Entropy and its conservation in expanding Universe

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See also Aoki-san's recent papers: 2305.09849, 2209.11357

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Summary

- Give further evidences of entropy interpretation of Noether charge Q_M in general relativity proposed by Aoki, Onogi, and Yokoyama ('21) in the expanding Universe

- We show that Q_M actually represents "entropy" in the expanding Universe

$$Q_M \propto \begin{cases} A_H/4G = \text{Bekenstein Hawking entropy} & \text{(for de Sitter Universe)} \\ \rho_R(t) \times a(t)^d = \text{Radiation entropy} & \text{(for radiation era)} \\ \rho_M(t) \times a(t)^{d-1} \propto \text{Total particle number} & \text{(for matter era)} \end{cases}$$

- We also numerically check the conservation of Q_M in a typical inflation Universe

Inflation → matter oscillation era → radiation era (=dynamical system)

- All the contents below are classical level

What is conservation law in gravity ?

- Einstein equation (=on-shell with respect to $g_{\mu\nu}(x)$)

$$G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}\frac{R}{2} = 8\pi G T_{\mu\nu} \;\rightarrow\; \nabla_\mu T^{\mu\nu} = 0 \;,$$

Using Bianchi identity
$\nabla_\mu G^{\mu\nu} = 0$

But, this does not always represents conservation laws i.e. $\partial_\mu(\sqrt{-g}T^{\mu\nu}) \neq 0$

∗ On the other hand, $\nabla_\mu F^{\mu\nu} = 0$ is always equivalent to $\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0$ when $F^{\mu\nu}$ is anti-symmetric tensor

- What are the consequences of Noether theorem for local gauge symmetries ?

We have to be very careful when we say “something is conserved” for local gauge symmetries
because it might be a trivial consequence of gauge symmetries
→ Noether’s 2nd theorem

Conserved charges form 2nd theorem

[Aoki, Onogi, Yokoyama, 2201.09557,]

- Komar charges

$$\partial_\mu J^\mu[\xi] = 0 \ , \quad J^\mu[\xi] = \frac{1}{8\pi G} \nabla_\nu \nabla^{[\mu}\xi^{\nu]}$$

without using any EOMs

$$Q_{\rm Komar}[\xi] = \int d^{d-1} J^0[\xi]$$

In particular, when space time has Killing vectors

This conserves for arbitrary metric $g_{\mu\nu}(x)$ and vector $\xi^\mu(x)$

$\xi^\mu = t^\mu = -\delta^\mu_0 \to$ Komar mass

$\xi^\mu = \phi^\mu = \delta^\mu_\phi \to$ Komar angular momentum

$\xi^\mu = t^\mu + \Omega_H \phi^\mu \to$ Wald entropy

- ADM mass = A specific case of Komar charge in asymptotically flat spacetime

$$E_{\rm ADM} = Q_{\rm Komar}[\xi = \eta]$$

where $\eta^\mu =$ asymptotic time-translation Killing

$\ast$ All the conventional charges are the trivial consequences of 2nd theorem !

What are the physical conservation laws ? (Physical=using EOM)

→ Let's focus on matter sector with arbitrary background $g_{\mu\nu}(x)$

Physical definition of conserved charge

[Aoki, Onogi, Yokoyama, KK, 2010.07660 ,2201.09557, 2210.03323]

- Consider the coordinate transformation $x^\mu \to x^\mu + \xi^\mu(x)$ of matter action

$$0 = \delta S_M = \int d^d x \left[\frac{\delta L_M}{\delta g^{\mu\nu}} \delta g_{\mu\nu} + E_\phi \delta\phi + \partial_\mu J^\mu \right]$$

In general, $J^\mu(x)$ is not conserved even if $E_\phi = 0$

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \qquad E_\phi = \frac{\delta L_M}{\delta \phi} - \nabla_\mu \frac{\delta L_M}{\delta \nabla_\mu \phi} \quad \text{(EOM)}$$

$$J^\mu(x) = T_M{}^\mu{}_\nu(x) \times \xi^\nu(x)$$

- However, if $\xi^\mu(x)$ satisfies

$$\frac{\partial L_M}{\partial g_{\mu\nu}} \delta g_{\mu\nu} \propto T_M^{\mu\nu} \nabla_\mu \xi_\nu \approx 0 \longrightarrow \partial_\mu J^\mu \approx 0$$

(On shell) (On shell)

On shell conservation law !

