# Quantum Chromodynamics 

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## some observations from first lecture ...

Some of you are already familiar with QCD ...


## some observations from first lecture ...

## Something during the time of this school that you're excited about

I am very curious about SUSY lectures and the exams at the end. I am too curious about
international lectures, its my first experience
History tour of the city

Long Lived Particles

## Even more physics

|  | Long Lived Particles |  |
| :--- | :--- | :--- |

## quantum chromodynamics - overview

- strong interaction part of the standard mode
- jet production
- internal structure of hadrons
- ingredients:
- 3 families of quarks/anti-quarks, come in 3 colours
- gluon, 8 colour states
- coupling constant $\alpha_{s} \sim 0.1$, relatively large $\rightarrow$ "strong" coupling



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## reading material

- much of these lectures based on previous HASCO lectures (Steffen Schumann 2012, Enrico Bothmann 2022)
- standard reference: Ellis, Stirling, Webber "QCD and Collider Physics"
- introductory material in general particle physics references, for example Griffiths "Introduction to Elementary Particles"



## hadron colliders in the real world



## hadron colliders for theorists

- Events factorised into
- Hard Process
- QCD radiation
- PDFs/Beams/Underlying event
- Hadrons



## hadron colliders for theorists

- Events factorised into
- Hard Process
- QCD radiation

Main goal of this lecture: understand and analyse this picture

## hadrons

- Hadrons = states observed in experiments
- basic examples: proton, neutron
- historic situation:
- there are many more hadrons than we encounter "every day"
$\rightarrow$ "zoo of hadrons"
- order, understanding from first principles?



## hadrons in the quark model

- observation: hadrons follow specific pattern - $S U(3)$ flavour symmetry
- structure of light hadrons can be explained by a model where hadrons are made up of three constituents, almost massless
"partons" - "up" (u), "down" (d) and "strange" $(s)$ quarks
-     + further experiments show there are three more quarks - "charm"

| $\Delta^{-}$ | $\Delta^{0}$ | $\Delta^{+}$ | $\Delta^{++}$ |
| :---: | :---: | :---: | :---: |
|  | $\Sigma^{-}$ | $\Sigma^{0}$ | $\Sigma^{+}$ |
|  | $\Xi^{-}$ | $\Xi^{0}$ |  |
|  |  | $\Omega^{-}$ |  |
|  | quar | k content |  |
| ddd | ddu | duu | uuu |
|  | dds | dus | uus |
|  | dss | uss |  |
|  |  | sss |  |

## properties of quarks

- to match the observed hadron spectrum: quarks should be spin-1/2 fermions
- up-type quarks with charge $Q_{u, c, t}=2 / 3$, down-type quarks $Q_{d, s, b}=-1 / 3$ (+ anti-quarks with opposite charges)
- $(u, d),(c, s),(t, b)$ form electroweak multiplets
- for example proton $|u u d\rangle$ or neutron $|u d d\rangle \rightarrow$ Baryons

Proton


## colour

- Historic puzzle: we observe hadrons in states like $\left|\Delta^{++}\right\rangle=\left|u_{\uparrow} u_{\uparrow} u_{\uparrow}\right\rangle \rightarrow$ three identical fermions, apparently in a completely symmetric wave functions $\rightarrow$ violation of Fermi-Dirac statistics?
- Solution: if we had an additional quantum number, lets call it colour, with three possible states, we could anti-symmetrise as

$$
\left|\Delta^{++}\right\rangle=\epsilon_{a b c}\left|u_{a, \uparrow} u_{b, \uparrow} u_{c, \uparrow}\right\rangle
$$



- We don't observe this quantum number (apart from theses statistics), so postulate: all physical, experimental observed states are colour-neutral $\rightarrow$ confinement


## parton model evidence - the R-ratio

- consider ratio of total cross sections $R=\sum_{q} \sigma\left(e^{+} e^{-} \rightarrow q \bar{q}\right) / \sigma\left(e^{+} e^{-} \rightarrow \mu^{+} \mu^{-}\right)$
- you saw the diagram for $e^{+} e^{-} \rightarrow \mu^{+} \mu^{-}$in the standard model lecture
- only differences for quarks: come in 3 colours and with fractional charges
- different mass thresholds, at low energies only $d, u, s$ quarks

$$
R=N_{c}\left[\left(\frac{2}{3}\right)^{2}+2\left(-\frac{1}{3}\right)^{2}\right]=N_{c} \frac{2}{3}
$$

- above charm, threshold, one more up-type quark

$$
R=N_{c}\left[2\left(\frac{2}{3}\right)^{2}+2\left(-\frac{1}{3}\right)^{2}\right]=N_{c} \frac{10}{9}
$$

## parton model evidence - the R-ratio



## QCD - gauge theory

- look back on Standard-Model lecture: QED gauge invariance $\psi \rightarrow e^{-i q \alpha(x)} \psi$


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- for QCD, we want to use colour as "charge" $\rightarrow \psi$ becomes vector in 3-d colour space, acted on by matrices $\psi \rightarrow e^{i t^{A} \alpha(x)} \psi$
- $t^{A} \ldots$ matrices representing gauge group


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Note: this is distinct from the $S U(3)$ flavour symmetry we saw earlier!

