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Soliton equations on symmetric spaces: reductions and applications

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PLAN

- Integrable MNLS and Lax representations $[L, M] = 0$
- Fundamental analytic solutions of L and Riemann-Hilbert problems
- Equivalence of RHP and Lax representations
- Singular solutions of RHP and the soliton solutions of NLEE. Dressing method
- Wronskian relations and 'squared solutions' of L
- Completeness of 'squared solutions' and generalized Fourier transforms.
- Fundamental properties of the NLEE
- Soliton interactions
- Conclusions

Symmetric spaces

A.III-type symmetric spaces: $SU(n)/S(U(n-1)\times U(1))$:

$$X \in SU(n)/S(U(n{-}1){\times}U(1)) \qquad \Leftrightarrow XJX^{-1} = J, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

Local coordinates:

$$\mathcal{Q} = -\mathcal{Q}^\dagger, \qquad \text{and} \qquad \mathcal{Q} = [J, Q],$$

i.e.

$$\mathcal{Q} = \begin{pmatrix} 0 & \vec{q}^{\,T} \\ -\vec{q}^{\,*} & 0 \end{pmatrix},$$

BD.I-type symmetric spaces: $SO(2r+1)/S(0(2r-1)\times O(2))$:
Define: $X \in SO(2r+1) \quad \Leftrightarrow \quad S_0 X^T S_0 = X^{-1}$

$$S_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$X \in SO(2r+1)/S(O(2r-1)\times O(2)) \qquad \Leftrightarrow \qquad XJX^{-1} = J,$$

$$\text{for } r=2 \quad J = \operatorname{diag}(1,0,0,0,-1).$$

Local coordinates:

$$\mathcal{Q} = -S_0 \mathcal{Q}^T S_0, \qquad \text{and} \qquad \mathcal{Q} = [J, Q],$$

$$Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix},$$

Integrable MNLS and Lax representations

A.III-type MNLS or the vector NLS (the Manakov model) – Manakov, 1974::

$$H_{\rm A.III} = \int_{-\infty}^{\infty} dx \ \left((\vec{q}_x{}^{\dagger}, \vec{q}_x) - (\vec{q}^{\,\dagger}, \vec{q})^2 \right),$$

$$i\vec{q}_{\,t} + \bar{q}_{\,xx} + 2(\vec{q}^{\,\dagger}\vec{q}(x,t))\vec{q}(x,t) = 0,$$

BD.I-type MNLS:

$$H_{\rm BD.I} = \int_{-\infty}^{\infty} dx \ \left((\vec{q}_x{}^{\dagger}, \vec{q}_x) - (\vec{q}^{\,\dagger}, \vec{q})^2 + \frac{1}{2}|(\vec{q}, \vec{q})|^2 \right)$$

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^{\dagger}, \vec{q})\vec{q} - (\vec{q}, s_0\vec{q})s_0\vec{q}^{*} = 0, \qquad s_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & \end{pmatrix}$$

$$[L(\lambda), M(\lambda)] = 0, \qquad \text{identically w.r. to } \lambda:$$

A.III-type MNLS

$$L\psi(x,\lambda) \equiv i\frac{d\psi}{dx} + q(x)\psi(x,\lambda) - \lambda J\psi(x,\lambda) = 0, \qquad (1)$$

$$Q(x,t) = \begin{pmatrix} 0 & \vec{q}^T(x,t) \\ -\vec{q}^*(x,t) & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix}.$$

$$\begin{aligned} M\psi &\equiv i\frac{d\psi}{dt} + \left(V_0(x,t) + 2\lambda Q(x,t) - 2\lambda^2 J\right)\psi(x,t,\lambda) \\ &= \psi(x,t,\lambda)C(\lambda), \\ V_0(x,t) &= \left[\mathrm{ad}_J^{-1}Q, Q(x,t)\right] + 2i\,\mathrm{ad}_J^{-1}Q_x, \end{aligned}$$

