UNIVERSITÉ DE GENÈVE

Quantum equations of motion in BF theory with sources

N. Ilieva

A. Alekseev

FNSNF

Introduction

Two-dimensional BF theory
- gauge theory
- (topological) Poisson σ-model

* star-product
* Kontsevich approach
* Torossian connection (correl. of B-exponentials)

$$\mathbb{A} = (\mathcal{A}^{reg}(z_1), \ldots, \mathcal{A}^{reg}(z_n))$$

connection on the space of configs. of points $(z_1, \ldots, z_n)$

* VP of gauge theory
* Sources at $(z_1, \ldots, z_n)$
* Expectations of quantum $\mathcal{A}$ and $\mathcal{B}$ fields; quantum eqs.
* regularization at $(z_1, \ldots, z_n)$

❖ A. Cattaneo, J. Felder, Commun. Math. Phys. **212** (2000) 591–611
❖ M. Kontsevich, Lett. Math. Phys. **66** (2003) 157–216
❖ C. Torossian, J. Lie Theory **12** (2001) 597-616

Classical action and equations of motion

$$S_{BF} = \mathrm{tr}\int BF, \qquad F = dA + \frac{1}{2}[A,A]$$

G	connected Lie group; $\mathcal{G}$
$tr(ab)$	inv scalar product on $\mathcal{G}$
A	gauge field on the G-bundle P over $\mathcal{M}_n$
B	$\mathcal{G}$ -valued n-2 form

$$dB + [A,B] = D_A B = 0\,,$$

$$dA + \frac{1}{2}[A,A] = F = 0\,.$$

$$A^g = g^{-1}dg + g^{-1}Ag$$

$$B^g = g^{-1}Bg$$

- E. Witten, *CMP* **117** (1988) 353; **118** (1988) 411; **121** (1989) 351
- *A.S. Schwarz, Lett. Math. Phys.* **2** (1978) 247–252
- D. Birmingham et al., *Phys. Rep.* **209** (1991) 129-340

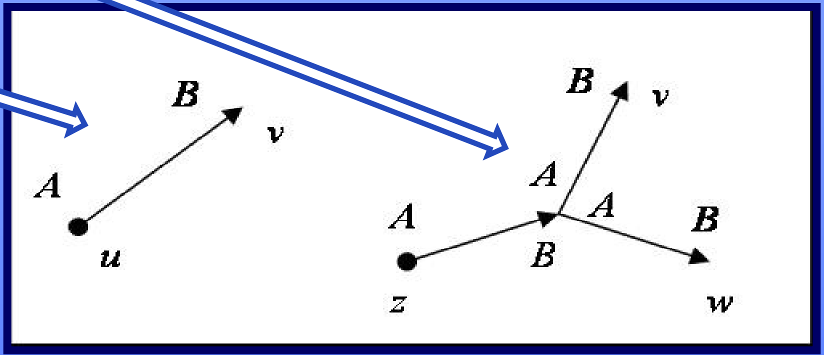
Classical action and equations of motion

