O(3) Non-linear Sigma model and Non-Abelian Bosonization duality

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Contents

What is Bosonization?

Ask ChatGPT

What is bosonization?

Bosonization is a method in theoretical physics used to describe interacting fermions (particles like electrons) in one-dimensional systems by transforming them into bosons (particles like photons or phonons). The idea behind bosonization is that certain types of fermion behavior, particularly in lower dimensions, can be more easily understood by treating them as bosons. This method is widely used in the study of condensed matter systems and quantum field theory.

- For fermionic theory $(S_F, \{\mathcal{O}_F\})$, find $(S_B, \{\mathcal{O}_B\})$ such that

$$\langle \mathcal{O}_{F1} \cdots \mathcal{O}_{Fn} \rangle_{S_F} = \langle \mathcal{O}_{B1} \cdots \mathcal{O}_{Bn} \rangle_{S_B}$$

Abelian Bosonization

In D=2

S. Coleman, Phys. Rev. D **11**, 2088

- **1 flavor** free fermion ↔ **1 flavor** free boson

$$S_F = \int \mathrm{d}^2 x \, \bar{\psi} i \gamma_\mu \partial_\mu \psi \quad \Longleftrightarrow \quad S_B = \int \mathrm{d}^2 x \, (\partial_\mu \phi)^2$$

- Operator correspondence

$$\bar{\psi}\psi \longleftrightarrow \mu \cos(\sqrt{4\pi}\phi) \qquad \bar{\psi}\gamma^5\psi \longleftrightarrow \mu \sin(\sqrt{4\pi}\phi)$$

- **U(1) symmetry** in fermion = **shift symmetry** in boson

Non-Abelian Bosonization

In D=2

E. Witten, Commun.Math. Phys. **92**, 455–472

- N **flavor** free fermion ↔ ***U(N)*** Wess-Zumino-Witten theory

$$S_F = \int \mathrm{d}^2x\, \bar{\psi}_i i\gamma_\mu \partial_\mu \psi_i \Longleftrightarrow S_B = \frac{1}{8\pi}\int \mathrm{d}^2x\, \mathrm{Tr}(\partial_\mu g \partial_\mu g^{-1}) - \frac{i}{12\pi}\int_B \mathrm{Tr}(\tilde{g}\mathrm{d}\tilde{g}^{-1})^3$$

- $g, \tilde{g} \in U(N),\ \tilde{g}\,|_{\partial B = \mathbb{R}^2} = g$

- Operator correspondence $\bar{\psi}\psi \longleftrightarrow \mu \mathrm{Tr}(g + g^\dagger)$

- $U(N)_L \times U(N)_R$ symmetry in fermion =

$$g \to \mathrm{g}_L \cdot g \cdot \mathrm{g}_R, \qquad \mathrm{g}_{L,R} \in U(N)$$

in WZW theory.

SU(2) WZW vs O(3) nonlinear σ model

Consider N=2

- $U(2) = U(1) \times SU(2)$
- $U(1)$ part is free boson theory.
- $SU(2)$ part: $SU(2)$ WZW model $\quad g,\ \tilde{g} \in SU(2)$

$$S_B = \frac{1}{8\pi}\int \mathrm{d}^2x\,\mathrm{Tr}(\partial_\mu g \partial_\mu g^{-1}) \ - \frac{i}{12\pi}\int_B \mathrm{Tr}(\tilde{g}\mathrm{d}\tilde{g}^{-1})^3$$