BD.I-type MNLS

$$\begin{aligned} L\psi(x,t,\lambda) &\equiv i\partial_x\psi + (Q(x,t) - \lambda J)\psi(x,t,\lambda) = 0. \\ M\psi(x,t,\lambda) &\equiv i\partial_t\psi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0, \\ V_1(x,t) &= Q(x,t), \qquad V_0(x,t) = i\,\mathrm{ad}\,_J^{-1}\frac{dQ}{dx} + \frac{1}{2}\left[\mathrm{ad}\,_J^{-1}Q, Q(x,t)\right]. \end{aligned}$$

where $J = \mathrm{diag}(1, 0, \dots 0, -1)$ and

$$Q = \begin{pmatrix} 0 & \vec{q}^T & 0 \\ \vec{p} & 0 & s_0\vec{q} \\ 0 & \vec{p}^T s_0 & 0 \end{pmatrix}, \quad S_0 = \sum_{k=1}^{2r+1} (-1)^{k+1} E_{k,2r+2-k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2}$$

Here $(E_{kn})_{ij} = \delta_{ik}\delta_{nj}$ and the $2r-1$-vectors $\vec{q}$ and $\vec{p} = \vec{q}^*$ take the form

$$\vec{q} = (q_1, \dots, q_{r-1}, q_0, q_{-1}, \dots, q_{-r+1})^T,$$

RHP with canonical normalization on $\mathbb{R}$

The Fundamental analytic solutions of L (FAS) for real λ satisfy:

$$\chi^{+}(x,\lambda) = \chi^{-}(x,\lambda) G_0(\lambda), \qquad \lambda \in \mathbb{R}.$$

Riemann-Hilbert problem:

$$\xi^{\pm}(x,\lambda) = \chi^{\pm}(x,\lambda) e^{i\lambda J x}.$$

$$\xi^{+}(x,\lambda) = \xi^{-}(x,\lambda) G(x,\lambda), \qquad \lambda \in \mathbb{R}.$$

$$\lim_{\lambda\to\infty} \xi^{\pm}(x,\lambda) = \mathbb{1}, \qquad \text{canonical normalization.}$$

$$G(x,\lambda) = e^{-i\lambda J x} G_0(\lambda) e^{i\lambda J x}.$$

Definition: $\xi^{\pm}(x,\lambda)$ is a regular solution of RHP if $\det \xi^{\pm}(x,\lambda) \neq 0$ for all $\lambda \in \mathbb{C}_{\pm}$.

Theorem: RHP with canonical normalization has unique regular solution.

Proof Let $\xi^{\pm}_{1,2}(x,\lambda)$ are two regular solutions of RHP. Then:

$$\xi_1^{+}(x,\lambda)\hat{\xi}_2^{+}(x,\lambda) = \xi_1^{-}(x,\lambda) G(x,\lambda)\hat{G}(x,\lambda)\hat{\xi}_2^{-}(x,\lambda) = \xi_1^{-}(x,\lambda)\hat{\xi}_2^{-}(x,\lambda),$$

$$\lim_{\lambda\to\infty} \xi_1^+(x,\lambda)\hat{\xi}_2^+(x,\lambda) = \mathbb{1}.$$

Liouville theorem:

$$\xi_1^+(x,\lambda)\hat{\xi}_2^+(x,\lambda) = \mathbb{1},$$

i.e.

$$\xi_1^+(x,\lambda) = \xi_2^+(x,\lambda).$$

Equivalence of RHP and Lax representations

Theorem [Zakharov, Shabat]. Let $\xi^{\pm}(x,\lambda)$ be solution to a RHP with canonical normalization and $G(x,\lambda)$ such that:

$$i\frac{dG}{dx} - \lambda[J, G(x,\lambda)] = 0.$$

Then

$$i\frac{d\xi^{\pm}}{dx} + q(x)\xi^{\pm}(x,\lambda) - \lambda[J, \xi^{\pm}(x,\lambda)] = 0, \qquad \text{i.e.}$$

$$i\frac{d\chi^{\pm}}{dx} + q(x)\chi^{\pm}(x,\lambda) - \lambda J\chi^{\pm}(x,\lambda) = 0.$$

