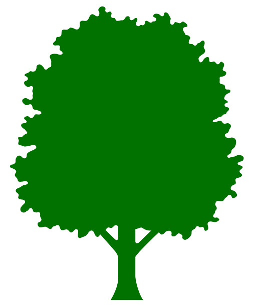
$$S(\rho_A || \rho_B) \equiv \mathrm{Tr}\left[\rho_A \ln \rho_A - \rho_A \ln \rho_B\right] \geq 0$$

- relative entropy is **non-negative** ※ equality holds if and only if $\rho_A = \rho_B$

- Relative entropy provides quantitative difference between two things defined by probability distribution functions

Ex.

 $\mapsto \rho_A$  $\mapsto \rho_B$

$S($  $||$  $) > 0$ $S($  $||$  $) = 0$

What about relative entropy b/w theories with and without interaction?

⇒ We have to define probability distribution for each theory.

Probability distributions of theories

- We define probability distributions of theory described by Euclidean action I as follows:

Probability distribution function: $P[\phi, \Phi] = e^{-I[\phi,\Phi]}/Z$

Partition function: $Z = \int d[\phi]d[\Phi]e^{-I[\phi,\Phi]}$

where I: Euclidean action, ϕ: light fields, Φ: heavy fields

- Relative entropy between two theories

$$S(P_A || P_B) \equiv \int d[\phi]d[\Phi]\left(P_A \ln P_A - P_A \ln P_B\right) \geq 0$$

where $P_A = e^{-I_A}/Z_A, \quad P_B = e^{-I_B}/Z_B$

Definition of two theories

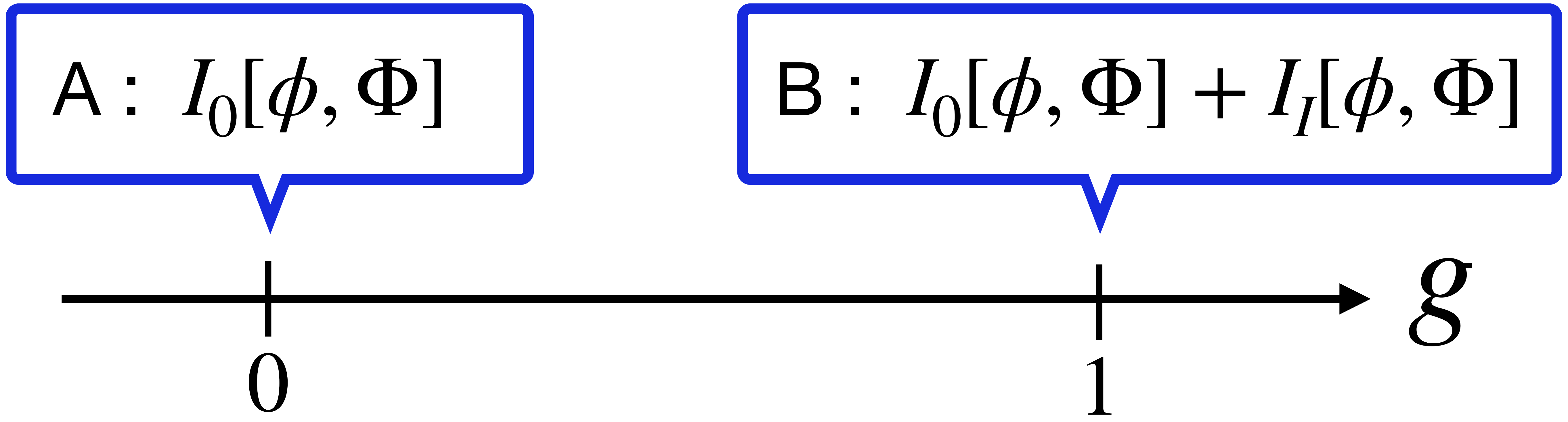
No interaction b/w ϕ and Φ

Interaction b/w ϕ and Φ

- We consider theories described by $I_0[\phi, \Phi] + I_I[\phi, \Phi]$

※ Φ: heavy fields, ϕ: light fields

- We define $I_0[\phi, \Phi] + g \cdot I_I[\phi, \Phi]$ by introducing parameter g