$$S_{BF} = \mathrm{tr}\int\left(BdA + \frac{1}{2}B[A,A]\right)$$

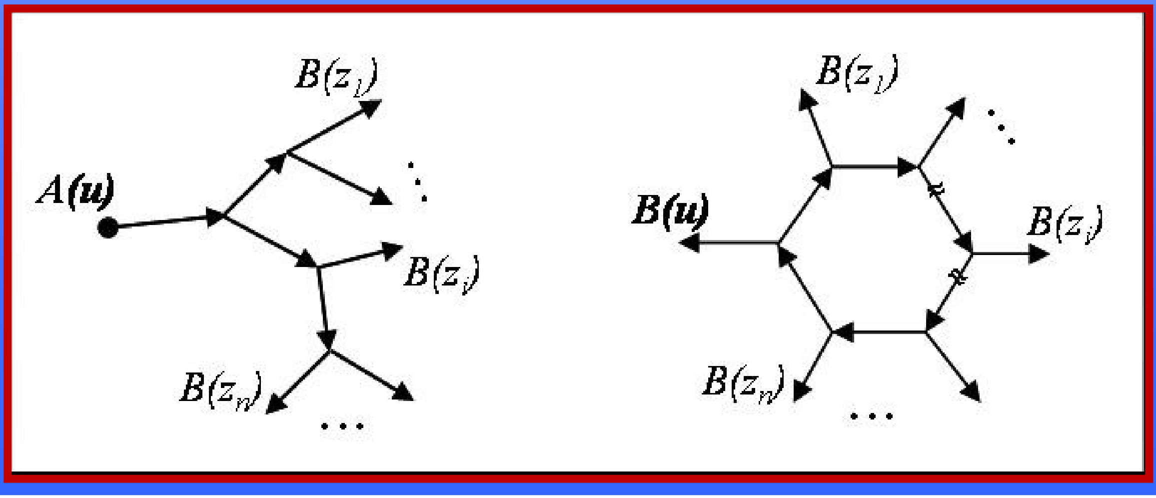
Propagator $\leftrightarrow \mathcal{M}$; gauge choice

Triple vertex $\leftrightarrow f_{abc}$

$$\langle A_a(u)B_b(v)\rangle = \frac{\delta_{ab}}{2\pi}\, d\arg(u-v)$$



Connected Feynman diagrams:



BF theory with sources

$$S_\eta = \mathrm{tr}\left(\int_{\mathcal{M}} BF + \sum_{i=1}^{n} \eta_i B(z_i)\right)$$

$$K_\eta(z_1,\ldots,z_n) = \int e^{S_\eta}$$

Correlation function / theory without sources

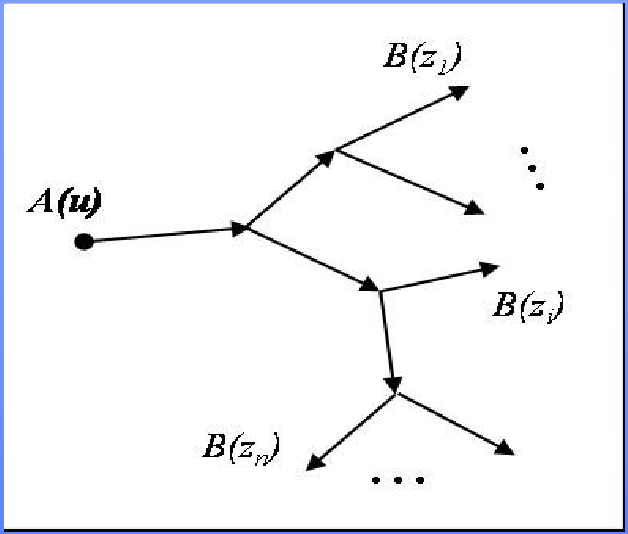
Partition function / theory with sources

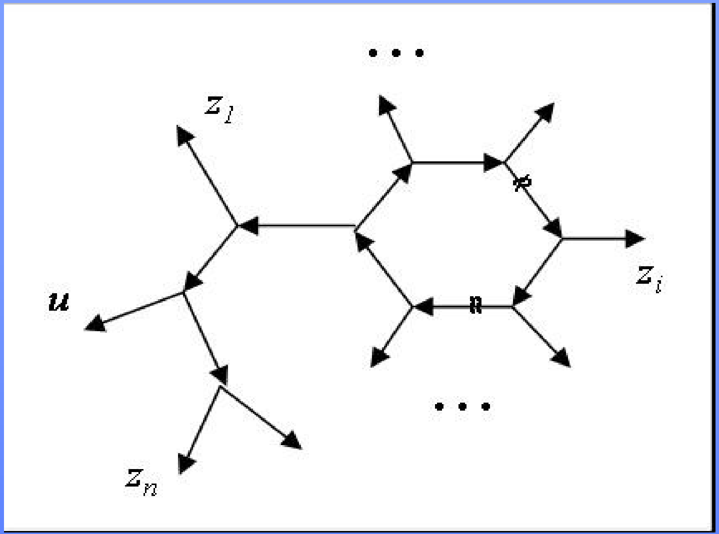
$$K_\eta(z_1,\ldots,z_n) = \left\langle e^{\mathrm{tr}\,(\eta_i B(z_i))} \ldots e^{\mathrm{tr}\,(\eta_i B(z_i))} \right\rangle$$