Proof:

$$g^{\pm}(x,t,\lambda) = i\frac{d\xi^{\pm}}{dx}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda\xi^{\pm}(x,t,\lambda)J\hat{\xi}^{\pm}(x,t,\lambda).$$

$$g^{+}(x,\lambda) = i\frac{d(\xi^{-}G)}{dx}\hat{G}\hat{\xi}^{-}(x,\lambda) + \lambda\xi^{-}GJ\hat{G}\hat{\xi}^{-}(x,\lambda)$$

$$
\begin{aligned}
&= i\frac{d\xi^-}{dx}\hat{\xi}^-(x,\lambda) + \xi^- \left(i\frac{dG}{dx}\hat{G} + \lambda GJ\hat{G}(x,\lambda) \right) \hat{\xi}^-(x,\lambda) \\
&= i\frac{d\xi^-}{dx}\hat{\xi}^-(x,\lambda) + \xi^- \left(\lambda[J,G]\hat{G} + \lambda GJ\hat{G}(x,\lambda) \right) \hat{\xi}^-(x,\lambda) \\
&= i\frac{d\xi^-}{dx}\hat{\xi}^-(x,\lambda) + \lambda\xi^- J\hat{\xi}^-(x,\lambda) \\
&\equiv g^-(x,\lambda), \qquad \lambda \in \mathbb{R}.
\end{aligned}
$$

Thus $g^+(x,\lambda) = g^-(x,\lambda)$ is analytic in the whole complex λ-plane except in the vicinity of $\lambda \to \infty$ where $g^+(x,\lambda)$ tends to λJ. Liouville theorem:

$$
g^+(x,\lambda) - \lambda J = \text{const}
$$

with respect to λ; denote it $-q(x)$ and get:

$$
g^+(x,\lambda) - \lambda J = -\tilde{Q}(x).
$$

$$
\tilde{Q}(x) \equiv [J, Q(x,t)] = \lim_{\lambda\to\infty} \lambda \left(J - \xi^{\pm}(x,\lambda)J\hat{\xi}^{\pm}(x,\lambda) \right).
$$

Similarly one treats also the time dependence. Therefore:
Let the sewing function $G(x,t,\lambda)$ satisfies:

$$i\frac{dG}{dx} - \lambda[J, G(x,t,\lambda)] = 0, \qquad i\frac{dG}{dt} - \lambda^2[J, G(x,t,\lambda)] = 0.$$

Then $Q(x,t)$ satisfies the MNLS equations:

$$i\left[J, \frac{\partial Q}{\partial t}\right] + \frac{\partial^2 Q}{\partial x^2} + Q^3 = 0.$$

For **A.III**:

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^\dagger, \vec{q})\vec{q} = 0,$$

For **BD.I**:

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^\dagger, \vec{q})\vec{q} - (\vec{q}s_0\vec{q})s_0\vec{q}^* = 0, \qquad s_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Singular solutions of RHP and the soliton solutions of MNLS. Dressing method

Starting from a given regular solutions $\xi_0^\pm(x,t,\lambda)$ to the RHP construct new singular solutions $\xi^\pm(x,t,\lambda)$ of RHP with zeroes and pole singularities at $\lambda_j^\pm \in \mathbb{C}_\pm$. Dressing factor $u_j(x,t,\lambda)$:

$$\xi^\pm(x,t,\lambda) = u_j(x,t,\lambda)\xi_0^\pm(x,t,\lambda)u_{j,-}^{-1}(\lambda),$$

$$\begin{aligned} u_j(x,t,\lambda) &= \mathbb{1} + (c_j(\lambda)-1)P_j(x,t), \qquad c_j(\lambda) = \frac{\lambda-\lambda_j^+}{\lambda-\lambda_j^-}, \\ u_{j,-}^{-1} &= \lim_{x\to-\infty} u_j(x,t,\lambda). \end{aligned}$$

$P_j(x,t)$ is a projector $P_j^2 = P_j$. If $\operatorname{rank} P_j = 1$, then:

$$P_j(x) = \frac{|n_j\rangle\langle m_j|}{\langle m_j|n_j\rangle},$$

The dressing factor $u(x,t,\lambda)$ satisfies the equation:

$$i\frac{du}{dx} + Q(x,t)u(x,t,\lambda) - u(x,t,\lambda)Q_0(x,t) - \lambda[J, u(x,t,\lambda)] = 0.$$

