Spatially Homogeneous Yang-Mills Theory: Instant, Light-Front and Point Forms of Dynamics

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Contents of the Talk

Forms of Relativistic Dynamics

The Poincaré Group

The generators of the Poincaré group are

P^μ **space-time translations,**

$M^{\mu\nu}$ **pure Lorentz transformations.**

The generators satisfy the commutation relations

$$
\begin{aligned}
[P^\mu, P^\nu] &= 0\,, \\
[M^{\mu\nu}\,, P^\alpha] &= i\,(P^\mu\, g^{\nu\alpha} - P^\nu\, g^{\mu\alpha})\;, \\
\left[M^{\mu\nu}\,, M^{\alpha\beta}\right] &= i\left(g^{\mu\beta}\, M^{\nu\alpha} - g^{\nu\beta}\, M^{\mu\alpha} + g^{\nu\alpha}\, M^{\mu\beta} - g^{\mu\alpha}\, M^{\nu\beta}\right)\;,
\end{aligned}
$$

which determine the Lie algebra of the Poincaré group.

The irreducible representations of the Poincaré group are characterized by the invariants:

the mass

$$P^{\mu}\,P_{\mu} = m^2\,,$$

the intrinsic spin

$$W^{\mu}\,W_{\mu} = -m^2\mathbf{S}^2$$

which is determined by the Pauli-Lubanski pseudovector

$$W^{\mu} = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_{\nu}S_{\alpha\beta}$$

In the pioneering paper
Forms of Relativistic Dynamics,
Reviews of Modern Physics 21 (1949) 392-399

P.A.M. Dirac stated:

- Invariance under generally covariant transformations.
- The equations of motion shall be expressible in the Hamiltonian form.

The evolution of a system with nonrelativistic dynamics can be completely determined by the Hamiltonian using the evolution operator.

$$U(t) = \mathbf{exp}(-iHt)\,.$$

The state specification at the surface $t = 0$, an instant in time, represents the initial conditions.

For the Galilei group the instant is the only appropriate initial surface.

For relativistic systems any hypersurface Σ in Minkowski-space that does not contain timelike directions can be used to formulate the initial conditions.

Here is very useful the concept of the stability group :

$$G_\Sigma : \Sigma \to \Sigma \,.$$

The generators of G_Σ are called kinematical operators.

The rest of the generators map Σ into another surface $\Sigma \to \Sigma'$. They are said to be dynamical operators.

If the surface Σ has the property

$$(\forall x \in \Sigma) \wedge (\forall y \in \Sigma) : \exists g \in G_\Sigma \to gx = y \,,$$

then it is said that the group G_Σ acts transitively and all points points in Σ are equivalent.

If we limit ourselves to consider only transitive actions of G_Σ on Σ, there exist just five inequivalent possibilities, corresponding to the five subgroups of the Poincaré group.

$$\left.\begin{array}{rr} \textbf{Instant Form} & x^0 = 0\,, \\ \textbf{Light- Front Form} & x^0 + x^3 = 0\,, \\ \textbf{Point Form} & x^2 = a^2 > 0,\, x^0 > 0\,. \end{array}\right\} \textbf{Dirac}$$

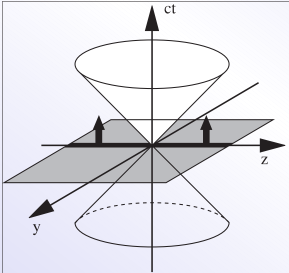
$$\left.\begin{array}{r} (x^0)^2 - (x^1)^2 - (x^2)^2 = a^2 > 0,\, x^0 > 0\,, \\ (x^0)^2 - (x^3)^2 = a^2 > 0,\, x^0 > 0\,. \end{array}\right\} \textbf{Leutwyler and Stern}$$

Comparison of Instant Form, Front Form, and Point Form

Instant Form	**Front Form**	**Point Form**
	Quantization Surface	
$x^0 = 0$	$x^0 + x^3 = 0$	$x^2 = a^2 > 0,\ x^0 > 0$
	Measure	
$\int \frac{d^3p}{2\,p^0}$	$\int \frac{d^2p^\perp dp^+}{2\,p^+}$	$\int \frac{d^3u}{2\,g^0}$

Instant Form	Front Form	Point Form
Kinematical Generators		
$\mathbf{P}$	$P^+, \mathbf{P}^\perp$	$M^{\mu\nu}$
$\mathbf{J}$	$E^1 = M^{+1} = \frac{K_x + J_y}{\sqrt{2}}$	
	$E^2 = M^{+2} = \frac{K_y - J_x}{\sqrt{2}}$	
	$J_z = M^{12}$	
	$K_z = M^{-+}$	

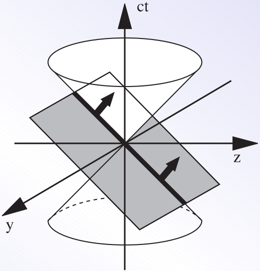
Instant Form	Front Form	Point Form
Dynamical Generators		
P^0 $\mathbf{K}$	P^- $F^1 = M^{-1} = \frac{K_x - J_y}{\sqrt{2}}$ $F^2 = M^{-2} = \frac{K_y + J_x}{\sqrt{2}}$	P^μ



The instant form

$$\tilde{x}^0 = ct$$
$$\tilde{x}^1 = x$$
$$\tilde{x}^2 = y$$
$$\tilde{x}^3 = z$$

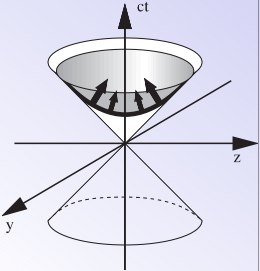
$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



The front form

$$\tilde{x}^0 = ct+z$$
$$\tilde{x}^1 = x$$
$$\tilde{x}^2 = y$$
$$\tilde{x}^3 = ct-z$$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$



The point form

$$\tilde{x}^0 = \tau \quad , \quad ct = \tau \cosh\omega$$
$$\tilde{x}^1 = \omega \quad , \quad x = \tau \sinh\omega \sin\theta \cos\phi$$
$$\tilde{x}^2 = \theta \quad , \quad y = \tau \sinh\omega \sin\theta \sin\phi$$
$$\tilde{x}^3 = \phi \quad , \quad x = \tau \sinh\omega \cos\theta$$

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\tau^2 & 0 & 0 \\ 0 & 0 & -\tau^2 \sinh^2\omega & 0 \\ 0 & 0 & 0 & -\tau^2 \sinh^2\omega \sin^2\theta \end{pmatrix}$$

