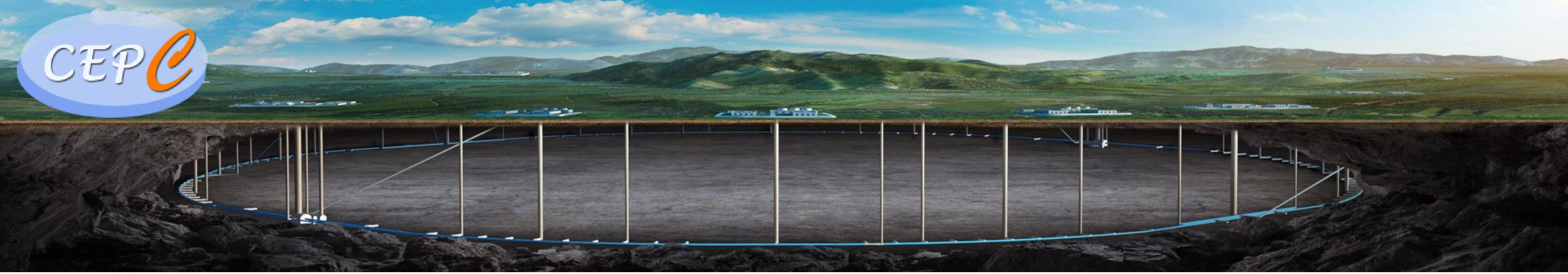

Polarization preservation issues at the CEPC

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E. Forest & D. Sagan on help with Bmad/PTC code.

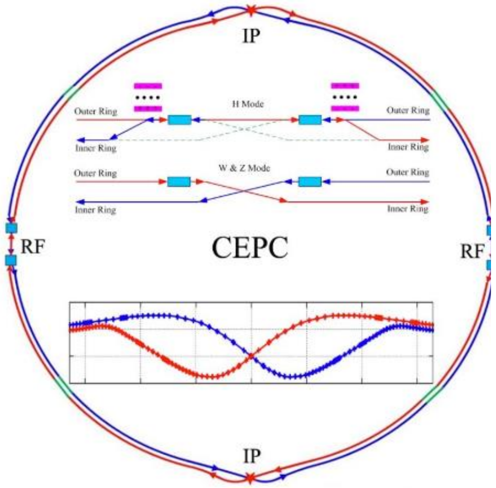
Content

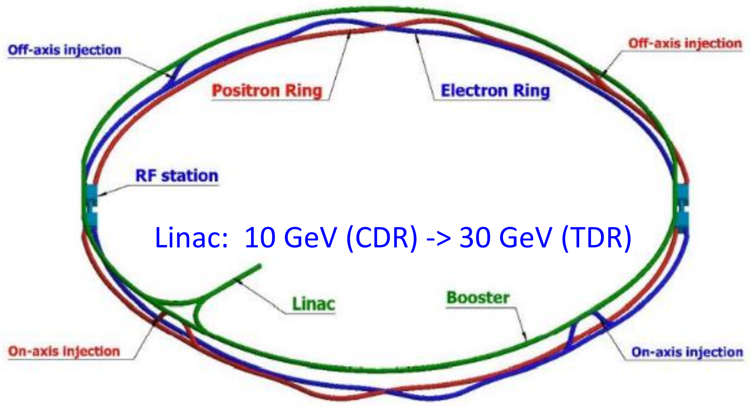
- Introduction
- Spin resonance structure
- Polarization transmission in the CEPC Booster
- Radiative depolarization at ultra-high energies
- Summary

References:
[1] T. Chen, Z. Duan, D. H. Ji, D. Wang, Phys. Rev. Accel. Beams, 26, 051003 (2023).
[2] W. H. Xia, Z. Duan, D. P. Barber, Y. W. Wang, B. Wang, J. Gao, accepted by Phys. Rev. Accel. Beams.

The Circular Electron Positron Collider (CEPC)

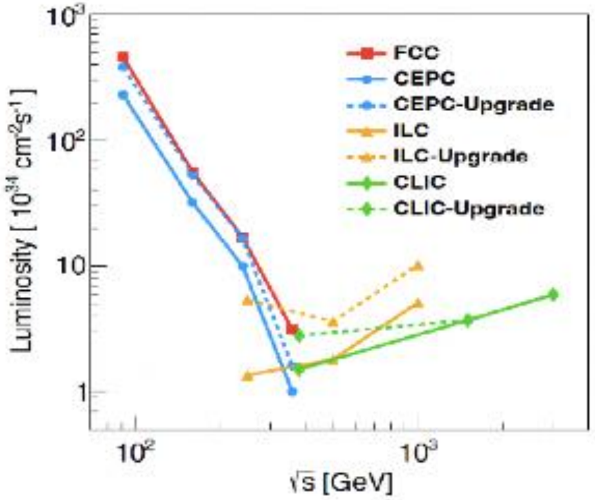
- Basic design
  - As a Higgs(120 GeV), Z (45.6GeV) & W(80GeV) Factory
  - Upgradable to High Lumi Z & ttbar(175 GeV)
  - Compatible with SppC
- Progress
  - CDR released in 2018
  - TDR to be delivered in 2023
  - Beam polarization as a chapter in Appendix
    - Transverse polarization for resonant depolarization at Z & W
    - Longitudinally polarized colliding beams at Z-pole (and beyond)





TDR, High luminosity (30MW)

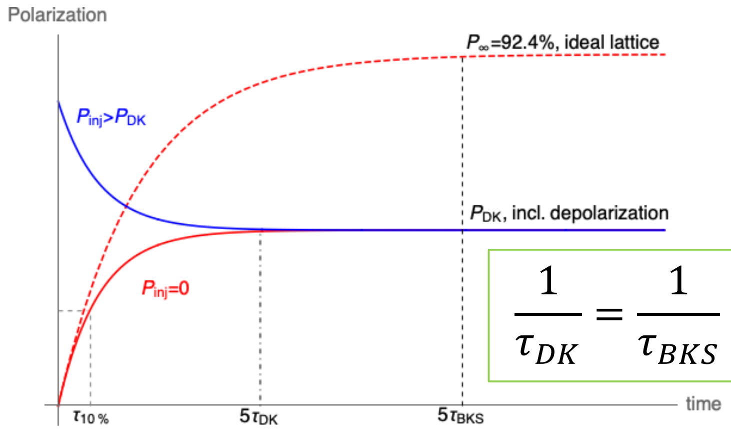
	Higgs	W	Z	ttbar
Number of IPs	2			
Circumference [km]	100.0			
SR power per beam [MW]	30	30	30	30
Energy [GeV]	120	80	45.5	180
Bunch number	**249**	**1297**	**11951**	**35**
Beam current [mA]	16.7	84.1	803.5	3.3
Beta functions at IP (βx/βy) [m/mm]	**0.33/1**	**0.21/1**	**0.13/0.9**	**1.04/2.7**
Emittance (εx/εy) [nm/pm]	**0.64/1.3**	**0.87/1.7**	**0.27/1.4**	**1.4/4.7**
Beam size at IP (σx/σy) [um/nm]	15/36	13/42	6/35	39/113
Bunch length (SR/total) [mm]	2.3/3.9	2.5/4.9	2.5/8.7	2.2/2.9
Energy spread (SR/total) [%]	0.10/0.17	0.07/0.14	0.04/0.13	0.15/0.20
Energy acceptance (DA/RF) [%]	1.7/2.2	1.2/2.5	1.3/1.7	2.3/2.6
Beam-beam parameters (ξx/ξy)	0.015/0.11	0.012/0.113	0.004/0.127	0.071/0.1
RF voltage [GV]	2.2 (2cell)	0.7 (2cell)	0.12 (1cell)	10 (5cell)
RF frequency [MHz]	650			
Beam lifetime [min]	20	55	80	18
Luminosity per IP[10^{34}/cm²/s]	**5.0**	**16**	**115**	**0.5**



[1] Slides of Beam Polarization Studies presented on CEPC Accelerator TDR Review Meeting 14/06/2023, Hong Kong
https://indico.ihep.ac.cn/event/19262/contributions/135019/attachments/69261/83123/CEPC_polarization_study_v5_uploaded.pptx