Main advantage: one can determine the x and t-dependence of $\langle m_j|$ and $|n_j\rangle$ through the regular solution $\chi_0^{\pm}(x,t,\lambda)$:

$$|n_j\rangle = \chi_{0j}^{+}(x,t)|n_j^0\rangle, \quad \langle m_j| = \langle m_j^0|\hat{\chi}_{0j}^{-}(x,t), \quad \chi_{0j}^{\pm}(x,t) = \chi_0^{\pm}(x,t,\lambda_j^{\pm})$$

If $q(x,t)$ is the potential corresponding to the singular solution $\chi^{\pm}(x,t,\lambda)$ then:

$$\begin{aligned} Q(x,t) &= Q_0(x,t) + \lim_{\lambda\to\infty} \lambda(J - u_j(x,t,\lambda)J\hat{u}_j(x,t,\lambda)) \\ &= Q_0(x,t) - (\lambda_j^{+} - \lambda_j^{-})[J, P_j(x,t)]. \end{aligned}$$

$$Q_0(x,t) = 0, \quad Q(x,t) = -(\lambda_j^+ - \lambda_j^-)[J, P_j(x,t)],$$

$$P_j(x) = \frac{|n_j\rangle\langle m_j|}{\langle m_j|n_j\rangle}, \quad |n_j\rangle = e^{i\lambda_j^+ J(x+\lambda_j^+ t)}|n_j^0\rangle, \quad \langle m_j| = \langle m_j^0|e^{-i\lambda_j^- J(x+\lambda_j^- t)},$$

One-soliton solutions of Vector NLS eqs.:

$$\vec{q}(x,t) = \frac{2\nu e^{i\phi}}{\cosh(z)}\vec{n}, \qquad \vec{n} = \begin{pmatrix} \kappa^{(1)} e^{i\gamma^{(1)}} \\ \vdots \\ \kappa^{(n)} e^{i\gamma^{(n)}} \end{pmatrix},$$

$$z = 2\nu(x - \xi(t)), \qquad \xi(t) = 2\mu t + \xi^{(0)},$$

$$\phi = \frac{\mu}{\nu} z_k + \delta(t), \qquad \delta(t) = 2(\mu^2 + \nu^2)t + \delta^{(0)}.$$

For the BD.I NLS

$$u_j(x,t,\lambda) = \mathbb{1} + (c_j(\lambda) - 1)P_j(x,t) + (c_j^{-1}(\lambda) - 1)S_0 P_j^T S_0(x,t),$$

One-soliton solutions of BD.I NLS with $n = 3$:

$$q_{1s;\pm 1} = -\frac{\sqrt{2|\nu_{01;1}\nu_{01;3}|}(\lambda_1^+ - \lambda_1^-)}{\Delta_1} e^{-i\phi_1 \pm i\beta_{13}} \times (\cosh(z_1 \mp \zeta_{01}) \cos(\alpha_{13}) - i \sinh(z_1 \mp \zeta_{01}) \sin(\alpha_{13})),$$

$$q_{1s;0} = -\frac{\sqrt{2}|\nu_{01;2}|(\lambda_1^+ - \lambda_1^-)}{\Delta_1} e^{-i\phi_1} (\sinh z_1 \cos(\alpha_{02}) + i \cosh z_1 \sin(\alpha_{02})),$$

$$\beta_{13} = \frac{1}{2}(\alpha_{03} - \alpha_{01}), \qquad \zeta_{01} = \frac{1}{2} \ln \frac{|\nu_{01;3}|}{|\nu_{01;1}|}, \qquad \alpha_{13} = \frac{1}{2}(\alpha_{03} + \alpha_{01}),$$

Dressing procedure adds pairs of discrete eigenvalues λ_j^+ and $\lambda_j^- = (\lambda_j^+)^*$ to the spectrum of L.

Repeat the dressing N times to get the N-soliton solution.