The conserved charge

$$Q_M = -\int_\Sigma (d^{d-1}\Sigma_\mu) T_M{}^\mu{}_\nu(x) \xi^\nu(x)$$

A conserved charge in expanding Universe

[S. Aoki, KK, arXiv:2210.03323]

- Consider perfect fluid $\quad T_M{}^{\mu}{}_{\nu} = (-\rho(t), p(t), \cdots, p(t))$
- Choose time-like vector $\xi^\mu = \beta(t)\delta^\mu_0$

$$\text{Conservation condition}: T^{\mu\nu}_M \delta g_{\mu\nu} \approx 0 \quad \rightarrow \quad \rho\dot{\beta} - (d-1)\frac{\dot{a}}{a} p\beta = 0$$

$$Q_M = -\int_\Sigma d^{d-1}x T_M{}^0{}_\nu(x)\xi^\nu(x) = V_{d-1} a(t)^{d-1}\rho(t) \times \beta(t) \qquad V_{d-1}: \text{comoving volume}$$

- For constant EoS $\omega = p/\rho = \text{constant}$, $\beta(t)$ can be explicitly solved (with $a_0 = 1$)

$$\beta(t) = \beta_0 \times \begin{cases} a(t)^{-(d-1)} & \text{de}-\text{Sitter} \\ a(t) & \text{radiation} \\ 1 & \text{matter} \end{cases}$$

→ We can check $\dot{Q}_M = 0$ explicitly

Q_M = Entropy

General expression $Q_M = V_{d-1} a(t)^{d-1} \rho(t) \times \beta(t)$

- Radiation era: $\beta(t) = \beta_0 a(t) := T(t)^{-1}$

$$Q_M = (\text{volume}) \times \underbrace{\rho(t) \times \beta(t)}_{\text{Entropy density}} = (\text{volume}) \times s(t)$$

When radiations are thermalized, Q_M is proportional to thermal entropy $S_R \sim (\text{volume}) \times T_R(t)^{d-1}$

- Matter era: $\beta(t) = \text{constant}$

$$Q_M = \underbrace{(\text{volume}) \times \rho(t)}_{\text{Total energy}} \times \beta_0 = E_M \times \beta_0$$

If we identify β_0^{-1} as the particle's mass m, Q_M corresponds to the total number of particles N_M

Q_M = Entropy

General expression $Q_M = V_{d-1} a(t)^{d-1} \rho(t) \times \beta(t)$

- De-Sitter spacetime: $\rho = \rho_I = \text{constant}\ , \quad \beta(t) = \beta_0 a(t)^{-(d-1)}$

$$\longrightarrow \quad Q_M = a_0^{d-1} V_{d-1} \times \rho_I \times \beta_0$$

How can we interpret this ?

- We can rewrite it in terms of Bekenstein Hawking entropy and number of Hubble patches

Horizon radius → $a_0 r_{\max} = \int_0^{+\infty} dt \frac{1}{e^{H_I t}} = \frac{1}{H_I}$ $\longrightarrow$ $N_H = (\text{number of Hubble patches}) = \frac{a_0^{d-1} V_{d-1}}{\frac{4\pi}{3} H_I^{-3}}$

$$\therefore \quad Q_M = N_H \times \frac{4\pi}{3} H_I^{-3} \times \rho_I \times \beta_0 = N_H \times \frac{A_H}{4G} \times T_H \beta_0$$

$\frac{A_H}{4G} = \text{Bekenstein Hawking entropy}$ $\qquad T_H = \frac{H_I}{2\pi} = \text{de Sitter temperature}$

If we identify $\beta_0 = T_H^{-1}$, Q_M coincides with the total entropy in de Sitter spacetime !