- relevant group for QCD: $S U\left(N_{c}\right)$ with $N_{c}=3 \ldots$ number of colours
- group of Special (determinant 1) Unitary $N_{c} \times N_{c}$ matrices
- relevant for physics: generators $t^{A}$ with $e^{i t^{A} \alpha} \in S U\left(N_{c}\right)$


## SU(3) colour group - Gell-Mann matrices

- special ( $\rightarrow$ generators traceless), unitary ( $\rightarrow$ generators hermitian) $3 \times 3$ matrices, explicit basis:

$$
\begin{aligned}
& \lambda^{1}=\left(\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \lambda^{2}=\left(\begin{array}{ccc}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{array}\right), \lambda^{3}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{array}\right), \lambda^{4}=\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{array}\right) \\
& \lambda^{5}=\left(\begin{array}{ccc}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{array}\right), \lambda^{6}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right), \lambda^{7}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -i \\
0 & i & 0
\end{array}\right), \lambda^{8}=\left(\begin{array}{ccc}
\frac{1}{\sqrt{3}} & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & \frac{-2}{\sqrt{3}}
\end{array}\right)
\end{aligned}
$$

Note: Gell-Man matrices analogous to Pauli matrices for $\operatorname{SU}(2)$

- conventionally, we actually work with $t_{a b}^{A}=\frac{1}{2} \lambda_{a b}^{A}$
- matrices acting on vectors $\left(\psi_{1}, \psi_{2}, \psi_{3}\right)$


## SU(3) group - colour algebra

- defining property $\left[t^{A}, t^{B}\right]=i f^{A B C} t^{C}$, and chosen normalisation $\operatorname{Tr}\left[t^{A} t^{B}\right]=T_{R} \delta_{A B}, T_{R}=1 / 2$
- $f^{A B C}$ are structure constants of the group, generate adjoint representation
- Casimir invariants

$$
t_{a b}^{A} t_{b c}^{A}=C_{F} \delta_{a c}, \quad C_{F}=\frac{N_{c}^{2}-1}{2 N_{c}} \quad \text { and } \quad f^{A C D} f^{B C D}=C_{A} \delta_{A B}, \quad C_{A}=N_{c}
$$

- Fierz identity

$$
t_{a b}^{A} t_{c d}^{A}=\frac{1}{2} \delta_{b c} \delta_{a d}-\frac{1}{2 N_{c}} \delta_{a b} \delta_{c d}
$$

## SU(3) group - colour algebra

- defining property $\left[t^{A}, t^{B}\right]=$ if ${ }^{A B C} t^{C}$,

Example use of Fierz identity and chosen normalisation $\operatorname{Tr}\left[t^{A} t^{B}\right]=T_{R} \delta$ re-calculate first Casimir:

- $f^{A B C}$ are structure constants of the group, $t_{a b}^{A} t_{b c}^{A}$
- Casimir operators

$$
t_{a b}^{A} b_{b c}^{A}=C_{F} \delta_{a c}, \quad C_{F}=\frac{N_{c}^{2}-1}{2 N_{c}} \quad \text { and }
$$

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$$

## SU(3) group - colour algebra

- definin $\quad[A \quad B] \quad . \quad A B C, C$
- $f^{A B C} a \delta_{i i}=\sum_{i=0}^{N_{c}} \delta_{i i}=\sum_{i=0}^{N_{c}} 1=N_{c}$
- Casimir operators

$$
t_{a b}^{A} t_{b c}^{A}=C_{F} \delta_{a c}, \quad C_{F}=\frac{N_{c}^{2}-1}{2 N_{c}} \quad \text { and }
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t_{a b}^{A} t_{c d}^{A}=\frac{1}{2} \delta_{b c} \delta_{a d}-\frac{1}{2 N_{c}} \delta_{a b} \delta_{c d}
$$

Example use of Fierz identity -
re-calculate first Casimir:

$$
\begin{aligned}
t_{a b}^{A} t_{b c}^{A} & =\frac{1}{2} \delta_{b b} \delta_{a c}-\frac{1}{2 N_{c}} \delta_{a b} \delta_{b c} \\
& =\frac{N_{c}}{2} \delta_{a c}-\frac{1}{2 N_{c}} \delta_{a c}
\end{aligned}
$$