Wronskian relations, 'squared solutions' of L and recursion operators $\Lambda_\pm$

Study the mappings: $\mathcal{M}_J \leftrightarrow T(\lambda)$. $\mathcal{M}_J \equiv \{q(x,t) = [J, Q(x,t)\}$, $Q(x,t)$ - smooth and tending to 0 for $x \to \pm\infty$.

$$(\hat{\chi} J \chi(x,\lambda) - J)\Big|_{x=-\infty}^{\infty} = -i \int_{-\infty}^{\infty} dx\, \hat{\chi}[Q(x), J]\chi(x,\lambda),$$

$$\operatorname{tr}\left((\hat{\chi} J \chi(x,\lambda) - J)\, E_{ab}\right)\Big|_{x=-\infty}^{\infty} = -i \int_{-\infty}^{\infty} dx\, \operatorname{tr}\left([Q(x), J] e_{ab}(x,\lambda)\right),$$

$$\rho_{lk}^{\pm}(\lambda) = -i [\![Q(y), e_{il}^{\pm}(y,\lambda)]\!], \qquad e_{ab}(x,\lambda) = P_{0J}(\chi E_{ab} \hat{\chi}(x,\lambda)),$$

$e_{ab}(x,\lambda)$ are the 'squared solutions' of L; P_{0J} is a projector $P_{0J} X = \operatorname{ad}_J^{-1} \operatorname{ad}_J X$ on the off-diagonal part of X.

$$[\![X(y), Y(y)]\!] \equiv \int_{-\infty}^{\infty} dy\, \operatorname{tr}\left(X(y), [J, Y(y)]\right),$$

Completeness of 'squared solutions' and generalized Fourier transforms.

Theorem The sets of 'squared solutions' $e_{il}^{\pm}(y,\lambda)$ form complete sets of functions in $\mathcal{M}_J$. The completeness relation has the form:

$$\delta(x-y)\sum_r (E_{1r}\otimes E_{r1} - E_{r1}\otimes E_{1r}) =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} d\lambda (G_1^+(x,y,\lambda) - G_1^-(x,y,\lambda)) - 2i\sum_{j=1}^{N}(G_{1;j}^+(x,y) + G_{1;j}^-(x,y)),$$

$$G_1^+(x,y,\lambda) = \sum_r e_{1r}^+(x,\lambda)\otimes e_{r1}^+(y,\lambda), \quad G_1^-(x,y,\lambda) = \sum_r e_{r1}^-(x,\lambda)\otimes e_{1r}^-(y,\lambda),$$

$$G_{1;j}^+(x,y) = \sum_r (e_{1r;j}^+(x)\otimes \dot{e}_{r1;j}^+(y) + \dot{e}_{1r;j}^+(x)\otimes e_{r1;j}^+(y)),$$

$$G_{1;j}^-(x,y) = \sum_r (\dot{e}_{r1;j}^-(x)\otimes e_{1r;j}^-(y) + e_{r1;j}^-(x)\otimes \dot{e}_{1r;j}^-(y)),$$

Idea of the proof: VSG (1981), (1986) –
Apply the contour integration method to the Green function.

$$
\begin{aligned}
G^{\pm}(x,y,\lambda) &= G_1^{\pm}(x,y,\lambda)\theta_{\pm}(x-y) - G_2^{\pm}(x,y,\lambda)\theta_{\pm}(y-x),\\
G_1^{+}(x,y,\lambda) &= \sum_r e_{1r}^{+}(x,\lambda)\otimes e_{r1}^{+}(y,\lambda), \quad G_1^{-}(x,y,\lambda) = \sum_r e_{r1}^{-}(x,\lambda)\otimes e_{1r}^{-}(y,\lambda),\\
G_2^{+}(x,y,\lambda) &= \sum_{2\leq i<r} e_{ir}^{+}(x,\lambda)\otimes e_{r1}^{+}(y,\lambda) + \sum_{i\geq r} e_{ir}^{+}(x,\lambda)\otimes e_{ri}^{+}(y,\lambda),\\
G_2^{-}(x,y,\lambda) &= \sum_{2\leq i<r} e_{ri}^{-}(x,\lambda)\otimes e_{ir}^{-}(y,\lambda) + \sum_{i\geq r} e_{ri}^{-}(x,\lambda)\otimes e_{ir}^{-}(y,\lambda),
\end{aligned}
\tag{3}
$$

and $\theta_{\pm}(z) = \theta(\pm z)$ is the step function.