$$\langle \mathcal{O} \rangle_\eta = \frac{\left\langle \mathcal{O}\, e^{\sum_{i=1}^{n} \mathrm{tr}\,(\eta_i B(z_i))} \right\rangle}{\left\langle e^{\sum_{i=1}^{n} \mathrm{tr}\,(\eta_i B(z_i))} \right\rangle}$$

- $\mathcal{O}$: $A(u)$, $B(u)$
- not gauge-inv observables
- source term breaks the gauge inv of the action

Feynman diagrams





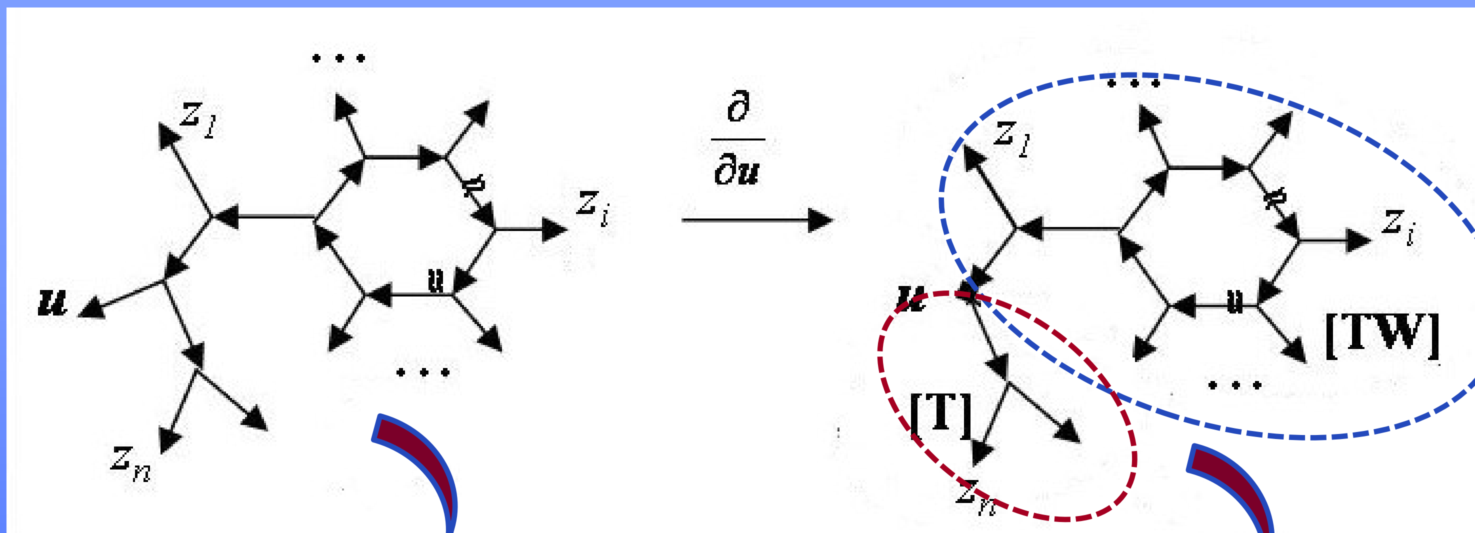
$$\mathcal{A}(u) = \langle A(u) \rangle_\eta = \sum_{all\ trees}$$

$$\mathcal{B}(u) = \langle B(u) \rangle_\eta = \sum_{all\ [\mathrm{TW}]\ compositions}$$

