DEFORMED SYMMETRIES AND EXACT SOLVABILITY

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Dedicated to the memory of Matey Mateev, Mag

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I feel indebted to Matey Mateev

for inspiring my teaching activity and for the

opportunity to actively contribute to the students education in

theoretical and mathematical physics at the University, Sofia.

SUPERSYMMETRY - New Symmetry, (1974)

found independently by

Gol'fand and Lichtmann   Wess and Zumino

relates particles of different spin and statistics

realized on an anticommutative space

"the first example of application of

noncommutative geometry in physics"   (Zumino, 1991)

DEFORMED SYMMETRIES   QUANTUM GROUPS

deformation of a Lie symmetry, depend on parameters

in the proper limit the original symmetry is restored

rich algebraic structures

Applications:

find sensible and nontrivial deformed versions of existing

theories which are difficult to study and apply

e.g. noncommutative Yang Mills theory

look for real processes where deformed symmetries naturally appear

e.g. stochastic processes in nonequilibrium physics

ASEP - the FUNDAMENTAL MODEL of NONEQUILIBRIUM PHYSICS

a lattice DIFFUSION model - a very good playground for

utility of QUANTUM GROUPS

the deformation parameter has direct physical meaning

the ratio of left/right diffusion probability rates

Main point in this talk

LATTICE MANY-BODY SYSTEMS EXACTLY

TRIDIAGONAL ALGEBRA APPROACH TO STOCHASTIC DYNAMICS

BASED ON BULK QUANTUM GROUP AND BOUNDARY

ASKEY-WILSON AND DEFORMED ONZAGER ALGEBRA

CHARACHTERIZING A CLASS OF SUPERINTEGRABLE MODELS

EXACT SOLUTION IN THE STATIONARY STATE AND

DESCRIPTION OF THE DYNAMICS

THROUGH NONLOCAL BOUNDARY CHARGES

The asymmetric simple exclusion process(ASEP)

a paradigm for nonequilibrium physics due to

simplicity, rich behaviour and wide range of applicability.

It is an exacltly solvable model of an open many-particle stochastic system interacting with hard core exclusion.

a simplified model of one dimensional transport

hopping conductivity and kinetics of biopolymerization,

applications - traffic flow, to interface growth, shock formation, hydrodynamic systems obeying the noisy Burger equation, problems of sequence alignment in biology.

At large time the ASEP exibits relaxation

to a steady state,

and even after the relaxation it has

a nonvanishing current.

An intriguing feature is the occurrence of

boundary induced phase transitions

and the fact that

the stationary bulk properties depend strongly

on the boundary rates.

The ASEP is a stochastic process described in terms of a probability distribution $P(s_i, t)$
of a stochastic variable $s_i = 0, 1$ at a site $i = 1, 2, \ldots .L$
A state on the lattice at a time t
is determined by the occupation numbers $\{s_i\}$
a transition to another configuration s_i' during
an infinitesimal time step dt is given by the probability $\Gamma(s, s')dt$.

Bulk rates $\Gamma \equiv \Gamma_{jl}^{ik}$ $(2^2 \times 2^2)$; Boundary rates L_i^j and R_i^j (2×2) at sites 1 and L

Due to probability conservation

$$\Gamma(s, s) = -\sum_{s' \neq s} \Gamma(s', s)$$

DIFFUSION - $\Gamma_{ki}^{ik} = g_{ik}$

Processes with exclusion - a site can be either empty or occupied by a particle of a given type.

In the set of occupation numbers $(s_1, s_2, ..., s_L)$ specifying a configuration 0f the system

$s_i = 0$ if a site i is empty,

$s_i = 1$ if there is a particle at a site i

- $g_{ik} dt$ - $i, k = 0, 1$ - with $i < k$,

g_{01} are the probability rates of hopping to the left,

g_{10} - to the right.

The event of exchange happens if out of two adjacent sites one is a vacancy and the other is occupied by a particle,

$g_{01} = 1, \quad g_{10} = q, \qquad \alpha, \quad \gamma, \quad \beta, \quad \delta$

EXACT SOLVABILITY of the ASYMMETRIC EXCLUSION MODEL within MATRIX PRODUCT STATES APPROACH

IN THE STATIONARY STATE (Derrida, Evans, Hakim, Pasquier)

ANZATZ: For a given configuration $(s_1, s_2, ..., s_L)$ the STATIONARY PROBABILITY is given by

$$P(s) = \frac{\langle w|D_{s_1} D_{s_2} ... D_{s_L}|v\rangle}{Z_L},$$

$D_{s_i} = D_1$ if a site $i = 1, 2, ..., L$ is occupied
$D_{s_i} = D_0$ if a site i is empty