Expansions over the 'squared solutions':

$$
\begin{aligned}
Q(x) &= -\frac{i}{\pi}\int_{-\infty}^{\infty} d\lambda \sum_r \left(\rho_{r1}^{+}(\lambda)e_{r1}^{+}(x,\lambda) - \rho_{1r}^{-}(\lambda)e_{1r}^{-}(x,\lambda)\right)\\
&-2\sum_{k=1}^{N}\sum_r \left(\rho_{r1;j}^{+}e_{r1;j}^{+}(x) + \rho_{1r;j}^{-}e_{1r;j}^{-}(x)\right),
\end{aligned}
$$

$$J\delta Q(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} d\lambda \sum_r \left(\delta'\rho_{r1}^{+}(\lambda)e_{r1}^{+}(x,\lambda) + \delta'\rho_{1r}^{-}(\lambda)e_{1r}^{-}(x,\lambda)\right)$$

$$+\sum_{k=1}^{N}\sum_r \left(\delta'\tilde{W}_{1r;j}^{+}(x) - \delta'\tilde{W}_{r1;j}^{-}(x)\right),$$

Consider variations of the type:

$$\delta Q \simeq Q(x,t+\delta t) - Q(x,t) \simeq Q_t \delta t.$$

$e_{ri}^{\pm}(x,\lambda)$ are generalizations of $e^{-i\lambda x}$. We need the analogs of id/dx for which $i(d/dx)e^{-i\lambda x} = \lambda e^{-i\lambda x}$

$$(\Lambda_+ - \lambda)e_{ri}^{+}(x,\lambda) = 0, \qquad (\Lambda_+ - \lambda)e_{ir}^{-}(x,\lambda) = 0,$$
$$(\Lambda_- - \lambda)e_{ir}^{+}(x,\lambda) = 0, \qquad (\Lambda_- - \lambda)e_{ri}^{-}(x,\lambda) = 0.$$

$$\Lambda_\pm X(x) \equiv \mathrm{ad}\,_J^{-1}\left(i\frac{dX}{dx} + i\left[[J,Q(x)], \int_{\pm\infty}^{x} dy\,[[J,Q(y)],X(y)]\right]\right).$$

Fundamental properties of the NLEE

Now we can prove that the principal series of NLEE has the form:

$$iJ\frac{dQ}{dt} + f(\Lambda)Q(x,t) = 0,$$

where Λ can be either Λ_+ or Λ_- and $f_0(\lambda)$ determines the dispersion law of the corresponding NLEE.

Theorem The NLEE are equivalent to the following linear evolution equations for the scattering data of L:

$$i\frac{d\rho_{1r}^{\pm}}{dt} \mp 2f(\lambda)\rho_{1r}^{\pm} = 0, \qquad \frac{d\lambda_j^{\pm}}{dt} = 0, \qquad i\frac{d\rho_{1r;j}^{\pm}}{dt} \mp 2f(\lambda_j^{\pm})\rho_{1r;j}^{\pm} = 0,$$

Series of integrals of motion generated by $m_k^{\pm}(\lambda)$.

$$i\frac{dm_k^{\pm}}{dt} = 0, \qquad k = 1, \dots n-1.$$

$$m_k^{\pm}(\lambda) = \sum_{s=1}^{\infty} \lambda^{-s} I_s^{(k)},$$

i.e.

$$\frac{dI_s^{(k)}}{dt} = 0, \qquad \text{for all} \quad s = 1, 2, 3, \dots .$$

$$I_1 \simeq \int_{-\infty}^{\infty} \operatorname{tr} Q^2, \quad I_2 \simeq i \int_{-\infty}^{\infty} \operatorname{tr}(QQ_x), \quad I_3 \simeq H_{\mathrm{MNLS}}$$

Hierarchies of Hamiltonian structures.

Hamiltonians are linear combinations of $I_s^{(k)}$.

$Q(x,t)$ is a local coordinate on the co-adjoint orbit passing through J.

For a generic NLEE:

$$H_0 = \sum_{l=1}^{n-1} b_l I_1^{(l)},$$

$$\Omega_0 = \frac{1}{i} \int_{-\infty}^{\infty} dx \operatorname{tr} \left(\delta Q(x) \wedge [J, \delta Q(x)) \right).$$

It is also expressed through the skew-scalar product by:

$$\Omega_0 = \frac{1}{i} [\![\operatorname{ad}_J^{-1} \delta Q \underset{,}{\wedge} \operatorname{ad}_J^{-1} \delta Q]\!].$$