Short trees
[T(l=1)]

$$\mathcal{A}(u) = \sum_{i=1}^{n} \eta_i \, d \arg(u - z_i) + a(u; z_1, \ldots, z_n)$$

Quantum equations of motion



$\mathcal{B}$

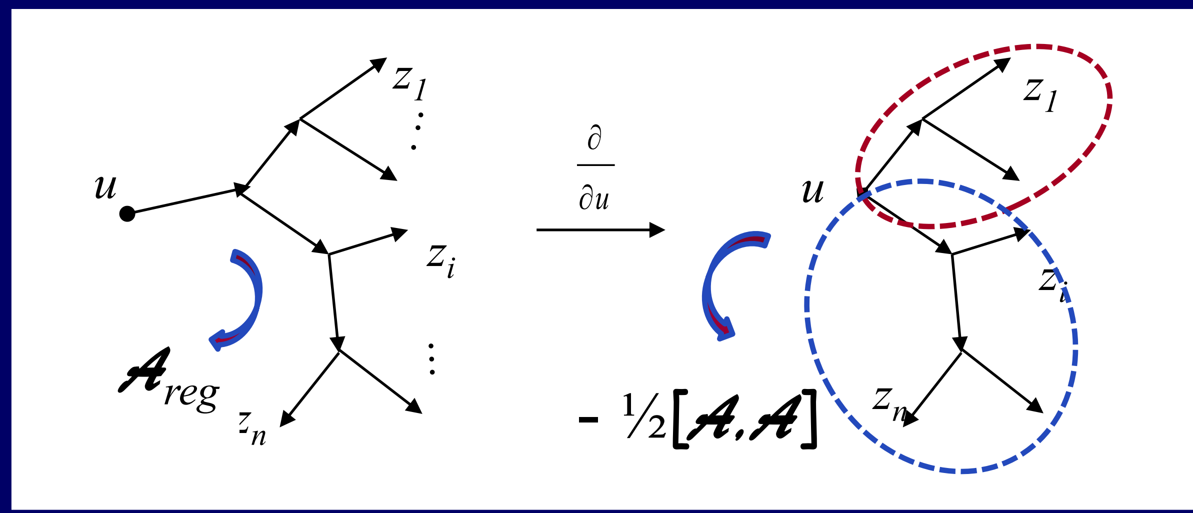
$-[\mathcal{A},\mathcal{B}]$

$$d\mathcal{B} = -[\mathcal{A},\mathcal{B}]$$

Coincides with the classical eqn.

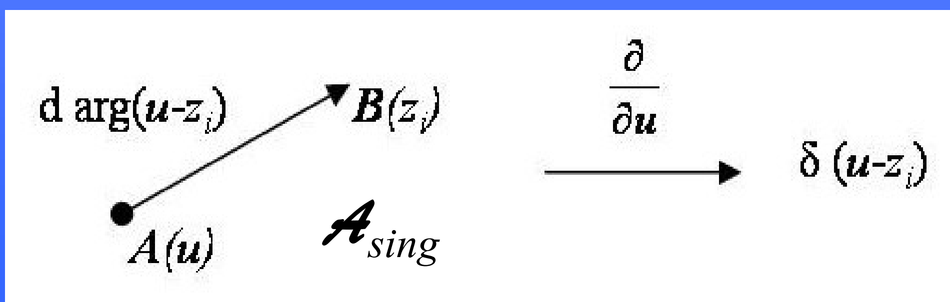
$$dB + [A,B] = D_A B = 0$$

Quantum equations of motion



$$dA + \frac{1}{2}[A, A] = 0$$

$$\mathcal{A}(u) = \sum_{i=1}^{n} \eta_i \, d\arg(u - z_i) + a(u; z_1, \ldots, z_n)$$



$$d\mathcal{A} = -\frac{1}{2}[\mathcal{A}, \mathcal{A}] + \sum_{i=1}^{n} \eta_i \, \delta(u - z_i)$$