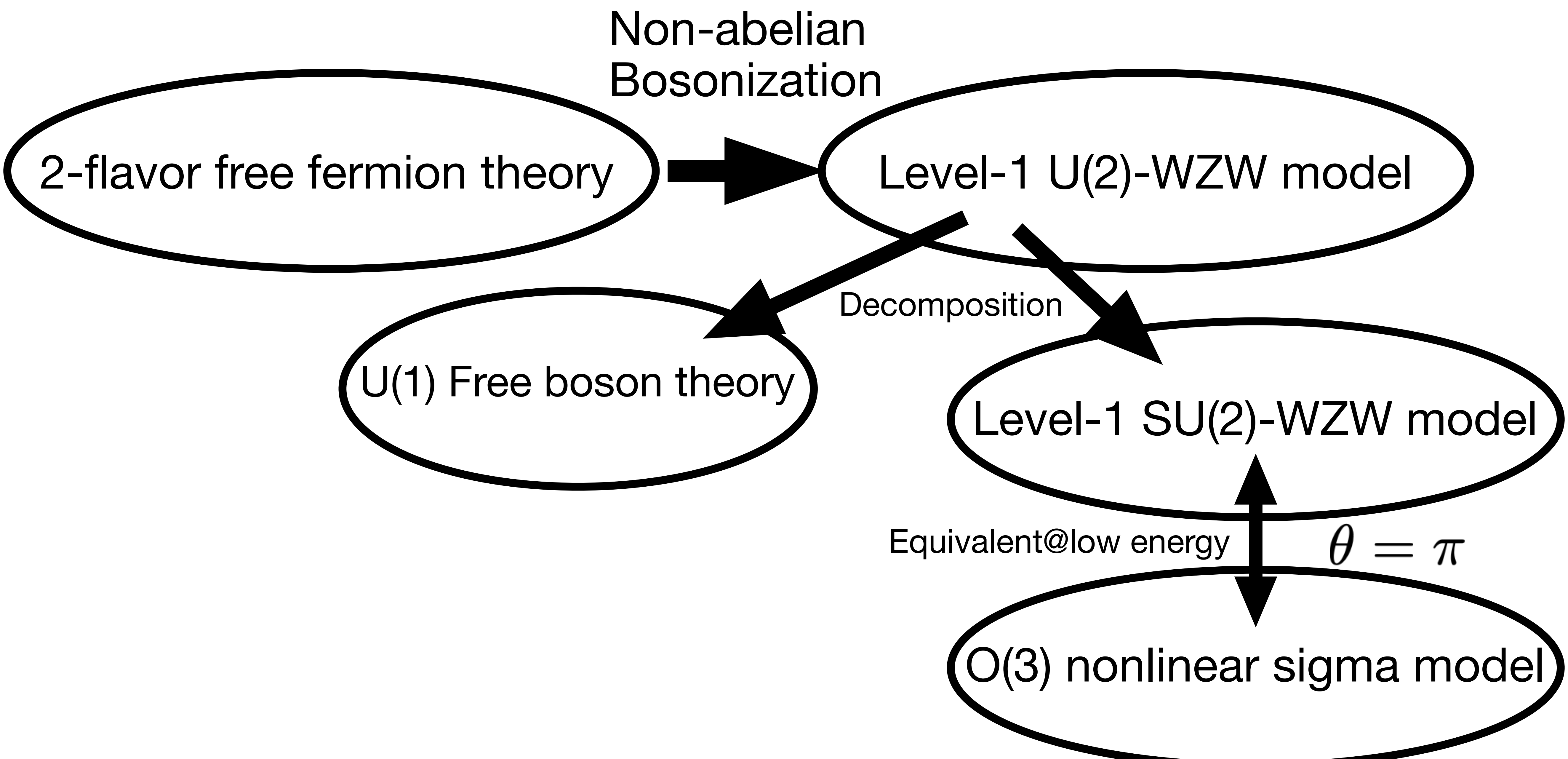
$$g = \exp\left(i\frac{\sigma^i}{2}\phi_i\right) \Big\downarrow \text{in low energy and } \theta = \pi$$

$$S_{\mathrm{NL}+\theta} = \frac{1}{16\pi}\int \mathrm{d}^2x(\partial_\mu\phi_i)^2 + \frac{i\theta}{4\pi}\int \mathrm{d}^2x\varepsilon_{\mu\nu}\varepsilon_{ijk}\phi_i(\partial_\mu\phi_j)(\partial_\nu\phi_k)$$