Instant form of the homogeneous $SU(2)$ Yang-Mills theory

The dynamics of the $SU(2)$ Yang-Mills 1-form **A** in 4-dimensional Minkowski space-time M_4 is governed by the conventional local functional

$$S_{YM} = \frac{1}{2} \int_{M_4} \mathbf{tr}\, F \wedge *F$$

defined in terms of the curvature 2-form $F = d\mathbf{A} + g\, \mathbf{A} \wedge \mathbf{A}$

After the supposition of the spatial homogeneity of the connection **A**

$$\mathcal{L}_{\partial_i} \mathbf{A} = 0\,,$$

the action reduces to the action for a finite dimensional model described by the degenerate matrix Lagrangian

$$L_{YMM} = \frac{1}{2}\,\mathbf{tr}\left((D_tA)(D_tA)^T\right) - V(A)\,,$$

where

$$\mathbf{A} := Y_a e_a dt + A_{ai} e_a dx^i\,, \qquad Y_a := A_0^a\,, A_{ai} := A_i^a\,,$$
$$(D_tA)_{ai} = \dot{A}_{ai} + g\varepsilon_{abc} Y_b A_{ci}$$

The part of the Lagrangian corresponding to the self-interaction of the gauge fields is gathered in the potential $V(A)$

$$V(A) = \frac{g^2}{4}\left(\mathbf{tr}^2(AA^T) - \mathbf{tr}(AA^T)^2\right)$$

To express the Yang-Mills mechanics in a Hamiltonian form, let us define the phase space endowed with the canonical symplectic structure and spanned by the canonical variables

$$(Y_a, P_{Y_a}; A_{ai}, E_{ai}) \,,$$

where

$$P_{Y_a} = \frac{\partial L}{\partial \dot{Y}_a} = 0 \,, \qquad E_{ai} = \frac{\partial L}{\partial \dot{A}_{ai}} = \dot{A}_{ai} + g\varepsilon_{abc} Y_b A_{ci}$$

According to these definitions of the canonical momenta, the phase space is restricted by the three primary constraints

$$P_{Y_a} = 0$$

and thus evolution of the system is governed by the total Hamiltonian

$$H_T = H_C + u_{Y_a}(t)\, P_{Y_a} \,,$$

where the canonical Hamiltonian is given by

$$H_C = \frac{1}{2}\mathbf{tr}(EE^T) + \frac{g^2}{4}\left(\mathbf{tr}^2(AA^T) - \mathbf{tr}(AA^T)^2\right) + gY_a\,\mathbf{tr}(J_aAE^T)$$

The conservation of the primary constraints in time entails the further condition on the canonical variables

$$\Phi_a = g\,\mathbf{tr}(J_aAE^T) = 0$$

They are the first class constraints obeying the Poisson brackets algebra

$$\{\Phi_a, \Phi_b\} = g\,\varepsilon_{abc}\Phi_c$$

In order to project onto the reduced phase space, we use the polar decomposition for an arbitrary 3×3 matrix

$$A_{ai}(\phi, Q) = O_{ak}(\phi)Q_{ki}$$

The field strength E_{ai} in terms of the new canonical pairs (Q_{ik}, P_{ik}) and (ϕ_i, P_i) is

$$E_{ai} = O_{ak}(\phi)\left(P_{ki} + \varepsilon_{kil}(\gamma^{-1})_{lj}\,(\xi^L_j - S_j\,)\right),$$

where ξ^L_a are three left-invariant vector fields on $SO(3)$

$$\xi^L_1 = \frac{\sin\phi_3}{\sin\phi_2}\,P_1 + \cos\phi_3\,P_2 - \cot\phi_2\sin\phi_3\;P_3\,,$$

$$\xi^L_2 = \frac{\cos\phi_3}{\sin\phi_2}\,P_1 - \sin\phi_3\,P_2 - \cot\phi_2\cos\phi_3\;P_3\,,$$

$$\xi^L_3 = P_3$$

Vector $S_j = \varepsilon_{jmn}(QP)_{mn}$ is the spin vector of the gauge field and $\gamma_{ik} = Q_{ik} - \delta_{ik}$ tr$Q$.

Reformulation of the theory in terms of these variables allows one to easily achieve the Abelianization of the secondary Gauss law constraints

$$\Phi_a = M_{ab}\, P_b = 0$$

Assuming nondegenerate character of the matrix

$$M = \begin{pmatrix} \frac{\sin\phi_1}{\sin\phi_2}\,, & \cos\phi_1\,, & -\sin\phi_1\cot\phi_2 \\ -\frac{\cos\phi_1}{\sin\phi_2}\,, & \sin\phi_1\,, & \cos\phi_1\cot\phi_2 \\ 0\,, & 0\,, & 1 \end{pmatrix}$$

we find the set of Abelian constraints equivalent to the Gauss law

$$P_a = 0$$

The resulting unconstrained Hamiltonian, defined as a projection of the total Hamiltonian onto the constraint shell

$$H_{YMM}(Q_{ab}\,, P_{ab}) := H_C\,\Big|_{P_a=0\,,\; P_{Y_a}=0}$$

can be written in terms of Q_{ab} **and** P_{ab} **as**

$$H_{YMM} \;=\; \frac{1}{2}\mathbf{tr}P^2 - \frac{1}{\det^2\gamma}\mathbf{tr}\left(\gamma\mathcal{M}\gamma\right)^2 + \frac{g^2}{4}\left(\mathbf{tr}^2Q^2 - \mathbf{tr}Q^4\right)\;,$$

where $\mathcal{M}_{mn} = (QP - PQ)_{mn}$ **denotes the gauge field spin tensor.**

To write down the Hamiltonian describing the motion on the principal orbit stratum we decompose the nondegenerate symmetric matrix Q **as**

$$Q = \mathcal{R}^T\left(\chi_1, \chi_2, \chi_3\right)\mathcal{D}\;\mathcal{R}(\chi_1, \chi_2, \chi_3)$$

with the $SO(3)$ matrix $\mathcal{R}$ parameterized by the three Euler angles $\chi_i := (\chi_1, \chi_2, \chi_3)$ and the diagonal matrix $\mathcal{D} = \operatorname{diag}(x_1, x_2, x_3)$. The Jacobian of this transformation is the relative volume of orbits

$$J := \left| \quad \det \left\| \frac{\partial Q}{\partial x_k}, \frac{\partial Q}{\partial \chi_k} \right\| \quad \right| = \prod_{i<k} |\, x_i - x_k \,|$$

and it is regular for the Principal stratum $x_1 < x_2 < x_3$.