We consider relative entropy $S(P_A || P_B)$

※ (Φ, ϕ) of A is the same as that of B

Relative entropy between two theories

$$S(P_A||P_B) = \int d[\phi]d[\Phi]\left[P_A \ln P_A - P_A \ln P_B\right]$$

$$P_A = e^{-I_0[\phi,\Phi]}/Z_0 \qquad P_B = e^{-(I_0[\phi,\Phi]+gI_I[\phi,\Phi])}/Z_g$$

$$= W_0 - W_g + g\left(\frac{\partial W_g}{\partial g}\right)_{g=0} \geq 0$$

Effective actions: $W_g = -\ln Z_g, \quad W_0 = -\ln Z_0$

$S(P_A||P_B)$ yields constraints on the Euclidean effective actions

even in quantum mechanical system

$$S(P_A||P_B) \to \mathrm{tr}\left[P_A \ln P_A - P_A \ln P_B\right]$$

$$P_A \to e^{-H_0}/Z_0 \qquad P_B \to e^{-(H_0+gH_I)}/Z_g$$

$$= W_0 - W_g + g\left(\frac{\partial W_g}{\partial g}\right)_{g=0} \geq 0$$

$$W_g = -\ln Z_g, \; W_0 = -\ln Z_0$$

Example: Tree level matching of Higgs-singlet model

- Consider the SM Higgs H coupled to a real singlet field s

Theory	Action in Minkowski space	Probability	Partition function
A No interaction b/w H and s	$I_0 = \int d^4x \Big[\lvert D_\mu H \rvert^2 + \frac{1}{2}(\partial_\mu s)^2 - \Big(\mu_0^2 \lvert H \rvert^2 + \lambda_0 \lvert H \rvert^4 + \frac{1}{2} M^2 s^2 + \frac{g_s \cdot A_1}{3} s^3 + \frac{\lambda_s}{4} s^4 \Big) \Big]$	$P_A = e^{-I_0[H,s]}/Z_0$	$Z_0 = \int d[H] d[s] e^{-I_0[H,s]}$
B With interaction b/w H and s	$I_g = I_0 + g \cdot \int d^4x \Big(\frac{\kappa}{2} \lvert H \rvert^2 s^2 - A_1 \lvert H \rvert^2 s \Big)$	$P_B = e^{-I_g[H,s]}/Z_g$	$Z_g = \int d[H] d[s] e^{-I_g[H,s]}$

- Relative entropy:

$$S(P_A || P_B) = W_0 - W_g + g \cdot \left(\frac{dW_g}{dg} \right)_{g=0} = \int (d^4x)_E \frac{g^2 \cdot A_1^2 \lvert H \rvert^4}{2M^2} + \mathcal{O}(g^3)$$

Effective potential:

$$W_g = \int (d^4x)_E \Big[\mu_0^2 \lvert H \rvert^2 + \lambda_0 \lvert H \rvert^4 - \frac{g^2 \cdot A_1^2 \lvert H \rvert^4}{2M^2} + \mathcal{O}(g^3)$$

Example: Tree level matching of Higgs-singlet model

- Consider the SM Higgs H coupled to a real singlet field s

Theory	Action	Probability	Partition function
A	$I_0 = \int d^4x \left[\lvert D_\mu H \rvert^2 + \frac{1}{2}(\partial_\mu s)^2 - \left(\mu_0^2 \lvert H \rvert^2 + \lambda_0 \lvert H \rvert^4 + \frac{1}{2} M^2 s^2 + \frac{g_s \cdot A_1}{3} s^3 + \frac{\lambda_s}{4} s^4 \right) \right]$	$P_A = e^{-I_0[H,s]}/Z_0$	$Z_0 = \int d[H]d[s]e^{-I_0[H,s]}$
B	$I_g = I_0 + g \cdot \int d^4x \left(\frac{\kappa}{2} \lvert H \rvert^2 s^2 - A_1 \lvert H \rvert^2 s \right)$	$P_B = e^{-I_g[H,s]}/Z_g$	$Z_g = \int d[H]d[s]e^{-I_g[H,s]}$