Conservation in dynamical process

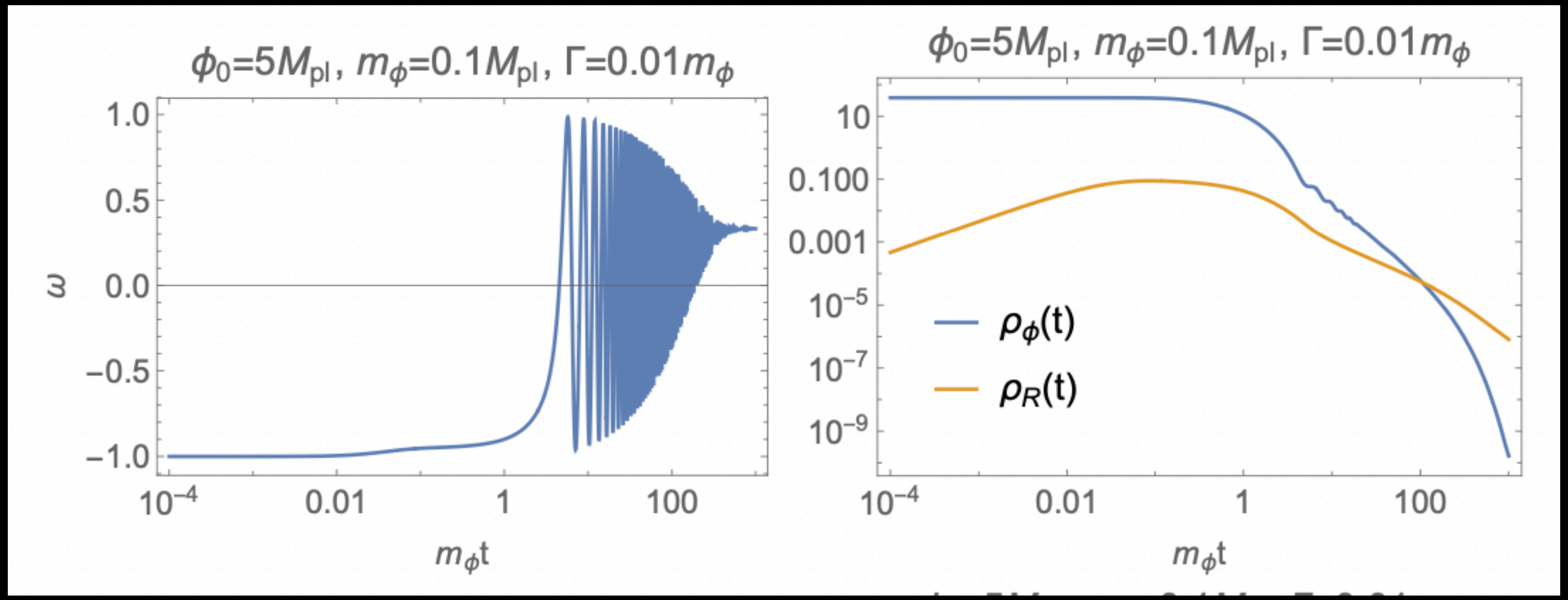
- In general, dominant energy component is changing in the expanding Universe
- But, Q_M is conserved by construction → Its carrier is changing in the transition processes
- As a toy model, we studied a typical inflation Universe

Inflation → matter oscillation era → radiation era

$$\ddot{\phi} + (3H + \Gamma)\dot{\phi} + \frac{\partial V}{\partial \phi} = 0 \,,$$
$$\dot{\rho}_R + 4H\rho_R = \Gamma(\rho_\phi + p_\phi) \,,$$
$$\dot{\beta} - 3H\frac{P}{\rho}\beta = 0 \,, \quad H^2 = \frac{8\pi G_N}{3}\rho \,,$$