Quantum flat connection

Dependence of the $\mathcal{B}$-field correlators $K_\eta\,(z_1, \ldots, z_n)$ on $(z_1, \ldots, z_n)$

$$\mathcal{A}(u) = \sum_{i=1}^{n} \eta_i \, d \arg(u - z_i) + a(u; z_1, \ldots, z_n)$$

Singularity → regularization by a new splitting

$$\mathcal{A}_{(i)}(u) = \frac{\eta_i}{2\pi} \, d \arg(u - z_i) + \mathcal{A}^{reg}_{(i)}(u)$$

$$\begin{aligned} \mathcal{A}^{reg}_{(i)}(u) &= \sum_{j \neq i} [T(l=1); \{u, z_j\}] + \sum_{all\ trees,\ l>1} [T] \\ &= \sum_{j \neq i} \frac{\eta_j}{2\pi} \, d \arg(u - z_j) + a(u; z_1, \ldots, z_n) \end{aligned}$$

no singularity at $u = z_i$ ⇒

$$a_i := \mathcal{A}^{reg}_{(i)}(u; z_1, \ldots, z_i, \ldots, z_n)\Big|_{u=z_i}$$

Quantum flat connection

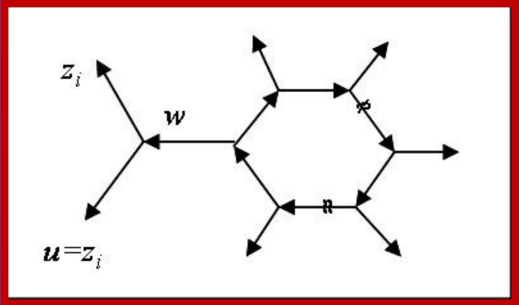
Quantum equation of motion for $\mathcal{B}$ field ⇒

$$\begin{aligned} d\,\mathrm{tr}\,(\eta\mathcal{B}(u)) &= -\mathrm{tr}\,(\eta[\mathcal{A}(u),\mathcal{B}(u)]) = -\mathrm{tr}\,([\eta,\mathcal{A}(u)]\mathcal{B}(u)) \\ &= -\mathrm{tr}\,[\eta,\mathcal{A}(u)]\,\frac{\partial}{\partial\eta}\,\mathrm{tr}\,(\eta\mathcal{B}(u))\,. \end{aligned}$$

Naïve expectation for $K_\eta\,(z_1,\ldots,z_n)$ equation (d_z being the de Rham diff.)

$$d_{z_i}K_\eta(z_1,\ldots,z_n) + \mathrm{tr}\,[\eta_i,\mathcal{A}(z_i)]\,\frac{\partial}{\partial\eta_i}\,K_\eta(z_1,\ldots,z_n) = 0$$

Contributing Feynman graphs



$$(d\arg(w-z_i))^2$$

$$d_{z_i}K_\eta + \mathrm{tr}\,[\eta_i,a_i]\,\frac{\partial}{\partial\eta_i}\,K_\eta = 0$$

Quantum flat connection

$$dK_\eta + \mathrm{tr}\sum_{i=1}^{n}[\eta_i, a_i]\frac{\partial}{\partial \eta_i}K_\eta = 0$$

d - the total de Rham differential for all variables $z_1, \dots, z_n$

Consider functions $\alpha_i(\eta_1, \dots, \eta_n) \in \mathcal{G}$, $i = 1, \dots, n$

$$D_\alpha = \mathrm{tr}\sum_{i=1}^{n}[\eta_i, \alpha_i]\frac{\partial}{\partial \eta_i}$$

$$\begin{cases} [D_\alpha, D_\beta] = D_{\{\alpha,\beta\}} & \textit{Lie algebra} \\ \{\alpha, \beta\}_i = D_\alpha \beta_i - D_\beta \alpha_i + [\alpha_i, \beta_i] \end{cases}$$