The hierarchy of Hamiltonian structures:

$$H_k = \sum_{l=1}^{n-1} b_l I_1^{(l+k)},$$

$$\Omega_k = \frac{1}{i} [\![\delta Q \mathop{\wedge}_{,} \Lambda^k \delta Q]\!], \qquad \Lambda = \frac{1}{2}(\Lambda_+ + \Lambda_-),$$

Calculate Ω_k in terms of the scattering data variations. The answer is

$$\Omega_k = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\lambda \, \lambda^k \left(\Omega_0^+(\lambda) - \Omega_0^-(\lambda) \right) - i \sum_{j=1}^{N} \left(\Omega_{k,j}^+ + \Omega_{k;j}^- \right),$$

$$\Omega_{k,j}^{\pm} = \operatorname*{Res}_{\lambda=\lambda_j^{\pm}} \lambda^k \Omega_0^{\pm}(\lambda).$$

This proves that all symplectic forms are compatible.
Besides all integrals of motion are in involution:

$$\Omega_k(dI_s^{(l)}, dI_p^{(m)}) = 0.$$

1 Soliton interactions

Scalar NLS

Consider the N-soliton solutions; the Lax operator L has $2N$ eigenvalues:

$$\lambda_k^{\pm} = \mu_k \pm i\nu_k$$

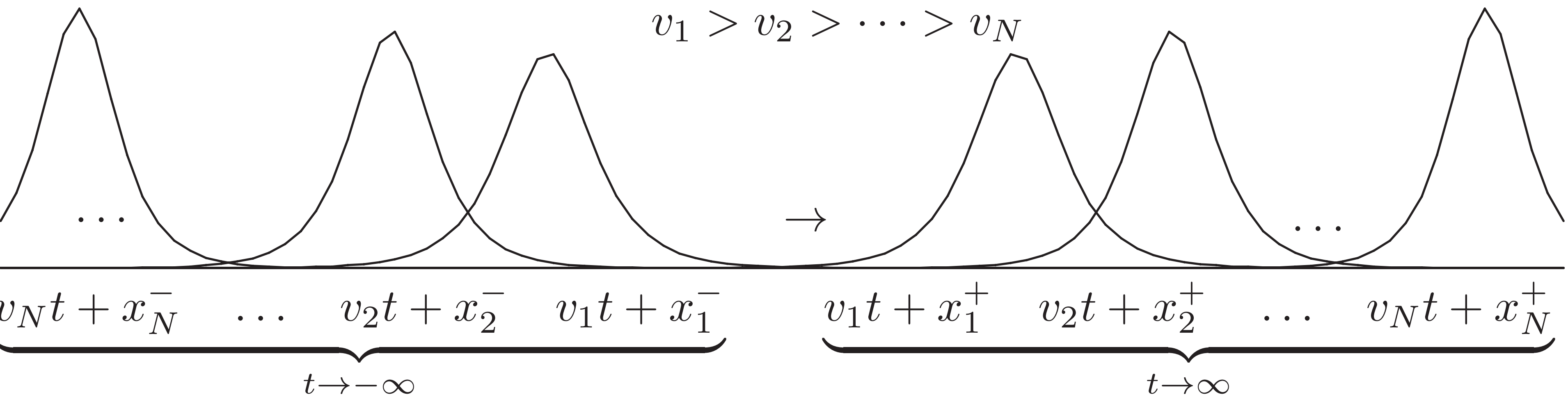
μ_k determine the asymptotic velocity of the k-th soliton; ν_k defines the amplitude of the k-th soliton.

If all μ_k are different one can evaluate

$$\lim_{t\to-\infty} Q_{\mathrm{Ns}}(x,t) \simeq \sum_{k=1}^{N} Q_{k,\mathrm{1s}}^{-}(x,t), \qquad \lim_{t\to\infty} Q_{\mathrm{Ns}}(x,t) \simeq \sum_{k=1}^{N} Q_{k,\mathrm{1s}}^{+}(x,t).$$

Generic case of soliton interaction: $v_k \neq v_j$ for $k \neq j$. Let

$$v_1 > v_2 > \cdots > v_N$$



For $\mu_k = \mu_j$ - one may expect that the solitons will form bound states.

