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Representations and q-Deformation of Anti de Sitter Symmetry

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Introduction

Anti de-Sitter symmetry plays an important role in modern mathematics and physics. In mathematics, the n-dimensional anti de Sitter space AdS_n, is a maximally symmetric Lorentzian manifold with constant negative scalar curvature. It is described as submerged in $(n+1)$-dimensional Euclidean space by the formula:

$$\sum_{i=1}^{n-1} x_i^2 - x_0^2 - x_n^2 = R^2 \tag{1}$$

In general relativity, anti de Sitter space is a maximally symmetric, vacuum solution of Einstein's field equation with a negative (attractive) cosmological constant.

The same space, though in momentum space, was the founding stepping stone of the quantum field theory with fundamental length developed by Kadyshevsky and Mateev for more than 40 years.

In group theory, the isometry group of AdS_n is the pseudo-orthogonal group SO(n-1,2). This group is also the conformal group of $(n-1)$-dimensional Minkowski space-time.

The latter properties were the group-theoretical foundation of the so-called AdS/CFT correspondence between objects (e.g., strings) in the n-dimensional AdS_n, called the bulk, and the $(n-1)$-dimensional Minkowski space-time as boundary of the bulk.

We discuss the simplest characteristic case $n = 4$ in the algebraic framework, namely, we discuss mostly the Lie algebra $so(3,2)$. That Lie algebra is the first that was called anti de Sitter algebra.

Preliminaries

The anti de Sitter algebra $\mathcal{G} = so(3,2)$ is ten-dimensional. It is a split form of its complexification $\mathcal{G}^{\mathbb{C}} = so(5,\mathbb{C})$ and we can use the same basis for both (though over different fields, $\mathbb{R},\mathbb{C}$, resp.).

For $\mathcal{G}^{\mathbb{C}}$ we use the standard (triangular) decomposition:

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}_- \oplus \mathcal{H} \oplus \mathcal{G}_+ \ , \tag{2}$$

where $\mathcal{H}$ is the diagonal two-dimensional Cartan subalgebra with generators denoted by H_1, H_2, $\mathcal{G}_+, \mathcal{G}_-$ are four-dimensional subalgebras of raising, lowering, generators X_i^+, X_i^-, resp., $i = 1,2,3,4$. For simplicity we give the following basis in terms of 4×4 matrices:

$$H_1 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \ , \quad H_2 = \begin{pmatrix} e_2 & 0 \\ 0 & -e_1 \end{pmatrix} \ , \tag{3}$$

$$X_1^+ = \begin{pmatrix} \sigma_+ & 0 \\ 0 & -\sigma_+ \end{pmatrix} \ , \quad X_1^- = \begin{pmatrix} \sigma_- & 0 \\ 0 & -\sigma_- \end{pmatrix} \ ,$$

$$X_2^+ = \begin{pmatrix} 0 & \sigma_- \\ 0 & 0 \end{pmatrix} \ , \quad X_2^- = \begin{pmatrix} 0 & 0 \\ \sigma_+ & 0 \end{pmatrix} \ ,$$
$$X_3^+ = \begin{pmatrix} 0 & 1_2 \\ 0 & 0 \end{pmatrix} \ , \quad X_3^- = \begin{pmatrix} 0 & 0 \\ 1_2 & 0 \end{pmatrix} \ ,$$
$$X_4^+ = \begin{pmatrix} 0 & \sigma_+ \\ 0 & 0 \end{pmatrix} \ , \quad X_4^- = \begin{pmatrix} 0 & 0 \\ \sigma_- & 0 \end{pmatrix} \ ,$$
$$e_1 \equiv \tfrac{1}{2}(1+\sigma_3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ e_2 \equiv \tfrac{1}{2}(1-\sigma_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\sigma_+ \equiv \tfrac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \sigma_- \equiv \ {}^t\sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where σ_i are the standard 2×2 Pauli matrices.