## SU(3) group - colour algebra

- defining property $\left[t^{A}, t^{B}\right]=$ if ${ }^{A B C} t^{C}$,

Example use of Fierz identity and chosen normalisation $\operatorname{Tr}\left[t^{A} t^{B}\right]=T_{R} \delta$ re-calculate first Casimir:

- $f^{A B C}$ are structure constants of the group, $t_{a b}^{A} t_{b c}^{A}=\frac{1}{2} \delta_{b b} \delta_{a c}-\frac{1}{2 N_{c}} \delta_{a b} \delta_{b c}$
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$$

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\begin{aligned}
& =\frac{N_{c}}{2} \delta_{a c}-\frac{1}{2 N_{c}} \delta_{a c} \\
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\end{aligned}
$$

## SU(3) group - colour algebra

- defining property $\left[t^{A}, t^{B}\right]=i f^{A B C} t^{C}$, and chosen normalisation $\operatorname{Tr}\left(t^{A} t^{B}\right)=T_{R} \delta_{A B}, T_{R}=1 / 2$
- $f^{A B C}$ a side note:
- Casimi
we can eliminate $f^{A B C}$ by using

$$
\left[t^{A}, t^{B}\right] t^{C}=i f^{A B D} t^{D} t^{C}
$$

and taking the trace

- Fierz io

$$
\begin{aligned}
& \operatorname{Tr}\left(\left[t^{A}, t^{B}\right] t^{C}\right)=i f^{A B D} \operatorname{Tr}\left(t^{D} t^{C}\right)=i f^{A B D} T_{R} \delta_{C D} \\
\Rightarrow & f^{A B C}=-\frac{i}{T_{R}} \operatorname{Tr}\left(\left[t^{A}, t^{B}\right] t^{C}\right)
\end{aligned}
$$

## SU(3) group - colour algebra

Together with Fierz:

- defining property $\left[t^{A}, t^{B}\right]=i f^{A B C} t^{C}$, and chosen normalisation $\operatorname{Tr}\left[t^{A} t^{B}\right]=T_{R} \delta_{A B}, T_{R}=1 / 2$
- $f^{A B C}$ a side note:
- Casimi we can eliminate $f^{A B C}$ by using
all colour factors become counting of in $\delta_{a b}(=0$ or 1)
$\Rightarrow$ trivial (though maybe cumbersome) calculation

$$
\left[t^{A}, t^{B}\right] t^{C}=i f^{A B D} t^{D} t^{C}
$$

and taking the trace

- Fierz ic

$$
\begin{aligned}
& \operatorname{Tr}\left(\left[t^{A}, t^{B}\right] t^{C}\right)=i f^{A B D} \operatorname{Tr}\left(t^{D} t^{C}\right)=i f^{A B D} T_{R} \delta_{C D} \\
\Rightarrow & f^{A B C}=-\frac{i}{T_{R}} \operatorname{Tr}\left(\left[t^{A}, t^{B}\right] t^{C}\right)
\end{aligned}
$$

## QCD as a gauge theory

- Fundamental particles: fermionic quark fields $\psi_{q}^{a}$, with flavour $q=u, d, s, c, b, t$ and colour charge $a \Rightarrow$ free quark lagrangian $\mathscr{L}_{\text {quark }}=\bar{\psi}_{q}^{a} i \gamma^{\mu} \partial_{\mu} \psi_{q}^{a}-m_{q} \bar{\psi}_{q}^{a} \psi_{q}^{a}$
$(\rightarrow$ generic Dirac for each colour and flavour)


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$(\rightarrow$ generic Dirac for each colour and flavour)
- analogous to QED: forces between (electrically) charged particles mediated by photons $\rightarrow$ forces between colour charged particles mediated by gluons $\rightarrow$ spin- 1 fields in adjoint representation $A_{\mu}^{A}$ with $A=1 \ldots 8$

$$
\mathscr{L}_{\text {gluon }}=-\frac{1}{4} F_{\mu \nu}^{A} F^{A \mu \nu} \quad \text { where } \quad F_{\mu \nu}^{A}=\partial_{\mu} A_{\nu}^{A}-\partial_{\nu} A_{\mu}^{A}-g_{s} f_{A B C} A_{\mu}^{A} A_{\nu}^{B}
$$

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$$
\mathscr{L}_{\text {gluon }}=-\frac{1}{4} F_{\mu \nu}^{A} F^{A \mu \nu} \text { where } F_{\mu \nu}^{A}=\underbrace{\text { you know this part }} \begin{aligned}
& \partial_{\mu} A_{\nu}^{A}-\partial_{\nu} A_{\mu}^{A}-g_{s} f_{A B C} A_{\mu}^{A} A_{\nu}^{B} \\
& \text { from QED/ED }
\end{aligned}
$$