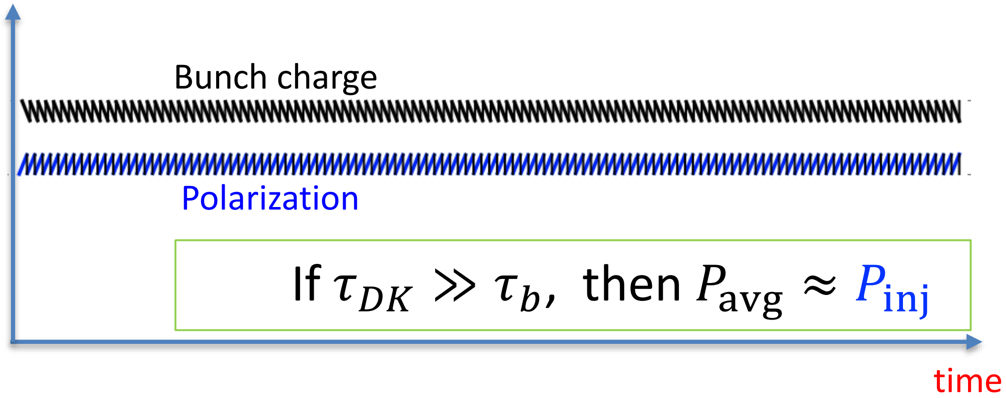
Beam polarization in the collider rings

- Non-colliding "pilot" bunches: decay mode



$$\frac{1}{\tau_{DK}} = \frac{1}{\tau_{BKS}} + \frac{1}{\tau_{\mathrm{dep}}}$$

- Colliding bunches: top-up injection



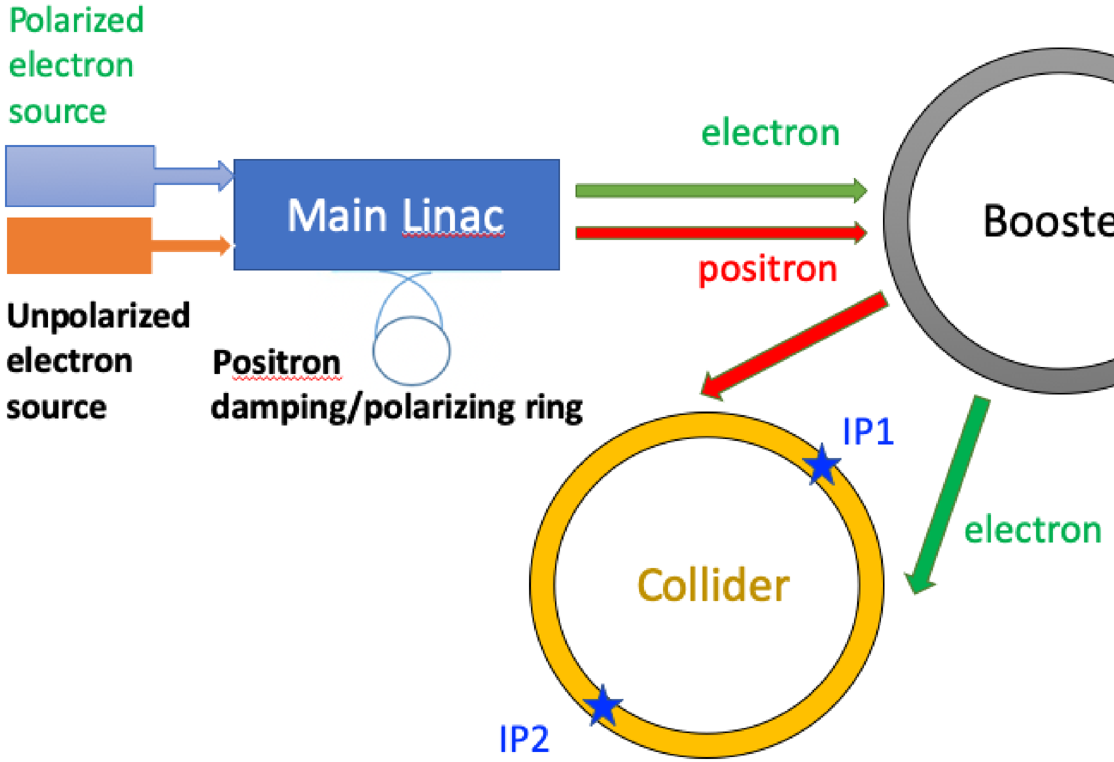
$$\text{If } \tau_{DK} \gg \tau_b, \text{ then } P_{\mathrm{avg}} \approx P_{\mathrm{inj}}$$

- Self-polarization can be utilized for Resonant Depolarization (RD) measurements using pilot bunches
  - Employ asymmetric wigglers to reduce the polarization time @Z [1]
- To achieve a high-level polarization for colliding bunches without significantly sacrificing luminosity [2]
  - Injection of polarized beams is mandatory

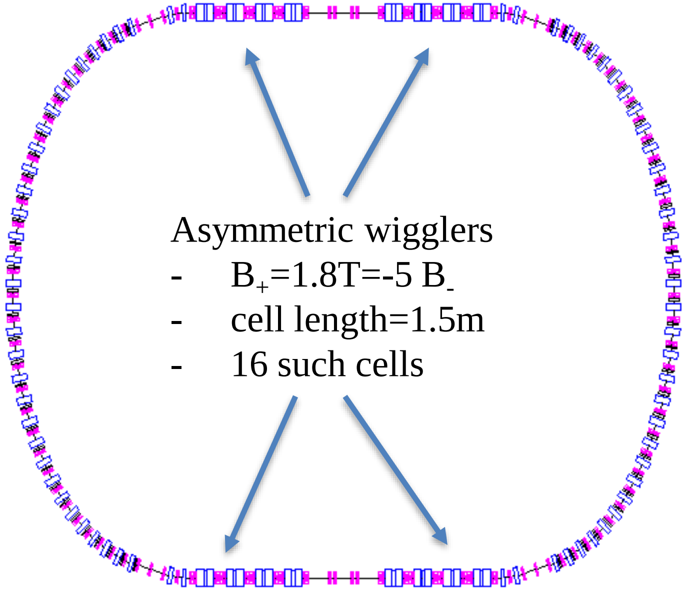
CEPC CDR parameters	45.6 GeV (Z, 2T)	80 GeV (W)	120 GeV (Higgs)
Polarization build-up time w/o radiative depolarization τ_{BKS} (hour)	253	15.2	2.0
Beam lifetime τ_b (hour)	2.5	1.4	0.43

[1] The FCC-ee Energy and Polarization Working Group, arXiv:1909.12245v1, 2019.
[2] Zhe Duan, talk on 2nd FCC EPOL Workshop, Sep 29, 2022

Modification of CEPC for RD measurements



Positron damping ring
1.542 GeV

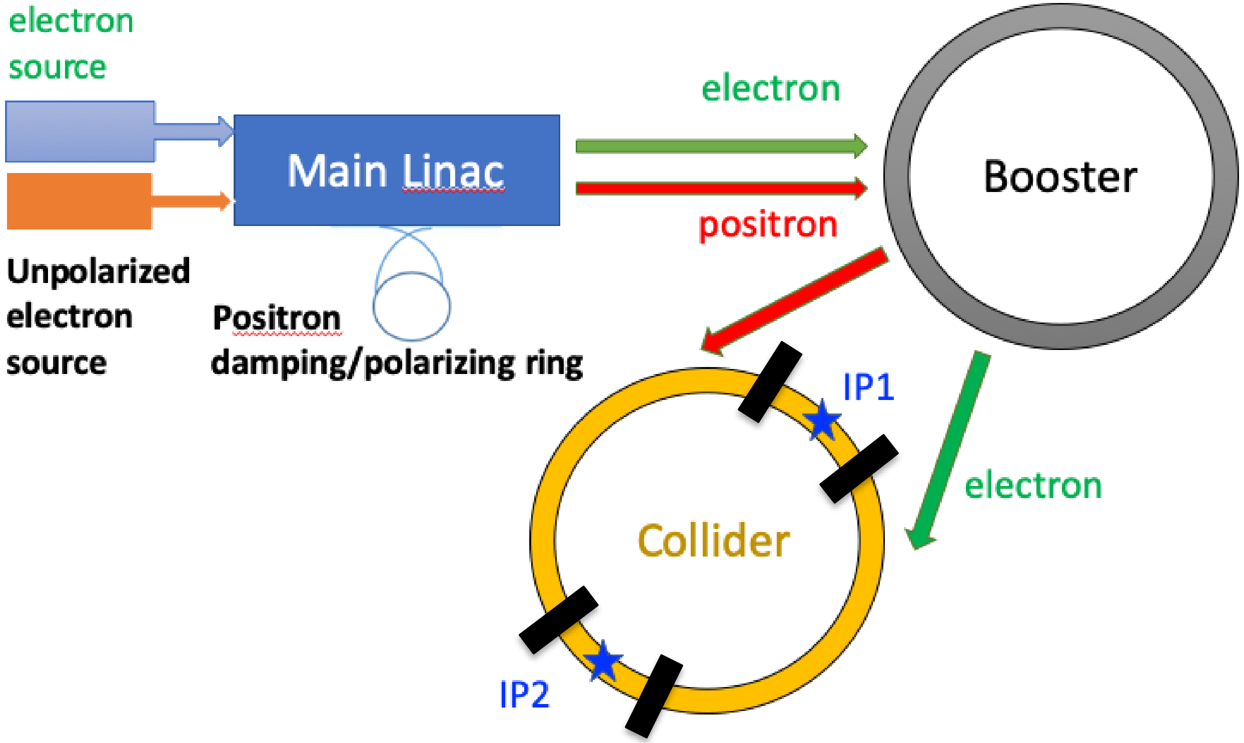


- Self-polarization of >20% achievable in 10 min [1]
- Consistent with the requirements of RD measurements, but not sufficient for top-up injection of colliding bunches

- Polarized e- source can supply ~ 85% polarized e- bunches that satisfy the needs of CEPC (SLC/ILC/EIC)

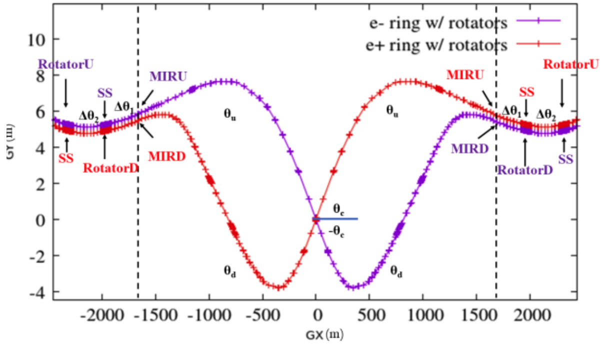
[1] Z Duan et al., Proc. eeFACT 2022 & IPAC 2023.

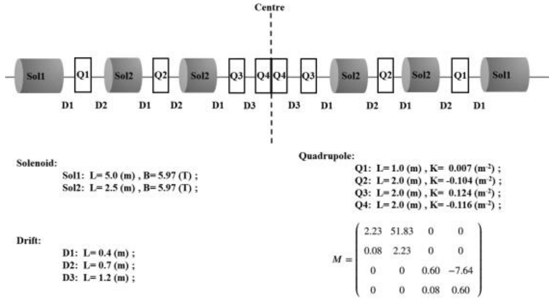
Modification of CEPC for longitudinal polarization

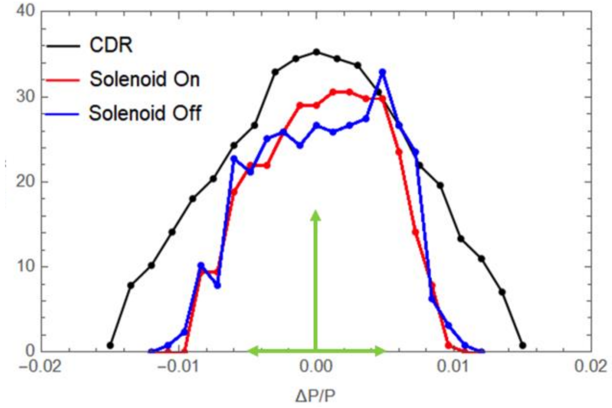


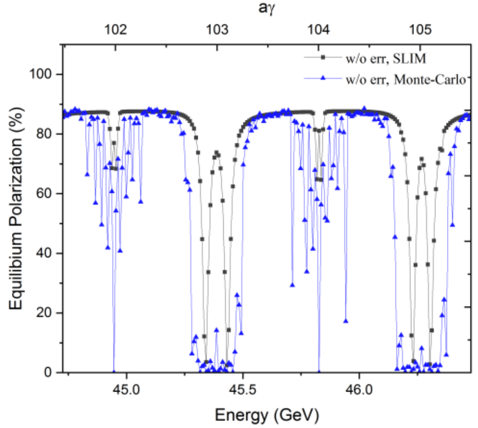
Spin rotators at Z-pole

W. Xia et al., RDTM (2022) doi: 10.1007/s41605-022-00344-2









- It is important to understand the depolarization effects at ultra-high beam energies.
  - Booster: depolarization due to spin resonance crossings during acceleration
  - Collider ring: radiative depolarization