EOMs

$\Gamma =$ decay rate of ϕ

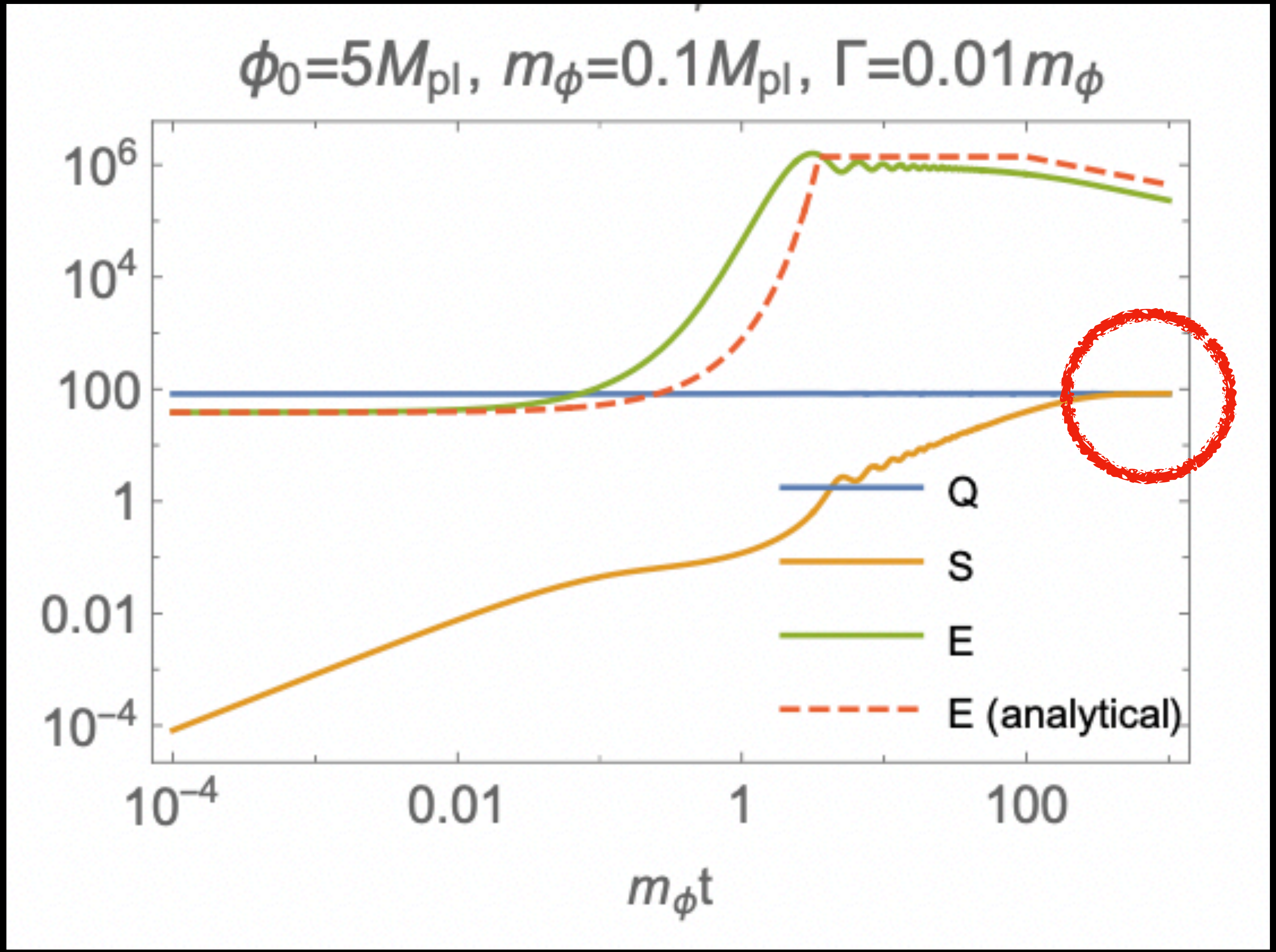


EoS

Energy densities

Conservation in dynamical process

Cont'd



$$Q_M = a(t)^3 \rho(t) \beta(t)$$

$$S_R = a(t)^3 \rho_R(t) \beta(t)$$

Radiation entropy

$$E = a(t)^3 \rho(t)$$

Total energy

Summary

- Gave further evidences of entropy interpretation of Noether charge Q_M in the expanding Universe
- Q_M actually represents "entropy" in the expanding Universe, (up to normalization factor)

$$Q_M \propto \begin{cases} A_H/4G = \text{Bekenstein Hawking entropy} \quad \text{(for de Sitter Universe)} \\ \rho_R(t) \times a(t)^d = \text{Radiation entropy} \quad \text{(for radiation era)} \\ \rho_M(t) \times a(t)^{d-1} \propto \text{Total particle number} \quad \text{(for matter era)} \end{cases}$$

- To be fair, these results themselves have been already well-known in the FRW Universe.
  We showed another way to obtain these results from the viewpoint of Noether method
- Studying more nontrivial dynamical systems (e.g. BH collapse) would be next subjects
- Any phenomenological implications ? (e.g. What happens during a first-order phase transition ?)

Backup

Essense of Noether's 2nd theorem

[Aoki, Onogi, Yokoyama, 2201.09557,]

- Consider the variation of total action under a coordinate transformation

$$0 = \delta S = \int_{\Sigma_d} d^d x \left(\xi^\mu F_\mu[g_{\mu\nu}, \phi, \cdots] + \partial_\mu J^\mu[\xi] \right) \quad , \quad F_\mu[g_{\mu\nu}, \phi, \cdots] = \text{ Some functional}$$

- First, let's consider $\xi^\mu(x)$ which vanishes at the boundary $\partial\Sigma_d$

$$0 = \delta S = \int_{\Sigma_d} d^d x \left(\xi^\mu F_\mu[g_{\mu\nu}, \phi, \cdots] + \cancel{\partial_\mu J^\mu[\xi]} \right) \quad \rightarrow \quad F^\mu[g_{\mu\nu}, \phi, \cdots] = 0$$

even for off-shell $g_{\mu\nu}$ and ϕ

$$\longrightarrow \quad 0 = \delta S = \int_{\Sigma_d} d^d x \left(\cancel{\xi^\mu F_\mu[g_{\mu\nu}, \phi, \cdots]} + \partial_\mu J^\mu[\xi] \right) \quad \rightarrow \quad \partial_\mu J^\mu[\xi] = 0$$