- Non-negativity of relative entropy:

$$S(P_A \| P_B) = W_0 - W_g + g \cdot \left(\frac{dW_g}{dg} \right)_{g=0} = \int (d^4x)_E \frac{g^2 \cdot A_1^2 \lvert H \rvert^4}{2M^2} + \mathcal{O}(g^3) \geq 0 \Rightarrow \frac{g^2 \cdot A_1^2 \lvert H \rvert^4}{2M^2} \geq 0$$

when $\mathcal{O}(g^3)$ is negligible

Non-negativity of relative entropy holds in Higgs-singlet model

※ Non-negativity always holds when $\mathcal{O}(g^4)$ is included

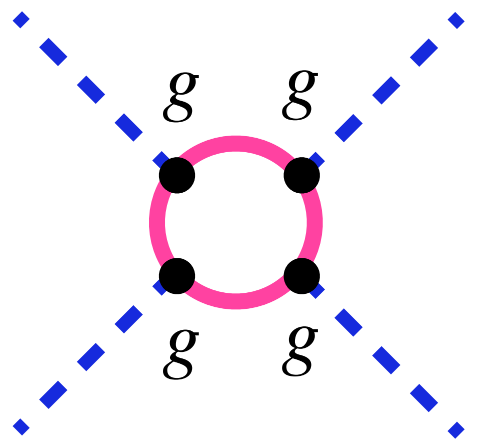
Example: Euler-Heisenberg theory

- Consider the U(1) gauge field A_μ coupled to a charged fermion ψ

Theory	Action in Minkowski space	Probability	Partition function
A	$I_0 = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi \right)$	$P_A = e^{-I_0[A_\mu,\psi]}/Z_0$	$Z_0 = \int d[A^\mu]d[\psi]d[\bar{\psi}]e^{-I_0[A^\mu,\psi]}$
B	$I_g = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi - g \cdot e(\bar{\psi}\gamma_\mu\psi)A^\mu \right)$	$P_B = e^{-I_g[A^\mu,\psi]}/Z_g$	$Z_g = \int d[A^\mu]d[\psi]d[\bar{\psi}]e^{-I_g[A^\mu,\psi]}$

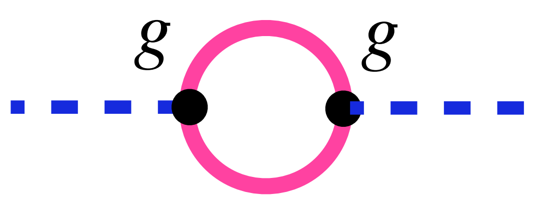
- Relative entropy:

$$S(P_A||P_B) = W_0 - W_g + g \cdot \left(\frac{dW_g}{dg} \right)_{g=0} = \int (d^4x)_E \left(\frac{1}{2} \frac{g^4 e^4}{6!\pi^2 m^4} (\bar{F}_{\mu\nu}\bar{F}^{\mu\nu})^2 + \frac{7}{8} \frac{g^4 e^4}{6!\pi^2 m^4} (\bar{F}_{\mu\nu}\widetilde{\bar{F}}^{\mu\nu})^2 + \mathcal{O}(m^{-6}) \right) \geq 0$$



$$W_g = \int (d^4x)_E \left(\frac{1}{4} \bar{F}_{\mu\nu}\bar{F}^{\mu\nu} - \frac{1}{2} \frac{g^4 e^4}{6!\pi^2 m^4} (\bar{F}_{\mu\nu}\bar{F}^{\mu\nu})^2 - \frac{7}{8} \frac{g^4 e^4}{6!\pi^2 m^4} (\bar{F}_{\mu\nu}\widetilde{\bar{F}}^{\mu\nu})^2 + \mathcal{O}(m^{-6}) \right)$$

where we choose $\partial\bar{F} = \text{const}.$ to remove dim-6 operators



$$\supset g^2 \cdot \int (d^4x)_E (\partial^2 \bar{F}\bar{F}), \cdots \Rightarrow 0, \text{ for } \partial\bar{F} = \text{const}.$$

Example: Euler-Heisenberg theory

- Consider the U(1) gauge field A_μ coupled to a charged fermion ψ

Theory	Action in Minkowski space	Probability	Partition function
A	$I_0 = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi \right)$	$P_A = e^{-I_0[A_\mu, \psi]}/Z_0$	$Z_0 = \int d[A^\mu]d[\psi]d[\bar{\psi}]e^{-I_0[A^\mu,\psi]}$
B	$I_g = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi - g \cdot e(\bar{\psi}\gamma_\mu \psi)A^\mu \right)$	$P_B = e^{-I_g[A^\mu, \psi]}/Z_g$	$Z_g = \int d[A^\mu]d[\psi]d[\bar{\psi}]e^{-I_g[A^\mu,\psi]}$