$$u_{\mathrm{Ns}}(x,t) = \frac{\det M_1(x,t)}{\det M_0(x,t)},$$

$$M_{ij}(x,t) = \frac{1+\gamma_j^*(x,t)\gamma_k(x,t)}{\lambda_j^- - \lambda_k^+} \qquad M_1(x,t) = \begin{pmatrix} M(x,t) & \vec{\gamma}(x,t) \\ \vec{c}_0^{\,T} & 0 \end{pmatrix},$$

$t \to -\infty$ $\qquad\qquad$ $t \to \infty$

$$u_{\mathrm{Ns}} \quad \simeq \sum_{k=1}^{N} u_{1\mathrm{s}}(x,t;\mu_k,\nu_k,\xi_k^-,\delta_k^-) \qquad \simeq \sum_{k=1}^{N} u_{1\mathrm{s}}(x,t;\mu_k,\nu_k,\xi_k^+,\delta_k^+)$$

- μ_k – asymptotic velocity of k-th soliton
- ν_k – asymptotic amplitude of k-th soliton
- $\xi_k^{\pm}$ – asymptotic relative center of mass position of k-th soliton
- $\delta_k^{\pm}$ – asymptotic relative phase of k-th soliton

Conclusion: The soliton interactions:

- preserve the number of solitons
- preserve their velocities and their amplitudes
- purely elastic in character.
- shifts of the relative center of masses and relative phases.

$$\xi_k^+ = \xi_k + \sum_{s=1}^{k-1} r_{sk}^+ - \sum_{s=k+1}^{N} r_{sk}^+, \qquad \phi_k^+ = \phi_k + \sum_{s=1}^{k-1} \alpha_{sk}^+ - \sum_{s=k+1}^{N} \alpha_{sk}^+$$

$$\xi_k^- = \xi_k - \sum_{s=1}^{k-1} r_{sk}^+ + \sum_{s=k+1}^{N} r_{sk}^+, \qquad \phi_k^- = \phi_k - \sum_{s=1}^{k-1} \alpha_{sk}^+ + \sum_{s=k+1}^{N} \alpha_{sk}^+$$

$$r_{sk}^+ = \ln \left| \frac{\lambda_s^+ - \lambda_k^+}{\lambda_s^+ - \lambda_k^-} \right|, \qquad \alpha_{sk}^+ = \arg \frac{\lambda_s^+ - \lambda_k^+}{\lambda_s^+ - \lambda_k^-}.$$

Pure elastic nature of the N-soliton interactions is due to the infinite set of integrals of motion.

BD.I MNLS:

Take the 2-soliton solution and calculate its asymptotics along the trajectory of the first soliton. Thus

$$\lim_{\tau\to\infty} \vec{Q}_{2\mathrm{s}}(x,t) = -\frac{i\sqrt{2}\nu_1 e^{-i(\phi_1-\alpha_+)}\left(e^{-z_1-r_+}s_0|\vec{\nu}_{01}\rangle + e^{z_1+r_+}|\vec{\nu}_{01}^*\rangle\right)}{\cosh(2(z_1+r_+)) + (\vec{\nu}_{01}^\dagger, \vec{\nu}_{01})},$$
$$\lim_{\tau\to-\infty} \vec{Q}_{2\mathrm{s}}(x,t) = \frac{i\sqrt{2}\nu_1 e^{-i(\phi_1+\alpha_-)}\left(e^{-z_1+r_-}s_0|\vec{\nu}_{01}\rangle + e^{z_1-r_-}|\vec{\nu}_{01}^*\rangle\right)}{\cosh(2(z_1-r_-)) + (\vec{\nu}_{01}^\dagger, \vec{\nu}_{01})}, \tag{4}$$

where

$$r_+ = \ln\left|\frac{\lambda_1^+ - \lambda_2^+}{\lambda_1^+ - \lambda_2^-}\right|, \qquad \alpha_+ = \arg\frac{\lambda_1^+ - \lambda_2^+}{\lambda_1^+ - \lambda_2^-}.$$

Conclusions

- RHP is an effective method to investigate MNLS and other soliton equations and derive their fundamental properties
- RHP provides effective tool to construct explicitly reflectionless potentials of the Lax operators and the soliton solutions of NLEE.
- RHP combined with the Wronskian relations allows to interprete the ISM as a Generalized Fourier transform
- The soliton interactions are purely elastic in nature