with $\phi_i\phi_i = 1$

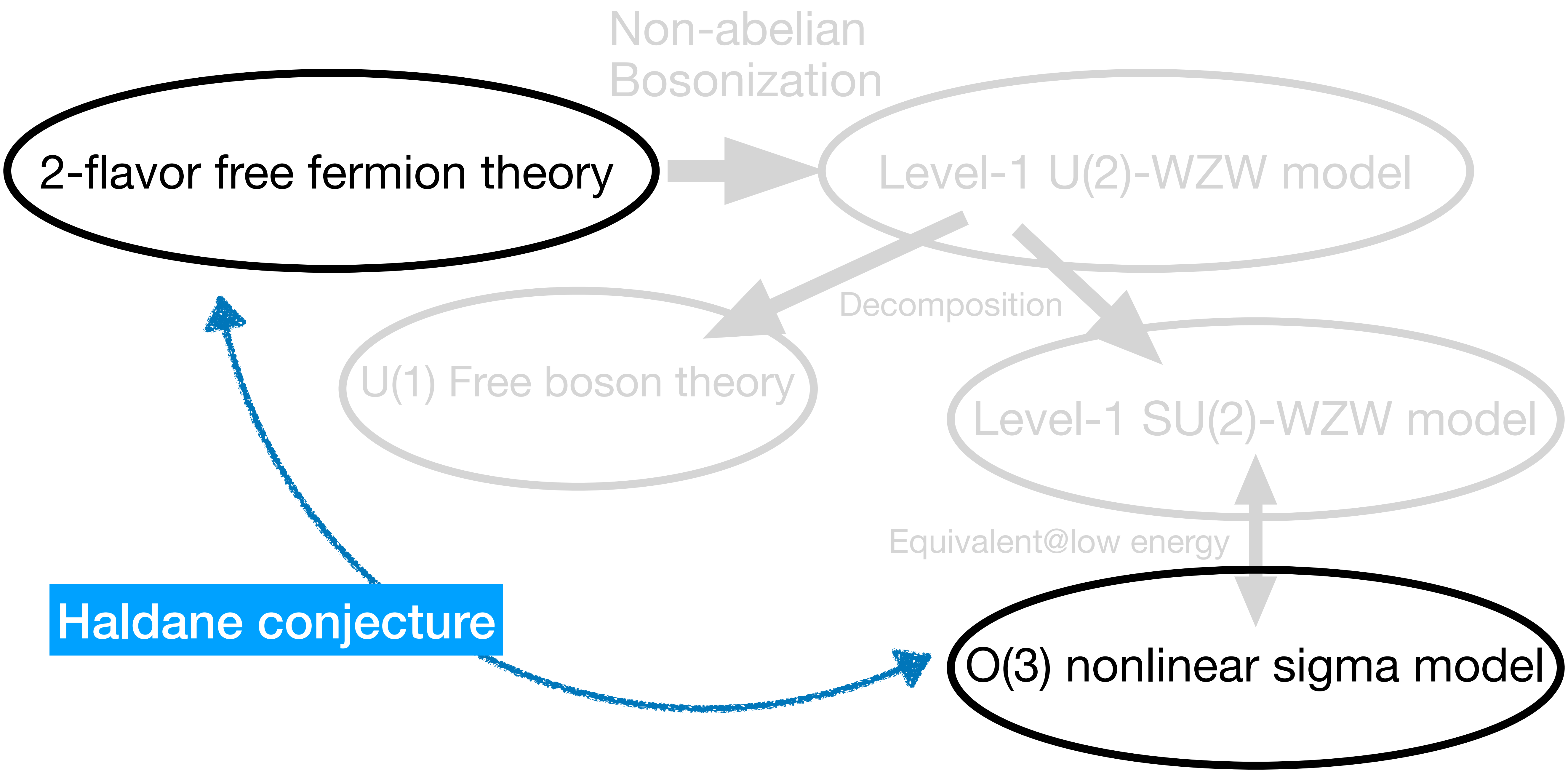
Relation between models with N=2

Whole structure



Relation between models with N=2

Haldane conjecture



How to understand the equivalence?

Free fermion with N=2 vs. O(3) nonlinear sigma model

- They looks so different.
  - **Fermionic side**: no interaction, trivial
  - **Sigma model side**: interactive, nontrivial
- How to understand their equivalence in terms of renormalization group flow?
- If there are equivalent, O(3) nonlinear sigma model should describe free fermion in low energy.
- Perturbative renormalization group analysis was given by Witten.
- This “duality” should be realized nonperturbatively.