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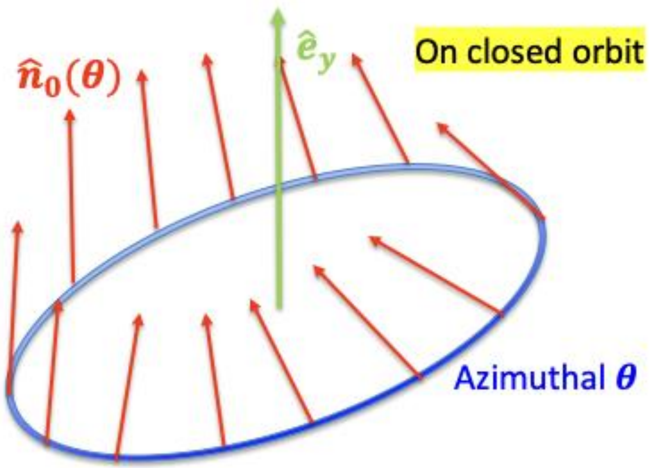
Spin-orbit coupling resonances in circular accelerators

- In a planar ring without solenoids, $\hat{n}_0$ could deviate from vertical near integer (imperfection) resonances $\nu_0 = K$, characterized by

$$\tilde{\epsilon}_K^{\mathrm{imp}} \approx -\frac{R(1+K)}{2\pi}\int_0^{2\pi}\frac{\Delta B_x(\theta')}{B\rho}e^{iK\Phi(\theta')}d\theta'$$

$$\frac{\Delta B_x}{B\rho} \approx \frac{\frac{\partial B_x}{\partial y}}{B\rho}y_{\mathrm{co}} + \boxed{(\frac{\Delta B_x}{B\rho})_0} \longrightarrow$$

- Misalignments: quad ΔY & dipole roll
  - Random: zero mean
  - Systematic: "smoothed" vertical positioning, uneven settling, etc
- Orbital correctors



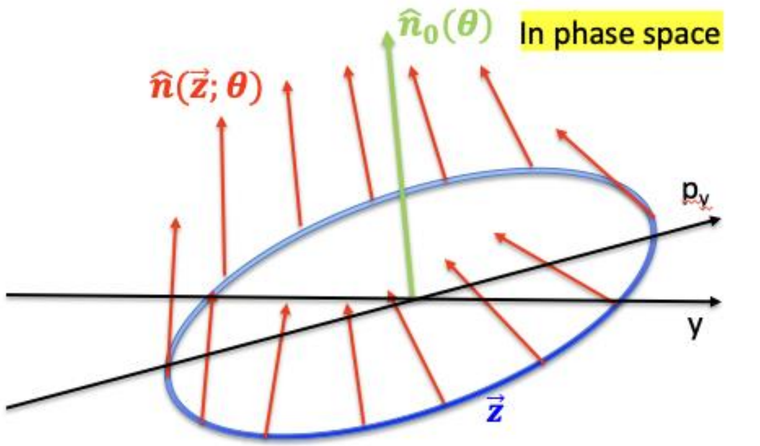
- $\hat{n}$ deviates from $\hat{n}_0$ near spin resonances

$$\nu_s = k + k_x\nu_x + k_y\nu_y + k_z\nu_z, \quad k, k_x, k_y, k_z \in \mathbb{Z}.$$

First-order parent spin resonances: $|k_x| + |k_y| + |k_z| = 1$

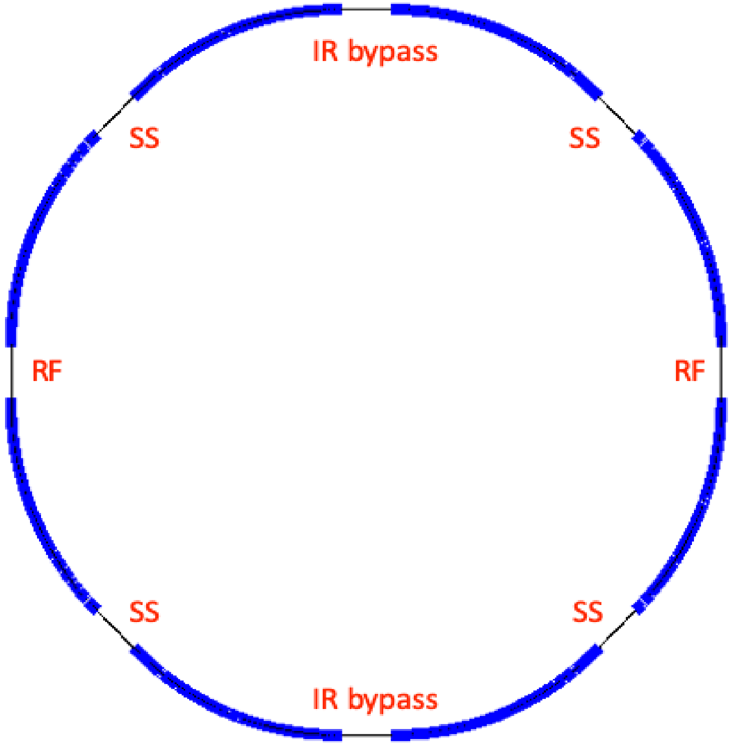
- For intrinsic resonances $\nu_0 = K = k \pm \nu_y$

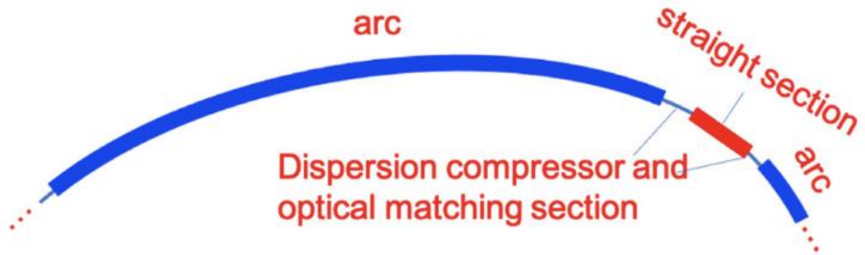
$$\tilde{\epsilon}_K^{\mathrm{intr},\pm}(I_y) \approx \frac{R(1+K)}{4\pi}\int_0^{2\pi}\boxed{\frac{\frac{\partial B_x}{\partial y}}{B\rho}\sqrt{2I_y\beta_y}}e^{i[K\Phi(\theta')\mp\nu_y\phi_y(\theta')\mp\psi_{y0}]}d\theta'$$



General lattice structure of CEPC booster & collider

CEPC booster





- Approximately period-8
- Each arc contains hundreds of standard (FODO) cells
- Arc region covers ~ 80% of circumference in both rings

Table 2: Parameters relevant for spin resonance structure

Parameter	Booster1 [11]	Booster2 [1]	Collider [1]
ν_y	353.28	261.2	365.22
P	8	8	8
M	140	97	145
$\eta_{\rm arc}$	140/142	97/99	145/147
ν_B	280	194	290
PM	1120	776	1160
$\nu_B/\eta_{\rm arc}$	284	198	294

- ν_B is the total ν_y in all standard arc cells
- $\eta_{\rm arc}$ is the fraction of total bending angle from arc standard cells over 2π

Spin resonance structure: Intrinsic resonance strength

$$\tilde{\epsilon}_K^{\mathrm{intr},\pm}(I_y) \approx \frac{R(1+K)}{4\pi}\int_0^{2\pi}\frac{\frac{\partial B_x}{\partial y}}{B\rho}\sqrt{2I_y\beta_y}\,\boxed{e^{i[K\Phi(\theta')\mp\nu_y\phi_y(\theta')\mp\psi_{y0}]}}\,d\theta'$$

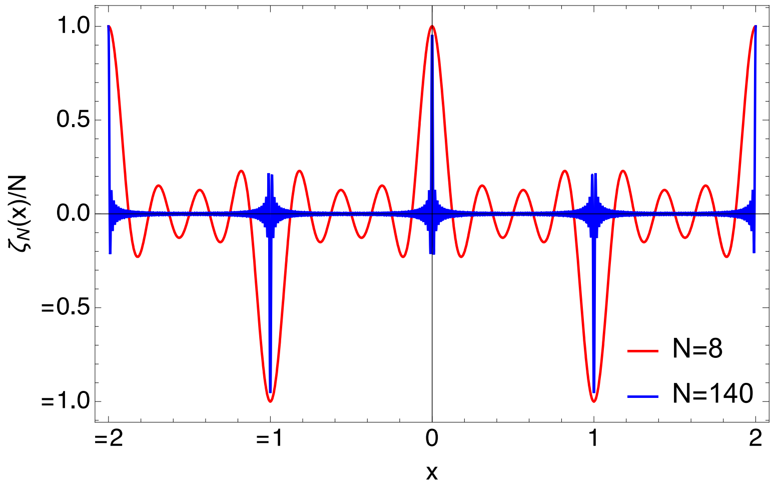
due to P superperiods

$$|\epsilon| \approx |\zeta_P|\,|\epsilon_{\mathrm{arc}} + \epsilon_{\mathrm{ss}} + \epsilon_{\mathrm{DOM}}|$$

due to M identical FODOs in each arc

$$\epsilon_{\mathrm{arc}} = |\epsilon_{FODO}|\,|\zeta_M|$$

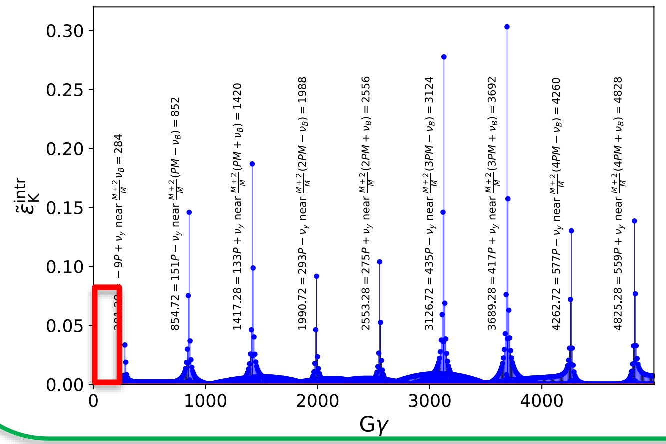
ϵ_{ss} = 0 for integer vertical phase advance & identical cells

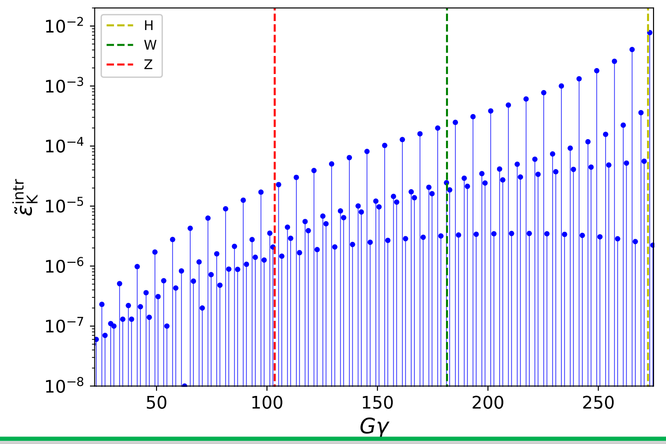


- Near super strong resonances, ϵ_{arc} dominates
- Away from super strong resonances, $\epsilon_{\mathrm{ss}} + \epsilon_{\mathrm{DOM}}$ becomes more important
  - For small K << ν_y the phasor includes a slow wave $K\Phi$ and a fast wave $\nu_y\phi_y$, leading to effective cancellation among all cells (arc, SS and DOM)
- Symmetry breaking leads to relatively weak resonances.