- Relative entropy:

$$S(P_A || P_B) = W_0 - W_g + g \cdot \left(\frac{dW_g}{dg} \right)_{g=0} = \int (d^4x)_E \left(\frac{1}{2} \frac{g^4 e^4}{6!\pi^2 m^4} (\bar{F}_{\mu\nu}\bar{F}^{\mu\nu})^2 + \frac{7}{8} \frac{g^4 e^4}{6!\pi^2 m^4} (\bar{F}_{\mu\nu} \widetilde{\bar{F}}^{\mu\nu})^2 + \mathcal{O}(m^{-6}) \right) \geq 0$$

dim-8 operators

Relative entropy constrains Wilson coefficients of dim-8 operator

⇒ Similar results for SU(N) gauge fields are obtained when dim-8 operators are generated through the interaction between heavy and light fields.

Example: SMEFT SU(N) gauge bosonic operators

- Relative entropy when dim-8 operators are generated by **interaction** b/w heavy and light fields:

※ assume the interaction doesn't involve higher-derivative terms

$$S(P_A||P_B) = W_0 - W_g + g \cdot (dW_g/dg)_{g=0} = \int (d^4x)_{\mathrm{E}} \frac{1}{\Lambda^4} \sum_i c_i \mathcal{O}_i \geq 0$$

$$\mathcal{O}_1^{F^4} = (F^a_{\mu\nu} F^{a,\mu\nu})(F^b_{\rho\sigma} F^{b,\rho\sigma})$$

$$\mathcal{O}_2^{F^4} = (F^a_{\mu\nu} \tilde{F}^{a,\mu\nu})(F^b_{\rho\sigma} \tilde{F}^{b,\rho\sigma})$$

$$\mathcal{O}_3^{F^4} = (F^a_{\mu\nu} F^{b,\mu\nu})(F^a_{\rho\sigma} F^{b,\rho\sigma})$$

$$\mathcal{O}_4^{F^4} = (F^a_{\mu\nu} \tilde{F}^{b,\mu\nu})(F^a_{\rho\sigma} \tilde{F}^{b,\rho\sigma})$$

$$\mathcal{O}_5^{F^4} = d^{abe} d^{cde} (F^a_{\mu\nu} F^{b,\mu\nu})(F^c_{\rho\sigma} F^{d,\rho\sigma})$$

$$\mathcal{O}_6^{F^4} = d^{abe} d^{cde} (F^a_{\mu\nu} \tilde{F}^{b,\mu\nu})(F^c_{\rho\sigma} \tilde{F}^{d,\rho\sigma})$$

$$\mathcal{O}_7^{F^4} = d^{ace} d^{bde} (F^a_{\mu\nu} F^{b,\mu\nu})(F^c_{\rho\sigma} F^{d,\rho\sigma})$$

$$\mathcal{O}_8^{F^4} = d^{ace} d^{bde} (F^a_{\mu\nu} \tilde{F}^{b,\mu\nu})(F^c_{\rho\sigma} \tilde{F}^{d,\rho\sigma})$$

$$\tilde{\mathcal{O}}_1^{F^4} = (F^a_{\mu\nu} F^{a,\mu\nu})(F^b_{\rho\sigma} \tilde{F}^{b,\rho\sigma})$$

$$\tilde{\mathcal{O}}_2^{F^4} = (F^a_{\mu\nu} F^{b,\mu\nu})(F^a_{\rho\sigma} \tilde{F}^{b,\rho\sigma})$$

$$\tilde{\mathcal{O}}_3^{F^4} = d^{abe} d^{cde} (F^a_{\mu\nu} F^{b,\mu\nu})(F^c_{\rho\sigma} \tilde{F}^{d,\rho\sigma})$$

$$\tilde{\mathcal{O}}_4^{F^4} = d^{ace} d^{bde} (F^a_{\mu\nu} F^{b,\mu\nu})(F^c_{\rho\sigma} \tilde{F}^{d,\rho\sigma})$$