Universality class

In terms of critical exponents

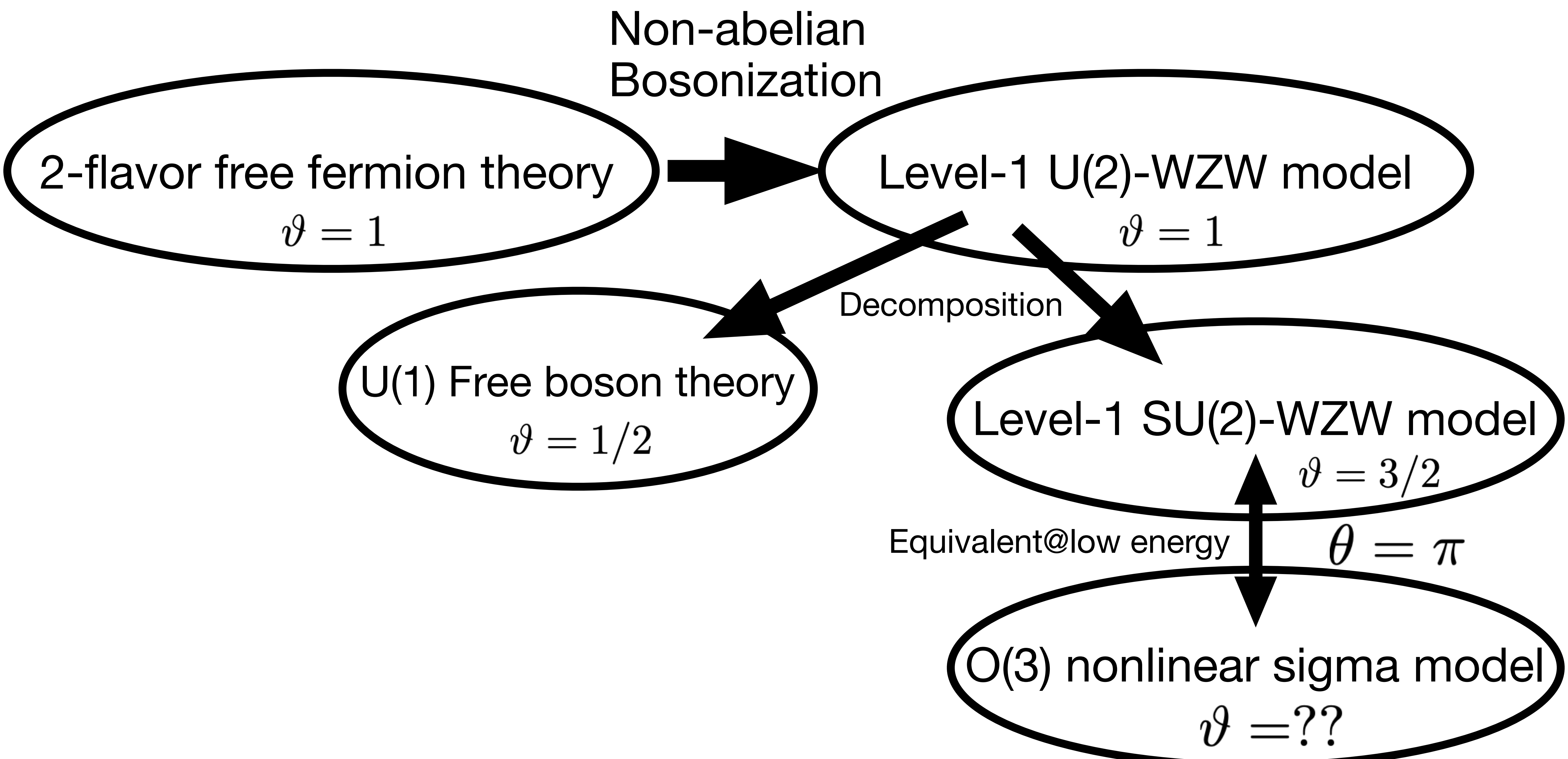
- **Free fermionic theory**
  - Gaussian fixed point
    - One relevant direction with critical exponent=1

Trivial

- **O(3) nonlinear sigma model**
  - At Gaussian fixed point
    - Several relevant directions
  - Is there a nontrivial fixed point?
  - Is the universality class (critical exponents) the same as free fermion case?

Nontrivial

Critical exponents



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Application of FRG for O(3) nonlinear σ model

- We study the O(3) nonlinear σ model with the θ term.

$$S_{\mathrm{NL}+\theta} = \frac{1}{16\pi}\int \mathrm{d}^2x(\partial_\mu\phi_i)^2 + \frac{i\theta}{4\pi}\int \mathrm{d}^2x\varepsilon_{\mu\nu}\varepsilon_{ijk}\phi_i(\partial_\mu\phi_j)(\partial_\mu\phi_k)$$

with $\phi_i\phi_i = 1$

- Purpose: study the universality class

- How to give a truncated effective action?

Application of FRG for O(3) nonlinear σ model

Technical difficulties

1. Target space is spherical S^2: $\phi_i\phi_i = 1$

- How to truncate the effective action with keeping this condition?

2. The θ term is topological.

- No RG flow.

Obstacle 1

Target space is spherical S^2: $\phi_i\phi_i = 1$

- **Possibility 1: Introduction of auxiliary field σ**

$$\int \mathcal{D}\phi\delta\left(\phi_i\phi_i - 1\right)e^{-S[\phi]+J_i\cdot\phi_i} = \int \mathcal{D}\phi\mathcal{D}\sigma e^{-S[\phi]+i\sigma\cdot(\phi_i\phi_i-1)+J_i\cdot\phi_i}$$

  - The auxiliary field becomes dynamical.
  - It becomes additional dynamical degree of freedom.
  - The effective action may be complicated.