Intrinsic resonances: $\nu_0 = K = k \pm \nu_y$ CEPC booster as an example

Super strong resonances: $K = nP \pm \nu_y, n \in \mathbb{Z}$ closest to $(mPM \pm \nu_B)/\eta_{\mathrm{arc}}, m \in \mathbb{Z}$



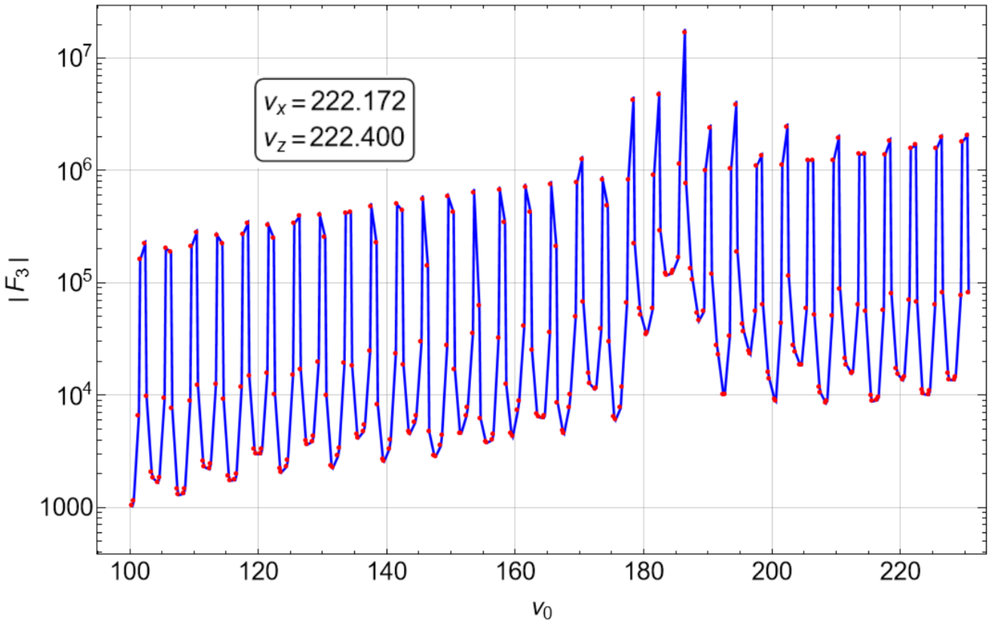


[1] S. Y. Lee, Spin dynamics and snakes in synchrotrons (World Scientific, 1997).
[2] T. Chen, Z. Duan, D. H. Ji, D. Wang, Phys. Rev. Accel. Beams, 26, 051003 (2023).

Taking FCC-ee collider ring as an example

I. Koop, Intrinsic resonances in FCC-ee, EPOL Meeting 15/12/2022.

Maximal F3 module energy dependence



Strong intrinsic spin resonances $\nu_z \pm \nu_0 = 4k$ are spaced with a period $\Delta k = 1$ due to 4-fold symmetry of FCC-ee lattice!

"Giant" resonance bump in a region $\nu_0 = 186 \pm 7$ due to synchronism of the spin rotation in arcs with the vertical kicks there from quads.

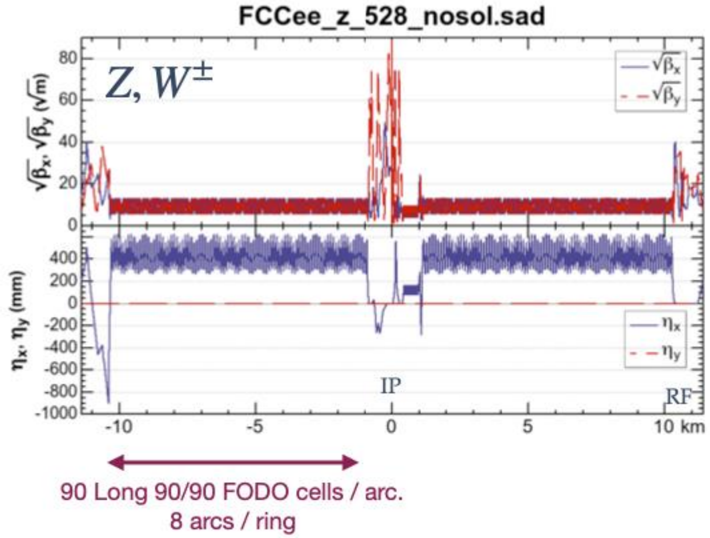
All points at fractional tunes: $\{\nu_0\}$=0.41 and 0.61 – near 0.4, 0.6.

At W the intrinsic resonances are much stronger than at Z : $w_k \approx 1.5 \cdot 10^{-4}$.

Conclusion for W energy region: a gap between the spin tune ν_0 and the vertical betatron tune ν_z needs to be chosen as large as:

$\nu_0 - \nu_z = \pm 0.25$.

K. Oide, 26/06/2023



Super strong intrinsic resonances near $\nu_0 = 8 * 90 * ¼ * 92/90 = 184$

Spin resonance structure: Imperfection resonance strength

$$\tilde{\epsilon}_K^{\rm imp} \approx -\frac{R(1+K)}{2\pi}\sum_{k=-\infty}^{\infty}\boxed{\frac{\nu_y^2 f_k}{\nu_y^2-k^2}}\oint \frac{\frac{\partial B_x}{\partial y}}{B\rho}\beta_y^{1/2} e^{i(k\phi_y+K\Phi)}d\theta.$$

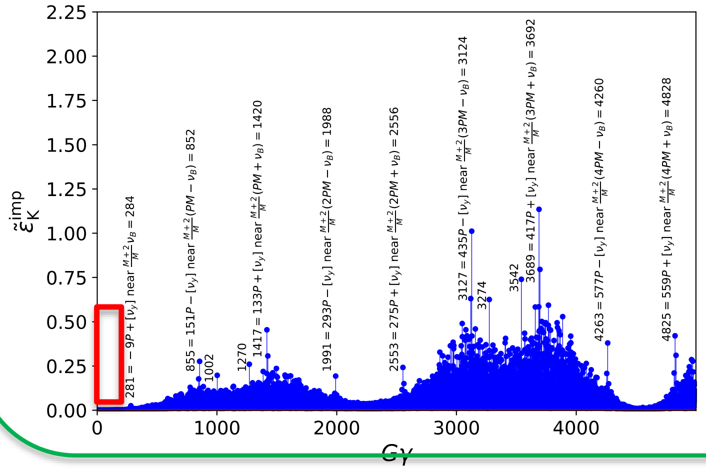
$$f_k = \frac{R}{2\pi\nu_y}\oint \sqrt{\beta_y}\left(\frac{\Delta B_x}{B\rho}\right)_0 e^{-ik\phi_y(\theta)}d\theta$$

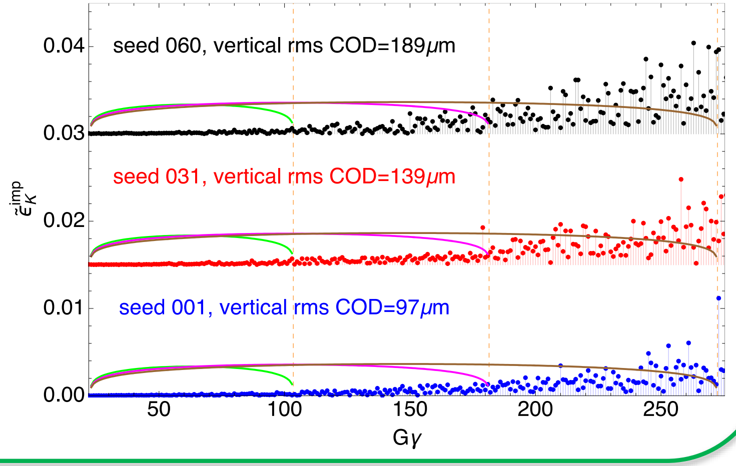
- For a specified *k*, its contribution follows a similar behavior of intrinsic resonance
- Spectrum of f_k depends on the error sources & closed-orbit correction scheme
- Most important terms are near k = [ν_y] leading to super strong resonances
- There tends to a wider plateau around each peak as a result of contributions from different *k*

Imperfection resonances: $\upsilon_0 = K$

CEPC booster as an example

Super strong resonances: $K = nP \pm [\nu_y], n \in \mathbb{Z}$ and $K = [(mPM \pm [\nu_y]\frac{\nu_B}{\nu_y})/\eta_{\rm arc}]$





[1] S. Y. Lee, Spin dynamics and snakes in synchrotrons (World Scientific, 1997).
[2] T. Chen, Z. Duan, D. H. Ji, D. Wang, Phys. Rev. Accel. Beams, 26, 051003 (2023).