T^a : generator of $SU(N)$ Lie algebra

$$[T^a, T^b] = i f^{abc} T^c$$

$$\{T^a, T^b\} = \delta^{ab} \hat{1}/N + d^{abc} T^c$$

Example: SMEFT SU(N) gauge bosonic operators

- Relative entropy when dim-8 operators are generated by **interaction** b/w heavy and light fields:

※ assume the interaction doesn't involve higher-derivative terms

$$S(P_A||P_B) = W_0 - W_g + g \cdot (dW_g/dg)_{g=0} = \int (d^4x)_{\rm E} \frac{1}{\Lambda^4} \sum_i c_i \mathcal{O}_i \geq 0$$

- Classical solution of $\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{\mu,b} F^c_{\mu\nu} = 0$: $A^a_\mu = u^a_1 \epsilon_{1\mu} w_1 + u^a_2 \epsilon_{2\mu} w_2$ with $f^{abc} u^a_1 u^b_2 = 0$, $\partial_\mu w_1 = l_\mu$, and $\partial_\mu w_2 = k_\mu$

※ l_μ, k_μ : constant vectors

- $U(1)_Y$: $c_1^{B^4} \geq 0$, $c_2^{B^4} \geq 0$, $4 c_1^{B^4} c_2^{B^4} \geq (\tilde{c}_1^{B^4})^2$,
- $SU(2)_L$: $c_1^{W^4} + c_3^{W^4} \geq 0$, $c_2^{W^4} + c_4^{W^4} \geq 0$, $4(c_1^{W^4} + c_3^{W^4})(c_2^{W^4} + c_4^{W^4}) \geq (\tilde{c}_1^{W^4} + \tilde{c}_2^{W^4})^2$,

U(1) and SU(2) bounds are the same as positivity bounds from unitarity and causality

[G.N. Remmen, and N.L. Rodd, arXiv:1908.09845]

- $SU(3)_C$: $2c_1^{G^4} + c_3^{G^4} \geq 0$, $3c_2^{G^4} + 2c_5^{G^4} \geq 0$, $3c_2^{G^4} + 3c_4^{G^4} + c_6^{G^4} \geq 0$, $3c_4^{G^4} + 2c_6^{G^4} \geq 0$,

$$4(3c_1^{G^4} + 3c_3^{G^4} + c_5^{G^4})(3c_2^{G^4} + 3c_4^{G^4} + c_6^{G^4}) \geq (3\tilde{c}_1^{G^4} + 3\tilde{c}_2^{G^4} + \tilde{c}_3^{G^4})^2$$

$$4(3c_3^{G^4} + 2c_5^{G^4})(3c_4^{G^4} + 2c_6^{G^4}) \geq (3\tilde{c}_2^{G^4} + 2\tilde{c}_3^{G^4})^2$$

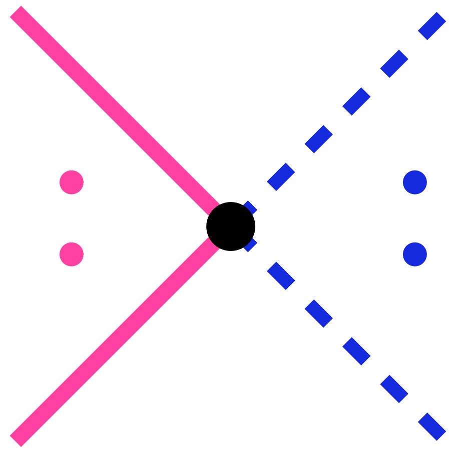
SU(3) bounds are stronger than positivity bounds from unitarity and causality

Summary

- Differences between theories with and without interaction characterize UV information

- We quantified their differences by relative entropy

- When EFTs are generated through interaction

$$I_I[\phi, \Phi] = \int (d^4x)_{\rm E}\, \mathcal{O}[\Phi] \otimes J[\phi] =$$ heavy :  : light

where we assume $J[\phi]$ does not involve higher-derivative terms

we found that the non-negativity of relative entropy constrains EFTs, e.g.,

SMEFT SU(N) gauge bosonic operators

- Comprehensive constraints on scalar field theories are on-going