The original physical momenta P_{ik} can then be expressed in terms of the new canonical pairs (x_i, p_i) and (χ_i, p_{χ_i}) as

$$P = \mathcal{R}^T \left(\sum_{s=1}^{3} \bar{\mathcal{P}}_s \, \overline{\alpha}_s + \sum_{s=1}^{3} \mathcal{P}_s \, \alpha_s \right) \mathcal{R}$$

with $\bar{\mathcal{P}}_s = p_s$**,**

$$\mathcal{P}_i = -\frac{1}{2}\frac{\xi_i^R}{x_j - x_k}, \quad (\textbf{cyclic permutation } i \neq j \neq k)\,,$$

where ξ^R **are** $SO(3,\mathbb{R})$ **right-invariant Killing vectors. In terms of these variables the physical Hamiltonian reads**

$$H_{YMM} = \frac{1}{2}\sum_{a=1}^{3} p_a^2 + \frac{1}{4}\sum_{a=1}^{3} k_a^2 \xi_a^2 + V^{(3)}(x)\,,$$

where

$$k_a^2 = \frac{1}{(x_b + x_c)^2} + \frac{1}{(x_b - x_c)^2}\,, \qquad \textbf{cyclic} \quad a \neq b \neq c$$

and

$$V^{(3)} = \frac{g^2}{2} \sum_{a<b} x_a^2 x_b^2$$

The potential $V^{(3)}(x_1, x_2, x_3)$ can be rewritten as

$$V^{(3)}(x_1, x_2, x_3) = \frac{\partial W^{(3)}}{\partial x_a} \frac{\partial W^{(3)}}{\partial x_a}, \qquad a = 1,\, 2,\, 3$$

with the superpotential $W^{(3)} = x_1 x_2 x_3$.

Hence the YMM can be represented as the following Matrix model

$$L_{YMM} = \frac{1}{2}\mathbf{tr}\dot{A}\dot{A}^T + \frac{g^2}{4}\left(\mathbf{tr}^2(AA^T) - \mathbf{tr}(AA^T)^2\right)$$

restricted on the invariant submanifold $\eta^R = 0$.

Reduction to Yang-Mills mechanics via the discrete symmetry

The Hamiltonian of the system defined as

$$H = \frac{1}{2}\mathbf{tr}P^2 + V^{(N)}(X)$$

describes a motion on the matrix configuration space and differs from the considered in preceding section by the inclusion of the external potential $V(X)$. We specify the external potential V in superpotential form

$$V^{(N)} = -\frac{1}{4}\,\mathbf{tr}\left(\frac{\partial W^{(N)}}{\partial X}\right)^2$$

with superpotential $W^{(N)}$ given as

$$W^{(N)} = i\sqrt{\mathbf{det}X}$$

After passing to the new variables one can convince that the Hamiltonian coincides with the Euler-Calogero-Moser Hamiltonian embedded in external potential

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{N} \frac{l_{ij}^2}{(x_i - x_j)^2} + V^{(N)}(x_1, x_2, \ldots, x_N)$$

For the description of discrete symmetries of the Hamiltonian it is convenient to use the Cartesian form of “angular variables”

$$l_{ab} = y_a \pi_b - y_b \pi_a$$

with canonically conjugated variables y_a, π_a**.**

One can easily check that the Hamiltonian possesses the following discrete symmetries
(A. Polychronakos, Nucl. Phys B 543 (1999) 485):

- **Parity** P

$$\begin{pmatrix} x_i \\ p_i \end{pmatrix} \mapsto \begin{pmatrix} -x_i \\ -p_i \end{pmatrix}, \qquad \begin{pmatrix} y_i \\ \pi_i \end{pmatrix} \mapsto \begin{pmatrix} -y_i \\ -\pi_i \end{pmatrix}$$

- **Permutation symmetry M**
  (M is the element of the permutation group S_N)

$$\begin{pmatrix} x_i \\ p_i \end{pmatrix} \mapsto \begin{pmatrix} x_{M(i)} \\ p_{M(i)} \end{pmatrix}, \qquad \begin{pmatrix} y_i \\ \pi_i \end{pmatrix} \mapsto \begin{pmatrix} y_{M(i)} \\ \pi_{M(i)} \end{pmatrix}$$

Let us consider the certain invariant submanifold of the phase space of the matrix model and find out the corresponding reduced system. One can verify that the submanifold defined by

the constraints

$$\chi_a := \frac{1}{\sqrt{2}}\left(x_a + x_{N-a+1}\right)\,, \qquad \bar{\chi}_a := \frac{1}{\sqrt{2}}\left(y_a + y_{N-a+1}\right)\,,$$

$$\Pi_a := \frac{1}{\sqrt{2}}\left(p_a + p_{N-a+1}\right)\,, \qquad \bar{\Pi}_a := \frac{1}{\sqrt{2}}\left(\pi_a + \pi_{N-a+1}\right)$$

is the invariant submanifold of the system. Because the Hamiltonian possesses the discrete symmetry mentioned above this manifold is invariant under the action

$$D = P \times M\,,$$

where M is specified as $M(a) = N - a + 1$. The nonvanishing Poisson brackets are

$$\{\chi_a, \Pi_b\} = \delta_{ab}\,, \qquad \{\bar{\chi}_a, \bar{\Pi}_b\} = \delta_{ab}$$

One can easy verify that for canonical constraints the corresponding fundamental Dirac brackets are

$$\{x_a, p_b\}_D = \frac{1}{2}\delta_{ab}\,, \qquad \{y_a, \pi_b\}_D = \frac{1}{2}\delta_{ab}$$

As result the system with Hamiltonian reduces to the following one

$$H_{red} = \frac{1}{2}\sum_{a=1}^{\frac{N}{2}} p_a^2 + \frac{1}{2}\sum_{a\neq b}^{\frac{N}{2}} l_{ab}^2 k_{ab}^2 + \frac{g^2}{2}\sum_{a\neq b}^{\frac{N}{2}} x_a^2 x_b^2\,,$$

where

$$k_{ab}^2 = \frac{1}{(x_a + x_b)^2} + \frac{1}{(x_a - x_b)^2}$$

Expression for $N = 6$ coincides with the Hamiltonian of the $SU(2)$ Yang-Mills mechanics.