Obstacle 1

Target space is spherical S^2: $\phi_i\phi_i = 1$

- **Possibility 2: Introduction of background field**
  - $\phi_i = \varphi_i + \xi_i$ and h_{ab} is spherical metric.

$$\Gamma_k[\varphi, \xi] = \Gamma_k[\varphi] + \frac{\zeta_k}{2}\int d^2x 2h_{ab}\left(\partial_\mu\varphi^a\right)\left(\nabla_\mu\xi^b\right) + \nabla_\mu\xi^a\nabla_\mu\xi_a + R_{abcd}\left(\partial_\mu\varphi^b\right)\left(\partial_\mu\varphi^c\right)\xi^a\xi^d$$
$$+\frac{i}{2\pi}\theta_k\int d^2x\varepsilon_{\mu\nu}\sqrt{h}\varepsilon_{ab}\alpha\left(2\left(\partial_\mu\varphi^a\right)\left(\nabla_\nu\xi^b\right) + \nabla_\mu\xi^a\nabla_\nu\xi^b + R^a_{cde}\left(\partial_\mu\varphi^e\right)\left(\partial_\nu\varphi^b\right)\xi^c\xi^d\right)$$

  - Highly complicated.
  - Truncation may spoil the condition.

Obstacle 2

The θ term is topological.

- The topological term is invariant under infinitesimal deformation.

$$\delta I_{\text{top}} = 0$$

under $\phi_i \to \phi_i + \delta\phi_i$ with $\phi_i(x)\Big|_{x\in\partial M} = 0$

- The functional derivative disappears:

$$\frac{\delta I_{\text{top}}}{\delta\phi_i} = 0$$

Our strategy

To overcome the obstacles

We extend the target space from S^2 to $\mathbb{R}^3$.

- The obstacles originate from target space S^2: $\phi_i\phi_i = 1$
- Extending to $\mathbb{R}^3$, in principle, the issues are resolved.
- It implies there is an additional degrees of freedom.

Fixed

$$\phi_i = (\sigma, \pi_i)$$

- How to justify this extension?
  - On the renormalized trajectory, they should have the same universality class.
  - The additional D.o.F. would decouple in low energy.

Our setup

- Effective action

$$\Gamma_k = \int_x \left[\frac{Z}{2} (\partial_\mu \phi_i)^2 + \frac{m^2}{2} (\phi_i)^2 \right] + i\theta \int_x \varepsilon_{\mu\nu} \varepsilon_{ijk} \phi_i (\partial_\mu \phi_j)(\partial_\nu \phi_k)$$

$$+ \int_x \left[\frac{\lambda_3}{24} (\phi_i \phi_i)^2 + \frac{\lambda_1}{24} (\phi_i)^2 (\partial_\mu \phi_j)^2 + \frac{\lambda_2}{24} (\phi_i \partial_\mu \phi_i)^2 \right]$$

- Litim cutoff function

$$R_k(p) = Z(k^2 - p^2)\theta(k^2 - p^2)$$

Flow equations

$$\beta_{\tilde{m}^2} = -(2-\eta)\tilde{m}^2 - \frac{3\tilde{\lambda}_1 + \tilde{\lambda}_2 + 20\tilde{\lambda}_3}{48\pi(1+\tilde{m}^2)^2}$$

$$\beta_{\tilde{\theta}} = \frac{3}{2}\eta\tilde{\theta} + \frac{\tilde{\theta}(8\tilde{\lambda}_1 - 3\tilde{\lambda}_2)}{32\pi(1+\tilde{m}^2)^3}$$

$$\eta = \frac{1}{2}\left(\frac{3\tilde{\lambda}_1 + \tilde{\lambda}_2}{12\pi(1+\tilde{m}^2)^2} + \frac{3\tilde{\theta}^2}{4\pi(1+\tilde{m}^2)^3}\right)$$

$$\beta_{\tilde{\lambda}_3} = -(2-2\eta)\tilde{\lambda}_3 + \frac{3\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 + 18\tilde{\lambda}_2\tilde{\lambda}_3 + 132\tilde{\lambda}_3^2 + 2\tilde{\lambda}_1\tilde{\lambda}_2 + 30\tilde{\lambda}_1\tilde{\lambda}_3}{72\pi(1+\tilde{m}^2)^3}$$