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\begin{aligned}
\mathscr{L}_{\text {gluon }}=-\frac{1}{4} F_{\mu \nu}^{A} F^{A \mu \nu} \quad \text { where } \quad & F_{\mu \nu}^{A}=\underbrace{\partial_{\mu} A_{\nu}^{A}-\partial_{\nu} A_{\mu}^{A}}_{\mu}-\underbrace{g_{s} f_{A B C} A_{\mu}^{A} A_{\nu}^{B}} \\
& \begin{array}{l}
\text { you know this part } \\
\\
\text { from QED/ED is new due to } \\
\text { non-abelian gauge } \\
\text { nomp }
\end{array}
\end{aligned}
$$

## QCD as a gauge theory

- interactions between quarks and gluons $\rightarrow$ minimal coupling, i.e. restoring local gauge symmetry $\partial_{\mu} \rightarrow\left(D_{\mu}\right)_{a b}=\delta_{a b} \partial_{\mu}-i g_{s} t_{a b}^{A} A_{\mu}^{A}$
- final lagrangian $\mathscr{L}_{Q C D}=\bar{\psi}_{q}^{a}\left(i \gamma^{\mu}\left(D_{\mu}\right)_{a b}-\delta_{a b} m_{q}\right) \psi_{q}^{b}-\frac{1}{4} F_{\mu \nu}^{A} F^{A \mu \nu}$
- generic form of "Yang-Mills" Lagrangian $\rightarrow$ general gauge theories, distinguished by group generated by $t^{A} s \rightarrow$ here $S U(3)$ and $t^{A} s$ as discussed before
- side note: consistent quantisation requires gauge-fixing terms (also present in QED) and ghost fields (decouple in abelian theory) $\rightarrow$ not necessary for our purposes now


## lattice QCD

- numerical solution to find most likely field configurations
- on discrete 4D-grid (lattice), then take limit of small lattice spacing $\rightarrow$ solution of QCD
- no approximation made (in principle)

- important result: mass spectrum of
[Durr et. al. '08]
hadrons $\rightarrow$ supports QCD as theory of hadrons/at low energies


## QCD at colliders

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## QCD - perturbation theory

- reminder: we want to calculate matrix elements as input for cross sections

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see Standard Model
lecture, Slides 51-54
```

$$
\frac{d \sigma}{d \Omega}=\left(\frac{1}{8 \pi}\right)^{2} \frac{S|\mathcal{M}|^{2}}{\left(E_{1}+E_{2}\right)^{2}} \cdot \frac{\left|\vec{p}_{f}\right|}{\left|\overrightarrow{p_{i}}\right|}
$$

- perturbation theory: expand $\mathscr{M}$ in powers of coupling constant $\mathscr{M}=\mathscr{M}_{0}+g \mathscr{M}_{1}+g^{2} \mathscr{M}_{2}+\ldots$
- e.g. in QED relevant coupling is $\alpha_{Q E D} \sim 1 / 137 \rightarrow$ first order naively accurate within $\sim 1 \%$
- how about QCD?


## QCD coupling constant

- Lagrangian includes coupling constant $g_{s} \sim$ analogous to electric charge, determines the coupling strength $\rightarrow$ needed as input for calculations
- we will use $\alpha_{s}=\frac{g_{s}^{2}}{4 \pi}$, compare to (in suitable units) $\alpha_{Q E D}=\frac{e^{2}}{4 \pi} \sim \frac{1}{137}$
- Typical behaviour in QFTs: higher order terms can be absorbed into coupling constant, introducing dependence on energy scale $\rightarrow$ differential equation $\frac{d \alpha_{s}\left(\mu^{2}\right)}{d \ln \mu^{2}}=\beta\left(\alpha_{s}\right)$, with $\beta\left(\alpha_{s}\right)$ computable in perturbation theory


## the running coupling

- $\frac{d \alpha_{s}\left(\mu^{2}\right)}{d \ln \mu^{2}}=\beta\left(\alpha_{s}\right)$ with $\quad \beta\left(\alpha_{s}\right)=-\alpha_{s}^{2}\left(\beta_{0}+\beta_{1} \alpha_{s}+\ldots\right)$
- important to note the overall minus sign $\Rightarrow \alpha_{s} \rightarrow 0$ at high energy scales, this is known as asymptotic freedom

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Nobel price 2004 for Gross, Polizer, Wilczek
```

- also provides evidence for validity of confinement $\rightarrow$ at small energies, interaction strength between quarks and gluons large $\rightarrow$ infinite energy required to separate two quarks