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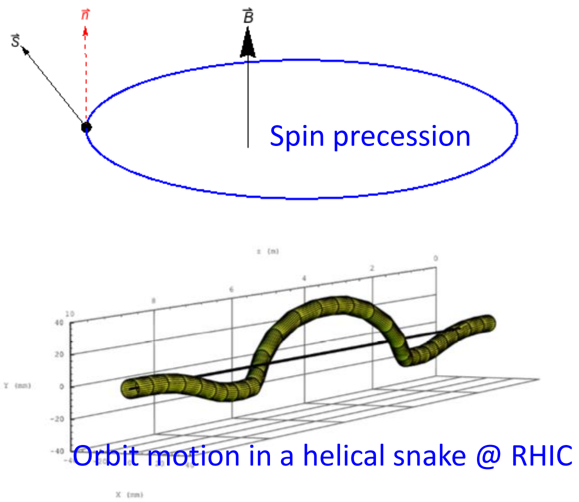
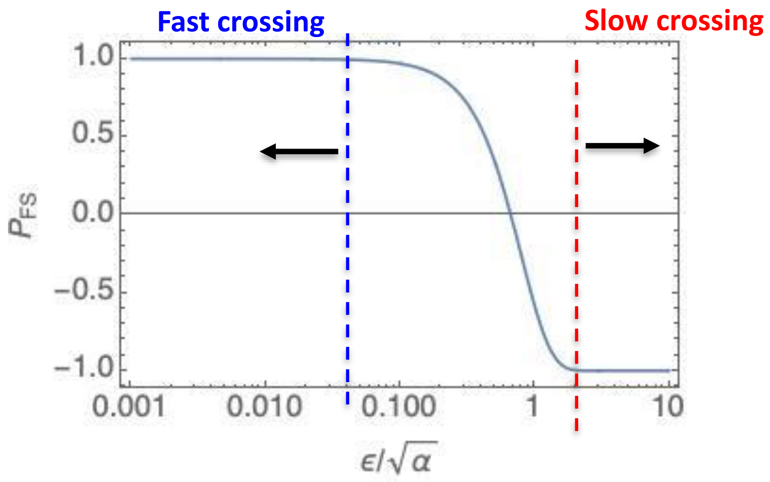
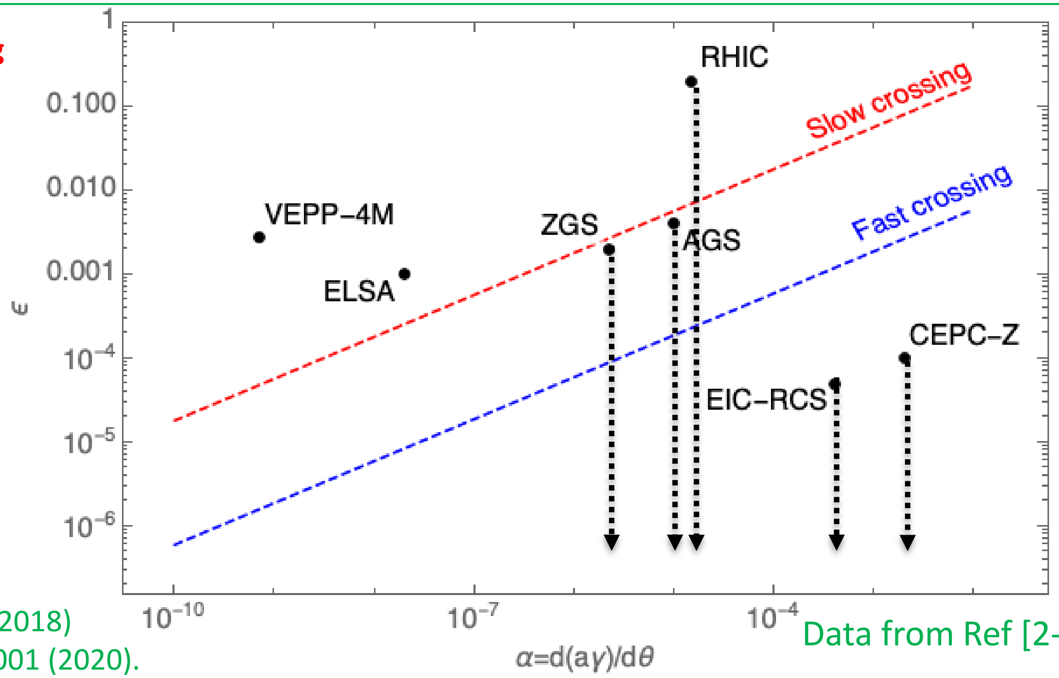
Depolarization in the booster

- The spin tune $\nu_s \approx \nu_0 \approx a\gamma$ changes and could cross spin resonances $\nu_s = k + k_x\nu_x + k_y\nu_y + k_z\nu_z$
  - The spin resonances $\nu_0 = k$ are spaced by 440 MeV for e+/e-
- The non-adiabatic crossing could vary $J_s = \vec{S} \cdot \vec{n}$ and lead to depolarization [1]
  - Spin resonance strength ε
  - Acceleration rate $\alpha \sim 10^{-6} \frac{dE}{dt}[\mathrm{GeV/s}]C[\mathrm{km}]$
  - Δ|P| < 1% in the regimes of fast crossing & slow crossing
- Previous studies suggested using Siberian snakes to maintain polarization for future 100km-scale boosters[7]







Data from Ref [2-6]

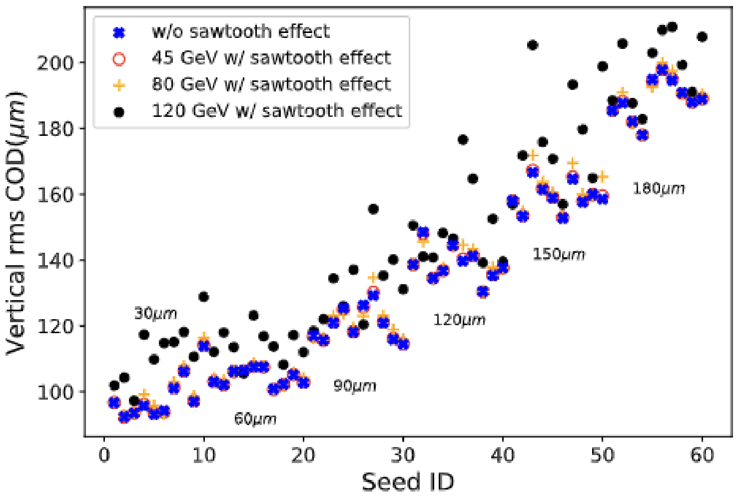
[1] Froissart and Stora, NIM 7, 297 (1960) [2] A. K. Barladyan, et al., PRAB 22, 112804, (2019)
[3] S. Nakamura, et al., NIM A 411, 93 (1998) [4] T. Khoe et al., Part. Accel. 6, 213 (1975)
[5] Configuration Manual: Polarized Proton Collider at RHIC, 2006 [6] V. Ranjbar, et al., PRAB 21, 111003 (2018)
[7] I. Koop et al., Phys. Part. Nucl. Lett. 13. 7 (2016); S. Nikitin, IJMPA 34, 1940004(2019); IJMPA 35, 2041001 (2020).

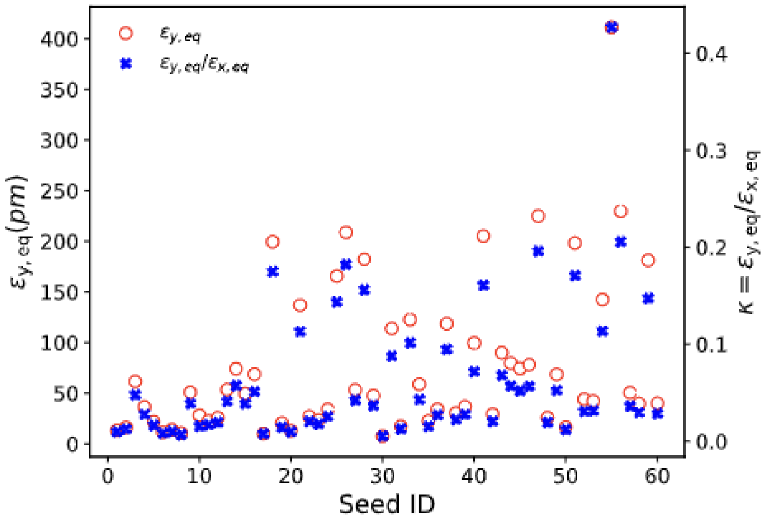
Setup of CEPC booster lattice

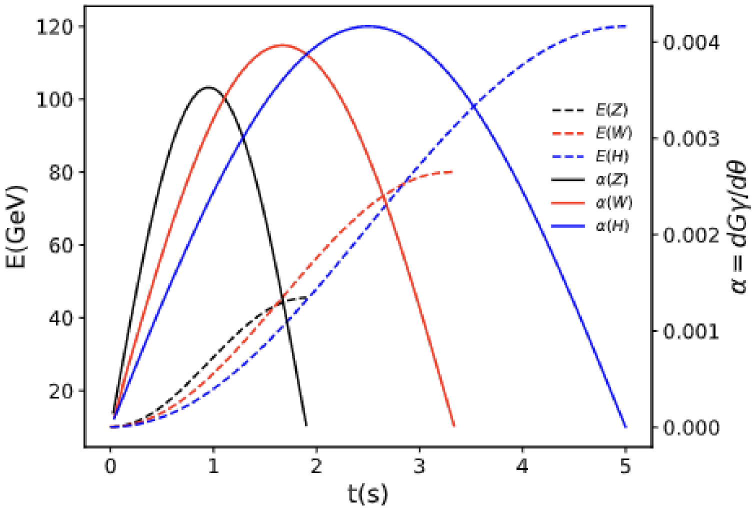
- 60 imperfection lattice seeds
  - Misalignment error & field error, scan BPM offset: 30μm ~ 180 μm
  - Closed orbit correction & tune correction
- Multi-particle tracking in Bmad
  - Energy and RF ramping in the whole process
  - Element-by-element tracking with radiation damping & quantum excitation

TABLE II. Magnet error settings.

	Misalignment error				
Component	Δx (μm)	Δy (μm)	Δz (μm)	$\Delta\theta_z$ (μrad)	Field error
Dipole	100	100	100	100	0.05%
Quadrupole	100	100	100	100	0.02%
Sextupole	100	100	100	100	0.03%







[1] T. Chen, Z. Duan, D. H. Ji, D. Wang, Phys. Rev. Accel. Beams, 26, 051003 (2023).