$$\beta_{\tilde{\lambda}_1} = 2\eta\tilde{\lambda}_1 + \frac{3\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2 + 12\tilde{\lambda}_2\tilde{\lambda}_3 + 2\tilde{\lambda}_1\tilde{\lambda}_2 + 20\tilde{\lambda}_1\tilde{\lambda}_3}{24\pi(1+\tilde{m}^2)^3} - \frac{2\tilde{\lambda}_3^2 - 36\tilde{\lambda}_3\tilde{\theta}^2}{3\pi(1+\tilde{m}^2)^4}$$

$$\beta_{\tilde{\lambda}_2} = 2\eta\tilde{\lambda}_2 + \frac{7\tilde{\lambda}_1^2 + 5\tilde{\lambda}_2^2 + 64\tilde{\lambda}_2\tilde{\lambda}_3 + 14\tilde{\lambda}_1\tilde{\lambda}_2 + 48\tilde{\lambda}_1\tilde{\lambda}_3}{24\pi(1+\tilde{m}^2)^3} - \frac{3\tilde{\lambda}_3^2 + 2\tilde{\lambda}_3\tilde{\theta}^2}{\pi(1+\tilde{m}^2)^4}$$

Contents

Fixed point structure

There exits a nontrivial fixed point.

Fixed Point (FP)	$\tilde{m}_*^2$	$\tilde{\theta}_*$	$\tilde{\lambda}_{1*}$	$\tilde{\lambda}_{2*}$	$\tilde{\lambda}_{3*}$	η^*
(1) Gaussian FP	0	0	0	0	0	0
(2) Nontrivial FP	-0.166948	± 0.0961412	0.168737	1.76901	1.59369	0.0453913

Fixed point structure

Critical exponents at the nontrivial fixed point

Fixed Point (FP)	$\tilde{m}_*^2$	$\tilde{\theta}_*$	$\tilde{\lambda}_{1*}$	$\tilde{\lambda}_{2*}$	$\tilde{\lambda}_{3*}$	η^*
(1) Gaussian FP	0	0	0	0	0	0
(2) Nontrivial FP	-0.166948	± 0.0961412	0.168737	1.76901	1.59369	0.0453913

		Relevant		Irrelevant	
Eigenvalue	ϑ_i	1.8	0.092	$-2.9 \pm 1.1i$	-0.50
Eigenvector	$\begin{pmatrix} \tilde{m}^2 \\ \tilde{\theta} \\ \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \\ \tilde{\lambda}_3 \end{pmatrix}$	$\begin{pmatrix} 1.00 \\ 0.0050 \\ 0.0034 \\ -0.0013 \\ 0.056 \end{pmatrix}$	$\begin{pmatrix} -0.011 \\ 1.0 \\ 0.039 \\ -0.0077 \\ 0.00038 \end{pmatrix}$	$\begin{pmatrix} 0.94 \\ -0.016 \pm 0.071i \\ 0.0010 \mp 0.10i \\ 0.046 \mp 0.087i \\ -0.31 \pm 0.034i \end{pmatrix}$	$\begin{pmatrix} -0.0092 \\ -0.98 \\ 0.20 \\ -0.042 \\ 0.000035 \end{pmatrix}$

Fixed point structure

Relevant direction

Eigenvalue	ϑ_i	1.8
Eigenvector	$\begin{pmatrix} \tilde{m}^2 \\ \tilde{\theta} \\ \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \\ \tilde{\lambda}_3 \end{pmatrix}$	$\begin{pmatrix} 1.00 \\ 0.0050 \\ 0.0034 \\ -0.0013 \\ 0.056 \end{pmatrix}$

- There is a relevant direction.
- This relevant direction may correspond to mass parameter.

Comparison of critical exponents

- $N=2$ free fermion theory: 1 relevant direction $\theta = 1$

$$\Big\downarrow \text{Bosonization} \qquad \bar{\psi}\psi$$

- (Level-1) $U(2)$ WZW model: 1 relevant direction $\theta = 1$

$$\Big\downarrow U(1)\times SU(2) \qquad \mathrm{tr}[g+g^\dagger]$$

- (Level-1) $SU(2)$ WZW model: 1 relevant direction $\theta = 3/2 = 1.5$

- **Our result for O(3) nonlinear σ model: 1 relevant direction** $\theta \simeq 1.8$

Good agreement!