Lax pair for Yang-Mills mechanics in zero coupling limit

The introduction of Dirac brackets allows one to use the Lax pair of higher dimensional Euler-Calogero-Moser model (namely A_6) for the construction of Lax pairs (L_{YMM}, A_{YMM}) of free Yang-Mills mechanics by performing the projection onto the constraint shell

$$L^{ECM}_{6\times6}|_{CS} = L_{YMM}\,, \qquad A^{ECM}_{6\times6}|_{CS} = A_{YMM}\,.$$

The explicit form of the Lax pair matrices for the free $SU(2)$ Yang-Mills mechanics is given by the following 6×6 matrices

$$L_{YMM} = \left(\begin{array}{ccc|ccc}
p_1 & -\frac{l_{12}}{x_1-x_2} & -\frac{l_{13}}{x_1-x_3} & \frac{l_{13}}{x_1+x_3} & \frac{l_{12}}{x_1+x_2} & 0 \\
-\frac{l_{12}}{x_1-x_2} & p_2 & -\frac{l_{23}}{x_2-x_3} & \frac{l_{23}}{x_2+x_3} & 0 & -\frac{l_{12}}{x_1+x_2} \\
-\frac{l_{13}}{x_1-x_3} & -\frac{l_{23}}{x_2-x_3} & p_3 & 0 & -\frac{l_{23}}{x_2+x_3} & -\frac{l_{13}}{x_1+x_3} \\
\hline
\frac{l_{13}}{x_1+x_3} & \frac{l_{23}}{x_1+x_2} & 0 & -p_3 & -\frac{l_{23}}{x_2-x_3} & -\frac{l_{13}}{x_1-x_3} \\
\frac{l_{12}}{x_1+x_2} & 0 & -\frac{l_{23}}{x_2+x_3} & -\frac{l_{23}}{x_2-x_3} & -p_2 & -\frac{l_{12}}{x_1-x_2} \\
0 & -\frac{l_{12}}{x_1+x_2} & -\frac{l_{13}}{x_1+x_3} & -\frac{l_{13}}{x_1-x_3} & -\frac{l_{12}}{x_1-x_2} & -p_1
\end{array}\right)$$

$$A_{YMM} = \left(\begin{array}{ccc|ccc}
0 & \frac{l_{12}}{(x_1-x_2)^2} & \frac{l_{13}}{(x_1-x_3)^2} & -\frac{l_{13}}{(x_1+x_3)^2} & -\frac{l_{12}}{(x_1+x_2)^2} & 0 \\
-\frac{l_{12}}{(x_1-x_2)^2} & 0 & \frac{l_{23}}{(x_2-x_3)^2} & -\frac{l_{23}}{(x_2+x_3)^2} & 0 & \frac{l_{12}}{(x_1+x_2)^2} \\
-\frac{l_{13}}{(x_1-x_3)^2} & -\frac{l_{23}}{(x_2-x_3)^2} & 0 & 0 & \frac{l_{23}}{(x_2+x_3)^2} & \frac{l_{13}}{(x_1+x_3)^2} \\
\hline
\frac{l_{13}}{(x_1+x_3)^2} & \frac{l_{23}}{(x_1+x_2)^2} & 0 & 0 & -\frac{l_{23}}{(x_2-x_3)^2} & -\frac{l_{13}}{(x_1-x_3)^2} \\
\frac{l_{12}}{(x_1+x_2)^2} & 0 & -\frac{l_{23}}{(x_2+x_3)^2} & \frac{l_{23}}{(x_2-x_3)^2} & 0 & -\frac{l_{12}}{(x_1-x_2)^2} \\
0 & -\frac{l_{12}}{(x_1+x_2)^2} & -\frac{l_{13}}{(x_1+x_3)^2} & \frac{l_{13}}{(x_1-x_3)^2} & \frac{l_{12}}{(x_1-x_2)^2} & 0
\end{array}\right.$$

The equations in the zero coupling constant limit read in a Lax form as

$$\dot{L}_{YMM} = [A_{YMM}, L_{YMM}], \qquad \dot{l}_{YMM} = [A_{YMM}, l_{YMM}],$$

where the matrix l_{YMM} is

$$l_{YMM} = \left(\begin{array}{ccc|ccc} 0 & l_{12} & l_{13} & -l_{13} & -l_{12} & 0 \\ -l_{12} & 0 & l_{23} & -l_{23} & 0 & l_{12} \\ -l_{13} & -l_{23} & 0 & 0 & l_{23} & l_{13} \\ \hline l_{13} & l_{23} & 0 & 0 & -l_{23} & -l_{13} \\ l_{12} & 0 & -l_{23} & l_{23} & 0 & -l_{12} \\ 0 & -l_{12} & -l_{13} & l_{13} & l_{12} & 0 \end{array}\right)$$

Light-cone form of the homogeneous $SU(2)$ Yang-Mills theory

Light-cone model and analysis of constraints

We start with the action of Yang-Mills field theory in four-dimensional Minkowski space M_4, endowed with a metric η and represented in the coordinate free form

$$I := \frac{1}{g^2} \int_{M_4} \mathbf{tr}\, F \wedge *F \,,$$

where g is a coupling constant and the $su(2)$ algebra valued curvature two-form

$$F := dA + A \wedge A$$

is constructed from the connection one-form A. The connection and curvature, as Lie algebra valued quantities, are expressed in terms of the antihermitian $su(2)$ algebra basis $\tau^a = \sigma^a/2i$ with the Pauli matrices $\sigma^a\,, a = 1, 2, 3$,

$$A = A^a\,\tau^a\,, \qquad F = F^a\,\tau^a\;.$$

The metric η enters the action through the dual field strength tensor defined in accordance with the Hodge star operation

$$*F_{\mu\nu} = \frac{1}{2}\,\sqrt{\eta}\,\epsilon_{\mu\nu\alpha\beta}\,F^{\alpha\beta}\;.$$

To formulate the light-cone version of the theory let us introduce the basis vectors in the tangent space $T_P(M_4)$

$$e_{\pm} := \frac{1}{\sqrt{2}}\left(e_0 \pm e_3\right) , \qquad e_{\perp} := \left(e_k\,, k = 1, 2\right) .$$

The first two vectors are tangent to the light-cone and the corresponding coordinates are referred usually as the light-cone coordinates $x^\mu = \left(x^+, x^-, x^\perp\right)$

$$x^{\pm} := \frac{1}{\sqrt{2}}\left(x^0 \pm x^3\right) , \qquad x^{\perp} := x^k , \quad k = 1, 2 .$$

The non-zero components of the metric η in the light-cone basis

(e_+, e_-, e_k) **are**

$$\eta_{+-} = \eta_{-+} = -\eta_{11} = -\eta_{22} = 1\,.$$

The connection one-form in the light-cone basis is given as

$$A = A_+\, dx^+ + A_-\, dx^- + A_k\, dx^k\,.$$

By definition the Lagrangian of light-cone Yang-Mills mechanics follows from the corresponding Lagrangian of Yang-Mills theory if one supposes that the components of the connection one-form A **depends on the light-cone "time variable"** x^+ **alone**

$$A_\pm = A_\pm(x^+)\,, \qquad A_k = A_k(x^+)\,.$$

Substitution this *ansatz* into the classical action defines the Lagrangian of light-cone Yang-Mills mechanics

$$L = \frac{1}{2g^2}\left(F^a_{+-}\,F^a_{+-} + 2\,F^a_{+k}\,F^a_{-k} - F^a_{12}\,F^a_{12}\right) ,$$

where the light-cone components of the field-strength tensor are given by

$$F^a_{+-} = \frac{\partial A^a_-}{\partial x^+} + \epsilon^{abc}\,A^b_+\,A^c_- \,,$$
$$F^a_{+k} = \frac{\partial A^a_k}{\partial x^+} + \epsilon^{abc}\,A^b_+\,A^c_k \,,$$
$$F^a_{-k} = \epsilon^{abc}\,A^b_-\,A^c_k \,,$$
$$F^a_{ij} = \epsilon^{abc}\,A^b_i\,A^c_j \,, \qquad i,j,k = 1,2 \,.$$