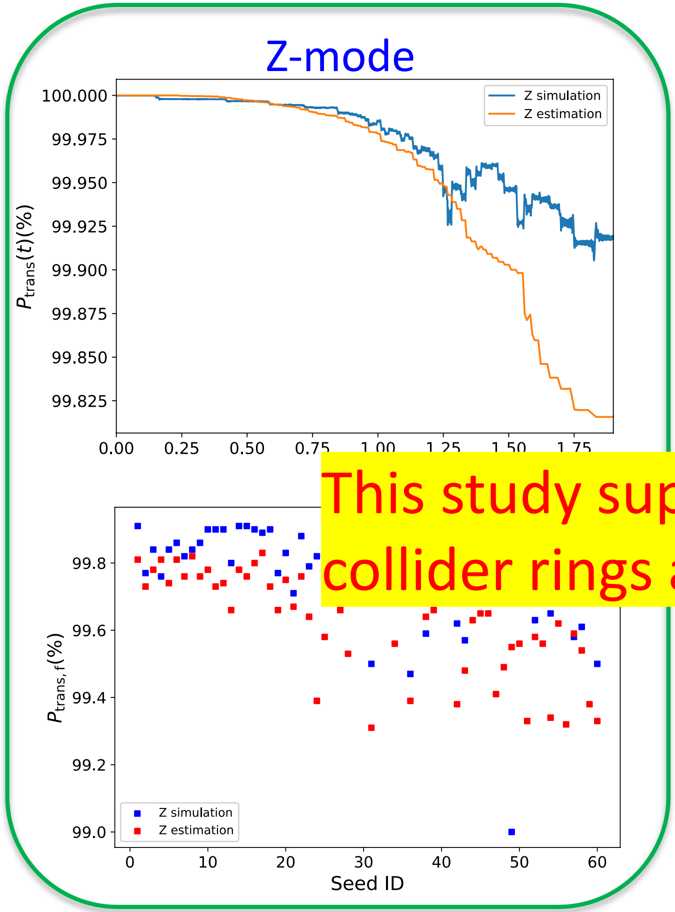
Depolarization effects: simulation vs. estimation

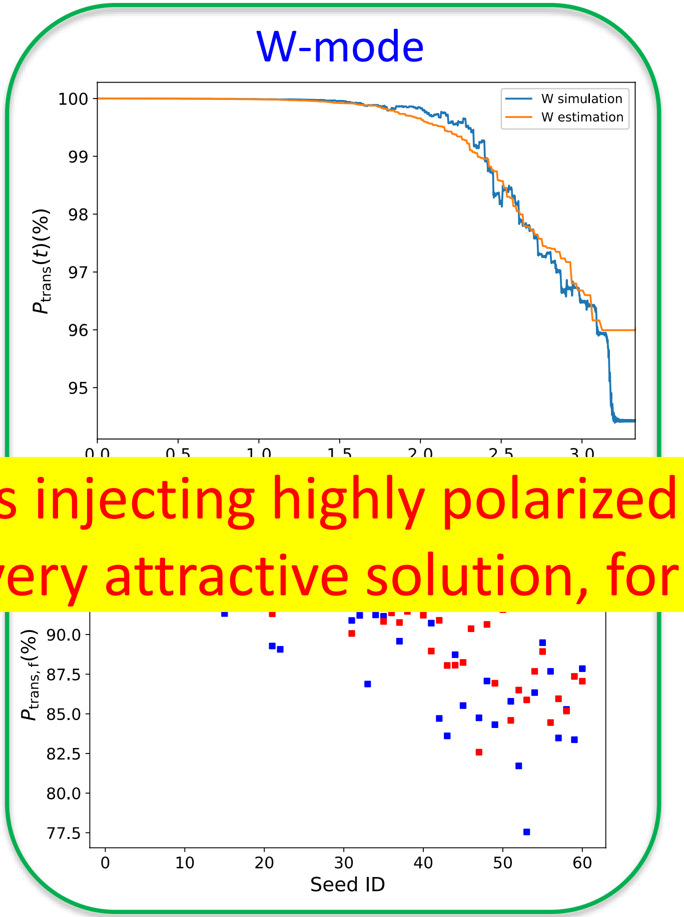
In the acceleration to Z & W

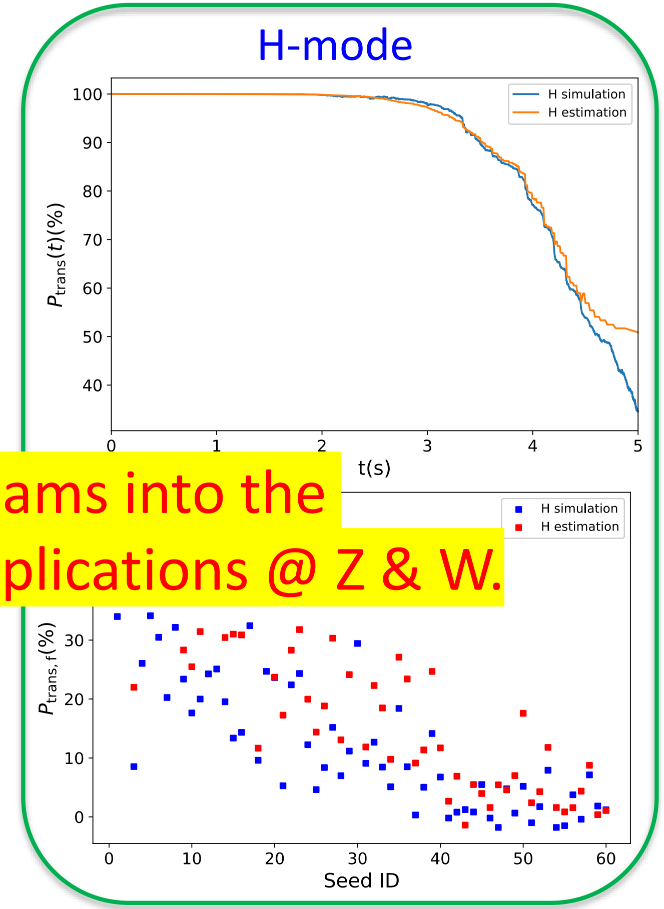
- The spin resonances are generally weak
- Polarization is mostly maintained
- Estimations agree fairly well with simulations

In the acceleration to H

- The spin resonances become stronger at higher energies
- Severe depolarization occurs
- Mitigation methods to be explored







This study supports injecting highly polarized beams into the collider rings as a very attractive solution, for applications @ Z & W.

[1] T. Chen, Z. Duan, D. H. Ji, D. Wang, Phys. Rev. Accel. Beams, 26, 051003 (2023).

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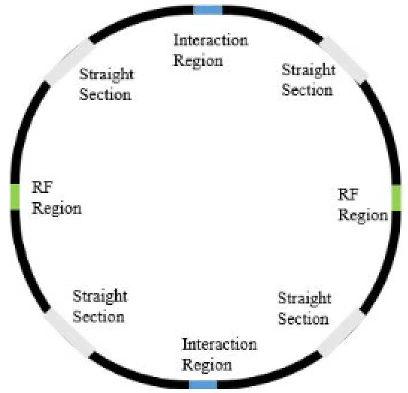
Setup of CEPC collider ring imperfect lattice

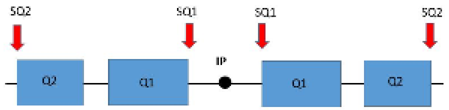
- W/ alignment and field errors, w/o BPM errors
- Closed orbit & optics correction in SAD & AT.
- The vertical emittance is adjusted to the design value
  - w/o solenoid fields
  - Quadrupoles in straight sections are artificially rotated
  - Skew quads inserted next to Q1 & Q2

TABLE I. CEPC magnets' errors.

Component	Misalignment error			Field error
	Δx(μm)	Δy(μm)	$\Delta\theta_z$(μrad)	
Dipole	-	-	-	0.01%
Arc quadrupole	100	100	100	0.02%
IR quadrupole	50	50	50	-
Sextupole	100	100	100	-

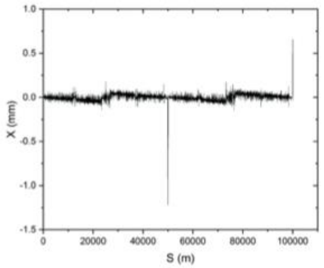


(a) Straight sections of the CEPC

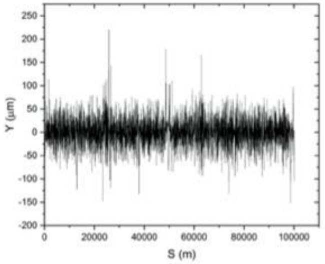


(b) Locations of SQ1 and SQ2 in one interaction region

rms closed orbit are 37μm/28μm

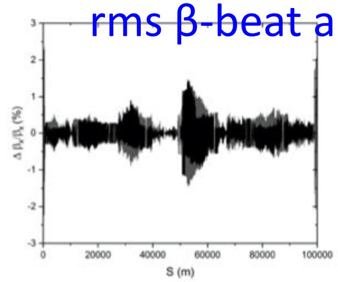


(a) Horizontal closed orbit .

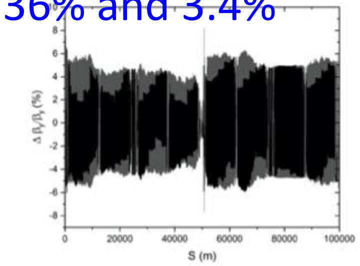


(b) Vertical closed orbit.

rms β-beat are 0.36% and 3.4%



(c) Horizontal beta beating .



(d) Vertical beta beating.

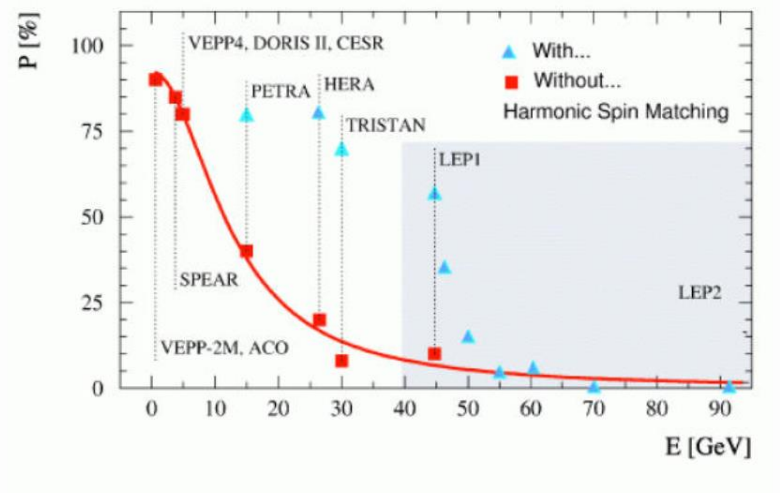
Radiative depolarization in electron storage rings

- $P_{\mathrm{DK}} \approx \frac{P_\infty}{1+\tau_{BKS}/\tau_{\mathrm{dep}}}$, $\frac{1}{\tau_{DK}} = \frac{1}{\tau_{BKS}} + \frac{1}{\tau_{\mathrm{dep}}}$, radiative depolarization characterized by $\tau_{BKS}/\tau_{\mathrm{dep}}$

- More difficult to achieve a high polarization at higher energies

Electron polarization measurements in different machines [1]

[1] R. Assmann, et al., AIP Conference Proceeding, 570, 169 (2001).