## the running coupling

- leading order solution: only include $\beta_{0}=\frac{11 C_{A}-2 n_{f}}{12 \pi}$
$\frac{d \alpha_{s}\left(\mu^{2}\right)}{d \ln \mu^{2}}=-\beta_{0} \alpha_{s}^{2} \Rightarrow \alpha_{s}\left(\mu^{2}\right)=\frac{\alpha_{s}\left(\mu_{0}^{2}\right)}{1+\alpha_{s}\left(\mu_{0}^{2}\right) \beta_{0} \ln \frac{\mu^{2}}{\mu_{0}^{2}}}$
- divergent (Landau pole) when $\mu^{2} \sim \mu_{0}^{2} e^{-\alpha_{s} \beta_{0}}$
- alternative representation $\alpha_{s}^{-1}\left(\mu^{2}\right)=\beta_{0} \ln \frac{\mu^{2}}{\Lambda_{Q C D}^{2}}$
- $\rightarrow \Lambda_{Q C D} \sim 200 \mathrm{GeV}$ fundamental scale for breakdown of perturbation theory


## the running coupling

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$\frac{d \alpha_{s}\left(\mu^{2}\right)}{d \ln \mu^{2}}=-\beta_{0} \alpha_{s}^{2} \Rightarrow \alpha_{s}\left(\mu^{2}\right)=\frac{\alpha_{s}\left(\mu_{0}^{2}\right)}{1+\alpha_{s}\left(\mu_{0}^{2}\right) \beta_{0} \ln \frac{\mu^{2}}{\mu_{0}^{2}}}$
- divergent (Landau pole) when $\mu^{2} \sim \mu_{0}^{2} e^{-\alpha_{s} \beta_{0}}$

$$
\begin{aligned}
& \text { But: } \\
& \alpha_{s} \rightarrow 0 \text { as } \mu \rightarrow \infty \\
& \Rightarrow \text { Perturbation } \\
& \text { theory valid at high } \\
& \text { energies (UV limit) }
\end{aligned}
$$

- alternative representation $\alpha_{s}^{-1}\left(\mu^{2}\right)=\beta_{0} \ln \frac{\mu^{2}}{\Lambda_{Q C D}^{2}}$
- $\rightarrow \Lambda_{Q C D} \sim 200 \mathrm{GeV}$ fundamental scale for breakdown of perturbation theory


## the running coupling

- So what is the value at relevant (for us) energies?
- The standard is to quote the value at the $Z$ mass $M_{Z} \sim 91.2 \mathrm{GeV}$, $\alpha_{s}\left(M_{Z}\right) \sim 0.118$
- typical jet $p_{T} \sim 50 \mathrm{GeV}$... 5 TeV
- expect much worse behaviour than QED, but should naively we are in the range where PQCD is applicable


## QCD - Feynman rules


gluon-quark interaction, analogous to photonfermion with $t^{A}$ as "charge"

## QCD - Feynman rules


(all momenta incoming, $\mathrm{p}+\mathrm{q}+\mathrm{r}=0$ )
gluon-quark interaction, analogous to photonfermion with $t^{A}$ as "charge"

+ interactions between gluons, as a result of the 'non-abelian' parts of $F_{\mu \nu}^{A}$

$$
\begin{array}{lll}
\mathrm{A}, \alpha & -\mathrm{ig}^{2} \mathrm{f}^{\mathrm{XAC}} \mathrm{f}^{\mathrm{XBD}} & {\left[\mathrm{~g}^{\alpha \beta} \mathrm{g}^{\gamma \delta}-\mathrm{g}^{\alpha \delta} \mathrm{g}^{\beta \gamma}\right]} \\
-\mathrm{ig}^{2} \mathrm{f}^{\mathrm{XAD}} \mathrm{f}^{\mathrm{XBC}} \boldsymbol{\rho}^{6^{\mathrm{B}, \beta}}\left[\mathrm{~g}^{\alpha \beta} \mathrm{g}^{\gamma \delta}-\mathrm{g}^{\alpha \gamma} \mathrm{g}^{\beta \delta}\right]
\end{array}
$$

## QCD - Feynman rules



$$
-\mathrm{ig}\left(\mathrm{t}^{\mathrm{A}}\right)_{\mathrm{cb}}\left(\gamma^{\alpha}\right)_{\mathrm{ji}}
$$

$$
q^{\{\mathrm{B}, \beta} \mathrm{r} \quad-\mathrm{g}^{\mathrm{ABC}}\left[(\mathrm{p}-\mathrm{q})^{\gamma} \mathrm{g}^{\alpha \beta}+(\mathrm{q}-\mathrm{r})^{\alpha} \mathrm{g}^{\beta \gamma}+(\mathrm{r}-\mathrm{p})^{\beta} \mathrm{g}^{\gamma \alpha}\right]
$$