Performing the Legendre transformation

$$\pi_a^+ = \frac{\partial L}{\partial \dot{A}_+^a} = 0\,,$$

$$\pi_a^- = \frac{\partial L}{\partial \dot{A}_-^a} = \frac{1}{g^2}\left(\dot{A}_-^a + \epsilon^{abc} A_+^b A_-^c\right)\,,$$

$$\pi_a^k = \frac{\partial L}{\partial \dot{A}_k^a} = \frac{1}{g^2}\,\epsilon^{abc} A_-^b A_k^c\,,$$

we obtain the canonical Hamiltonian

$$H_C = \frac{g^2}{2}\pi_a^-\pi_a^- - \epsilon^{abc} A_+^b\left(A_-^c \pi_a^- + A_k^c \pi_a^k\right) + V(A_k)$$

with a potential term

$$V(A_k) = \frac{1}{2g^2}\left[\left(A_1^b A_1^b\right)\left(A_2^c A_2^c\right) - \left(A_1^b A_2^b\right)\left(A_1^c A_2^c\right)\right] .$$

The non-vanishing Poisson brackets between the fundamental canonical variables are

$$\{A_{\pm}^a\,, \pi_b^{\pm}\} = \delta_b^a\,,$$
$$\{A_k^a\,, \pi_b^l\} = \delta_k^l \delta_b^a\,.$$

The Hessian of the Lagrangian system (??) is degenerate, $\det || \frac{\partial^2 L}{\partial \dot{A} \partial \dot{A}} ||$ 0, and as a result there are primary constraints

$$\varphi_a^{(1)} := \pi_a^+ = 0\,,$$
$$\chi_k^a := g^2\, \pi_k^a + \epsilon^{abc}\, A_-^b\, A_k^c = 0\,,$$

satisfying the following Poisson brackets relations

$$\{\varphi^{(1)}_a, \varphi^{(1)}_b\} = 0\,,$$
$$\{\varphi^{(1)}_a, \chi^b_k\} = 0\,,$$
$$\{\chi^a_i, \chi^b_j\} = -2\, g^2 \epsilon^{abc}\, A^c_-\, \eta_{ij}\,.$$

According to the Dirac prescription, the presence of primary constraints affects the dynamics of the degenerate system. Now the generic evolution is governed by the total Hamiltonian

$$H_T = H_C + U_a(\tau)\varphi^{(1)}_a + V^a_k(\tau)\chi^a_k\,,$$

where $U_a(\tau)$ and $V^a_k(\tau)$ are unspecified functions of the light-cone time τ. Using this Hamiltonian the dynamical self-consistence of the primary constraints may be checked. From the requirement of conservation of the primary constraints $\varphi_a^{(1)}$ it follows

$$0 = \dot{\varphi}_a^{(1)} = \{\pi_a^+, H_T\} = \epsilon^{abc}\left(A_-^b \pi_c^- + A_k^b \pi_c^k\right) .$$

Therefore there are three secondary constraints $\varphi_a^{(2)}$

$$\varphi_a^{(2)} := \epsilon_{abc}\left(A_-^b \pi_c^- + A_k^b \pi_c^k\right) = 0 ,$$

which obey the $so(3,\mathbb{R})$ algebra

$$\{\varphi_a^{(2)}, \varphi_b^{(2)}\} = \epsilon_{abc}\, \varphi_c^{(2)} .$$

The same procedure for the primary constraints χ^a_k gives the following self-consistency conditions

$$0 = \dot{\chi}^a_k = \{\chi^a_k\,, H_C\} - 2\,g^2\,\epsilon^{abc}\,V^b_k\,A^c_-\,.$$

Hereinafter we shall consider the subspace of configuration space where $\mathrm{rank}||\mathcal{C}|| = 2$. For those configurations we are able to introduce the unit vector

$$N^a = \frac{A^a_-}{\sqrt{(A^1_-)^2 + (A^2_-)^2 + (A^3_-)^2}}\,,$$

which is a null vector of the matrix $\|\,\epsilon^{abc}\,A^c_-\,\|$, and to decompose the set of six primary constraints χ^a_k as

$$\psi_k := N^a \chi^a_k \,,$$
$$\chi^a_{k\perp} := \chi^a_k - \left(N^b \chi^b_k\right) N^a \,.$$

In this decomposition the first two constraints ψ_k are functionally independent and satisfy the Abelian algebra

$$\{\psi_i \,, \psi_j\} = 0 \,,$$

while the constraints $\chi^a_{k\perp}$ are functionally dependent due to the conditions

$$N^a \, \chi^a_{k\perp} = 0 \,.$$

Choosing among them any four independent constraints we can determine four Lagrange multipliers $V^{\ k}_{b\perp}$.

The Poisson brackets of the constraints ψ_k and $\varphi_a^{(2)}$ with the total Hamiltonian vanish after projection on the constraint surface (CS) defined by equations $\psi_k = 0$ and $\varphi_a^{(2)} = 0$

$$\{\psi_k\,, H_T\}\,|_{CS} = 0\,,$$
$$\{\varphi_a^{(2)}\,, H_T\}\,|_{CS} = 0$$

and thus there are no ternary constraints.

Summarizing, we arrive at the set of constraints $\varphi_a^{(1)}, \psi_k, \varphi_a^{(2)}, \chi_{k\perp}^b$. The Poisson brackets algebra of the first three is

$$\begin{aligned}
&\{\varphi_a^{(1)}, \varphi_a^{(1)}\} = 0\,, \\
&\{\psi_i\,, \psi_j\} = 0\,, \\
&\{\varphi_a^{(2)}, \varphi_b^{(2)}\} = \epsilon_{abc}\,\varphi_c^{(2)}\,, \\
&\{\varphi_a^{(1)}, \psi_k\} = \{\varphi_a^{(1)}, \varphi_b^{(2)}\} = \{\psi_k\,, \varphi_a^{(2)}\} = 0\,.
\end{aligned}$$

The constraints $\chi_{k\perp}^b$ satisfy the relations

$$\{\chi_{i\perp}^a\,, \chi_{j\perp}^b\} = -2\,g^2\,\epsilon^{abc}\,A_-^c\,\eta_{ij}\,,$$

and the Poisson brackets between these two sets of constraints are

$$\{\varphi^{(2)}_a \,, \chi^b_{k\perp}\} = \epsilon^{abc}\, \chi^c_{k\perp} \,,$$
$$\{\varphi^{(1)}_a \,, \chi^b_{k\perp}\} = \{\psi_i \,, \chi^b_{j\perp}\} = 0 \,.$$