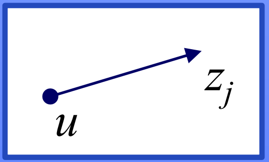
1-forms $(a_1, \dots, a_n)$ as components of a connection $\mathbb{A}(a_1, \dots, a_n)$, with values in this *LA*

$$dK_\eta + D_{\mathbb{A}}K_\eta = 0$$

Similarly for the differential of gauge field $\mathcal{A}$

$$d_{z_i}\mathcal{A}(u) = -\mathrm{tr}\,[\eta_i, a_i]\,\frac{\partial}{\partial \eta_i}\mathcal{A}(u) = -D_{a_i}\mathcal{A}(u)$$

$j \neq i: \quad \mathcal{A}(u) \;\rightarrow\; \mathcal{A}_{(j)}^{reg}$



does not contribute

$u = z_j \quad \Longrightarrow \quad d_{z_i} a_j = -D_{a_i} a_j$

$$\mathbb{F} = d\mathbb{A} + \frac{1}{2}\{\mathbb{A}, \mathbb{A}\}$$

- ∗ holomorphic components
- ∗ anti-holomorphic components
- ∗ mixed components

($\mathbb{F}_{ij}$ corresponds to $\{z_i, z_j\}$)

Quantum flat connection

The curvatute $\mathbb{F}$ of $\mathbb{A}$ vanishes.

$$(\mathbb{F}_{ij})_k\,,\; k = 1,\dots,n$$

(i) Components with $k \neq i, j$ vanish identically

(ii) $(\mathbb{F}_{ij})_i = d_{z_j} a_i + D_{a_j} a_i = 0\,, \qquad (\mathbb{F}_{ij})_j = d_{z_i} a_j + D_{a_i} a_j = 0.$

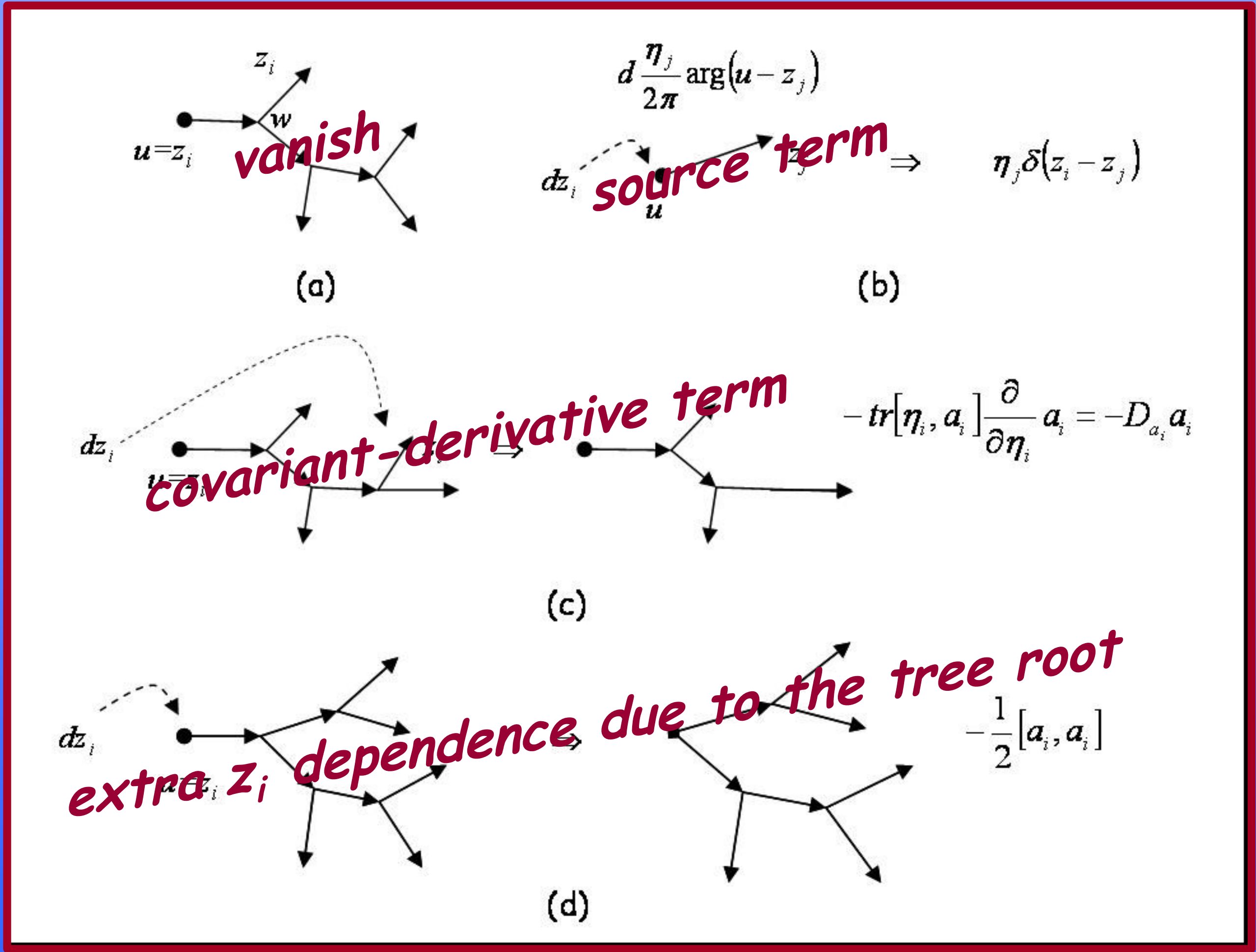
(iii) The only non-vanishing component is $(\mathbb{F}_{ii})_i$