(all momenta incoming, $\mathrm{p}+\mathrm{q}+\mathrm{r}=0$ )


$$
\begin{aligned}
& -\mathrm{ig}^{2} \mathrm{f}^{\mathrm{XAC}} \mathrm{f}^{\mathrm{XBD}}\left[\mathrm{~g}^{\alpha \beta} \mathrm{g}^{\gamma \delta}-\mathrm{g}^{\alpha \delta} \mathrm{g}^{\beta \gamma}\right] \\
& -\mathrm{ig}^{2} \mathrm{f}^{\mathrm{AAD}} \mathrm{f}^{\mathrm{BC}}\left[\mathrm{~g}^{\alpha \beta} \mathrm{g}^{\gamma \delta}-\mathrm{g}^{\alpha \gamma} \mathrm{g}^{\prime}\right] \\
& -\mathrm{ig}^{2} \mathrm{f}^{\mathrm{AB}} \mathrm{f}^{\mathrm{CDD}}\left[\mathrm{~g}^{\alpha \gamma} \mathrm{g}^{\beta-}-\mathrm{g}^{\alpha \delta} \mathrm{g}^{\beta \gamma}\right]
\end{aligned}
$$

of $F_{\mu \nu}^{A}$
gluon-quark interaction, analogous to photonfermion with $t^{A}$ as "charge"

## + interactions between gluons, as a result of the 'non-abelian' parts

+ not shown here: propagators, with colour conserving deltas, couplings to EW bosons


## QCD - colour flow



- gluons carry both colour and anti-colour
- at interaction vertices they can "carry away" color and change color of patron they interact with


## QCD - colour factors

- Colour factors (Casimirs) we saw earlier indeed appear in common diagrams:

$$
\begin{aligned}
& \operatorname{Tr}\left\{t^{A} t^{B}\right\}=T_{R} \delta^{A B}, \quad T_{R}=\frac{1}{2} \\
& \sum_{A} t_{a b}^{A} t_{b c}^{A}=C_{F} \delta_{a c}, \quad C_{F}=\frac{N_{c}^{2}-1}{2 N_{c}} \\
& \sum_{C, D} f^{A C D} f^{B C D}=C_{A} \delta^{A B}, \quad C_{A}=N_{c}
\end{aligned}
$$

- Observation: $g \rightarrow g g$ is enhanced relative to $q \rightarrow q g$ by factor $C_{A} / C_{F}$


## QCD calculations

- simplest process involving quarks: $e^{+} e^{-} \rightarrow \gamma^{*} \rightarrow q \bar{q}$
- not actually involving QCD, generic fermion production (but $N_{c}$ times)

- next higher order (+ gluon attached to other quark + virtual corrections)



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## QCD - Feynman rules


gluon-quark interaction, analogous to photonfermion with $t^{A}$ as "charge"

(all momenta incoming, $\mathrm{p}+\mathrm{q}+\mathrm{r}=0$ )

+ interactions between gluons, as a result of the 'non-abelian' parts of $F_{\mu \nu}^{A}$


$$
\begin{aligned}
& -\mathrm{ig}^{2} \mathrm{f}^{\mathrm{XAC}} \mathrm{f}^{\mathrm{XBD}}\left[\mathrm{~g}^{\alpha \beta} \mathrm{g}^{\gamma \delta}-\mathrm{g}^{\alpha \delta} \mathrm{g}^{\beta \gamma}\right] \\
& -\mathrm{ig}^{2} \mathrm{f}^{\mathrm{XAD}} \mathrm{f}^{\mathrm{XBC}}\left[\mathrm{~g}^{\alpha \beta} \mathrm{g}^{\gamma \delta}-\mathrm{g}^{\alpha \gamma} \mathrm{g}^{\beta \delta}\right] \\
& -\mathrm{i}^{2} \mathrm{f}^{\mathrm{XAB}} \mathrm{f}^{\mathrm{XCD}}\left[\mathrm{~g}^{\alpha \gamma} \mathrm{g}^{\beta \delta}-\mathrm{g}^{\alpha \delta} \mathrm{g}^{\beta \gamma}\right]
\end{aligned}
$$

+ not shown here: propagators, with colour conserving deltas, couplings to EW bosons


## QCD - Feynman rules



$-\mathrm{g} \mathrm{f}^{\mathrm{ABC}}\left[(\mathrm{p}-\mathrm{q})^{\gamma} \mathrm{g}^{\alpha \beta}+(\mathrm{q}-\mathrm{r})^{\alpha} \mathrm{g}^{\beta \gamma}+(\mathrm{r}-\mathrm{p})^{\beta} \mathrm{g}^{\gamma \alpha}\right]$
(all momenta incoming, $\mathrm{p}+\mathrm{q}+\mathrm{r}=0$ )


$$
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& -\mathrm{ig}^{2} \mathrm{f}^{\mathrm{XAC}} \mathrm{f}^{\mathrm{XBD}}\left[\mathrm{~g}^{\alpha \beta} \mathrm{g}^{\gamma \delta}-\mathrm{g}^{\alpha \delta} \mathrm{g}^{\beta \gamma}\right] \\
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& -i g^{2} f^{X A B} f^{X C D}\left[g^{\alpha \gamma} g^{\beta \delta}-g^{\alpha \delta} g^{\beta \gamma}\right]
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## QCD calculations - soft limit



$$
\sim \bar{u}\left(p_{1}\right)\left(-i g_{s}\right) t^{A} \gamma^{\alpha} \frac{i\left(\not p_{1}+\not k\right)}{\left(p_{1}+k\right)^{2}}(-i e) \gamma^{\mu} v\left(p_{2}\right) \epsilon_{\alpha}
$$