Unconstrained version of light-cone mechanics

Let us organize the configuration variables A^a_i and A^a_- in one 3×3 matrix A_{ab} whose entries of the first two columns are A^a_i

and third column is composed by the elements A^a_-

$$A_{ab} := \| A^a_1 \,, A^a_2 \,, A^a_- \| \,,$$

and the momentum variables similarly

$$\Pi_{ab} := \| \pi^{a1} \,, \pi^{a2} \,, \pi^{a-} \| .$$

In order to find an explicit parametrization of the orbits with respect to the gauge symmetry action, it is convenient to use a polar decomposition [?] for the matrix A_{ab}

$$A = OS \,,$$

where S is a positive definite $3{\times}3$ symmetric matrix, $O(\phi_1, \phi_2, \phi_3) = e^{\phi_1 J_3} e^{\phi_2 J_1} e^{\phi_3 J_3}$ is an orthogonal matrix parameterized by the three

Euler angles (ϕ_1, ϕ_2, ϕ_3). The matrices $(J_a)_{ij} = \epsilon_{iaj}$ are the $SO(3, \mathbb{R})$ generators in adjoint representation.

It is in order to make a few remarks on the change of variables. It is well-known that the polar decomposition is valid for an arbitrary matrix. However, the orthogonal matrix uniquely determined only for an invertible matrix A

$$O = AS^{-1}\,, \qquad S = \sqrt{AA^T}\ .$$

It is worth to note here that in virtue of the constraints the determinant of the matrix A is related to the third component of the gauge field spin

$$2\,\det A - g^2 \epsilon_{3ik}\, A^a_k\, \pi^a_i = 0\,.$$

The polar decomposition induces the point canonical transformation from the coordinates A_{ab} and Π_{ab} to new canonical pairs (S_{ab}, P_{ab}) and (ϕ_a, P_a) with the following non-vanishing Poisson brackets

$$\{S_{ab}\,, P_{cd}\} = \frac{1}{2}\,(\delta_{ac}\,\delta_{bd} + \delta_{ad}\,\delta_{bc})\;,$$
$$\{\phi_a\,, P_b\} = \delta_{ab}\,.$$

The expression of the old Π_{ab} as a function of the new coordinates is

$$\Pi = O\left(P - k_a J_a\right) \,,$$

where

$$k_a = \gamma^{-1}_{ab}\left(\eta^L_b - \varepsilon_{bmn}\,(SP)_{mn}\right) \,,$$

$\gamma_{ik} = S_{ik} - \delta_{ik}\,\mathrm{tr}S$ and η^L_a are three left-invariant vector fields on the $SO(3,\mathbb{R})$ group

$$\eta^L_1 = \frac{\sin\phi_3}{\sin\phi_2}\,P_1 + \cos\phi_3\,P_2 - \cot\phi_2\sin\phi_3\;P_3 \,,$$
$$\eta^L_2 = \frac{\cos\phi_3}{\sin\phi_2}\,P_1 - \sin\phi_3\,P_2 - \cot\phi_2\cos\phi_3\;P_3 \,,$$
$$\eta^L_3 = P_3 \,.$$

In terms of the new variables the constraints take the form

$$\varphi_a^{(2)} = O_{ab}\,\eta_b^L\,,$$
$$\chi_{am} = O_{ab}\,\left(P_{bm} + \epsilon_{bmc}\,k_c + \epsilon_{bij}\,S_{i3}\,S_{jm}\right)\,.$$

Thus one can pass to the equivalent set of constraints

$$\eta_a^L = 0\,,$$
$$\widetilde{\chi}_{ai} = P_{ai} + \epsilon_{aij}\,\gamma_{jk}^{-1}\,\epsilon_{kmn}(SP)_{mn} + \epsilon_{amn}\,S_{m3}\,S_{ni} = 0$$

with vanishing Poisson brackets

$$\{\eta_a^L, \widetilde{\chi}_{bi}\} = 0\,.$$

In order to proceed further in resolution of the remaining constraints we introduce the main-axes decomposition for the symmetric 3×3 matrix S

$$S = R^T(\chi_1, \chi_2, \chi_3) \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix} R(\chi_1, \chi_2, \chi_3)\,,$$

with orthogonal matrix $R(\chi_1, \chi_2, \chi_3) = e^{\chi_1 J_3} e^{\chi_2 J_1} e^{\chi_3 J_3}$, parameterized by three Euler angles (χ_1, χ_2, χ_3). The Jacobian of this transformation is

$$\frac{\partial(\ S_{i<j}\ )}{\partial(q_a, \chi_b)} \sim \prod_{a \neq b}^{3} |\ q_a - q_b\ |\,.$$

The momenta p_a and p_{χ_a}, canonically conjugated to the diagonal q_a and angular variables χ_a, can be found using the canonical invariance of the symplectic one-form

$$\sum_{a,b=1}^{3} P_{ab}\, dS_{ab} = \sum_{a=1}^{3} p_a\, dq_a + \sum_{a=1}^{3} p_{\chi_a}\, d\chi_a \,.$$

The original momenta P_{ab}, expressed in terms of the new canonical variables, read

$$P = R^T \sum_{a=1}^{3} (p_a\, \overline{\alpha}_a + \mathcal{P}_a\, \alpha_a)\, R \,.$$

Here $\overline{\alpha}_a$ and α_a denote the diagonal and off-diagonal basis elements of the space of symmetric matrices with orthogonality relations

$$\mathbf{tr}\ (\overline{\alpha}_a\overline{\alpha}_b) = \delta_{ab}\,, \qquad \mathbf{tr}\ (\alpha_a\alpha_b) = 2\delta_{ab}\,, \qquad \mathbf{tr}\ (\overline{\alpha}_a\alpha_b) = 0$$

and

$$\mathcal{P}_a = -\frac{1}{2}\,\frac{\xi_a^R}{q_b - q_c} \quad (\textbf{cyclic permutations}\, a \neq b \neq c)\,.$$

The ξ_a^R are three $SO(3,\mathbb{R})$ right-invariant vector fields given in terms of the angles χ_a and their conjugated momenta p_{χ_a} via

$$\xi_a^R = M_{ba}^{-1} p_{\chi_b}\,,$$

where the matrix M is given by

$$M_{ab} = -\frac{1}{2}\,\mathbf{tr}\left(J_a \frac{\partial R}{\partial \chi_b} R^T\right) .$$