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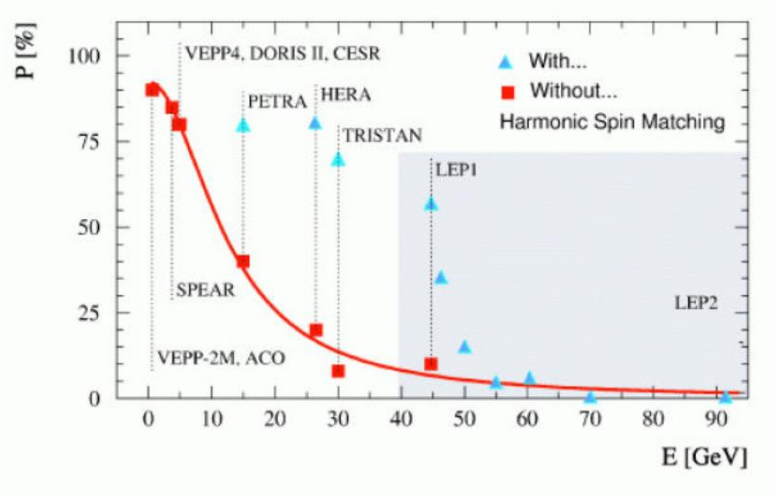
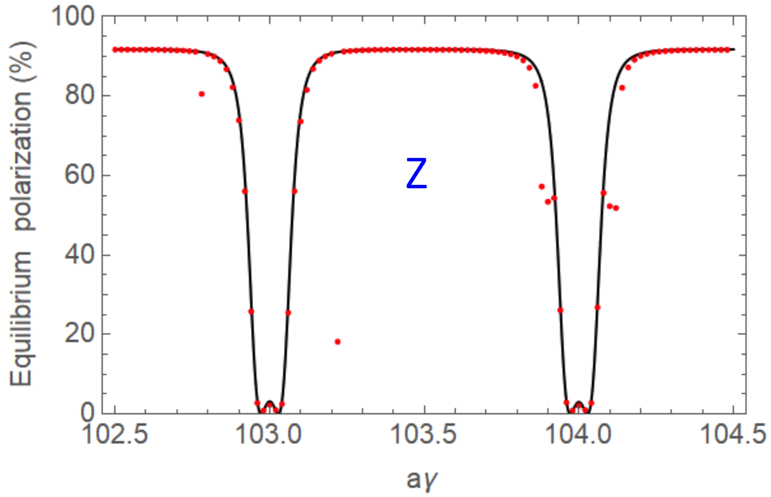
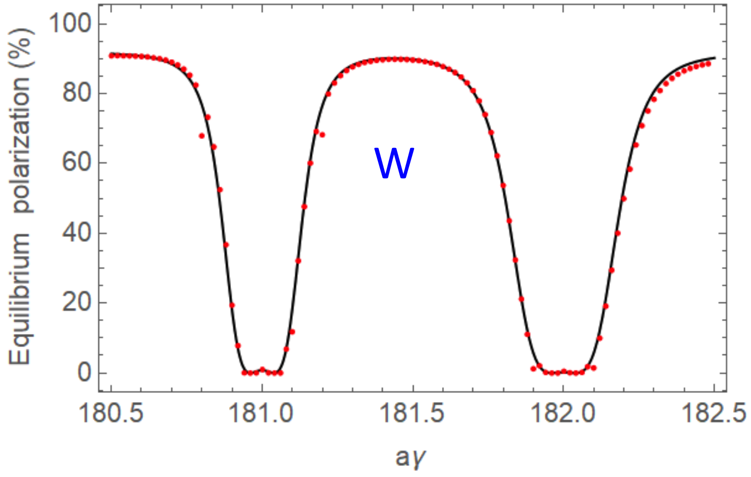


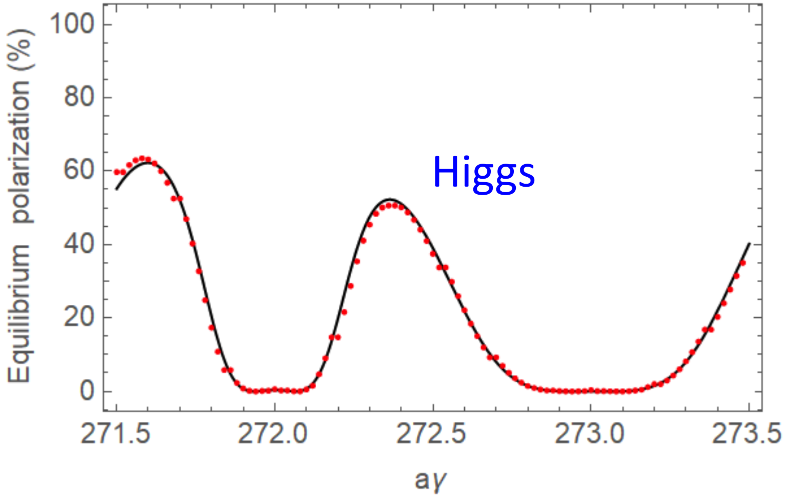
- More difficult to achieve a high polarization at higher energies
  - Stronger first-order spin resonances $\nu_0 = K = k \pm \nu_z$

$$\frac{\tau_{\mathrm{BKS}}}{\tau_{\mathrm{dep}}} \approx \frac{11}{18} \sum_{k=n-l}^{n+l} \frac{\nu_0^2 |\tilde{\epsilon}_k|^2}{[(\nu_0 - k)^2 - \nu_z^2]^2}$$

  - Agree well with SLIM simulations, can be alleviated w/ harmonic spin matching (Yi Wu's talk)







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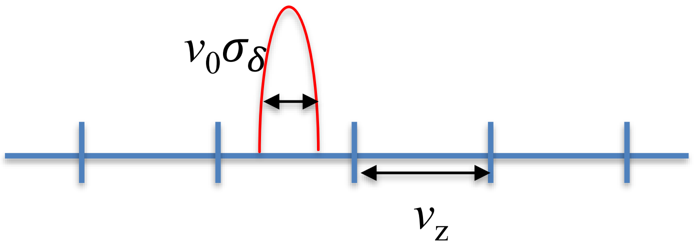
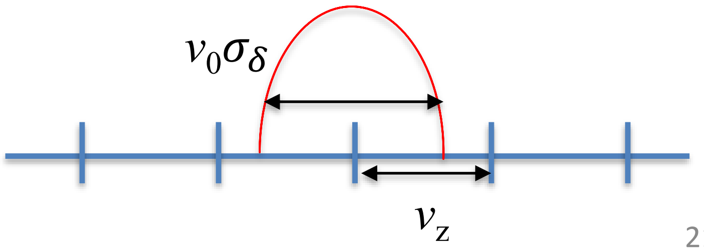
  - Enhanced higher-order synchrotron-sideband resonances

$$\frac{\tau_{\mathrm{BKS}}}{\tau_{\mathrm{dep}}} \approx \frac{11}{18} \sum_{k=n-l}^{n+l} \sum_{m=-\infty}^{\infty} \frac{\nu_0^2 |\tilde{\epsilon}_k|^2 e^{-\sigma^2} I_m(\sigma^2)}{\left[(\nu_0 - k - m\nu_z)^2 - \nu_z^2\right]^2}$$





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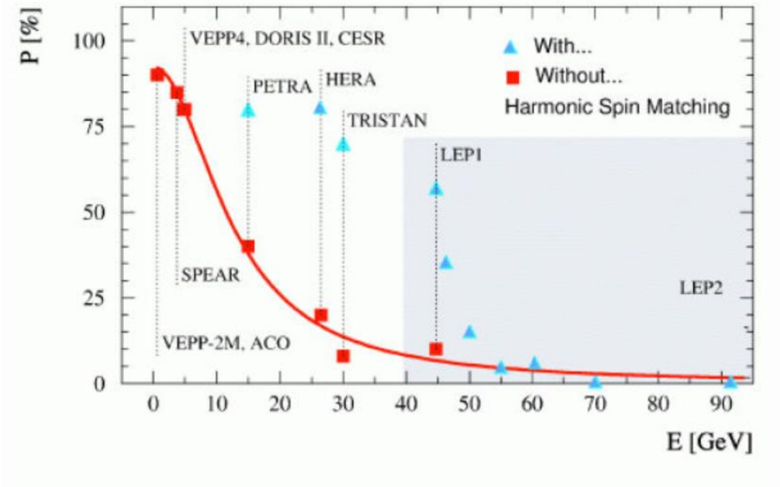


- Size of safe region in v_0 shrinks
- What about even higher energies?

Radiative depolarization in ultra-high-energy storage rings

- Two distinct spin diffusion mechanisms were proposed in [1] in 1970s, regarding the regimes of spin resonance crossing, in the combined effects of synchrotron oscillation and synchrotron radiation

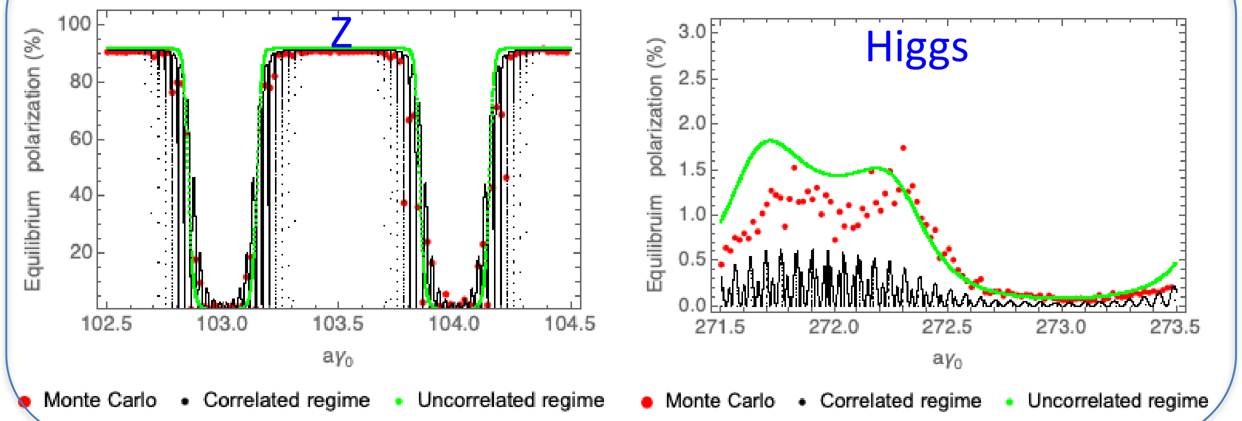
Regime	Correlated regime (consistent with existing measurements)	Uncorrelated regime (not yet confirmed by experiments)
Condition	$\kappa = \frac{v_0^2 \lambda_p}{v_z^3} \ll 1$	$\kappa = \frac{v_0^2 \lambda_p}{v_z^3} \ll 1$ is violated and $\frac{v_0 \sigma_\delta}{v_z} \gg 1$
Theory	Non-resonant spin diffusion & perturbative treatment of $\frac{\partial \hat{n}}{\partial \delta}$	Resonant spin diffusion
Depolarization effect	Higher-order synchrotron sideband spin resonances	No dependence on v_z, weaker depolarization