## QCD calculations - soft limit



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& \sim \bar{u}\left(p_{1}\right)\left(-i g_{s}\right) t^{A} \gamma^{\alpha} \frac{i\left(\not p_{1}+\mathbb{k}\right)}{\left(p_{1}+k\right)^{2}}(-i e) \gamma^{\mu} v\left(p_{2}\right) \epsilon_{\alpha} \\
& \sim \bar{u}\left(p_{1}\right)\left(-i g_{s}\right) t^{A} \epsilon \frac{i p_{1}}{2 p_{1} \cdot k}(-i e) \gamma^{\mu} v\left(p_{2}\right)
\end{aligned}
$$

assume massless partons, $p_{1}^{2}=0, k^{2}=0$ and analyse the soft gluon $k \rightarrow 0$ limit

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& \sim \bar{u}\left(p_{1}\right)\left(-i g_{s}\right) t^{A} \frac{i p_{1} \cdot \epsilon}{p_{1} \cdot k}(-i e) \gamma^{\mu} v\left(p_{2}\right)
\end{aligned}
$$

use $\notin p_{1}=2 \epsilon \cdot p_{1}-\not p_{1} \notin$ and the Dirac equation $\bar{u}\left(p_{1}\right) \not p_{1}=0$

## QCD calculations - soft limit



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$$

$$
\sim \bar{u}\left(p_{1}\right)\left(-i g_{s}\right) t^{A} \notin \frac{i p_{1}}{2 p_{1} \cdot k}(-i e) \gamma^{\mu} v\left(p_{2}\right)
$$

$$
\sim \bar{u}\left(p_{1}\right)\left(-i g_{s}\right) t^{A} \frac{i p_{1} \cdot \epsilon}{p_{1} \cdot k}(-i e) \gamma^{\mu} v\left(p_{2}\right)
$$

$$
\sim g_{s} s^{A} \frac{p_{1} \cdot \epsilon}{p_{1} \cdot k} \times
$$

soft gluon emissions factorise!

## QCD calculations - soft limit




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perform sum over gluon polarisations $\epsilon_{\mu} \epsilon_{\nu} \rightarrow-g_{\mu \nu}$, and colours $t^{A} t^{B} \rightarrow C_{F}$


## QCD calculations - soft limit


perform sum over gluon polarisations $\epsilon_{\mu} \epsilon_{\nu} \rightarrow-g_{\mu \nu}$, and colours $t^{A} t^{B} \rightarrow C_{F}$


Note: phase space factorises as well
$d \phi_{q \bar{q} g}=d \phi_{q \bar{q}} d \phi_{+1}$
Factorisation with "eikonal" factor!

## eikonal

$$
\frac{2 p_{1} \cdot p_{2}}{\left(p_{1} \cdot k\right)\left(p_{2} \cdot k\right)}
$$

observation: divergent if $k \| p_{1}$ or $k \| p_{2}$ or $k \rightarrow 0$
$\Rightarrow$ collinear and soft/infrared limits

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Explicitly in some reference frame, use $p_{i} \cdot k=E_{i} E_{k}\left(1-\cos \theta_{i k}\right)$

$$
\frac{2 p_{1} \cdot p_{2}}{\left(p_{1} \cdot k\right)\left(p_{2} \cdot k\right)} \sim \frac{1}{E_{k}^{2}} \frac{1}{\left(1-\cos \theta_{1 k}\right)\left(1-\cos \theta_{2 k}\right)}
$$

$\Rightarrow$ divergencies visible for $\theta_{i k} \rightarrow 0$ (collinear) and $E_{k} \rightarrow 0$ (soft)

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$$

$\Rightarrow$ divergencies visible for $\theta_{i k} \rightarrow 0$ (collinear) and $E_{k} \rightarrow 0$ (soft)
General structure, divergencies and factorisation in the soft and collinear limits is a universal property of QCD amplitudes!

## divergencies ...?

- If amplitudes in QCD are divergent in the infrared, how can we ever calculated meaningful results?


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- Answer: we did not yet consider the full $\mathcal{O}\left(\alpha_{s}\right)$ correction, also have to take into account virtual terms

- on their own divergent as well $\Rightarrow$ sum turns out to be finite!