Fixed point structure

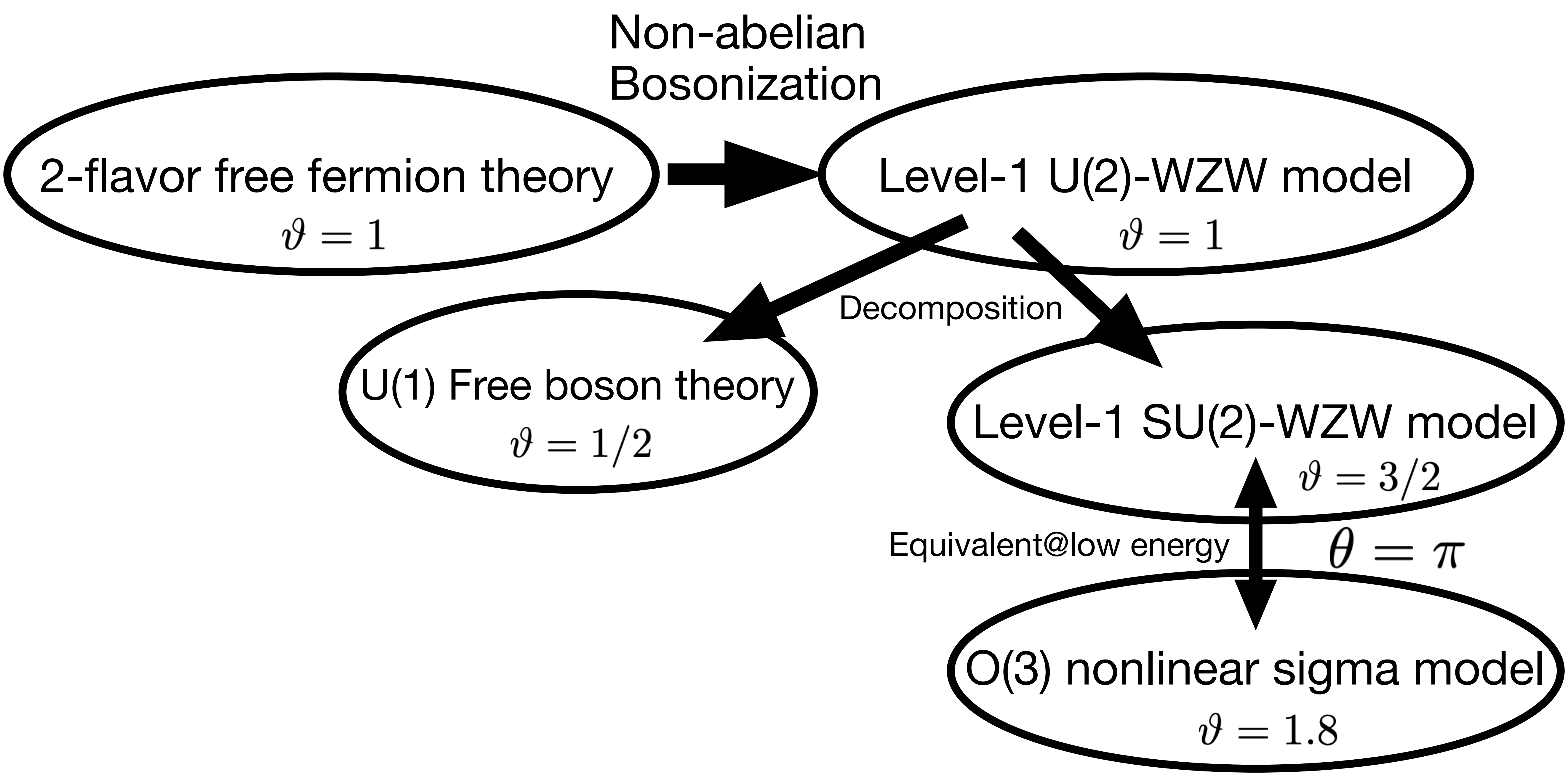
Marginal relevant direction

Eigenvalue	ϑ_i	0.092
Eigenvector	$\begin{pmatrix} \tilde{m}^2 \\ \tilde{\theta} \\ \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \\ \tilde{\lambda}_3 \end{pmatrix}$	$\begin{pmatrix} -0.011 \\ 1.0 \\ 0.039 \\ -0.0077 \\ 0.00038 \end{pmatrix}$

- This marginally relevant direction may correspond to the θ parameter.
- Critical exponent is relatively smaller than others.
- In topological sense, it should be $\theta = 0$ (no running effect).
    - Breaking of topological feature due to cutoff, extension of target space.