Using the same basis over $\mathbb{R}$ the anti de Sitter algebra $\mathscr{G}$ has the following (Bruhat) decomposition

$$\mathscr{G} \ = \ \mathscr{N}_- \oplus \mathscr{M} \oplus \mathscr{A} \oplus \mathscr{N}_+ \qquad (4)$$

in which the four subalgebras have physical meaning related to the fact that $\mathscr{G}$ is also the conformal algebra of three-dimensional Minkowski space-time M^3. Namely, the subalgebra $\mathscr{M} \cong$

$so(2,1)$ is the *Lorentz* algebra of M^3, the subalgebras $\mathscr{N}_\pm$ (with basis $X_k^\pm$, $k=2,3,4$) are abelian and represent the *translations* of M^3 and *special conformal transformations* of M^3, and the algebra $\mathscr{A}$ (spanned by $H_3 \equiv H_1 + 2H_2$) represents the *dilatations* of M^3.

Representations and invariant operators

We work with two kind of representations. Both are characterized by two quantum numbers E_0, s_0 called energy and spin, $s_0 = 0, \frac{1}{2}, \ldots$.

The first kind are the so-called *elementary representations* (ERs), denoted by C^Λ, where Λ is a weight $\Lambda \in \mathscr{H}^*$ depending on E_0, s_0. These can be realized as complex-valued C^∞ functions on $G^{\mathbb{C}}/B$, where $G = SO_0(3,2)$, $B = \exp(\mathscr{H})\exp(\mathscr{G}_+)$ is a Borel subgroup of $G^{\mathbb{C}}$. Since $G^{\mathbb{C}}/B$ is a completion of $G_- = \exp(\mathscr{G}_-)$ we shall use the four local coordinates of G_- denoted by: z, u, v, x. The functions of C^Λ, denoted $\hat{\varphi}(z,u,v,x)$, are polynomials in the variable z of degree $2s_0$ and smooth functions in the other three variables.

The other kind of representations are highest weight modules, in particular, Verma modules.

Verma modules give explicit realization of the so-called *positive energy* UIRs $D(E_0,s_0)$ of $\mathscr{G}$ given as follows [Dirac,Fronsdal,Evans,FF]:

$$D(E_0,s_0) = D(1/2,0) \ , \quad D(E_0,s_0) = D(1,1/2) \ ,$$
$$D(E_0 > 1/2+s_0, \ s_0 = 0,1/2),$$
$$D(E_0 \geq s_0+1, \ s_0 \geq 1) \ . \tag{5}$$

The first two are the singleton representations discovered by Dirac, and called later "Rac", "Di", resp. The last ones for $E_0 = s_0+1$ correspond to the spin-s_0 massless representations.

Pictorially, all positive energy UIRs are given in Figures 1 and 2:

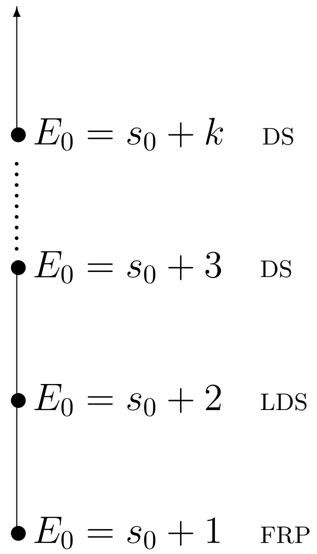

Fig. 1. $so(3,2)$, $s_0 = 1, \frac{3}{2}, ...$
black dots denote representations - irreducible factors of reducible C^Λ and V^Λ,
DS means discrete series of unitary representations,
LDS means limit of discrete series representations,
FRP means First reduction point - the end point of the PE representations