## divergencies ... ?

- If amplitudes in QCD are divergent in the infrared, how can we ever calculated meaningful results?
- Answer: we did not yet consider the full $\mathcal{O}\left(\alpha_{s}\right)$ correction, also have to take into account virtual terms

- on their own divergent as well $\Rightarrow$ sum turns out to be finite!
- $\Rightarrow$ we can calculate at least inclusive (enough) cross sections (e.g.
$e^{+} e^{-} \rightarrow$ hadrons, but not $e^{+} e^{-} \rightarrow$ exactly 2 quarks)


## summary of pQCD so far

- Feynman rules derived from Lagrangian for $S U(3)$ gauge theory.
- If we attempt to calculate higher order corrections to QCD cross sections, we encounter soft and collinear divergencies.
- These cancel after adding real and virtual corrections, rendering inclusive (enough) cross sections finite.
- Both matrix elements and phase space factorise in these limits $\Rightarrow$ we can think of matrix elements with soft gluons as some hard "core" matrix element with additional emissions of soft gluons


## What is "inclusive enough"?

- We have seen that not all observables are well defined in QCD, since we must not disturb the cancellation of real and virtual singularities
- We must exclude anything that is sensitive to arbitrarily soft and/or collinear emissions
- typical example: multiplicities
- Observables that are not affected by a soft/collinear emission are called infrared-collinear (IRC) safe


## IRC safe observables: Jets

- A typical example for the construction of IRC safe quantities are sequential recombination algorithms used to define jets

1. compute distance measure $d_{i j}$ for each pair of final-state particles and the beam distance $d_{i B}$ for each particle
2. determine minimum of all $d_{i j}, d_{i B}$
A. if one of the $d_{i j}$ is smallest, combine those particles $i, j$
B. if one of the beam distances $d_{i B}$ is smallest, $i$ is a jet and removed from the procedure
3. go back to 1 , repeat until all objects are clustered

## IRC safe observables: Jets

- distance measures are a matter of choice
- only formal requirement is IRC safety, i.e. a soft/collinear emission should not change the jets obtained from the algorithm
$k_{T}$ - algorithm/Durham - algorithm:

$$
\begin{aligned}
& d_{i j}=\min \left(k_{T, i}^{2}, k_{T, j}^{2}\right) \frac{\Delta R^{2}}{R^{2}}, \quad d_{i B}=\min \left(k_{T, i}^{2}, k_{T, j}^{2}\right) \\
& \Delta R_{i j}^{2}=\left(y_{i}-y_{j}\right)^{2}+\left(\phi_{i}-\phi_{j}\right)^{2}
\end{aligned}
$$

## IRC safe observables: Jets

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anti- $k_{T}$ - algorithm:

$$
\begin{aligned}
& d_{i j}=\min \left(k_{T, i}^{-2}, k_{T, j}^{-2}\right) \frac{\Delta R^{2}}{R^{2}}, d_{i B}=\min \left(k_{T, i}^{-2}, k_{T, j}^{-2}\right) \\
& \Delta R_{i j}^{2}=\left(y_{i}-y_{j}\right)^{2}+\left(\phi_{i}-\phi_{j}\right)^{2}
\end{aligned}
$$

## IRC safe observables: Jets

- distance measures are a matter of choice
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Cambridge/Aachen - algorithm:

$$
\begin{aligned}
& d_{i j}=\frac{\Delta R^{2}}{R^{2}}, \quad d_{i B}=1 \\
& \Delta R_{i j}^{2}=\left(y_{i}-y_{j}\right)^{2}+\left(\phi_{i}-\phi_{j}\right)^{2}
\end{aligned}
$$

## IRC safe observables: Jets

- Standard reference for implementations: FastJet program
- Generalised $k_{t}$ - algorithm

$$
\begin{aligned}
& d_{i j}=\min \left(k_{T, i}^{2 p}, k_{T, j}^{2 p}\right) \frac{\Delta R^{2}}{R^{2}}, d_{i B}=\min \left(k_{T, i}^{2}, k_{T, j}^{2}\right) \\
& \Delta R_{i j}^{2}=\left(y_{i}-y_{j}\right)^{2}+\left(\phi_{i}-\phi_{j}\right)^{2}
\end{aligned}
$$

Durham
$p=1$
closest match to structure of QCD matrix elements, theoretical interest

Cambridge/Aachen
$p=0$
angular ordered splitting sequence, close match to QCD coherence

| anti- $k_{t}$ |
| :--- |
| $p=-1$ |
| closest to defining jets as |
| "cones" with radius $R$ around |
| hard particles, default choice |
| in LHC experiments |

## Jet cross sections at the LHC

- jets are infrared save, so we can compute their production cross section in perturbation theory
- powerful tests of perturbative QCD
- here: compared to measurement by ATLAS at $\sqrt{s}=13 \mathrm{TeV}$



## QCD at colliders

- Events factorised into
- Hard Process
- QCD radiation
- PDFs/Beams/Underlying event
- Hadrons



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