- quadratic algebra $D_1 D_0 - q D_0 D_1 = x_1 D_0 - x_0 D_1$
- boundary conditions:

$(x_0 + x_1 = 0, \quad x_0 = -x_1 = \zeta)$

$$\begin{aligned} (\beta D_1 - \delta D_0)|v\rangle &= \zeta|v\rangle \\ \langle w|(\alpha D_0 - \gamma D_1) &= \zeta\langle w|. \end{aligned}$$

$$Z_L = \langle w|(D_0 + D_1)^L|v\rangle$$

is the normalization factor to the stationary probability distribution

the current

$$J = \zeta \frac{Z_{L-1}}{Z_L}$$

TRIDIAGONAL ALGEBRA APPROACH

BULK TRIDIAGONAL ALGEBRA

The matrices D_0 and D_1, and the q-commutator $[D_0, D_1]_q \equiv q^{1/2} D_0 D_1 - q^{-1/2} D_1 D_0$

form a tridiagonal algebra

$$[D_1, [D_1, [D_1, D_0]_q]_{q^{-1}}] = 0$$

$$[D_0, [D_0, [D_0, D_1]_q]_{q^{-1}}] = 0$$

the q-Serre relations for the level zero $U_q(\hat{su}(2))$

QUADRATIC ALGEBRA OF THE BULK QUANTUM GROUP

R-MATRIX satisfying YANG-BAXTER EQUATION

ASKEY-WILSON BOUNDARY ALGEBRA

K-MATRIX satisfying BOUNDARY YANG-BAXTER EQUATION

TRIDIAGONAL ALGEBRA

EXACT SPECTRAL PROBLEM SOLUTION and
(INFINITE SET) of CONSERVED CHARGES

BOUNDARY ASKEY - WILSON ALGEBRA

of the ASYMMETRIC EXCLUSION PROCESS with

incoming and outgoing particles at

the left and right boundaries

4 boundary parameters $\alpha, \beta, \gamma, \delta$

and bulk parameter $0 < q < 1$

The two linearly independant boundary operators

$$B^R = \beta D_1 - \delta D_0, \qquad B^L = -\gamma D_1 + \alpha D_0$$

are coideal elements of $U_q(\hat{s}u(2))$ algebra

in the evaluation representation

REPRESENTATION of the BOUNDARY OPERATORS

$$\beta D_1 - \delta D_0 =$$
$$-\frac{\beta}{\sqrt{1-q}} q^{N/2} A_+ - \frac{\delta}{\sqrt{1-q}} A_- q^{N/2}$$
$$-\frac{\beta q^{1/2} + \delta}{1-q} q^N - \frac{\beta - \delta}{1-q}$$

$$\alpha D_0 - \gamma D_1 =$$
$$\frac{\alpha}{\sqrt{1-q}} q^{-N/2} A_+ + \frac{\gamma}{\sqrt{1-q}} A_- q^{-N/2}$$
$$+\frac{\alpha q^{-1/2} + \gamma}{1-q} q^{-N} + \frac{\alpha - \gamma}{1-q}$$

SEPARATE the SHIFT PARTS and DENOTE the REST by A and A^*

$$\beta D_1 - \delta D_0 = A + \beta - \delta, \qquad \alpha D_0 - \gamma D_1 = A^* + \alpha - \gamma$$

the OPERATORS A and A^*

form a closed linear algebra - the ASKEY-WILSON ALGEBRA

$$[[A, A^*]_q, A]_q = -\rho A^* - \omega A - \eta$$
$$[A^*, [A, A^*]_q]_q = -\rho^* A - \omega A^* - \eta^*$$

with REPRESENTATION-DEPENDENT STRUCTURE CONSTANTS

TRIDIAGONAL ALGEBRA

$$TD \rightarrow AW \rightarrow U_q(\hat{su}(2))$$

$$[A,[A[A,A^*]_q]_{q^{-1}}] = \rho[A,A^*]$$
$$[A^*,[A^*[A^*,A]_q]_{q^{-1}}] = \rho^*[A^*,A]$$

$$\begin{aligned} \rho &= -\beta\delta q^{-1}(q^{1/2}+q^{-1/2})^2, \\ \rho^* &= -\alpha\gamma q^{-1}(q^{1/2}+q^{-1/2})^2 \end{aligned}$$

INFINITE-DIMENSIONAL REPRESENTATION - AW AND TD

with the AW polynomials as basis $(p_0, p_1, \ldots)$

$A^* \equiv \mathcal{D}$ - the second order q-difference operator for the Askey-Wilson polynomials p_n - DIAGONAL MATRIX $\mathrm{diag}(\lambda_0^*, \lambda_1^*, \lambda_2^*, \ldots)$

A - operator mulltiplication by x - TRIDIAGONAL matrix with matrix elements from the three-term recurrence relation for the Askey-Wilson polynomials

$$xp_n = b_n p_{n+1} + a_n p_n + c_n p_{n-1}, \qquad p_{-1} = 0$$

the Askey-Wilson polynomials $p_n = p_n(x; a, b, c, d), n = 0, 1, 2, \ldots$ depend on four parameters a, b, c, d

$$p_n = {}_4\Phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{matrix} \; | q; q \right)$$

with $p_0 = 1$, $x = z + z^{-1}$ and $0 < q < 1$.