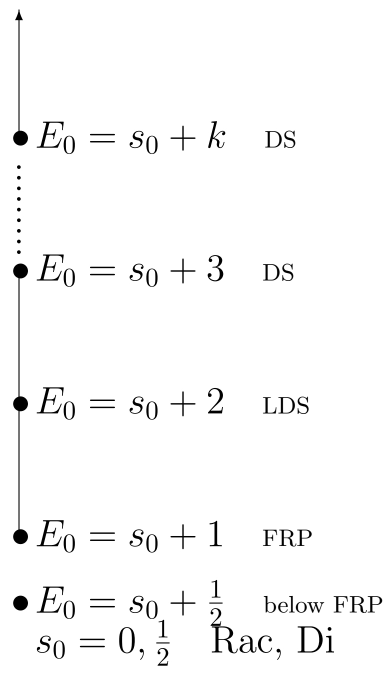

Fig. 2. $so(3,2)$, $s_0 = 0, \frac{1}{2}$

The other property of Verma modules that we use, is the fact that it is easy to find when they are reducible. In our situation this happens every time when at least one of the following four numbers:

$$
\begin{aligned}
m_1 &= \Lambda(H_1)+1 = 2s_0+1 \ , \\
m_2 &= \Lambda(H_2)+1 = 1-E_0-s_0 \ , \\
m_3 &= m_1+2m_2 \ , \quad m_4 = m_1+m_2
\end{aligned}
\tag{6}
$$

is a positive integer.

An important application of the above reducibility is that when the Verma module V^Λ is reducible, then also the ER C^Λ is reducible. Furthermore, in these cases there exist invariant differential operators between the ERs, and these operators are determined in a straightforward way from special objects in the corresponding Verma modules, called singular vectors [Dob].

All singular vectors and the corresponding invariant differential operators and equations are known [Dob], but for the lack of time we shall show only some operators that are related to the positive energy cases.

For instance, since m_1 is always a positive integer, there is always the following invariant operator: $(\partial_z)^{m_1}$, and in order to obtain irreducibility w.r.t. to the variable z, (recall that our functions $\hat{\varphi}$ are polynomials in z of degree $2s_0 = m_1 - 1$), there is always the invariant equation:

$$(\partial_z)^{m_1} \hat{\varphi}(z,u,v,x) = 0 \ . \tag{7}$$

Rac: $E_0 = \frac{1}{2}$, $s_0 = 0$. There are two equations:

$$\begin{aligned} &\partial_z \hat{\varphi} = 0 \ , \quad m_1 = 1 \ , \\ &\left(\partial_x^2 - 4\partial_u\partial_v\right) \hat{\varphi} \ = \left(\partial_0^2 - \partial_1^2 - \partial_2^2\right) \hat{\varphi} \ = \\ &\quad = \Box \hat{\varphi} \ = \ \hat{\varphi}' \ , \quad m_3 = 2 \ , \end{aligned} \tag{8}$$

where we have introduced new variables $y_0 = x$, $y_1 = \frac{1}{2}(u+v)$, $y_2 = i\frac{1}{2}(u-v)$, in terms of which we get the d'Alembert operator $\Box$ in M^3.
The functions $\hat{\varphi}'$ belong to the target space $C^{\Lambda'}$, also unitary of scalar functions (since $E_0' = \frac{3}{2}$, $s_0' = 0$), but fulfilling only the first equation in (8).

Di: $E_0 = 1$, $s_0 = \frac{1}{2}$. The two equations are:

$$\begin{aligned} &\partial_z^2 \hat{\varphi} = 0 \ , \quad m_1 = 2 \ ; \\ &\{ \ \tfrac{1}{2}(\partial_x - 2z\partial_u) \ + (\partial_v - z\partial_x + z^2\partial_u)\,\partial_z \ \}\,\hat{\varphi} \ = \ \hat{\varphi}' \ , \\ &\qquad m_3 = 1 \ . \end{aligned} \tag{9}$$

Massless representations: $E_0 = s_0 + 1$, $s_0 = 1, \frac{3}{2}, \ldots$ The equations are:

$$\begin{aligned} &\partial_z^{p+1}\,\hat{\varphi} = 0 \ , \qquad p = 2s_0 = 2, 3, \ldots \ , \\ &\{\, p(p-1)\,\partial_u \ + \ (p-1)(\partial_x - 2z\partial_u)\,\partial_z \ + \\ &+ \ (\partial_v - z\partial_x + z^2\partial_u)\,\partial_z^2\,\}\,\hat{\varphi} \ = \ \hat{\varphi}' \ , \quad m_4 = 1 \ . \end{aligned} \tag{10}$$