The explicit form of the three $SO(3,\mathbb{R})$ right-invariant Killing vector fields is

$$\xi_1^R = -\sin\chi_1 \cot\chi_2 \; p_{\chi_1} + \cos\chi_1 \; p_{\chi_2} + \frac{\sin\chi_1}{\sin\chi_2} \; p_{\chi_3} ,$$

$$\xi_2^R = \;\cos\chi_1 \cot\chi_2 \; p_{\chi_1} + \sin\chi_1 \; p_{\chi_2} - \frac{\cos\chi_1}{\sin\chi_2} \; p_{\chi_3} ,$$

$$\xi_3^R = p_{\chi_1} .$$

Using these formulas the constraints $\widetilde{\chi}$ may be rewritten in

terms of the main-axes variables as

$$\widetilde{\chi} = \sum_{a=1}^{3} R^T \left[\pi_a \, \overline{\alpha}_a - \frac{1}{2} \, \rho_a^- \, \alpha_a + \frac{1}{2} \, \rho_a^+ \, J_a \right] R \, ,$$

where

$$\rho_a^{\pm} = \frac{\xi_a^R}{q_b \ \pm q_c} \pm \frac{1}{g^2} \, q_a n_a (q_b \ \pm \ q_c) \, ,$$

and $n_a = R_{a3}$**.**

Note that the constraint on the determinant of the matrix A **now takes the form**

$$2 \, q_1 \, q_2 \, q_3 - g^2 \xi_3^L = 0 \, ,$$

where ξ_3^L is the third left-invariant Killing vector field, $\xi_a^L = R_{ab}\,\xi_b^R$

$$\xi_1^L = \frac{\sin\chi_3}{\sin\chi_2}\,p_{\chi_1} + \cos\chi_3\,p_{\chi_2} - \cot\chi_2\sin\chi_3\ p_{\chi_3}\,,$$
$$\xi_2^L = \frac{\cos\chi_3}{\sin\chi_2}\,p_{\chi_1} - \sin\chi_3\,p_{\chi_2} - \cot\chi_2\cos\chi_3\ p_{\chi_3}\,,$$
$$\xi_3^L = p_{\chi_3}\,.$$

The expression for the Abelian constraints ψ_i dictates the appropriate gauge fixing condition

$$\overline{\psi}_i := N^a A_i^a = 0\,,$$

which is the canonical one in the sense that

$$\{\overline{\psi}_i,\ \psi_j\} \ = \ \delta_{ij}\,.$$

The constraints $\psi_i = 0$ rewritten in terms of the main-axes variables may be identified with the nullity of the momenta

$$p_{\chi_1} = 0\,, \qquad p_{\chi_2} = 0\,,$$

while the canonical gauge-fixing condition fixes the corresponding angular variables χ_1 and χ_2

$$\chi_1 = \frac{\pi}{2}\,, \qquad \chi_2 = \frac{\pi}{2}\,.$$

Projection of the canonical Hamiltonian to the surface described by constraints gives

$$H_{LC} := H_C(\chi_1 = \frac{\pi}{2}\,, p_{\chi_1} = 0\,, \chi_2 = \frac{\pi}{2}\,, p_{\chi_2} = 0) = \frac{g^2}{2}\left(p_1^2 + \frac{q_2^2\, q_3^2}{g^4}\right)\,.$$

Furthermore, taking into account the constraint the projected Hamiltonian may be rewritten as

$$H_{LC}\Big|_{2\,q_1\,q_2\,q_3 - g^2\xi_3^L = 0} = \frac{g^2}{2}\left(p_1^2 + \left(\frac{\xi_3^L}{2q_1}\right)^2\right)\,.$$

It may be checked that the constraints $\chi^a_{i\,\perp}$ lead to the conditions on the "diagonal" canonical pairs $(q_i\,, p_i)$. Namely, the

canonical momenta p_2 and p_3 are vanishing

$$p_2 = 0\,, \quad p_3 = 0\,,$$

while the corresponding coordinates q_2 and q_3 are subject to the constraint

$$q_2^2 + q_3^2 = 0$$

as well the constraint.

Let us consider the analytic continuation of the constraint into a complex domain and explore its *complex* solution

$$q_2 = \pm\, i\, q_3\,.$$

Expressing q_3 from equation

$$q_3 = \frac{1 \mp i}{2} \sqrt{\frac{g^2 \xi_3^L}{q_1}} \,,$$

we find that (q_1, p_1) and (χ_3, p_{χ_3}) remain *real* unconstrained variables whose Dirac brackets are the canonical ones

$$\{q_1, p_1\}_D = 1 \,, \qquad \{\chi_3, p_{\chi_2}\}_D = 1 \,.$$

Therefore the dynamics of the unconstrained pairs (q_1, p_1) and (χ_3, p_{χ_3}) is given by the standard Hamilton equations with the Hamiltonian. Remarking that the ξ_3^L is conserved we conclude that coincides with the Hamiltonian of conformal mechanics

$$H = \frac{g^2}{2} \left(p_1^2 + \frac{\kappa^2}{q_1^2} \right) \,,$$

with "coupling constant" $\kappa^2 = \left(\xi_3^L/2\right)^2$ determined by the value of the gauge spin, while the gauge field coupling constant g controls the scale for the evolution parameter.

The dynamical $SL(2,R)$ symmetry

The action for conformal mechanics

$$S := \frac{1}{2}\int \mathrm{d}t\left(\dot{q}^2 - \frac{\kappa}{q^2}\right),$$

is invariant under the three parameters time reparametrization

$$t \to t' := \frac{\alpha t + \beta}{\gamma t + \delta}\,, \qquad \alpha\delta - \beta\gamma = 1\,,$$

accompanied with following transformation of independent variable

$$q'(t') := \frac{1}{\gamma t + \delta}\, q(t)\,.$$

These transformation represent the conformal transformations in $0{+}1$ dimensions and can be build with the aid of the explicitly time dependent Noether generators

$$H := \frac{1}{2}\left(p^2 + \frac{\kappa}{q^2}\right)\,, \quad D := tH - \frac{1}{2}\, qp\,, \qquad K := t^2 H - t\, qp + \frac{1}{2}\, q^2\,,$$

which obey the $SL(2,R)$ algebra with respect to the Poisson brackets

$$\{H\,,K\} = 2D\,, \qquad \{H\,,D\} = H\,, \qquad \{K\,,D\} = -K\,.$$

The generators H, D and K correspond to the *time translations*, *dilations* **and** *special conformal* **transformations**

$$t \to t+\beta\,, \qquad t \to \alpha^2 t\,, \qquad t \to \frac{1}{\gamma t+1}\,.$$

respectively.