Result

: A representation π with basis $(p_0, p_1, p_2, ...)^t$

$\pi(D_1 - \frac{\delta}{\beta} D_0)$ is diagonal with eigenvalues

$$\lambda_n = \frac{1}{1-q}\left(bq^{-n} + dq^{n-1}\right) + \frac{1}{1-q}(1+bd)$$

and $\pi(D_0 - \frac{\gamma}{\alpha} D_1)$ is tridiagonal

$$\pi(D_0 - \frac{\gamma}{\alpha} D_1) = \frac{1}{1-q} b\mathcal{A}^t + \frac{1}{1-q}(1+ac)$$

The dual representation π^* has a basis $p_0, p_1, p_2, ...$

with $\pi^*(D_0 - \frac{\gamma}{\alpha} D_1)$ diagonal

and $\pi^*(D_1 - \frac{\delta}{\beta} D_0)$ tridiagonal

The choice
$\langle w| = h_0^{-1/2}(p_0, 0, 0, \ldots)$, $|v\rangle = h_0^{-1/2}(p_0, 0, 0, \ldots)^t$
(h_0 is a normalization)

as eigenvectors of the diagonal matrices

$\pi(D_1 - \frac{\delta}{\beta} D_0)$ and $\pi^*(D_0 - \frac{\gamma}{\alpha} D_1)$

yields a solution to the boundary equations which uniquely relate a, b, c, d to $\alpha, \beta, \gamma, \delta$.

Namely
$a = \kappa_+^*(\alpha, \gamma), \quad b = \kappa_+(\beta, \delta), \quad c = \kappa_-^*(\alpha, \gamma), \quad d = \kappa_-(\beta, \delta)$

where $\kappa_\pm^{(*)}(\nu, \tau)$ ($\equiv \kappa_\pm^{(*)}$) is

$$\kappa_\pm^{(*)} = \frac{-(\nu - \tau - (1-q)) \pm \sqrt{(\nu - \tau - (1-q))^2 + 4\nu\tau}}{2\nu}$$

EACH BOUNDARY OPERATOR

and the

TRANSFER MATRIX $D_0 + D_1$

form an ISOMORPHIC AW algebra

HENCE $(D_0 + D_1)p_n = (2 + x)p_n$

BOUNDARY ASKEY-WILSON ALGEBRA OF THE OPEN ASEP

ALLOWS FOR THE EXACT SOLVABILITY

in the STATIONARY STATE.

- normalization factor

$$Z = \langle w|(D_0 + D_1)^L|v\rangle$$

- mean density $\langle s_i \rangle$ at a site i

$$\langle s_i \rangle = \frac{\langle w|(D_0 + D_1)^{i-1} D_1 (D_0 + D_1)^{L-i}|v\rangle}{Z_L}$$

- current J through a bond between site i and site $i+1$,

$$J = \langle s_i(1 - s_{i+1}) - q(1 - s_i)s_{i+1} \rangle$$

$$J = \zeta \frac{Z_{L-1}}{Z_L}$$

- two-point correlation function $\langle s_i s_j \rangle$

$$\frac{\langle w|(D_0 + D_1)^{i-1} D_1 (D_0 + D_1)^{j-i-1} D_1 (D_0 + D_1)^{L-j}|v\rangle}{Z_L}$$

- higher correlation functions

SPECTRAL PROBLEM EXACT SOLUTION

the transition rate matrix Γ of the ASEP is identified with the tridiagonal representation of the left (right) boundary operator.

It can be diagonalized

in the auxiliary space

of the finite-dimensional representation

of the tridiagonal algebra

generated by the ASEP boundary operators

THE

ASKEY-WILSON POLYNOMIALS

ARE INTIMATELY RELATED TO

THE OPEN ASEP AND PROVIDE

A SCHEME FOR UNIFIED DESCRIPTION

ALLOWING FOR

EXACT SOLVABILITY IN

THE STATIONARY STATE AND

EXACT SPECTRUM of TRANSITION MATRIX.