This must hold for arbitrary $\xi^\mu(x)$

Conserved current by 2nd theorem

$$\partial_\mu J^\mu[\xi] = 0 \ , \quad J^\mu[\xi] = \frac{1}{8\pi G} \nabla_\nu \nabla^{[\mu}\xi^{\nu]}$$

- More explicitly, the current can be written as

$$J^\mu = A^\mu{}_\nu \xi^\nu + B^\mu{}_\nu{}^\rho \xi^\nu_{,\rho} + C^\mu{}_\nu{}^{\rho\lambda} \xi^\nu_{,\rho\lambda}$$

- Since $\partial_\mu J^\mu[\xi] = 0$ holds for any $\xi^\mu(x)$, we have

Represents some portions of conservations

$$\partial_a A^a{}_b = 0,$$
$$A^a{}_b + \partial_c B^c{}_b{}^a = 0,$$
$$B^a{}_b{}^c + B^c{}_b{}^a + 2\partial_d C^d{}_b{}^{ac} = 0, ,$$
$$C^a{}_b{}^{cd} + C^d{}_b{}^{ac} + C^c{}_b{}^{da} = 0$$

→

$$A^\mu{}_\nu = -\partial_\rho \tilde{B}^\rho{}_\nu{}^\mu$$

$$\tilde{B}^c{}_b{}^a := \frac{1}{2} B^{[c}{}_b{}^{a]} - \frac{1}{3}\partial_d C^{[c}{}_b{}^{a]d},$$

Anti-symmetric for $a \leftrightarrow c$

Einstein pseudo tensor

$$\sqrt{-g}\,\nabla_\mu T^\mu{}_\nu = \partial_\mu(\sqrt{-g}T^{\mu\nu}) + \Gamma_\mu{}^\mu{}_\alpha(\sqrt{-g}T^\alpha{}_\nu) - \Gamma_\mu{}^\alpha{}_\nu(\sqrt{-g}T^\mu{}_\alpha) = 0$$

- We can add new tensor $t^\mu{}_\nu$ to cancel the yellow terms $\quad \partial_\mu(\sqrt{-g}(T^\mu{}_\nu + t^\mu{}_\nu)) = 0$
- But, $t^\mu{}_\nu$ is not covariant because it contains $\Gamma_\mu{}^\nu{}_\lambda$ explicitly → Pseudo tensor
- Moreover, this conservation is actually the consequence of 2nd theorem

$$A^\mu{}_\nu = \sqrt{-g}(2R^\mu{}_\nu + g^{\mu\lambda}\Gamma_\nu{}^\rho{}_{\rho,\lambda} - g^{\alpha\beta}\Gamma_\alpha{}^\mu{}_{\beta,\nu}) \quad \approx \sqrt{-g}(T^\mu{}_\nu + t^\mu{}_\nu)$$

$\partial_\mu A^\mu{}_\nu = 0$ always holds without using any EOMs

↑ Using Einstein equation

∴ Einstein pseudo tensor method is not physical one, but just a trivial consequence of 2nd theorem !

ADM mass

$$E_{\mathrm{ADM}} = \int_{r=\infty} d^{d-2}S^{0i}(\partial_j h_{ij} - \partial_i h_{jj}) \ , h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$$

5.2.1 Alternative Formula for ADM Energy

Subtract (5.21) from (5.20) to get

$$\partial_i \left(\partial_j h_{ij} - \partial_i h_{jj}\right) = -2\nabla^2 h_{00} \tag{5.25}$$

This allows us to rewrite ADM formula as

$$E = -\frac{1}{8\pi G} \oint_\infty dS_i \, \partial_i h_{00} \tag{5.26}$$

But (Exercise)

$$g^{ij}\Gamma_{0j}{}^0 = -\frac{1}{2}\partial_i h_{00} + \mathcal{O}\left(\frac{1}{r^3}\right) \quad (\Gamma = \text{affine connection}) \tag{5.27}$$

and hence

$$E = \frac{1}{4\pi G} \oint_\infty dS_i \, g^{ij}\Gamma_{0j}{}^0 \tag{5.28}$$

$$= \frac{1}{4\pi G} \oint_\infty dS_{0i} \, D^i k^0 \quad \text{where } k = \frac{\partial}{\partial t}, \ dS_i \equiv dS_{0i} \tag{5.29}$$

But k is asymptotically Killing, i.e.

$$D^\mu k^\nu + D^\nu k^\mu = \mathcal{O}\left(\frac{1}{r^3}\right) \tag{5.30}$$

[P.K. Townsend ('97)]