It is useful to write out (10) in components using: $\hat{\varphi} = \sum_{j=0}^{p} z^j \hat{\varphi}_j$, $\hat{\varphi}' = \sum_{j=0}^{p-2} z^j \hat{\varphi}'_j$. The resulting equations are:

$$(p-j)(p-j-1)\,\partial_u \hat{\varphi}_j \;+\; (j+1)(p-j-1)\,\partial_x \hat{\varphi}_{j+1} \;+$$
$$+(j+1)(j+2)\,\partial_v \hat{\varphi}_{j+2} \;=\; \hat{\varphi}'_j \;, \qquad j = 0,1,\dots,p-2 \;.$$

If we restrict to the kernel then the last equations may be rewritten as equations for $p-1$ independent conserved currents $J^{p,j}$ in M^3 :

$$\partial_0 J_0^{p,j} \;-\; \partial_1 J_1^{p,j} \;-\; \partial_2 J_2^{p,j} \;=\; 0 \;, \qquad j = 0,\dots,p-2 \;, \tag{11}$$

where the components are given as follows:

$$\begin{aligned} J_0^{p,j} &= -(j+1)(p-j-1)\,\hat{\varphi}_{j+1} \;, \qquad (12)\\ J_1^{p,j} &= \tfrac{1}{2}\{\,(j+1)(j+2)\,\hat{\varphi}_{j+2} \;+\; (p-j)(p-j-1)\,\hat{\varphi}_j\} \;,\\ J_2^{p,j} &= \tfrac{i}{2}\{\,(j+1)(j+2)\,\hat{\varphi}_{j+2} \;-\; (p-j)(p-j-1)\,\hat{\varphi}_j\} \;. \end{aligned}$$

Character formulae

Let $\mathcal{G}$ be any simple Lie algebra. Let Γ_+ be the set of all integral dominant elements of $\mathcal{H}^*$, i.e., $\lambda \in \mathcal{H}^*$ such that $\lambda(H_i) \in \mathbb{Z}$, (resp. $\mathbb{Z}_+$), for all H_i of the basis of $\mathcal{H}$. We recall that for each invariant subspace V of a Verma module V^Λ we have the following decomposition

$$V = \bigoplus_{\mu\in\Gamma_+} V_\mu \ , \quad V_\mu = \{u \in V \mid H_k u = (\Lambda+\mu)(H_k)u, \ \forall\, H_k\}$$

The character of V is defined by [Dixmier]:

$$ch\ V = \sum_{\mu\in\Gamma_+} (\dim\ V_\mu) e(\Lambda+\mu)$$

where the formal exponents $e(\mu)$ have the properties $e(0) = 1,\ e(\mu)e(\nu) = e(\mu+\nu)$. Furthermore for the character of the Verma module we have [Dixmier] :

$$ch\ V^\Lambda = e(\Lambda) \prod_{\alpha\in\Delta^+} (1-e(\alpha))^{-1} \qquad (13)$$

where $\Delta^+ \subset \mathscr{H}^*$ is the positive root system of $\mathscr{G}$.

The Weyl character formula for the finite-dimensional irreducible representations has the form [Dixmier]:

$$ch\ L_\Lambda \ = \ \sum_{w\in W} (-1)^{\ell(w)}\ ch\ V^{w\cdot\Lambda}\ , \qquad (14)$$

where W is the Weyl group of $\mathscr{G}$ and we use the fact that W acts on Λ.