On the other hand the classical action of pure 4-dimensional Yang-Mills theory in the Minkowski space-time is invariant un-

der the conformal transformations

$$x'^{\mu} = x^{\mu} + \xi^{\mu}\,,$$
$$A'_{\mu}(x) = A_{\mu}(x) + \delta_{\xi} A_{\mu}\,,$$

where the vector ξ satisfies the conformal Killing equation

$$\partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} = \frac{1}{2}\, g^{\mu\nu}\partial_{\sigma}\xi^{\sigma}\,,$$

and the infinitesimal change $\delta_{\xi} A_{\mu}$ is given by the Lie derivative $\pounds_{\xi}$ of a gauge potential

$$\delta_{\xi} A_{\mu} = \pounds_{\xi} A^{a}_{\mu} = \partial_{\mu}\xi^{\nu} A^{a}_{\nu} + \xi^{\nu}\partial_{\nu} A^{a}_{\mu}\,,$$

For the standard cartesian Minkowski metric tensor $g = (1, -1, -1, -$

the general solution to the equation reads:

$$\xi^{\mu} := a^{\mu} + bx^{\mu} + \omega^{\mu}{}_{\nu}x^{\nu} + 2x^{\mu}c_{\nu}x^{\nu} - c^{\mu}x_{\nu}x^{\nu} \,,$$

where $a^{\mu}, b\,, c^{\mu}$ and $\omega^{\mu}{}_{\nu} = -\omega^{\nu}{}_{\mu}$ are 15 independent parameters.

Now we construct integrals of motion for the light-cone mechanics that are inherited from the conformal symmetry of the underlining field theory.

The conserved symmetric traceless energy momentum tensor gives rise to the differential conservation law

$$\partial_{\mu}\left(\xi^{\nu}\mathrm{T}^{\mu}{}_{\nu}\right) = 0$$

Supposing now the dependence of fields on light-cone time only the charges corresponding to the conformal group symmetry can be defined as follows. The identity

$$0 = \int \mathrm{d}x^- \mathrm{d}\mathbf{x}_\perp^2 \, \partial_\mu \left(\xi^\nu \mathrm{T}^\mu{}_\nu\right) = \int \mathrm{d}x^- \mathrm{d}\mathbf{x}_\perp^2 \, \partial_+ \left(\xi^\nu \mathrm{T}^+{}_\nu\right) + \sum_{\alpha=-,\, i} \mathrm{T}^\alpha{}_\nu \int \mathrm{d}x^- \mathrm{d}\mathbf{x}_\perp^2 \, \partial_\alpha \xi$$

after integration gives

$$\frac{\partial}{\partial \tau} \left(\int \mathrm{d}x^- \mathrm{d}\mathbf{x}_\perp^2 \, \xi^\nu \mathrm{T}^+{}_\nu \right) = \left(\mathrm{T}^-{}_\nu \omega^\nu{}_- + \mathrm{T}^-{}_+ c^+ + \mathrm{T}^-{}_i c^i \right) \times \mathrm{Vol}$$

where $\mathrm{Vol} := \int \mathrm{d}x^- \mathrm{d}\mathbf{x}_\perp^2$ denotes a 3-dimensional volume.

Therefore if the vector ξ is specified as

$$\xi^+ = a^+ + b\tau + 2c\tau^2 \, ,$$
$$\xi^- = a^- \, ,$$
$$\xi^i = a^i \, .$$

the right hand side of the equation vanishes and we arrive at the following integrals of motion

$$I(\tau) = a^{\nu}\mathrm{T}^{+}{}_{\nu} + b\tau\mathrm{T}^{+}{}_{+} + c_{+}\tau^{2}\mathrm{T}^{+}{}_{+}\,,$$

Now having these in mind consider three functions T_+, T_0, and T_- defined on the phase space of our model

$$\begin{aligned} T_+ &= \tfrac{1}{2}\left(\pi^-_a\pi^-_a + \pi^+_a\pi^+_a + \pi^a_i\pi^i_a\right)\,, \\ T_0 &= -\tfrac{1}{2}\left(A^a_-\pi^-_a + A^a_+\pi^+_a + A^a_i\pi^i_a\right)\,, \\ T_- &= \tfrac{1}{2}\left(A^a_-A^a_- + A^a_+A^a_+ + A^i_aA^a_i\right)\,. \end{aligned}$$

Note that these functions obey the $SL(2,R)$ algebra and can be rewritten as Indeed, noting that

$$T_+ = H_c + \pi^i_a \chi^i_a + A^a_+ \varphi^{(2)}_a$$

With the aid of these functions one can construct three integrals of motion as follows. Straightforward calculation shows that the function

$$I = 2f(\tau)T_+ + \dot{f}(\tau)T_0 + \ddot{f}(\tau)T_-$$

with quadratic function $f(\tau)$ of light-cone time

$$f(\tau) = a + b\tau + c\tau^2\,, \qquad a, b, c - \text{ constants}\,,$$

represents the 3-parameter integral of motion. I one can verify that the total derivative

$$\frac{dI}{d\tau} = \frac{\partial I}{\partial \tau} + \{I\,, H_T\}\,.$$

vanishes on the primary constraint surface.

This integral of motion generates the rigid 3-parameter infinitesimal symmetry transformation $A'(\tau) = A(\tau) + \delta_f A(\tau)$

$$\begin{aligned}
\delta_f A^a_+(\tau) &= f(\tau)\dot{A}^a_+ + \dot{f}(\tau)A_+\,,\\
\delta_f A^a_-(\tau) &= \qquad f(\tau)\dot{A}^a_-\,,\\
\delta_f A^a_i(\tau) &= \qquad f(\tau)\dot{A}^a_i
\end{aligned}$$

induced by the infinitesimal time reparameterization

$$\tau' = \tau + f(\tau) \, .$$

Therefore we conclude that the dynamical algebra of light-cone Yang-Mills mechanics include the $SL(2, R)$ algebra.

In order to clarify the meaning of the 5- parameter gauge symmetry group, let us define 4-vector $\xi = (\xi^+ , \xi^- , \xi^i)$, whose $\pm$ components coincide with the function f the transverse components of which are two arbitrary functions of light-cone time $\xi^i(\tau) \, , i = 1, 2$

$$\xi = \left(f(\tau) \, , f(\tau) \, , \xi^i(\tau)\right)$$

One can convince that the change of the dynamical variables represented by the action of Lie derivative with respect vector field ξ

$$\delta_\xi A^a_\mu = \pounds_\xi A^a_\mu = \partial_\mu \xi^\nu A^a_\nu + \xi^\nu \partial_\nu A^a_\mu$$

is a combination of rigid $SL(2,R)$ transformations and Abelian subgroup of gauge transformation defined by the $\varepsilon^a(\tau) = 0$ with $v^i = \dot{\xi}^i$.

Concluding remarks

Instant Form	Front Form	Point Form
Reduced Systems		
Spin Calogero-Moser-Sutherland model with external potential	Free particle motion or Conformal mechanics	?
Dynamics		
Chaotic behavior	Exactly integrable system	?

Many thanks

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