$$(\mathbb{F}_{ii})_i = d_{z_i} a_i + D_{a_i} a_i + \frac{1}{2}[a_i, a_i]$$

with $a_i = \alpha_i dz_i + \bar{\alpha}_i d\bar{z}_i$

$$(\mathbb{F}_{ii})_i = \partial_{z_i}\bar{\alpha}_i - \bar{\partial}_{z_i}\alpha_i + D_{\alpha_i}\bar{\alpha}_i - D_{\bar{\alpha}_i}\alpha_i + [\alpha_i, \bar{\alpha}_i]$$

Quantum flat connection



Quantum flat connection

$$d_{z_i} a_i(z_i) + D_{a_i} a_i + \frac{1}{2}[a_i, a_i] = \sum_{j \neq i} \eta_j \, \delta(z_i - z_j)$$

$$d\mathbb{A} + \frac{1}{2}\{\mathbb{A}, \mathbb{A}\} = 0 \qquad \mathbb{F}_i = \sum_{j \neq i} \eta_j \delta(z_i - z_j)$$

Outlook

Torossian connection

Knizhnik-Zamolodchikov connection / WZW

$$d\Psi + \mathbb{A}_{KZ}\Psi = 0$$

$$\mathbb{A}_{KZ} = \frac{1}{2\pi i}\sum_{i,j} t_{i,j}\, d\ln(z_i - z_j),$$

$$t_{i,j} = \sum_{\alpha} e^{i}_{\alpha} \otimes e^{j}_{\alpha}$$ irreps of $\mathcal{G}$

play the role of T(l=1)

propagator $d \ln (z_i\text{-}z_j)/2\pi i$

- KZ connection as eqn on the wave funtion of the CS TFT with *n* time-like Wilson lines (corr. primary fields)
- holonomy matrices of the flat connection $\mathbb{A}_{KZ}$ → braiding of Wilson lines
- 3D TFT with TC as a wave function eqn (?)
- non-local observables → insertions of exp(tr $\eta B(z)$) in 2D theory

Thank you!

Feynman diagrams