Let $\mathscr{G} = so(3,2)$. The positive root system Δ^+ has four roots: α_i, $i=1,2,3,4$, so that $\alpha_3 = \alpha_1+\alpha_2$, $\alpha_4 = 2\alpha_1+\alpha_2$. Denote $t_i \equiv e(\alpha_i)$, $i=1,2$, then $e(\alpha_3) = t_1 t_2$, $e(\alpha_4) = (t_1)^2 t_2$. Then (13) and (14) can be rewritten, respectively, as

$$ch\ V^\Lambda \ = \ \frac{e(\Lambda)}{(1-t_1)(1-t_2)(1-t_1t_2)(1-(t_1)^2t_2)}\ ,$$

$$ch\ L_\Lambda \ = \ ch\ V^\Lambda\ (1\ -\ t_1^{m_1}\ -\ t_2^{m_2}\ +\ t_1^{m_1}t_2^{m_1+m_2}\ +$$

$$+\ t_1^{m_1+2m_2} t_2^{m_2} \ -\ t_1^{2(m_1+m_2)} t_2^{m_1+m_2}\ -$$
$$-\ (t_1 t_2)^{m_1+2m_2}\ +\ t_1^{2(m_1+m_2)} t_2^{m_1+2m_2}) \quad (15)$$

In the last formula we have used the fact that the Weyl group for $\mathscr{G}$ has eight elements given explicitly in terms of the generating elements w_1, w_2 as follows:

$$W \quad = \quad \{1,\ w_1,\ w_2,\ w_1w_2,\ w_2w_1,\ w_1w_2w_1,\ w_2w_1w_2,$$
$$w_1w_2w_1w_2 = w_2w_1w_2w_1\ \}\ . \quad (16)$$

The character formulae for the infinite-dimensional irreducible highest weight representations over $so(3,2)$ involve less terms than in (15) since the maximal invariant submodules I^Λ of V^Λ are smaller. These character formulae can be given in terms of reduced Weyl groups [Dob]. This means that the character formulae for the infinite-dimensional irreducible highest weight representations over $so(3,2)$ will look like the Weyl character formula (14), however with W replaced by certain subgroups of W, called reduced Weyl groups.

For the singletons the reduced Weyl group $W_R = W^s$ is given by:

$$W^s = \{ 1 , w_1 , w_2w_1w_2 , w_1w_2w_1w_2 \} . \quad (17)$$

Then the character formula for the singletons is:

$$\begin{aligned} ch\ L^s &= ch\ V^\Lambda \sum_{w\in W^s} (-1)^{\ell(w)} e(w\cdot\Lambda - \Lambda) = \\ &= ch\ V^\Lambda \ (1 - t_1^{m_1} - (t_1t_2)^{m_1+2m_2} + \\ &\quad + t_1^{2(m_1+m_2)} t_2^{m_1+2m_2}). \end{aligned} \quad (18)$$

More explicitly for the Rac we have $m_1 = 1$, $m_2 = 1/2$ and we obtain:

$$\begin{aligned} ch\ L_{\rm Rac} &= ch\ V^\Lambda \ (1 - t_1 - (t_1t_2)^2 + t_1^3t_2^2) = \\ &= \frac{e(\Lambda)\ (1 + t_1t_2)}{(1-t_2)(1-(t_1)^2t_2)} = \\ &= e(\Lambda) \sum_{n=0}^\infty (t_1t_2)^n \sum_{p=-n}^n t_1^p = \end{aligned}$$

$$= \quad e(\Lambda) \sum_{n=0}^{\infty} \sum_{p=-n}^{n} (t_1 t_2)^{n-|p|} \; t'^{|p|} \; , \quad (19)$$

where

$$t' \quad = \quad \begin{cases} t_1^2 t_2 \; = \; e(\alpha_4) & \text{for} \quad p > 0 \quad , \\ t_2 \; = \; e(\alpha_2) & \text{for} \quad p < 0 \quad . \end{cases}$$

Analogously the Di we have $m_1 = 2$, $m_2 = -1/2$ and we obtain:

$$\begin{aligned} ch\ L_{\rm Di} \quad &= \quad ch\ V^{\Lambda} \left(1 - t_1^2 - t_1 t_2 + t_1^3 t_2 \right) = \\ &= \quad \frac{e(\Lambda) \left(1 + t_1 \right)}{(1-t_2)(1-(t_1)^2 t_2)} = \\ &= \quad e(\Lambda) \sum_{n=0}^{\infty} (t_1 t_2)^n \sum_{p=-n}^{n+1} t_1^p = \qquad (20) \\ &= \quad e(\Lambda) \sum_{n=0}^{\infty} t_2^n \sum_{r=0}^{n} \left(t_1^{2r} + t_1^{2r+1} \right) \end{aligned}$$

Clearly, the terms in (19) have different weights, i.e., each weight is represented only once. The same is true for (20). That is why these representations are called singletons. These formulae were known, but this derivation is original.