- Monte-Carlo simulations were compared with these theories in the energy range of CEPC [2]

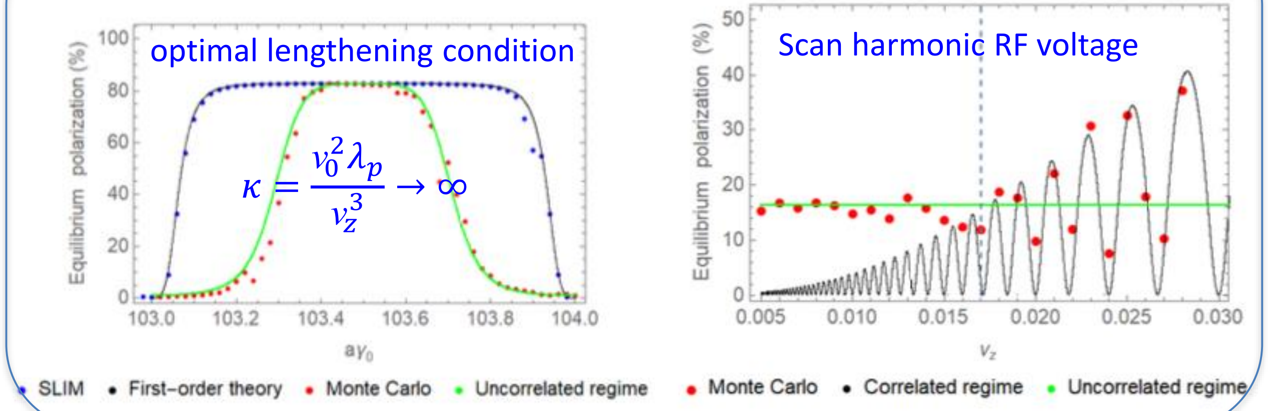
This study suggests existing theories are incomplete, requiring further development

  - showing a gradual evolution from the correlated regime to the uncorrelated regime in parameter scan

Case A: dependence on beam energy



Case B: influence of harmonic RF cavity



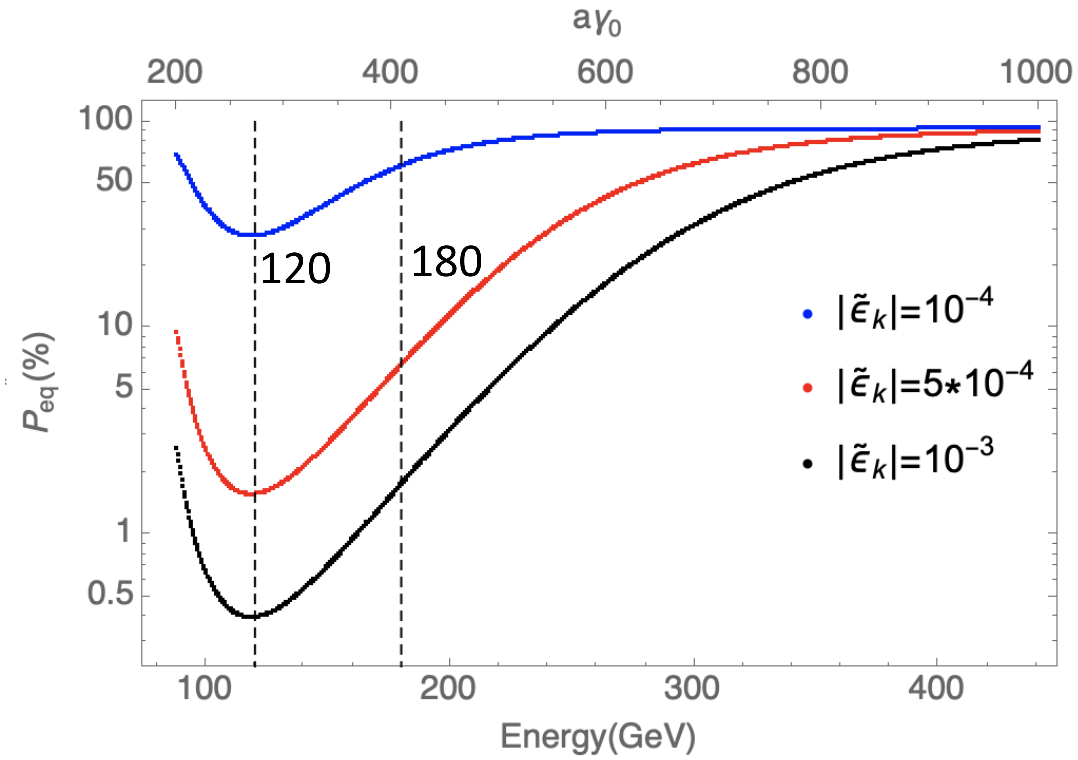
[1] Derbenev, Kondrantenko and Skrinsky, Part. Accel. 9, 247 (1979)

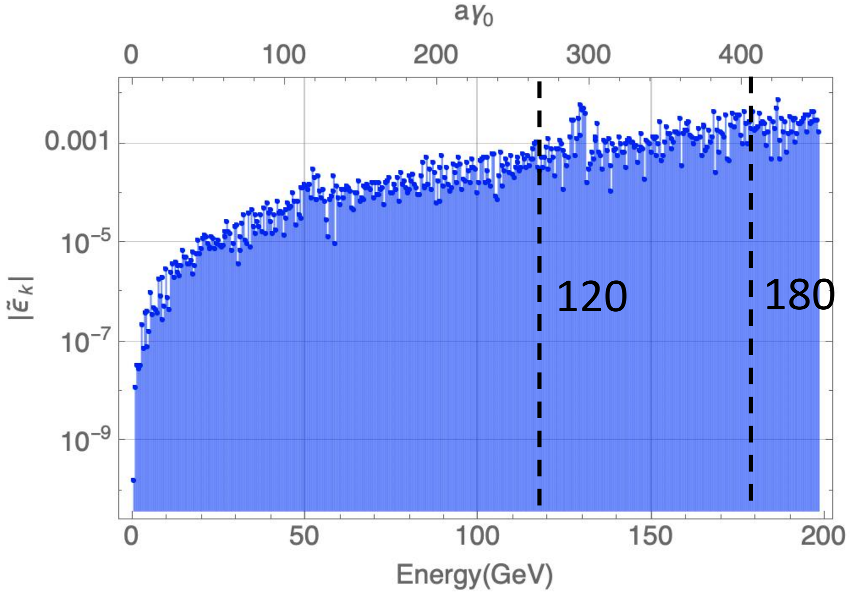
[2] W. H. Xia, Z. Duan, D. P. Barber, Y. W. Wang, B. Wang, J. Gao, accepted by Phys. Rev. Accel. Beams.

Uncorrelated regime of spin resonance crossing

- Prediction of the theory of resonant spin diffusion [1]
  - Assume the adjacent two integer resonances have the same strength
  - Influence of first-order betatron spin resonances are not included

$$\frac{\tau_{\mathrm{BKS}}}{\tau_{\mathrm{dep}}} \approx \frac{\sqrt{\pi/2}}{\lambda_{\mathrm{BKS}}} \sum_{k=n-l}^{n+l} \frac{|\tilde{\epsilon}_k|^2}{\sigma_0} \exp\left[-\frac{(\nu_0 - k)^2}{2\sigma_0^2}\right]$$





Is it possible to have a few percent polarization at Higgs or even ttbar energies?

- In collaboration with Yi Wu on spin bumps for ultra-high energies.

[1] Derbenev, Kondrantenko and Skrinsky, Part. Accel. 9, 247 (1979)

Summary

- Spin resonance structure featured in highly periodic lattices
  - Enable polarization maintenance in the booster
  - Helpful to avoid super strong resonances in the collider ring for working beam energies

- Comparison between simulations with the theories of (radiative) depolarization at ultra-high energies.

- Better understanding of the strengths of integer spin resonances is needed
  - Lattice error sensitivity (S. Liuzzo) in terms of spin resonance strength ?
  - Influence of “systematic” alignment errors & consequent corrector patterns ?
  - How well can harmonic spin matching work -> percent-level self-polarization at H & ttbar ?

- Collaboration on these aspects are welcome!

Your comments and suggestions are highly appreciated!

Cancellation at small K

Intrinsic resonances: $\upsilon_0 = K = k \pm \upsilon_y$

$$\tilde{\epsilon}_K^{\text{intr},\pm}(I_y) \approx \frac{R(1+K)}{4\pi} \int_0^{2\pi} \frac{\frac{\partial B_x}{\partial y}}{B\rho} \sqrt{2I_y\beta_y} \times e^{i[K\Phi(\theta') \mp \nu_y\phi_y(\theta') \mp \psi_{y0}]} d\theta', \qquad (24)$$

from DOM sections more significant. Additionally, there can be cancelations between the contributions from the arc sections and the DOM sections, depending on the lattice parameters. In particular, when ν_y is large and $K \ll \nu_y$, the exponential factor $e^{i[K\Phi(\theta') \mp \nu_y\phi_y(\theta') \mp \psi_{y0}]}$ in Eq. (24) includes a fast wave with a phase $\nu_y\phi_y(\theta')$ modulated by the slow wave with a phase $K\Phi(\theta')$ so that the contributions from all FODO cells in each superperiod tend to cancel out. Such a cancelation generally becomes more incomplete as K increases.

Imperfection resonances: $\upsilon_0 = K$

$$\tilde{\epsilon}_K^{\text{imp}} \approx -\frac{R(1+K)}{2\pi} \sum_{k=-\infty}^{\infty} \frac{\nu_y^2 f_k}{\nu_y^2 - k^2} \oint \frac{\frac{\partial B_x}{\partial y}}{B\rho} \beta_y^{1/2} e^{i(k\phi_y + K\Phi)} d\theta. \qquad (34)$$

To summarize, the structure of imperfection resonances, with the contribution from only one harmonic k, is quite similar to the structure of the intrinsic resonances. Besides the peaks when the contributions from all arc FODO cells add up coherently, there is also cancelation among all FODO cells if $k \gg 1$ and $K \ll k$. Nevertheless, the strength of an imperfection resonance is the sum of various harmonics k modulated by $\frac{\nu_y^2 f_k}{\nu_y^2 - k^2}$, with varying locations of enhancement and thus strongly depends on the spectrum of f_k. In general, after the closed-orbit correction, the f_k terms with $|k|$ near ν_y become weaker, the terms with $|k|$ further away from ν_y are less reduced, forming a plateau around the original peak. Generally, we expect that the strength of imperfection resonances increases with K until reaching the plateau near the first superstrong imperfection resonance, after which it oscillates as adjacent superstrong imperfection resonances are approached and left behind.