Summary

Duality is realized at the fixed point

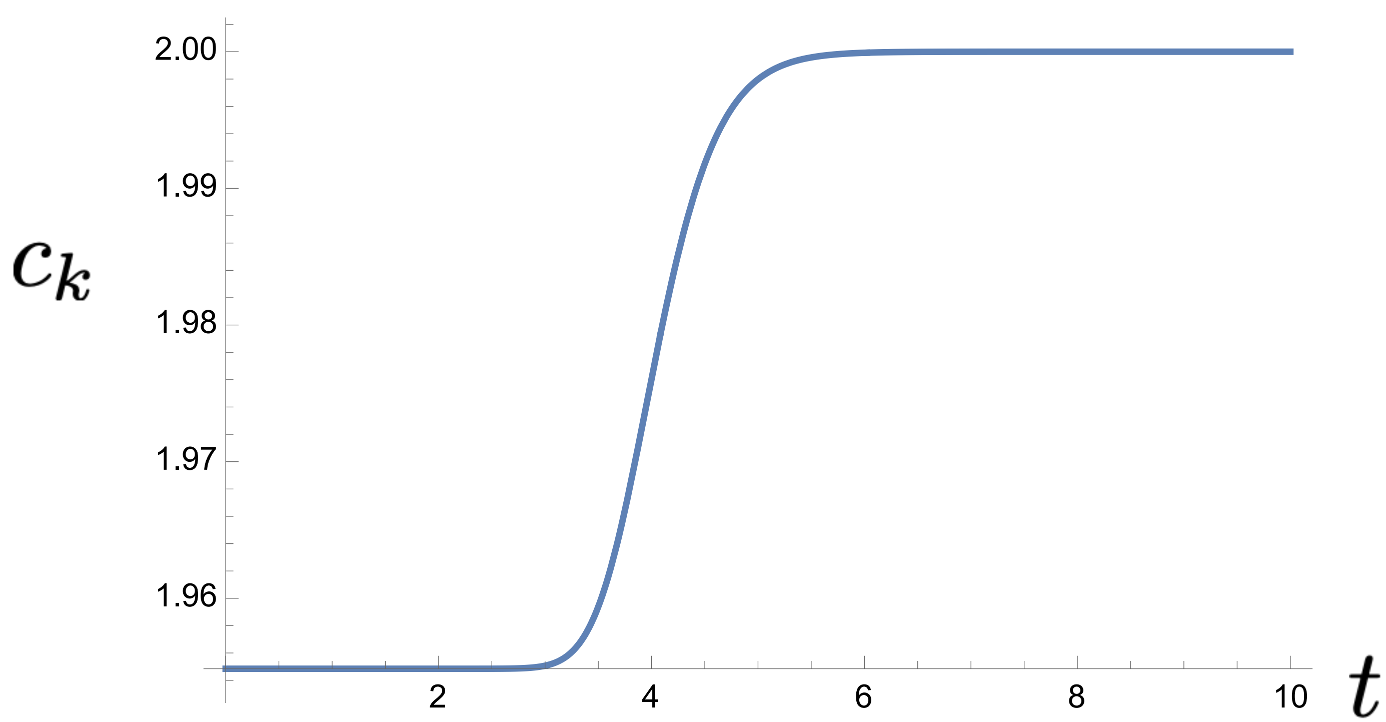


Working in progress

Prospect

RG flow of Central Charge…

- SU(2) WZW model has $c = 1$.
- O(3) nonlinear sigma model should have $c = 1$ at the nontrivial fixed point.
- RG flow of central charge is discussed in [A.Codello et al, JHEP07(2014)040].



$$k\partial_k c_k = \frac{4k^2m^2}{(k^2+m^2)^3}$$

- Not converge to $c = 1$…
- We need to reformulate RG flow equation for c_k.

Prospect

Working in progress

Gravity from Fermion?

- Can we understand a gravitational dynamics from fermion as a low energy effective theory?

  - **SYK model**: one-dimensional fermionic QM

$$S_{\mathrm{SYK}} = \int \mathrm{d}t \left[\frac{1}{2}\chi_i \dot{\chi}_i - \frac{1}{4!}\sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i\chi_j\chi_k\chi_l \right]$$

Cf. R.L. Smit, D. Valentinis, J. Schmalian, and P. Kopietz, Phys. Rev. Res. **3**, 033089 (2021)

  - **Jackiw-Teitelboim (JT) gravity** (2D dilation gravity)

$$S_{\mathrm{JT}} = -\frac{1}{16\pi G^{(2)}} \int \mathrm{d}^2 x \sqrt{g}\phi(R - 2\Lambda)$$

  - $G^{(4)} = 4\pi G^{(2)} \sim J$