Finally we consider the massless representations with $E_0 = s_0 + 1$, $s_0 \geq 1$. Let us define $W' \equiv \{ 1,\ w_1,\ w_1 w_2 w_1,\ w_2 w_1 \}$. Note that the set W' is not a subgroup of W, but nevertheless, the character formula is valid:

$$\begin{aligned} ch\ L_{massless} &= ch\ V^{\Lambda} \sum_{w \in W'} (-1)^{\ell(w)} e(w \cdot \Lambda - \Lambda) = \\ &= ch\ V^{\Lambda}\ (1 - t_1^{m_1} - t_1^2 t_2 + t_1^{m_1} t_2)\ , \end{aligned} \tag{21}$$

$m_1 = 1 + 2s_0 \geq 3$.

Quantum group case

The quantum group associated to a Lie algebra $\mathscr{G}$ is given by a deformation $U_q(\mathscr{G})$ of the universal enveloping algebra $U(\mathscr{G})$ depending on a parameter q (there are also multiparameter deformations!). Explicitly, the deformation occurs by replacing some commutators by their q-analogs, e.g., in our case

$$[X^+, X^-] = H \tag{22}$$

is replaced by:

$$[X^+, X^-] = [H]_q \equiv \frac{q^H - q^{-H}}{q - q^{-1}} \tag{23}$$

Matters are arranged so that the undeformed case is obtained from the deformed by setting $q = 1$. Indeed, setting $q = e^h$ and taking the limit $h \to 0$, we see that (23) passes to (22).

Generically, the main features of the representations of the quantum group deformation are the same as for the undeformed case, the only difference being that in matrix elements numbers, e.g., n, are replaced by q-numbers $[n]_q$, defined as above.

There is one interesting situation, where things differ drastically. First, we restrict the parameter q on the unit circle, namely, we set $q = e^{i\tau}$, $\tau \in I\!R$, $(|q| = 1)$. Then the quantum number $[n]_q$ becomes:

$$[n]_q = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{\sin(\tau n/2)}{\sin(\tau/2)} \tag{24}$$

changing to a more refined definition. The interesting situation is when q is a root of unity: $q^N = 1$, $q = e^{2\pi i/N}$, where $N = 2, 3, \ldots$ Then the quantum number becomes:

$$[n]_q = \frac{\sin(\pi n/N)}{\sin(\pi/N)} \tag{25}$$

Obviously,

$$[N]_q = 0$$

This means that many matrix elements become zero. Furthermore, elements in the enveloping algebra like $(X^{\pm})^N$ become central, and in fact *all representations become finite-dimensional*, including the unitary ones!

For instance, the singleton representations become finite-dimensional and with Moylan have found the formula for their dimensions. Our result is:

$$dim\ \mathrm{Rac}_N = \begin{cases} \frac{N^2+1}{2}\ , & \text{for } N \text{ odd} \\ \frac{N^2}{2}\ , & \text{for } N \text{ even} \end{cases} \tag{26}$$

$$dim\ \mathrm{Di}_N = \begin{cases} \frac{N^2-1}{2}\ , & \text{for } N \text{ odd} \\ \frac{N^2}{2}\ , & \text{for } N \text{ even} \end{cases} \tag{27}$$

One may ask whether these finite-dimensional representations coincide with some classical finite-dimensional representations of the anti de Sitter algebra $so(3,2)$. The answer is that with two exceptions there are no coincidences. The exceptions happen for third root of unity, $N = 3$, then:

$$\begin{aligned} dim \ \mathrm{Rac}_{N=3} &= 5 \ , & (28) \\ dim \ \mathrm{Di}_{N=3} &= 4 \ , & (29) \end{aligned}$$

and these are the dimensions of the two fundamental irreps of $so(3,2)$.